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Pricing Swaptions under the LIBOR Market Model of Interest Rates with Local-Stochastic Volatility Models*

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Abstract

This paper presents a new approximation formula for pricing swaptions and caps/floors under the LIBOR market model of interest rates (LMM) with the local and affine-type stochastic volatility.

In particular, two approximation methods are applied in pricing, one of which is so called "drift-freezing" that fixes parts of the underlying stochastic processes at their initial values. Another approximation is based on an asymptotic expansion approach. An advantage of our method is that those approximations can be applied in a unified manner to a general class of local-stochastic volatility models of interest rates.

To demonstrate effectiveness of our method, the paper takes CEV-Heston LMM and Quadratic-Heston LMM as examples; it confirms sufficient flexibility of the models for calibration in a caplet market and enough accuracies of the approximation method for numerical evaluation of swaption values under the models.

Keywords: Asymptotic Expansion, Local Volatility, Affine-type Stochastic Volatility, LIBOR Market Model, Approximation Formula

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1 Introduction

This paper proposes a new analytical approximation method for pricing swaptions under the LIBOR market model of interest rates (LMM) with the local and affine-type stochastic volatility. Especially, our scheme is general enough to be applied in a unified way to a general class of local-stochastic volatility models of interest rates, which is distinct from other existing methods.

After the epoch making papers such as Brace, Gatarek and Musiela (1997) and Jamshidian (1997), LMM with deterministic volatilities has become a standard model in interest rate derivative markets. Thanks to LMM, practitioners can not only obtain consistent prices of ATM caps/floors and exotic interest rate derivatives, but also hedge the exotic derivatives by using ATM caps/floors as Vega hedging tools. Moreover, by joint calibration to cap/floor and swaption markets, traders are able to execute relative value trading between ATM caps/floors and ATM swaptions.

Thereafter, many researchers and practitioners have been trying to develop extended LMMs in order to calibrate them to volatility smiles/skews that are observed in cap/floor and swaption markets. If an extended LMM model can be calibrated to volatility smiles/skews perfectly, exotic derivatives are evaluated consistently with market prices of caps/floors and hedged by caps/floors at different strikes as hedging tools. Many papers focusing on extended LMMs have been published; for instance, LMM with jumps (Glasserman and Kou (2003)), LMM with local volatilities (Andersen and Andreasen (2000)), and LMM with stochastic volatilities (Andersen and Brotherton-Ratcliffe (2001), Piterbarg (2003), Wu and Zhang (2006)).

More recently, SABR-LMM developed by Labordere (2007), Rebonato and White (2007), Rebonato (2007), Hagan and Lesniewski (2008), Mercurio and Morini (2009), and Rebonato, et al. (2009) comes under the spotlight among practitioners. While the original SABR proposed by Hagan, et al. (2002) is a local-stochastic volatility model without the term structure of interest rates, SABR-LMM is a unified model of LMM and SABR. SABR-LMM is gradually getting popularity in practice since the original SABR has been an industry standard for interpolating and extrapolating the prices of plain-vanilla caps/floors and swaptions, and with so called *freezing techniques*, the well-known Hagan's formula can be applied to pricing swaptions as well as caps/floors.

Next, let us recall some features of existing researches for SABR-LMM: The first one is on its volatility modeling. In SABR-LMM, the volatility

process V is given by

$$dV(t) = vV(t)dW_t^Q,$$

where W^Q is a Brownian motion under the spot measure Q and v is constant. However, the process might not be suitable for modeling volatility dynamics because many empirical studies reported that the observed volatility dynamics has mean-reverting property. For example, Rebonato, et al. (2009) pointed out that for pricing exotic derivatives through Monte Carlo simulations, there are some problems for numerical convergence and stability due to the diffusion process of the SABR volatility.

The second one is on the freezing techniques used for derivation of approximation formulas under SABR-LMM. In order to keep the SABR framework even after the change of a numéraire, not only well-known freezing techniques such as "drift-freezing", but also some peculiar freezing techniques are needed. For example, Mercurio and Morini (2009) starts with the volatility process under the forward measure Q^k:

$$dV(t) = -v\mu_0(\gamma(t), k; V)V^2(t)dt + vV(t)dW_t^{Q^k},$$

where W^{Q^k} is a Brownian motion under the forward measure Q^k. Then, for the application of the Hagan's analytical pricing formula, it applies a new freezing method such that $V^2(t)$ in the drift coefficient is changed to $V(0)V(t)$. That is, the volatility process is approximated as

$$dV(t) = -v\mu_0(\gamma(t), k; V)V(0)V(t)dt + vV(t)dW_t^{Q^k}.$$

The third one is related to the flexibility of the existing methods. It seems not easy for the same or similar methods to be applied to extensions or modifications of SABR-LMM; some other special ideas seems necessary for the applications to extended or modified models. For example, many existing works highly rely on the Hagan's SABR formula. On the other hand, the Hagan's formula cannot be directly applied to other types of local-stochastic models such as CEV-Heston LMM and Quadratic-Heston LMM. Also, Labordere (2007) proposed the heat kernel expansion approach to develop the approximation formula for pricing swaptions under SABR-LMM. However, it seems not easy for this approach to be applied except to the one-dimensional stochastic volatility model.

Comparing with the existing models and approximation techniques, our extended LMMs and approximation scheme have the following features:

1. Volatility Modeling: appropriate volatility processes with a mean-reverting property are introduced in the model.

2. Model Flexibility: all parameters are time-dependent, and multi-dimensional stochastic volatility processes can be applied.

3. Generality of Approximation Techniques: a general approximation scheme by an asymptotic expansion with standard freezing techniques is proposed for pricing swaptions and caps/floors; it can be applied to a broad class of the underlying models in a unified manner.

4. Analytical Tractability: the same approximation formulas except concrete specifications of the coefficients can be applied to different models, which is very useful for testing various models, for example in calibration.

The organization of the paper is as follows; the next section describes the basic setup and LMM with the local and affine-type stochastic volatility. It also presents an approximation of swap rate processes. After Section 3 briefly explains the framework of an asymptotic expansion method, Section 4 applies the method to deriving an approximation formula for swaption prices. Section 5 gives numerical examples. Section 6 concludes. Appendix lists up the conditional expectation formulas used in the approximation.

2 LIBOR Market Model with Local and Affine-type Stochastic Volatility

This section introduces a LIBOR market model (LMM) of interest rates with with the local and affine-type stochastic volatility after briefly describing basics on the framework of LMM. Then, it discusses on the changes of numéraires among the equivalent martingale measure (EMM) to the spot, forward and swap measures as well as on the swap rate processes. Moreover, it shows that an appropriate approximation makes LMM with the local and affine-type stochastic volatility included in the same class as before after the changes of measure.

2.1 Basic Setup

This subsection defines basic concepts such as tenor structures, discount bond prices, the money market account (MMA) and forward LIBOR rates. First a tenor structure is given by a finite set of dates:

$$0 = T_0 < T_1 < \cdots < T_N,$$

where $T_i (i = 0, 1, \cdots, N)$ are pre-specified dates. $P_j(t)$ denotes the price of the discount bond with maturity T_j at time t, where $P_j(T_j) = 1$ and $P_j(t) = 0$ for $t \in (T_j, T_N]$.

The forward LIBOR rate at time $t (\leq T_j)$ with the term $[T_{j-1}, T_j]$ is defined as

$$F_j(t) = \frac{1}{\delta_j} \left(\frac{P_{j-1}(t)}{P_j(t)} - 1 \right), \quad \text{for any } j = 1, 2, \ldots, N, \tag{1}$$

where $\delta_j := T_j - T_{j-1}$.

On the other hand, $P_j(T_k)$ is expressed by the forward LIBOR as

$$P_j(T_k) = \prod_{i=k}^{j} \frac{1}{1+\delta_i F_i(T_k)}. \tag{2}$$

The money market account (MMA)'s price $B_d(t)$ is defined as

$$B_d(t) = \frac{P_{\gamma(t)}(t)}{\prod_{j=1}^{\gamma(t)} P_j(T_{j-1})} = P_{\gamma(t)}(t) \prod_{j=1}^{\gamma(t)} \{1+\delta_j F_j(T_{j-1})\}, \tag{3}$$

where $\gamma(t) = \min\{i \in \{1,2,\cdots,N\} : T_i \geq t\}$.

2.2 LMM with Affine-type Local-Stochastic Volatility Model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T_N]}, Q)$ denote a complete probability space satisfying the usual conditions where Q is the spot measure; it stands for uncertainty of the market.

Under the spot measure Q, it is assumed that forward LIBOR $F_j,\ j = 1,2,\ldots,N$ follows a SDE having a unique strong solution;

$$\begin{aligned} dF_j(t) &= \mu_j^Q dt + \phi(t,F_j(t))\sigma_j(t)^{'}\Sigma(t,V(t))dW_t^{Q,1} \\ F_j(0) &= f_j \in (0,\infty), \end{aligned} \tag{4}$$

where $W_t^{Q,1}$ is a D-dimensional Brownian motion under Q, each element of D-dimensional stochastic volatility process $V = (V_1, V_2, \ldots, V_D)^{'}$ is given by

$$\begin{aligned} dV_d(t) &= \{\alpha_{1d}(t) + \alpha_{2d}(t)^{'}V(t)\}dt \\ &\quad + \sum_{l=1}^{2} \theta_{dl}(t)^{'}\Sigma(t,V(t))dW_t^{Q,l}, \\ V_d(0) &= 1,\ d = 1,2,\ldots,D, \end{aligned} \tag{5}$$

$W^Q = (W^{Q,1}, W^{Q,2})^{'}$ is a $2D$-dimensional Brownian motion under Q ($W^{Q,2}$ is a D-dimensional Brownian motion), and μ_j^Q is an appropriate drift term[1] of F_j under Q. Here, $x^{'}$ denotes transpose of vector x.

The matrix $\Sigma(t,x) : [0,\infty) \times \mathbf{R}^D \mapsto \mathbf{R}^{D\times D}$ is assumed to be a diagonal matrix such that its diagonal elements are given by

$$\Sigma_{dd}(t,x) := \sqrt{\beta_{1d}(t) + \beta_{2d}(t)'x},\ \ d = 1,2,\ldots,D,$$

where

$$\begin{aligned} &\beta_{1d}(t) : [0,\infty) \mapsto \mathbf{R} \\ &\beta_{2d}(t) : [0,\infty) \mapsto \mathbf{R}^D \\ &\beta_{1d}(t) + \beta_{2d}(t)^{'}x > 0 \end{aligned}$$

[1]Hereafter, the drift terms $\mu_j^Q,\ j = 1,2,\ldots,N$ will not appear explicitly due to the changes of numéraires.

$\phi(t,x) : [0,\infty) \times \mathbf{R} \mapsto \mathbf{R}$ represents a local volatility function.

In addition to $\beta_{jd}(t)$, $j = 1, 2$, all the other coefficients in the processes are assumed to be deterministic functions of the time parameter:

$$\begin{aligned}
&\sigma_j(t) : [0,\infty) \mapsto \mathbf{R}^D \\
&\alpha_{1d}(t) : [0,\infty) \mapsto \mathbf{R} \\
&\alpha_{2d}(t) : [0,\infty) \mapsto \mathbf{R}^D \\
&\theta_{dl}(t) : [0,\infty) \mapsto \mathbf{R}^D, \quad l = 1, 2.
\end{aligned}$$

Therefore, our extended LMM has a D-dimensional and mean-reverting stochastic volatility combined with a local volatility, and this model is equipped with time-dependent parameters.

Next, we apply the method of *the change of a numéraire* to our setting. First, we recall the measure-change from the spot measure Q to a *forward measure*: Under the forward measure Q^k where the numéraire is the discount bond $P_k(t)$, $W^k = (W^{k,1}, W^{k,2})'$ is a $2D$-dimensional Brownian motion given as

$$W_t^{k,1} = W_t^{Q,1} + \int_0^t \Sigma(s, V_s)\mu(s, \gamma(s), k)ds, \tag{6}$$

and

$$W_t^{k,2} = W_t^{Q,2}, \tag{7}$$

where $\mu(t, \gamma(t), k)$ is a $\mathbf{R}^D$-valued process defined by

$$\mu(t, \gamma(t), k) := \sum_{i=\gamma(t)+1}^{k} \frac{\delta_i \phi(t, F_i(t))}{1 + \delta_i F_i(t)} \sigma_i(t). \tag{8}$$

Next, we apply the measure-change from the spot measure Q to an *annuity measure*: Under the annuity measure $Q^{(a,b)}$ where the numéraire is the annuity $\mathcal{N}^{(a,b)}(t) = \sum_{i=a+1}^{b} \delta_i P_i(t)$, $W^k = (W^{(a,b),1}, W^{(a,b),2})'$ is a $2D$-dimensional Brownian motion given as

$$W_t^{(a,b),1} = W_t^{Q,1} + \int_0^t \Sigma(s, V_s)\mu^{(a,b)}(s, \gamma(s))ds, \tag{9}$$

and

$$W_t^{(a,b),2} = W_t^{Q,2}. \tag{10}$$

Here, a $\mathbf{R}^D$-valued process $\mu^{(a,b)}(t, \gamma(t))$ is defined by

$$\mu^{(a,b)}(t, \gamma(t)) := \sum_{k=a+1}^{b} w_k^{(a,b)}(t)\mu(t, \gamma(t), k), \tag{11}$$

$$w_k^{(a,b)}(t) := \frac{\delta_k P_k(t)}{\sum_{i=a+1}^{b} \delta_i P_i(t)}. \tag{12}$$

We will derive a swap rate dynamics under the annuity measure. Note first that a time-t forward swap rate $S_{a,b}(t)$ with effective date T_a and terminate date T_b is given by

$$S_{a,b}(t) = \sum_{j=a+1}^{b} \frac{\delta_j P_j(t)}{\sum_{i=a+1}^{b} \delta_i P_i(t)} F_j(t) = \sum_{j=a+1}^{b} w_j^{(a,b)}(t) F_j(t), \tag{13}$$

Thus, under the annuity measure $Q^{(a,b)}$ its dynamics follows a stochastic differential equation (SDE):

$$dS_{a,b}(t) = \phi(t, S_{a,b}(t)) \sum_{j=a+1}^{b} \lambda_j^{(a,b)}(t) \sigma_j(t)' \Sigma(t, V(t)) dW_t^{(a,b),1}, \tag{14}$$

where

$$\lambda_j^{(a,b)}(t) := \frac{\partial S_{a,b}(t)}{\partial F_j(t)} \frac{\phi(t, F_j(t))}{\phi(t, S_{a,b}(t))}, \tag{15}$$

and

$$\frac{\partial S_{a,b}(t)}{\partial F_j(t)} = w_j^{(a,b)}(t) + \frac{\delta_j}{1 + \delta_j F_j(t)} \left[\sum_{l=a}^{j-1} w_l^{(a,b)}(t) \{F_l(t) - S_{a,b}(t)\} \right], \quad a+1 \le j \le b. \tag{16}$$

Also, under the annuity measure $Q^{(a,b)}$, each element of stochastic volatility $V = (V_1, V_2, \ldots, V_D)$ is given by

$$\begin{aligned} dV_d(t) &= \{\eta_{1d}^{(a,b)}(t) + \eta_{2d}^{(a,b)}(t)' V(t)\} dt \\ &\quad + \sum_{l=1}^{2} \theta_{dl}(t)' \Sigma(t, V(t)) dW_t^{(a,b),l}, \\ V_d(0) &= 1, \ d = 1, 2, \ldots, D, \end{aligned} \tag{17}$$

where

$$\eta_{1d}^{(a,b)}(t) := \alpha_{1d}(t) - [\theta_{d1}(t)' \mu^{(a,b)}(t, \gamma(t))] \beta_{1d}(t) \tag{18}$$

$$\eta_{2d}^{(a,b)}(t) := \alpha_{2d}(t) - [\theta_{d1}(t)' \mu^{(a,b)}(t, \gamma(t))] \beta_{2d}(t). \tag{19}$$

Although an asymptotic expansion technique introduced in the next section can be directly applied to the above equations for an approximation of swaption prices, this paper will derive a simpler analytical approximation formula. Hence, before the application of the asymptotic expansion method, the so called *freezing technique* is used for the swap rate dynamics. That is, the variation of $\lambda_j^{(a,b)}(t)$ is so small that the standard freezing technique is applied to $\lambda_j(t)$ such that

$$\lambda_j^{(a,b)}(t) \approx \lambda_j^{(a,b),0}(t) := \frac{\partial S_{a,b}(0)}{\partial F_j(0)} \frac{\phi(t, F_j(0))}{\phi(t, S_{a,b}(0))}. \tag{20}$$

More precisely, $\lambda_j^{(a,b)}(t)$ is approximated as

$$\begin{aligned}\lambda_j^{(a,b),0}(t) &= \left\{ w_j^{(a,b)}(0) + \frac{\delta_j}{1+\delta_j F_j(0)} \left[\sum_{l=a}^{j-1} w_l^{(a,b)}(0)\{F_l(0) - S_{a,b}(0)\} \right] \right\} \\ &\quad \times \frac{\phi(t, F_j(0))}{\phi(t, S_{a,b}(0))}. \end{aligned} \tag{21}$$

Therefore, the approximated swap rate process is obtained as

$$dS_{a,b}(t) \approx \phi(t, S_{a,b}(t)) \bar{\sigma}^{(a,b)}(t)' \Sigma(t, V(t)) dW_t^{(a,b),1}, \tag{22}$$

where $\bar{\sigma}^{(a,b)}(t)$ is a $\mathbf{R}^D$-valued deterministic process given by

$$\bar{\sigma}^{(a,b)}(t) := \sum_{j=a+1}^{b} \lambda_j^{(a,b),0}(t) \sigma_j(t). \tag{23}$$

Moreover, the standard freezing technique is also applied to the stochastic volatility process. That is, set

$$\begin{aligned}\mu^{(a,b)}(t, \gamma(t)) &\approx \mu_0^{(a,b)}(t, \gamma(t)) \\ &= \sum_{k=a+1}^{b} w_k^{(a,b)}(0) \sum_{i=\gamma(t)+1}^{k} \frac{\delta_i \phi(t, F_i(0))}{1+\delta_i F_i(0)} \sigma_i(t), \end{aligned} \tag{24}$$

and hence, $\mu_0^{(a,b)}(t, \gamma(t))$ becomes a $\mathbf{R}^D$-valued deterministic process. Then, the approximated stochastic volatility process is obtained as

$$dV_d(t) \approx \{\eta_{1d}(t) + \eta_{2d}(t)' V(t)\} dt + \sum_{l=1}^{2} \theta_{dl}(t)' \Sigma(t, V(t)) dW_t^{(a,b),l}, \tag{25}$$

where $\eta_{jd}(t)$, $j = 1, 2$ also become deterministic processes:

$$\eta_{1d}(t) := \alpha_{1d}(t) - [\theta_{d1}(t)' \mu_0^{(a,b)}(t, \gamma(t))] \beta_{1d}(t) \tag{26}$$

$$\eta_{2d}(t) := \alpha_{2d}(t) - [\theta_{d1}(t)' \mu_0^{(a,b)}(t, \gamma(t))] \beta_{2d}(t). \tag{27}$$

In sum, the approximated swap rate process is re-written as follows:

$$dS_{a,b}(t) = \phi(t, S_{a,b}(t)) \bar{\sigma}^{(a,b)}(t)' \Sigma(t, V(t)) dW_t^{(a,b),1}, \tag{28}$$

$S_{a,b}(0)$ given,

$$\begin{aligned} dV_d(t) &= \{\eta_{1d}(t) + \eta_{2d}(t)' V(t)\} dt \\ &\quad + \sum_{l=1}^{2} \theta_{dl}(t)' \Sigma(t, V(t)) dW_t^{(a,b),l}, \end{aligned} \tag{29}$$

$V_d(0) = 1, \ d = 1, 2, \cdots, D.$

In particular, when $V(t)$ is one dimensional, we are able to re-express the forward swap rate process by using a one-dimensional Brownian motion W^1 independent of $W^{(a,b),2}$ and a one-dimensional deterministic process $\sigma^{a,b}(t)$ as

$$dS_{a,b}(t) = \phi(t, S_{a,b}(t))\sqrt{\beta_1(t) + \beta_2(t)V(t)}\sigma^{(a,b)}(t)dW_t^1 \tag{30}$$

$$dV(t) = \{\eta_1(t) + \eta_2(t)V(t)\}dt + \sqrt{\beta_1(t) + \beta_2(t)V(t)}[\theta_1(t)dW_t^1 + \theta_2(t)dW_t^2], \tag{31}$$

where

$$W_t^1 = \frac{1}{\|\bar{\sigma}^{(a,b)}(t)\|^2}\int_0^t \bar{\sigma}^{(a,b)}(t)' dW_t^{(a,b),1}, \tag{32}$$

$$W_t^2 = W_t^{(a,b),2}, \tag{33}$$

$$\sigma^{(a,b)}(t) = \|\bar{\sigma}^{(a,b)}(t)\|. \tag{34}$$

3 Asymptotic Expansion Method in a General Markovian Setting

This section briefly describes an asymptotic expansion method in a general Markovian setting, which will be applied to the derivation of swaption prices under the approximated swap rate process above in the next section. See Takahashi (1999), Kunitomo and Takahashi (2003) and references therein for the detail of the theory and applications of the method from finance perspective. Also, see Takahashi, et al. (2009) for the detail of its computational aspect.

Let (Z, P) be the r-dimensional Wiener space. We consider a d-dimensional diffusion process $X_t^{(\epsilon)} = (X_t^{(\epsilon),1}, \cdots, X_t^{(\epsilon),d})$ which is the solution to the following stochastic differential equation:

$$dX_t^{(\epsilon)} = V_0(X_t^{(\epsilon)})dt + \epsilon V(X_t^{(\epsilon)})dZ_t;\ X_0^{(\epsilon)} = x_0,\ t \in [0, T], \tag{35}$$

where $Z = (Z^1, \cdots, Z^m)$ is a m-dimensional Brownian motion and $\epsilon \in [0, 1]$ is a known parameter. Also, $V_0 : \mathbf{R}^d \mapsto \mathbf{R}^d$, $V : \mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^m$ satisfy some regularity conditions.(e.g. V_0 and V are smooth functions with bounded derivatives of all orders.)

Next, suppose that a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders. Then, a smooth Wiener functional $g(X_T^{(\epsilon)})$ has its asymptotic expansion;

$$g(X_T^{(\epsilon)}) = g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \epsilon^3 g_{3T} + o(\epsilon^3). \tag{36}$$

in L^p for every $p > 1$(or in $\mathbf{D}^\infty$) as $\epsilon \downarrow 0$. The coefficients in the expansion $g_{nT} \in \mathbf{D}^\infty (n = 0, 1, \cdots)$ can be obtained by Taylor's formula and represented based on multiple Wiener-Itô integrals. Here, $\mathbf{D}^\infty$ denotes the set of

smooth Wiener functionals. See chapter V of Ikeda and Watanabe (1989) for the detail.

In particular, let $D_t = \frac{\partial X_t^{(\epsilon)}}{\partial \epsilon}|_{\epsilon=0}$, $E_t = \frac{\partial^2 X_t^{(\epsilon)}}{\partial \epsilon^2}|_{\epsilon=0}$ and $F_t = \frac{\partial^3 X_t^{(\epsilon)}}{\partial \epsilon^3}|_{\epsilon=0}$. Then, g_{0T}, g_{1T}, g_{2T} and g_{3T} are expressed as follows:

$$g_{0T} = g(X_T^{(0)}),\ g_{1T} = \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) D_T^i, \tag{37}$$

$$g_{2T} = \frac{1}{2}\sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^{(0)}) D_T^i D_T^j + \frac{1}{2}\sum_{i=1}^{d} \partial_i g(X_T^{(0)}) E_T^i, \tag{38}$$

$$\begin{aligned} g_{3T} = {} & \frac{1}{6}\sum_{i,j,k=1}^{d} \partial_i \partial_j \partial_k g(X_T^{(0)}) D_T^i D_T^j D_T^k + \frac{1}{2}\sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^{(0)}) E_T^i D_T^j \\ & + \frac{1}{6}\sum_{i=1}^{d} \partial_i g(X_T^{(0)}) F_T^i, \end{aligned} \tag{39}$$

where D_t^i, E_t^i and F_t^i, $(i = 1, \cdots, d)$ denote the i-th element of D_t, E_t and F_t, respectively. D_t, E_t and F_t are given by

$$D_t = \int_0^t Y_t Y_u^{-1} V(X_u^{(0)}) dZ_u, \tag{40}$$

$$\begin{aligned} E_t = {} & \int_0^t Y_t Y_u^{-1} \left(\sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)}) D_u^j D_u^k du \right. \\ & \left. + 2\sum_{j=1}^{d} \partial_j V(X_u^{(0)}) D_u^j dZ_u \right), \end{aligned} \tag{41}$$

$$\begin{aligned} F_t = {} & \int_0^t Y_t Y_u^{-1} \left(\sum_{j,k,l=1}^{d} \partial_j \partial_k \partial_l V_0(X_u^{(0)}, 0) D_u^j D_u^k D_u^l du \right. \\ & + 3\sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)}, 0) E_u^j D_u^k du + 3\sum_{j,k=1}^{d} \partial_j \partial_k V(X_u^{(0)}) D_u^j D_u^k dZ_u \\ & \left. + 3\sum_{j=1}^{d} \partial_j V(X_u^{(0)}) E_u^j dZ_u \right). \end{aligned} \tag{42}$$

Here, Y is the solution to the following ordinary differential equation:

$$dY_t = \partial V_0(X_t^{(0)}) Y_t dt;\ Y_0 = I_d,$$

where ∂V_0 is a $d \times d$ matrix whose (j,k) element is given by $\partial_k V_0^j$. ($\partial_k = \frac{\partial}{\partial x_k}$, V_0^j denotes the j-th component of V_0.) Also, I_d represents the $d \times d$ identity matrix.

Next, normalize $g(X_T^{(\epsilon)})$ to

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}. \tag{43}$$

Moreover, let

$$a_t = a_t^{(0)} = (\partial g(X_T^{(0)}))' [Y_T Y_t^{-1} V(X_t^{(0)})],$$

and make an assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T a_t a_t' dt > 0. \tag{44}$$

Note that Σ_T is the variance of a random variable g_{1T} following a normal distribution. Thus, (Assumption 1) means that the distribution of g_{1T} does not degenerate.

Then, $\psi_{G^{(\epsilon)}}(\xi)$, the characteristic function of $G^{(\epsilon)}$ is approximated as

$$\begin{aligned}
\psi_{G^{(\epsilon)}}(\xi) &= \mathbf{E}[\exp(i\xi G^{(\epsilon)})] \\
&= \mathbf{E}[\exp(i\xi g_{1T})] + \epsilon(i\xi)\mathbf{E}[\exp(i\xi g_{1T}) g_{2T}] \\
&\quad + \epsilon^2 (i\xi)\mathbf{E}[\exp(i\xi g_{1T}) g_{3T}] + \frac{\epsilon^2}{2}(i\xi)^2 \mathbf{E}[\exp(i\xi g_{1T}) g_{2T}^2] + o(\epsilon^2) \\
&= \exp\left(\frac{(i\xi)^2 \Sigma_T}{2}\right) + \epsilon(i\xi)\mathbf{E}\left[\exp(i\xi g_{1T})\mathbf{E}[g_{2T}|g_{1T}]\right] \\
&\quad + \epsilon^2 (i\xi)\mathbf{E}\left[\exp(i\xi g_{1T})\mathbf{E}[g_{3T}|g_{1T}]\right] \\
&\quad + \frac{\epsilon^2}{2}(i\xi)^2 \mathbf{E}\left[\exp(i\xi g_{1T})\mathbf{E}[g_{2T}^2|g_{1T}]\right] + o(\epsilon^2),
\end{aligned} \tag{45}$$

where $\mathbf{E}[g_{2T}|g_{1T}]$, $\mathbf{E}[g_{2T}^2|g_{1T}]$ and $\mathbf{E}[g_{3T}|g_{1T}]$ become some polynomials of g_{1T}.

Hence, the inversion of the approximated characteristic function provides an approximation of the density function of G^ϵ, f_{G^ϵ}:

$$\begin{aligned}
f_{G^\epsilon} &= n[x; 0, \Sigma_T] + \epsilon\left[-\frac{\partial}{\partial x}\{h_2(x) n[x; 0, \Sigma_T]\}\right] \\
&\quad + \epsilon^2\left[-\frac{\partial}{\partial x}\{h_3(x) n[x; 0, \Sigma_T]\}\right] \\
&\quad + \frac{1}{2}\epsilon^2\left[\frac{\partial^2}{\partial x^2}\{h_{22}(x) n[x; 0, \Sigma_T]\}\right] + o(\epsilon^2),
\end{aligned} \tag{46}$$

where $h_2(x) = \mathbf{E}[g_{2T}|g_{1T} = x]$, $h_{22}(x) = \mathbf{E}[g_{2T}^2|g_{1T} = x]$, $h_3(x) = \mathbf{E}[g_{3T}|g_{1T} = x]$. Also, $n[x; 0, \Sigma_T]$ represents the density function of a normal distribution with mean 0 and variance Σ_T:

$$n[x; 0, \Sigma_T] = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left\{\frac{-x^2}{2\Sigma_T}\right\}. \tag{47}$$

Let $\phi : \mathbf{R} \mapsto \mathbf{R}$ a smooth function of which all derivatives have polynomial growth orders. Then, the expectation $\mathbf{E}[\phi(G^{(\epsilon)})I_B(G^{(\epsilon)})]$ has an asymptotic expansion with respect to ϵ:

$$\mathbf{E}[\phi(G^{(\epsilon)})I_B(G^{(\epsilon)})] = \Phi_0 + \epsilon\Phi_1 + \epsilon^2\Phi_2 + o(\epsilon^2), \tag{48}$$

where $\mathcal{B}$ stands for a Borel set on $\mathbf{R}$. $I_B(G^{(\epsilon)}) = 1$ when $G^{(\epsilon)} \in \mathcal{B}$ and $I_B(G^{(\epsilon)}) = 0$ otherwise.

Especially, Φ_0, Φ_1, Φ_2 are obtained by

$$\Phi_0 = \int_B \phi(x)n[x; 0, \Sigma_T]dx, \tag{49}$$

$$\Phi_1 = -\int_B \phi(x)\partial_x\{\mathbf{E}[g_{2T}|g_{1T} = x]n[x; 0, \Sigma_T]\}dx, \tag{50}$$

$$\begin{aligned}\Phi_2 = \int_B \Big(&\frac{1}{2}\phi(x)\partial_x^2\{\mathbf{E}[g_{2T}^2|g_{1T} = x]n[x; 0, \Sigma_T]\} \\ &-\phi(x)\partial_x\{\mathbf{E}[g_{3T}|g_{1T} = x]n[x; 0, \Sigma_T]\}\Big)dx.\end{aligned} \tag{51}$$

Finally, when the underlying asset value at maturity T and the strike price are given by $g(X_T^\epsilon)$ and $K = g(X_T^0) - \epsilon y$ for an arbitrary $y \in \mathbf{R}$, respectively the payoff of the call option is expressed as

$$\max\{g(X_T^\epsilon) - K, 0\} = \epsilon\phi(G^\epsilon)I_B(G^\epsilon), \tag{52}$$

where $\phi(x) = (x + y)$ and $B = \{G^\epsilon \geq -y\}$.

Remark 1. $\mathbf{E}[g_{2T}|g_{1T} = x]$, $\mathbf{E}[g_{2T}^2|g_{1T} = x]$, $\mathbf{E}[g_{3T}|g_{1T} = x]$ *are some polynomial functions of* x *and those conditional expectations are evaluated by the formulas in Appendix.*

4 Approximation Formula of Swaption Price

Given the approximated swap rate process in Section 2, this section derives an analytical approximation formula for swaption prices by using the asymptotic expansion technique.

For simplicity, let us consider a swap rate process with parameter $\epsilon(\in [0, 1])$ under one-dimensional stochastic volatility environment described as the models (30) and (31) in Section 2. However, even for the case of multi-dimensional stochastic volatility models, the swaption pricing formula can be derived in the similar manner as in the one-dimensional model.

The forward swap rate model under our asymptotic expansion setting is given as follows:

$$dS_{a,b}^{(\epsilon)}(t) = \epsilon\phi(t, S_{a,b}^{(\epsilon)}(t))\sigma^{(a,b)}(t)\sqrt{\beta_1(t) + \beta_2(t)V^{(\epsilon)}(t)}dW_t^1, \tag{53}$$

$$\begin{aligned}dV^{(\epsilon)}(t) = &\{\eta_1(t) + \eta_2(t)V^{(\epsilon)}(t)\}dt \\ &+ \epsilon\sum_{l=1}^{2}\theta_l(t)\sqrt{\beta_1(t) + \beta_2(t)V^{(\epsilon)}(t)}dW_t^l,\end{aligned} \tag{54}$$

where (W^1, W^2) is a two-dimensional Brownian motion, $\theta_1(t) = \rho^{(a,b)}(t)\theta(t)$, $\theta_2(t) = \sqrt{1-\rho^{(a,b)}(t)^2}\theta(t)$ and $\rho^{(a,b)}(t) \in [-1,1]$ denotes the correlation between $S^{(\epsilon)}_{a,b}(t)$ and $V(t)$.

Then, based on the discussion in Section 3, the swap rate process $S^{(\epsilon)}_{a,b}(T)$ and variance process $V^{(\epsilon)}(T)$ described by (53) have asymptotic expansions;

$$\begin{aligned} S^{(\epsilon)}_{a,b}(T) &\sim S^{(0)}_{a,b}(T) + \epsilon S^{(1)}_{a,b}(T) + \epsilon^2 S^{(2)}_{a,b}(T) + \epsilon^3 S^{(3)}_{a,b}(T) + \cdots, \quad (55) \\ V^{(\epsilon)}(T) &\sim V^{(0)}(T) + \epsilon V^{(1)}(T) + \epsilon^2 V^{(2)}(T) + \epsilon^3 V^{(3)}(T) + \cdots, \end{aligned}$$

as $\epsilon \downarrow 0$, where the coefficients in the expansions are given by the next proposition. $S^{(0)}_{a,b}(T)$, $S^{(1)}_{a,b}(T)$, $S^{(2)}_{a,b}(T)$ and $S^{(3)}_{a,b}(T)$ correspond to $X^{(0)}_T$, D_T, E_T and F_T in Section 3, respectively.

Proposition 1. *The coefficients $S^{(0)}_{a,b}(T)$, $S^{(1)}_{a,b}(T)$, $S^{(2)}_{a,b}(T)$ and $S^{(3)}_{a,b}(T)$ in (55) are given by:*

$$S^{(0)}_{a,b}(T) = S_{a,b}(0), \quad S^{(1)}_{a,b}(T) = \int_0^T f_{11}(s)' dW_s, \quad (56)$$

$$S^{(2)}_{a,b}(T) = \sum_{k=1}^{2} \int_0^T \int_0^s f_{2k}(u)' dW_u \; g_{2k}(s)' dW_s, \quad (57)$$

$$\begin{aligned} S^{(3)}_{a,b}(T) &= \sum_{k=1}^{3} \int_0^T \int_0^s \int_0^u f_{3k}(v)' dW_v \; g_{3k}(u)' dW_u \; h_{3k}(s)' dW_s \\ &\quad + \sum_{k=1}^{3} \int_0^T \left(\int_0^s g_{4k}(u)' dW_u \right) \left(\int_0^s f_{4k}(u)' dW_u \right) h_{4k}(s)' dW_s. \end{aligned} \quad (58)$$

Here, integrands f, g, h above are obtained as follows:

$$\begin{aligned} f_{11}(t) &= f_{21}(t) = f_{31}(t) = f_{41}(t) = g_{41}(t) = g_{42}(t) \\ &= \begin{pmatrix} \sigma(t)\sqrt{\beta_1(t) + \beta_2(t)V^{(0)}(t)}\phi(t, S(0)) \\ 0 \end{pmatrix}, \quad (59) \\ f_{22}(t) &= f_{32}(t) = f_{33}(t) = f_{42}(t) = f_{43}(t) = g_{43}(t) \\ &= \begin{pmatrix} \theta_1(t) e^{-\int_0^t \eta_2(s)ds} \sqrt{\beta_1(t) + \beta_2(t)V^{(0)}(t)} \\ \theta_2(t) e^{-\int_0^t \eta_2(s)ds} \sqrt{\beta_1(t) + \beta_2(t)V^{(0)}(t)} \end{pmatrix}, \quad (60) \\ g_{21}(t) &= g_{31}(t) = h_{31}(t) = 2h_{32}(t) \\ &= \begin{pmatrix} \sigma(t)\sqrt{\beta_1(t) + \beta_2(t)V^{(0)}(t)}\partial\phi(t, S(0)) \\ 0 \end{pmatrix}, \quad (61) \end{aligned}$$

$$2g_{22}(t) = g_{32}(t) = 4h_{33}(t) = \begin{pmatrix} \frac{\sigma(t)\beta_2(t)e^{\int_0^t \eta_2(s)ds}}{\sqrt{\beta_1(t)+\beta_2(t)V^{(0)}(t)}}\phi(t, S(0)) \\ 0 \end{pmatrix}, \quad (62)$$

$$g_{33}(t) = \begin{pmatrix} \frac{\theta_1(t)\beta_2(t)}{\sqrt{\beta_1(t)+\beta_2(t)V^{(0)}(t)}} \\ \frac{\theta_2(t)\beta_2(t)}{\sqrt{\beta_1(t)+\beta_2(t)V^{(0)}(t)}} \end{pmatrix}, \tag{63}$$

$$h_{41}(t) = \begin{pmatrix} \frac{\sigma(t)\sqrt{\beta_1(t)+\beta_2(t)V^{(0)}(t)}}{2}\partial^2\phi(t,S(0)) \\ 0 \end{pmatrix}, \tag{64}$$

$$h_{42}(t) = \begin{pmatrix} \frac{\sigma(t)\beta_2(t)e^{\int_0^t \eta_2(s)ds}}{2\sqrt{\beta_1(t)+\beta_2(t)V^{(0)}(t)}}\partial\phi(t,S(0)) \\ 0 \end{pmatrix}, \tag{65}$$

and

$$h_{43}(t) = \begin{pmatrix} -\frac{\sigma(t)\beta_2(t)^2e^{2\int_0^t \eta_2(s)ds}}{8\left[\beta_1(t)+\beta_2(t)V^{(0)}(t)\right]^{3/2}} \\ 0 \end{pmatrix}, \tag{66}$$

where

$$V^{(0)}(t) := e^{\int_0^t \eta_2(s)ds}\left(\int_0^t \eta_1(s)e^{-\int_0^s \eta_2(u)du}ds + V(0)\right), \tag{67}$$

$$\partial_x\phi(t,S(0)) := \frac{\partial\phi(t,x)}{\partial x}\Big|_{x=S(0)},$$

$$\partial_x^2\phi(t,S(0)) := \frac{\partial^2\phi(t,x)}{\partial x^2}\Big|_{x=S(0)},$$

and we use the abbreviated notation $S(0)$ for $S_{a,b}(0)$ and σ for $\sigma^{(a,b)}$.

Proof. We derive coefficients, $S_{a,b}^{(0)}(T)$, $S_{a,b}^{(1)}(T)$ and $S_{a,b}^{(2)}(T)$ explicitly. $S_{a,b}^{(3)}(T)$ can be derived in the similar manner and hence the detail is omitted. Also, we use the abbreviated notation $S^{(i)}(\cdot)$ for $S_{a,b}^{(i)}(\cdot)$ below.

First, we calculate $S^{(0)}(T)$.

$$\begin{aligned} S^{(0)}(T) &= \Bigg(S(0) + \epsilon\int_0^T \phi\left(t, (S^{(0)}(t) + \epsilon S^{(1)}(t) + \cdots)\right) \\ &\quad \times\sigma(t)\sqrt{\beta_1(t) + \beta_2(t)\left(V^{(0)}(t) + \epsilon V^{(1)}(t) + \cdots\right)}dW_t^1\Bigg)\Bigg|_{\epsilon=0} \\ &= S(0). \end{aligned}$$

Next, we calculate $V^{(0)}(T)$ and $S^{(1)}(T)$.

$$\begin{aligned} V^{(0)}(T) &= \Bigg(V(0) + \int_0^T \eta_1(t) + \eta_2(t)(V^{(0)}(t) + \epsilon V^{(1)}(t) + \cdots)dt \\ &\quad + \epsilon\sum_{l=1}^{2}\int_0^T \theta_l(t)\sqrt{\beta_1(t) + \beta_2(t)(V^{(0)}(t) + \epsilon V^{(1)}(t) + \cdots)}dW_t^l\Bigg)\Bigg|_{\epsilon=0} \\ &= V(0) + \int_0^T \left(\eta_{1d}(t) + \eta_2(t)V^{(0)}(t)\right)dt, \end{aligned}$$

$$
\begin{aligned}
S^{(1)}(T) &= \left(\int_0^T \phi\left(t, (S^{(0)}(t) + \epsilon S^{(1)}(t) + \cdots)\right) \right. \\
&\quad \times \sigma(t)\sqrt{\beta_1(t) + \beta_2(t)\left(V^{(0)}(t) + \epsilon V^{(1)}(t) + \cdots\right)} dW_t^1 \\
&\quad + \epsilon \int_0^T \partial\phi\left(t, (S^{(0)}(t) + \epsilon S^{(1)}(t) + \cdots)\right)(S^{(1)}(t) + 2\epsilon S^{(2)}(t) + \cdots) \\
&\quad \times \sigma(t)\sqrt{\beta_1(t) + \beta_2(t)\left(V^{(0)}(t) + \epsilon V^{(1)}(t) + \cdots\right)} dW_t^1 \\
&\quad + \epsilon \int_0^T \phi\left(t, (S^{(0)}(t) + \epsilon S^{(1)}(t) + \cdots)\right) \\
&\quad \left. \times \sigma(t) \frac{\beta_2(t)(V^{(1)}(t) + 2\epsilon V^{(2)}(t) + \cdots)}{\sqrt{\beta_1(t) + \beta_2(t)\left(V^{(0)}(t) + \epsilon V^{(1)}(t) + \cdots\right)}} dW_t^1 \right) \Bigg|_{\epsilon=0} \\
&= \int_0^T \phi\left(t, S^{(0)}(t)\right) \sigma(t)\sqrt{\beta_1(t) + \beta_2(t)V^{(0)}(t)} dW_t^1.
\end{aligned}
$$

$V^{(0)}(T)$ can be solved as follows:

$$
V^{(0)}(t) = e^{\int_0^t \eta_2(s)ds}\left(\int_0^t \eta_1(s)e^{-\int_0^s \eta_2(u)du}ds + V(0)\right).
$$

Then, substituting $V^{(0)}(t)$ into $S^{(1)}(T)$, we obtain the coefficient $f_{11}(t)$.

In the similar manner, we get the following equations for calculation of $V^{(1)}(t)$ and $S^{(2)}(T)$.

$$
\begin{aligned}
V^{(1)}(T) &= \frac{\partial V^{(\epsilon)}(T)}{\partial \epsilon}\Big|_{\epsilon=0} \\
&= \int_0^T \eta_2(t)V^{(1)}(t)dt + \sum_{l=1}^{2}\int_0^T \theta_l(t)\sqrt{\beta_1(t) + \beta_2(t)V^{(0)}(t)} dW_t^l, \\
S^{(2)}(T) &= \frac{\partial^2 S^{(\epsilon)}(T)}{\partial \epsilon^2}\Big|_{\epsilon=0} \\
&= 2\int_0^T \partial_x\phi\left(t, S^{(0)}(t)\right) S^{(1)}(t)\sigma(t)\sqrt{\beta_1(t) + \beta_2(t)V^{(0)}(t)} dW_t^1 \\
&\quad + \int_0^T \phi\left(t, S^{(0)}(t)\right)\sigma(t)\frac{\beta_2(t)V^{(1)}(t)}{\sqrt{\beta_1(t) + \beta_2(t)V^{(0)}(t)}} dW_t^1.
\end{aligned}
$$

Those equations are solved as follows:

$$
V^{(1)}(t) = \sum_{l=1}^{2} e^{\int_0^t \eta_2(s)ds}\left(\int_0^t e^{-\int_0^s \eta_2(u)du}\theta_l(s)\sqrt{\beta_1(s) + \beta_2(s)V^{(0)}(s)} dW_s^l\right),
$$

$$
\begin{aligned}
S^{(2)}(T) &= 2\int_0^T \partial_x\phi\left(t, S^{(0)}(t)\right)\sigma(t)\sqrt{\beta_1(t)+\beta_2(t)V^{(0)}(t)} \\
&\quad \times \int_0^t \phi\left(t, S^{(0)}(s)\right)\sigma(s)\sqrt{\beta_1(s)+\beta_2(s)V^{(0)}(s)}dW_s^1 dW_t^1 \\
&\quad + \int_0^T \phi\left(t, S^{(0)}(t)\right)\sigma(t)\frac{\beta_2(t)\sum_{l=1}^2 e^{\int_0^t \eta_2(s)ds}}{\sqrt{\beta_1(t)+\beta_2(t)V^{(0)}(t)}} \\
&\quad \times \int_0^t e^{-\int_0^s \eta_2(u)du}\theta_l(s)\sqrt{\beta_1(s)+\beta_2(s)V^{(0)}(s)}dW_s^l dW_t^1.
\end{aligned}
$$

Thus, we obtain $f_{2i}(t)$ and $g_{2i}(t)$, $i = 1, 2$.

□

Therefore, applying the general result in the previous section to the current setting, the European payers-swaption price;

$$Swptn(a,b) := \mathcal{N}_{a,b}(0)\mathbf{E}^{(a,b)}[(S_{a,b}(T)-K)^+]$$

is obtained by

$$
\begin{aligned}
\frac{Swptn(a,b)}{\mathcal{N}_{a,b}(0)} &= \epsilon\left(y\int_{-y}^{\infty} n(x,\Sigma)dx + \int_{-y}^{\infty} xn(x,\Sigma)dx\right) \\
&\quad + \epsilon^2\int_{-y}^{\infty}\mathbf{E}^{(a,b)}\left[S_{a,b}^{(2)}(T)|S_{a,b}^{(1)}(T)=x\right]n(x,\Sigma)dx \\
&\quad + \epsilon^3\left(\int_{-y}^{\infty}\mathbf{E}^{(a,b)}\left[S_{a,b}^{(3)}(T)|S_{a,b}^{(1)}(T)=x\right]n(x,\Sigma)dx\right. \\
&\quad \left. + \frac{1}{2}\mathbf{E}^{(a,b)}\left[(S_{a,b}^{(2)}(T))^2|S_{a,b}^{(1)}(T)=y\right]n(y,\Sigma)\right) + o(\epsilon^3),
\end{aligned}
\tag{68}
$$

where $\mathcal{N}_{a,b}(0) = \sum_{i=a+1}^{b}\delta_i P_i(0)$, $y := \{S_{a,b}(0)-K\}/\epsilon$ and $\Sigma := \int_0^T f_{11}'(t)f_{11}(t)dt$.

We remark that $\left(\frac{S_{a,b}^{(\epsilon)}(T)-S_{a,b}^{(0)}(T)}{\epsilon}\right)$ corresponds to $G^{(\epsilon)}$ in Section 3. Note also that the equation (48) with (52) in Section 3 is applied. Then, the equation (68) is obtained after some calculation of (49), (50) and (51).

Finally, the following theorem is obtained through evaluations of the conditional expectations in the above equation by the formulas in Appendix, as well as applications of formulas below:

$$\int_{-y}^{\infty} n[x;0,\Sigma]dx = N\left(\frac{y}{\sqrt{\Sigma}}\right), \tag{69}$$

$$\int_{-y}^{\infty} xn[x;0,\Sigma]dx = \Sigma n[y;0,\Sigma], \tag{70}$$

$$\int_{-y}^{\infty} x^2 n[x;0,\Sigma]dx = \Sigma N\left(\frac{y}{\sqrt{\Sigma}}\right) - y\Sigma n[y;0,\Sigma], \tag{71}$$

$$\int_{-y}^{\infty} x^3 n[x;0,\Sigma]dx = \left(2\Sigma^2 + \Sigma y^2\right) n[y;0,\Sigma], \tag{72}$$

where $N(x)$ denotes the distribution function of the standard normal distribution, and

$$n[x;0,\Sigma] = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left\{\frac{-x^2}{2\Sigma}\right\}. \tag{73}$$

Theorem 1. *The European payers-swaption price $Swptn(a,b)$ at time 0 with strike rate K and maturity T is evaluated by the following formula, where the underlying forward swap's effective date and terminate date are given by T_a and T_b, respectively($T \leq T_a < T_b$, $a, b \in \{1, 2, \cdots, N\}$):*

$$\begin{aligned} Swptn(a,b) &= \mathcal{N}_{a,b}(0)\Bigg\{\epsilon\left(yN\left(\frac{y}{\sqrt{\Sigma}}\right) + \Sigma n[y;0,\Sigma]\right) \\ &\quad +\epsilon^2 C_1\left(-\frac{yn[y;0\Sigma]}{\Sigma}\right) \\ &\quad +\epsilon^3\Bigg(C_2\left(\frac{-1}{\Sigma} + \frac{y^2}{\Sigma^2}\right) n[y;0,\Sigma] + C_3 n[y;0,\Sigma] \\ &\quad + \left(C_4\left(\frac{y^4}{\Sigma^4} - \frac{6y^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right) + C_5\left(\frac{y^2}{\Sigma^2} - \frac{1}{\Sigma}\right) + C_6\right) n[y;0,\Sigma]\Bigg)\Bigg\} \\ &\quad +o(\epsilon^3). \end{aligned} \tag{74}$$

where $\epsilon(\in [0,1])$ is a constant, $\mathcal{N}_{a,b}(0) = \sum_{i=a+1}^{b} \delta_i P_i(0)$, and y, Σ and $C_i (i = 1,2,3,4,5,6)$ are given as follows:

$$y = \frac{S_{a,b}(0) - K}{\epsilon}, \tag{75}$$

$$\Sigma = \int_0^T f_{11}(s)' f_{11}(s) ds, \tag{76}$$

$$C_1 = \sum_{i=1}^{2} \int_0^T f_{11}(s)' g_{2i}(s) \int_0^s f_{11}(u)' f_{2i}(u) du ds, \tag{77}$$

$$\begin{aligned} C_2 &= \sum_{i=1}^{3} \int_0^T f_{11}(s)' h_{3i}(s) \int_0^s f_{11}(u)' g_{3i}(u) \int_0^u f_{11}(v)' f_{3i}(v) dv du ds \\ &\quad + \sum_{i=1}^{3} \int_0^T f_{11}(s)' h_{4i}(s) \int_0^s f_{11}(u)' g_{4i}(u) du \int_0^s f_{11}(u)' f_{4i}(u) du ds, \end{aligned} \tag{78}$$

$$C_3 = \sum_{i=1}^{3} \int_0^T f_{11}(s)' h_{4i}(s) \int_0^s g_{4i}(u)' f_{4i}(u) du ds, \tag{79}$$

$$
\begin{aligned}
C_4 &= \frac{1}{2}\sum_{i=1}^{3}\left(\int_0^T f_{11}(s)'g_{5i}(s)\int_0^s f_{11}(s)'f_{5i}(u)duds\right) \\
&\quad \times\left(\int_0^T f_{11}(s)'k_{5i}(s)\int_0^s f_{11}(s)'h_{5i}(u)duds\right), \qquad (80)\\
C_5 &= \frac{1}{2}\sum_{i=1}^{3}\Bigg(\int_0^T f_{11}(s)'k_{5i}(s)\int_0^s f_{11}(u)'g_{5i}(u)\int_0^u f_{5i}(v)'h_{5i}(v)dvduds \\
&\quad + \int_0^T f_{11}(s)'g_{5i}(s)\int_0^s f_{11}(u)'k_{5i}(u)\int_0^u f_{5i}(v)'h_{5i}(v)dvduds \\
&\quad + \int_0^T f_{11}(s)'g_{5i}(s)\int_0^s f_{5i}(u)'k_{5i}(u)\int_0^u f_{11}(v)'h_{5i}(v)dvduds \\
&\quad + \int_0^T g_{5i}(s)'k_{5i}(s)\int_0^s f_{11}(u)'h_{5i}(u)du\int_0^s f_{11}(u)'f_{5i}(u)duds \\
&\quad + \int_0^T f_{11}(s)'k_{5i}(s)\int_0^s g_{5i}(u)'h_{5i}(u)\int_0^u f_{11}(v)'f_{5i}(v)dvduds\Bigg), \qquad (81)\\
C_6 &= \frac{1}{2}\sum_{i=1}^{3}\int_0^T g_{5i}(s)'k_{5i}(s)\int_0^s f_{5i}(u)'h_{5i}(u)duds. \qquad (82)
\end{aligned}
$$

Here, $f_{11}(t)$*,* $f_{2i}(t)$ $(i=1,2)$*,* $f_{3i}(t)$ $(i=1,2,3)$*,* f_{4i} $(i=1,2,3)$*,* $g_{2i}(t)$ $(i=1,2)$*,* $g_{3i}(t)$ $(i=1,2,3)$*,* $g_{4i}(t)$ $(i=1,2,3)$*,* $h_{3i}(t)$ $(i=1,2,3)$*, and* $h_{4i}(t)$ $(i=1,2,3)$ *are given as equations (59)-(66) in Proposition 1.* $f_{5i}(t)$*,* $g_{5i}(t)$*,* $h_{5i}(t)$*, and* $k_{5i}(t)$ $(i=1,2,3)$ *are defined as follows:*

$$
\begin{aligned}
&f_{51}(t) = f_{53}(t) = h_{51}(t) = f_{21}(t),\\
&f_{52}(t) = h_{52}(t) = h_{53}(t) = f_{22}(t),\\
&g_{51}(t) = g_{53}(t) = k_{51}(t) = g_{21}(t),\\
&g_{52}(t) = k_{52}(t) = \frac{1}{2}k_{53}(t) = g_{22}(t).
\end{aligned}
$$

Remark 2. On the computational complexity and speed for the swaption formula (74) in Theorem 1[2]

First of all, note that ϵ*,* $\mathcal{N}_{a,b}(0)$ *and* y *are constants and that there are no problems for evaluations of the standard normal distribution* $N(y)$ *and the normal density function* $n[y;0,\Sigma]$*, given* Σ*.*

When Σ *and* $C_i(i=1,\cdots,6)$ *are obtained as closed-forms, we have obviously no problems in terms of computational complexity and speed. Thus, let us discuss about the cases that their closed-forms are not available and numerical integrations are necessary.*

[2]This remark discusses about the multi-dimensional case that is the model described by the equations (28) and (29), because the same formula (74) is applied to the case.

As $f_{11}(t)$ whose concrete expression is found in Proposition 1 is a D-dimensional vector given t, which is equal to the dimension of Brownian motion in the swap process (28), $f'_{11}(t)f_{11}(t)$ is obtained by D-times addition. Hence, the order of the computational effort for $\Sigma = \int_0^T f'_{11}(t)f_{11}(t)dt$ is at most DM, where M is the number of time-steps for the discretization in the numerical integral.

Note also that all the multiple integrals appearing in $C_i, (i = 1, \cdots, 6)$ are computed by the program code with only one loop against the time parameter. For instance, look at the following term in C_5 in Theorem 1:

$$\int_0^T f_{11}(s)'k_{5i}(s) \int_0^s f_{11}(u)'g_{5i}(u) \int_0^u f_{5i}(v)'h_{5i}(v)dvduds.$$

Let $f(s) = f'_{11}(s)k_{5i}(s)$, $g(u) = f'_{11}(u)g_{5i}(u)$ and $h(v) = f'_{5i}(v)h_{5i}(v)$. Then, the above integral is approximated for the numerical integration as follows:

$$\begin{aligned}
&\int_0^T f(s) \int_0^t g(u) \int_0^s h(v)dvduds \\
&\approx \sum_{i=1}^{M} \Delta_{t_i} f(t_i) \sum_{j=1}^{i} \Delta_{t_j} g(t_j) \sum_{k=1}^{j} \Delta_{t_k} h(t_k) \\
&= \sum_{i=1}^{M} \Delta_{t_i} f(t_i) \left(G(t_{i-1}) + \Delta_{t_i} g(t_i) \left(H(t_{i-1}) + \Delta_{t_i} h(t_i)\right)\right),
\end{aligned}$$

where $\Delta_{t_i} = (t_i - t_{i-1})$, $H(t_i) = H(t_{i-1}) + \Delta_{t_i} h(t_j)$ and $G(t_i) = G(t_{i-1}) + \Delta_{t_i} g(t_i) H(t_i)$.

Here, each of $h(t_i)$, $g(t_i)$ and $f(t_i)$ is obtained by at most $2D$-times addition since the dimension of each vector is equal to $2D$, the Brownian motion's dimension under our setting. Hence, the order of the computational effort is at most $(2D)M$, where M is the number of time-steps for the discretization in the numerical integral. Note that we have no problems in terms of computational complexity and speed since various fast numerical integration methods are available such as the extrapolation method: In fact, we enjoy pretty much fast calibrations and pricings such as within $1/1000$ seconds per pricing a swaption for numerical examples reported in Section 6.

5 Applications

This section provides concrete applications of the general approximation formula developed in the previous section to CEV-Heston LMM and Quadratic-Heston LMM.

Let us start with the stochastic volatility process specified by the Heston model (Heston (1993)):

$$dV(t) = \xi(\eta - V(t))dt + \theta\sqrt{V(t)}dW_t^Q, \tag{83}$$

where $\xi \geq 0$, $\eta > 0$ and $\theta > 0$ are some constants satisfying $\xi\eta \geq \theta^2/2$.

Applying this model, we describe the forward LIBOR process as in Section 2:

$$\begin{aligned} dF_k(t) &= \phi(t, F_k(t))\sigma_k(t)\sqrt{V(t)}dW_t^{k,1}, & (84)\\ dV(t) &= \xi\left(\eta - \frac{\xi + \theta\mu(t,\gamma(t),k)}{\xi}\right)dt + \theta\rho_k\sqrt{V(t)}dW_t^{k,1} \\ &\quad + \theta\sqrt{1-\rho_k^2}\sqrt{V(t)}dW_t^{k,2}, & (85)\end{aligned}$$

where,

$$\mu(t,\gamma(t),k) = \sum_{j=\gamma(t)+1}^{k} \frac{\delta_j\rho_j\sigma_j(t)\phi(t,F_j(t))}{1+\delta_j F_j(t)}, \quad (86)$$

and ρ_j denotes the correlation between j-th forward LIBOR and the stochastic volatility. After applying the change of a numéraire and the freezing technique discussed in Section 2.2, the forward swap rate process is expressed as follows:

$$\begin{aligned} dS_{a,b}(t) &= \sigma^{(a,b)}(t)\sqrt{V(t)}\phi(t,S_{a,b}(t))dW_t^1, & (87)\\ dV(t) &= \xi(\eta - \nu(t)V(t))dt + \theta_1(t)\sqrt{V(t)}dW_t^1 \\ &\quad + \theta_2(t)\sqrt{V(t)}dW_t^2. & (88)\end{aligned}$$

where $\theta_1(t) = \theta\rho^{(a,b)}(t)$, $\theta_2(t) = \theta\sqrt{1-(\rho^{(a,b)}(t))^2}$, and W_t^l, $l = 1, 2$ are independent Brownian motions. By (23), (27), (32), (33), (34) and (85), the parameters $\sigma^{(a,b)}(t)$, $\rho^{(a,b)}(t)$ and $\nu(t)$ are expressed as follows:

$$\begin{aligned} \sigma^{(a,b)}(t) &= \sqrt{\sum_{k=a+1}^{b}\sum_{h=a+1}^{b}\lambda_k^{(a,b),0}(0)\sigma_k(t)\lambda_h^{(a,b),0}(t)\sigma_h(t)\rho_{k,h}}, & (89)\\ \rho^{(a,b)}(t) &= \frac{\sum_{j=a+1}^{b}\lambda_j^{(a,b),0}(t)\sigma_j(t)\rho_j}{\sigma^{(a,b)}(t)}, & (90)\\ \nu(t) &= 1 + \frac{\theta}{\xi}\mu_0^{(a,b)}(t,\gamma(t)), & (91)\\ \mu_0^{(a,b)}(t,\gamma(t)) &= \sum_{k=a+1}^{b} w_k^{(a,b)}(0)\sum_{j=\gamma(t)+1}^{k}\frac{\delta_j\rho_j\sigma_j(t)\phi(t,F_j(0))}{1+\delta_j F_j(0)}, & (92)\\ w_k^{(a,b)}(0) &= \frac{\delta_k P_k(0)}{\sum_{i=a+1}^{b}\delta_i P_i(0)}, & (93)\\ \gamma(t) &= \min\{i \in \{1,2,\cdots,N\} : T_i \geq t\}, & (94)\end{aligned}$$

where $\rho_{k,h}$ represents the correlation between k-th forward LIBOR and h-th forward LIBOR.

5.1 CEV-Heston LMM

The first example is the *CEV-Heston LMM*, where the local volatility function is given by the constant elasticity of variance (CEV) form and the stochastic volatility process is specified by the Heston model (83). That is, $\phi(t,F) = F^{\beta}$. Based on the discussion in Section 2, in the CEV-Heston LMM the dynamics of a forward swap rate $S_{a,b}(t)$ under the swap measure is given as

$$\begin{aligned} dS_{a,b}(t) &= \sigma^{(a,b)}(t)\sqrt{V(t)}S_{a,b}(t)^{\beta}dW_t^1, \quad (95) \\ dV(t) &= \xi(\eta - \nu(t)V(t))dt + \theta_1(t)\sqrt{V(t)}dW_t^1 \\ &\quad + \theta_2(t)\sqrt{V(t)}dW_t^2. \quad (96) \end{aligned}$$

An approximation formula for the swaption price in the CEV-Heston LMM is obtained by the formula (74) where $f_{11}(t)$, $f_{2i}(t)$ $(i=1,2)$, $f_{3i}(t)$ $(i=1,2,3)$, f_{4i} $(i=1,2,3)$, $g_{2i}(t)$ $(i=1,2)$, $g_{3i}(t)$ $(i=1,2,3)$, $g_{4i}(t)$ $(i=1,2,3)$, $h_{3i}(t)$ $(i=1,2,3)$, and $h_{4i}(t)$ $(i=1,2,3)$ appearing in the equations (76)-(82) are specified as follows:

$$\begin{aligned} &f_{11}(t) = f_{21}(t) = f_{31}(t) = f_{41}(t) = g_{41}(t) = g_{42}(t) \\ &= \begin{pmatrix} \sigma(t)(S(0))^{\beta}\sqrt{V^{(0)}(t)} \\ 0 \end{pmatrix}, \quad (97) \\ &f_{22}(t) = f_{32}(t) = f_{33}(t) = f_{42}(t) = f_{43}(t) = g_{43}(t) \\ &= \begin{pmatrix} \theta_1(t)e^{\xi\int_0^t \nu(s)ds}\sqrt{V^{(0)}(t)} \\ \theta_2(t)e^{\xi\int_0^t \nu(s)ds}\sqrt{V^{(0)}(t)} \end{pmatrix}, \quad (98) \\ &g_{21}(t) = g_{31}(t) = h_{31}(t) = 2h_{32}(t) \\ &= \begin{pmatrix} \sigma(t)\beta(S(0))^{\beta-1}\sqrt{V^{(0)}(t)} \\ 0 \end{pmatrix}, \quad (99) \\ &2g_{22}(t) = g_{32}(t) = 4h_{33}(t) = \begin{pmatrix} \frac{\sigma(t)(S(0))^{\beta}\left(e^{-\xi\int_0^t \nu(s)ds}\right)}{\sqrt{V^{(0)}(t)}} \\ 0 \end{pmatrix}, \quad (100) \\ &g_{33}(t) = \begin{pmatrix} \frac{\theta_1(t)}{\sqrt{V^{(0)}(t)}} \\ \frac{\theta_2(t)}{\sqrt{V^{(0)}(t)}} \end{pmatrix}, \quad (101) \\ &h_{41}(t) = \begin{pmatrix} \frac{\sigma(t)\beta(\beta-1)(S(0))^{\beta-2}\sqrt{V^{(0)}(t)}}{2} \\ 0 \end{pmatrix}, \quad (102) \\ &h_{42}(t) = \begin{pmatrix} \frac{\sigma(t)\beta(S(0))^{\beta-1}\left(e^{-\xi\int_0^t \nu(s)ds}\right)}{2\sqrt{V^{(0)}(t)}} \\ 0 \end{pmatrix}, \quad (103) \\ &h_{43}(t) = \begin{pmatrix} -\frac{\sigma(t)(S(0))^{\beta}\left(e^{-2\xi\int_0^t \nu(s)ds}\right)}{8\left(V^{(0)}(t)\right)^{\frac{3}{2}}} \\ 0 \end{pmatrix}, \quad (104) \end{aligned}$$

where $S(0)$ stands for $S_{a,b}(0)$, $\sigma(t)$ stands for $\sigma^{(a,b)}(t)$ and

$$V^{(0)}(t) = e^{-\xi \int_0^t \nu(s)ds} \left(\int_0^t \xi\eta e^{\xi \int_0^s \nu(u)du} ds + V(0) \right). \tag{105}$$

Moreover, $f_{5i}(t)$, $g_{5i}(t)$, $h_{5i}(t)$, and $k_{5i}(t)$ $(i = 1, 2, 3)$ in the equations (76)-(82) are given as follows:

$$\begin{aligned} &f_{51}(t) = f_{53}(t) = h_{51}(t) = f_{21}(t), \\ &f_{52}(t) = h_{52}(t) = h_{53}(t) = f_{22}(t), \\ &g_{51}(t) = g_{53}(t) = k_{51}(t) = g_{21}(t), \\ &g_{52}(t) = k_{52}(t) = \frac{1}{2}k_{53}(t) = g_{22}(t). \end{aligned} \tag{106}$$

We remark that in this approximation, the parameter $\nu(t)$ standing for the mean-reversion seed of the volatility is made time-dependent for the reduction of the approximation error as much as possible; the effect of this parameter seems large for a long-tenor swap while it seems small for a short-tenor swap.

5.2 Quadratic-Heston LMM

The second example is the *Quadratic-Heston LMM*, where the stochastic volatility process is given by the Heston model and the local volatility function is specified as a quadratic function:

$$\phi(t, F) = (1 - b(t))F(0) + b(t)F + \frac{c(t)}{2F(0)}(F - F(0))^2, \tag{107}$$

where $b(t)$ and $c(t)$ are some (deterministic) functions of the time-parameter t.

Based on the discussion in Section 2, in the Quadratic-Heston LMM the dynamics of a forward swap rate $S_{a,b}(t)$ under the swap measure is given as

$$\begin{aligned} dS_{a,b}(t) &= \sigma^{(a,b)}(t)\sqrt{V(t)} \Big((1 - b(t))S_{a,b}(0) + b(t)S_{a,b}(t) \\ &\qquad + \frac{c(t)}{2S(0)}(S_{a,b}(t) - S_{a,b}(0))^2 \Big) dW_t^1, \end{aligned} \tag{108}$$

$$\begin{aligned} dV(t) &= \xi(\eta - \nu(t)V(t))dt + \theta_1(t)\sqrt{V(t)}dW_t^1 \\ &\quad + \theta_2(t)\sqrt{V(t)}dW_t^2. \end{aligned} \tag{109}$$

Next, set $X_{a,b}(t) := S_{a,b}(t)/S_{a,b}(0)$, and then the swaption price is expressed as

$$\begin{aligned} Swptn(a, b) &= \mathcal{N}_{a,b}(0)\mathbf{E}\left[\max\left\{S_{a,b}(T) - K, 0\right\}\right] \\ &= \mathcal{N}_{a,b}(0)S_{a,b}(0)\mathbf{E}\left[\max\left\{X_{a,b}(T) - \frac{K}{S_{a,b}(0)}, 0\right\}\right], \end{aligned} \tag{110}$$

and the dynamics of $X_{a,b}(t)$ is given by

$$\begin{aligned} dX_{a,b}(t) &= \sigma^{(a,b)}(t)\sqrt{V(t)}\Big(1 - b(t) + b(t)X_{a,b}(t) \\ &\quad + \frac{1}{2}c(t)(X_{a,b}(t) - 1)^2\Big)dW^1(t). \end{aligned} \tag{111}$$

We note that as the local volatility function in (111) can be regarded as an approximation by the second-order Taylor expansion around the initial value $X_{a,b}(0) = 1$ of an arbitrary twice differentiable function, this quadratic form is considered as a rather general local volatility function.[3]

An approximation formula for the swaption price in the Quadratic-Heston LMM is obtained by the formula (74) where $f_{11}(t)$, $f_{2i}(t)$ $(i = 1, 2)$, $f_{3i}(t)$ $(i = 1, 2, 3)$, f_{4i} $(i = 1, 2, 3)$, $g_{2i}(t)$ $(i = 1, 2)$, $g_{3i}(t)$ $(i = 1, 2, 3)$, $g_{4i}(t)$ $(i = 1, 2, 3)$, $h_{3i}(t)$ $(i = 1, 2, 3)$, and $h_{4i}(t)$ $(i = 1, 2, 3)$ appearing in the equations (76)-(82) are specified as follows:

$$\begin{aligned} &f_{21}(t) = f_{31}(t) = f_{41}(t) = g_{41}(t) = g_{42}(t) \\ &= \begin{pmatrix} \sigma(t)\sqrt{V^{(0)}(t)} \\ 0 \end{pmatrix}, \end{aligned} \tag{112}$$

$$\begin{aligned} &f_{22}(t) = f_{32}(t) = f_{33}(t) = f_{42}(t) = f_{43}(t) = g_{43}(t) \\ &= \begin{pmatrix} \theta_1(t)e^{\xi\int_0^t \nu(s)ds}\sqrt{V^{(0)}(t)} \\ \theta_2(t)e^{\xi\int_0^t \nu(s)ds}\sqrt{V^{(0)}(t)} \end{pmatrix}, \end{aligned} \tag{113}$$

$$f_{11}(t) = S(0)f_{21}(t), \tag{114}$$

$$g_{31}(t) = \begin{pmatrix} \sigma(t)b(t)\sqrt{V^{(0)}(t)} \\ 0 \end{pmatrix}, \tag{115}$$

$$g_{21}(t) = h_{31}(t) = 2h_{32}(t) = S(0)g_{31}(t), \tag{116}$$

$$g_{32}(t) = \begin{pmatrix} \frac{\sigma(t)e^{-\xi\int_0^t \nu(s)ds}}{\sqrt{V^{(0)}(t)}} \\ 0 \end{pmatrix}, \tag{117}$$

$$2g_{22}(t) = 4h_{33}(t) = S(0)g_{32}(t), \tag{118}$$

$$g_{33}(t) = \begin{pmatrix} \frac{\theta_1(t)}{\sqrt{V^{(0)}(t)}} \\ \frac{\theta_2(t)}{\sqrt{V^{(0)}(t)}} \end{pmatrix}, \tag{119}$$

$$h_{41}(t) = \begin{pmatrix} \frac{S(0)\sigma(t)c(t)\sqrt{V^{(0)}(t)}}{2} \\ 0 \end{pmatrix}, \tag{120}$$

$$h_{42}(t) = \begin{pmatrix} \frac{S(0)\sigma(t)b(t)e^{-\xi\int_0^t \nu(s)ds}}{2\sqrt{V^{(0)}(t)}} \\ 0 \end{pmatrix}, \tag{121}$$

[3]The asymptotic expansion of $X_{a,b}(t)$ gives the simpler expression than that of $S_{a,b}(t)$.

$$h_{43}(t) = \begin{pmatrix} -\frac{S(0)\sigma(t)e^{-2\xi\int_0^t \nu(s)ds}}{8\left(V^{(0)}(t)\right)^{\frac{3}{2}}} \\ 0 \end{pmatrix}, \tag{122}$$

where $S(0)$ stands for $S_{a,b}(0)$ and σ stands for $\sigma^{(a,b)}$. Moreover, $f_{5i}(t)$, $g_{5i}(t)$, $h_{5i}(t)$, and $k_{5i}(t)$ $(i = 1, 2, 3)$ in the equations (76)-(82) are given as follows:

$$\begin{aligned} &f_{51}(t) = f_{53}(t) = h_{51}(t) = f_{21}(t), \\ &f_{52}(t) = h_{52}(t) = h_{53}(t) = f_{22}(t), \\ &g_{51}(t) = g_{53}(t) = k_{51}(t) = g_{21}(t), \\ &g_{52}(t) = k_{52}(t) = \frac{1}{2}k_{53}(t) = g_{22}(t). \end{aligned} \tag{123}$$

6 Numerical Examples

This section provides two numerical examples: the calibration test and the accuracy test. First, let us set LSV-LMM as the CEV-Heston LMM and the Quadratic-Heston LMM for the numerical examples. Under the spot measure Q, the local volatility functions of CEV-Heston LMM and Quadratic-Heston LMM are given by

$$\phi(t, F) = F^{\beta} \quad \text{and} \quad \phi(t, F) = (1 - b)F(0) + bF + c\frac{(F - F(0))^2}{2F(0)}, \tag{124}$$

respectively, where b, c and σ_j are some constants. Then the one-dimensional Heston-type stochastic volatility in (83) is equipped with the two models. All model parameters are assumed to be constant for simplicity. We set the parameter $\epsilon = 1$.

6.1 Calibration Test

This subsection examines the calibration ability of the CEV-Heston LMM and the Quadratic-Heston LMM with our approximation formula. In particular, because a caplet is regarded as a special case of a swaption[4],Theorem 1 with specifications in Section 5.1 or 5.2 is applied to the evaluation of cap prices in calibration of each model: Formula (74) with equations (97)-(106) is applied to CEV-Heston LMM, while the one with equations (112)-(123) is applied to Quadratic-Heston LMM.

The US cap market data[5] as of April 1, 2008 downloaded from Bloomberg are employed for the calibration test. The two models are calibrated to the

[4]For a caplet, the underlying forward rate's effective and terminal dates are given by T_a and $T_{a+1}(T \leq T_a,\, a \in \{1, 2, \cdots, N-1\})$, respectively. Hence, setting $b = a + 1$ in the formula (74) provides the formula for the caplet.

[5]In our calibration test, we calibrated the models to the cap market data solely, because suitable swaption data are not available in our circumstance.

market caplet implied volatilities with 1, 2, 3, 5, 7, 10, 15, and 20-year maturities, simultaneously.

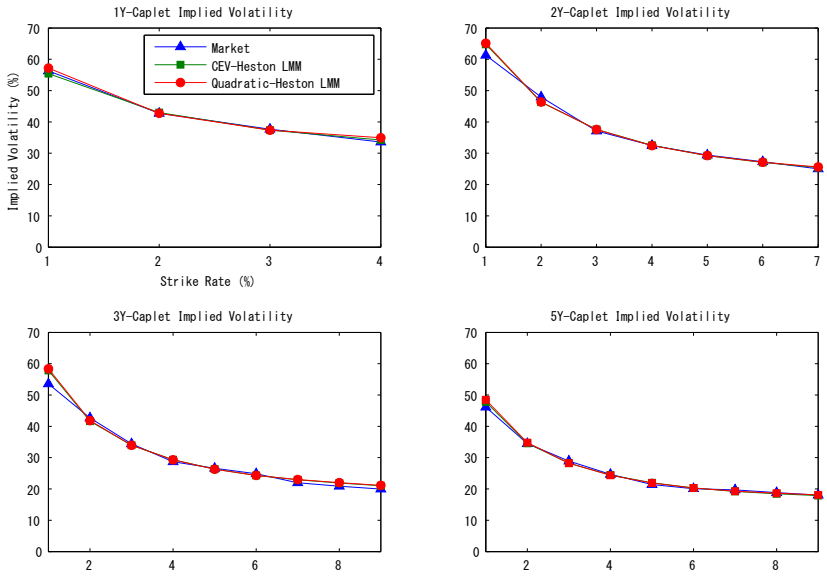
The parameters $V(0)$ and η are fixed as 1. The other parameters of the local stochastic volatilities and the correlations between LIBORs and the volatilities are obtained by calibration. The calibrated parameters of the local stochastic volatilities are listed in Table 1. The number of parameters, σ_k, ρ_k and forward LIBORs is so many that those values are not reported here.[6]

Table 1: Local Stochastic Volatility Parameters

	ξ	θ	β	b	c
CEV-Heston	0.0987	0.4442	0.0100	-	-
Quadratic-Heston	0.0488	0.3124	-	0.2438	1.2919

Figure 1 and 2 plot the market and model-based caplet implied volatilities.

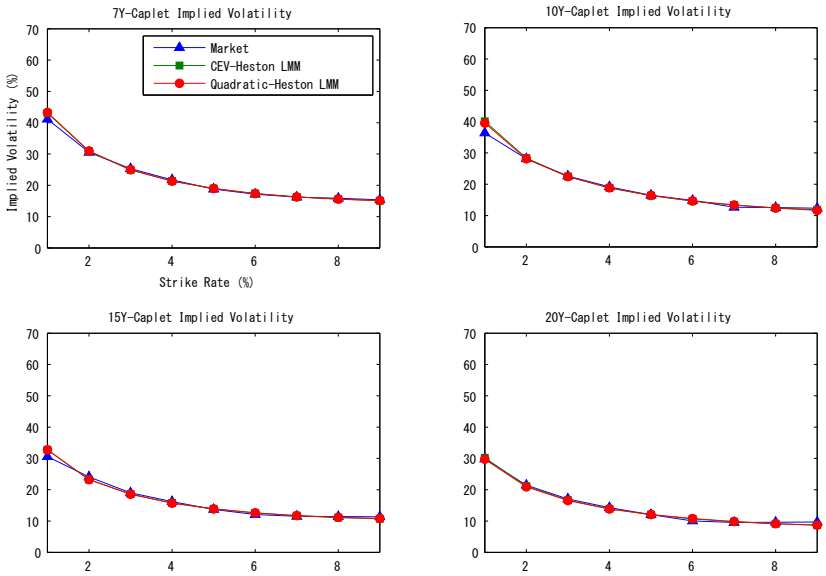
Figure 1: Caplet Implied Volatilities with 1, 2, 3, and 5-Year Maturities



These figures show that the model-based caplet implied volatilities generated by both the CEV-Heston LMM and the Quadratic-Heston LMM are fitted into the market ones very well. This calibration test implies that the CEV-Heston LMM and the Quadratic-Heston LMM have sufficient calibration ability to cap markets, and that our approximation formula is a very

[6]They will be given upon request.

Figure 2: Caplet Implied Volatilities with 7, 10, 15, and 20-Year Maturities



powerful tool because such a fast caplet pricing scheme is necessary for implementing the calibration. In fact, it only takes less than 1/1000 seconds with core i7-870 processor to evaluate each caplet by applying our formula.

6.2 Accuracy Test

This subsection provides the accuracy test of our approximate swaption pricing formula in Theorem 1. Setting the calibrated parameters in section 5.1 and historically estimated correlations among forward LIBOR rates in the CEV-Heston LMM and the Quadratic-Heston LMM, we compute 5-year×5-year and 10-year×10-year payers swaption prices by our formula. In order to calculate $\sigma^{(a,b)}$, the total correlation structure in LMM with the stochastic volatility should remain positive semi-definite. For that reason, we use the parameterization method proposed by Mercurio and Morini (2007) for the correlation matrix. Then, we compare our approximate swaption prices with exact ones.

The parameters used for calculating swaption price are reported in Table 1 to 3, where C-H and Q-H stand for CEV-Heston and Quadratic-Heston, respectively.

Table 4 to 7 display the prices of 5-year×5-year and 10-year×10-year payers swaption under the CEV-Heston LMM and the Quadratic-Heston LMM, respectively. In the tables, the values of (a) Full MC denote swaption prices computed by the Monte Carlo simulation with 1,000,000 sample

Table 2: The Value of $\nu(t)$

t	C-H 5x5	Q-H 5x5	C-H 10x10	Q-H 10x10	t	C-H 10x10	Q-H 10x10
0.25	1.035	0.853	1.000	0.716	5.25	0.998	0.811
0.5	1.029	0.851	1.000	0.714	5.5	0.998	0.817
0.75	1.023	0.850	1.000	0.713	5.75	0.998	0.822
1	1.018	0.850	1.000	0.713	6	0.998	0.827
1.25	1.012	0.850	0.999	0.714	6.25	0.998	0.831
1.5	1.009	0.855	0.999	0.718	6.5	0.998	0.836
1.75	1.007	0.862	0.999	0.725	6.75	0.998	0.841
2	1.006	0.870	0.999	0.733	7	0.998	0.846
2.25	1.006	0.879	0.999	0.743	7.25	0.998	0.850
2.5	1.004	0.885	1.000	0.749	7.5	0.998	0.858
2.75	1.003	0.892	0.999	0.755	7.75	0.999	0.866
3	1.002	0.899	0.999	0.762	8	0.999	0.874
3.25	1.002	0.906	0.999	0.770	8.25	0.999	0.883
3.5	1.000	0.912	0.999	0.775	8.5	0.999	0.889
3.75	0.999	0.918	0.999	0.781	8.75	0.999	0.895
4	0.999	0.924	0.999	0.787	9	0.999	0.902
4.25	0.998	0.929	0.999	0.793	9.25	0.999	0.908
4.5	0.997	0.934	0.999	0.798	9.5	0.999	0.915
4.75	0.996	0.939	0.998	0.803	9.75	0.999	0.921
5	0.995	0.944	0.998	0.807	10	0.999	0.928

Table 3: Other parameters

	Forward Swap	Annuity	$\rho^{(a,b)}$ (C-H)	$\rho^{(a,b)}$ (Q-H)	$\sigma^{(a,b)}$ (C-H)	$\sigma^{(a,b)}$ (Q-H)
10x10	5.413%	5.037	-0.1535	-0.5239	0.0048	0.0877
5x5	5.049%	3.720	-0.1378	-0.5443	0.0068	0.1370

paths without any approximation techniques. We consider these prices as the exact values of swaption prices. The values of (b) FT + MC are the Monte Carlo prices with the freezing techniques. The values of (c) FT + AE are the swaption prices by the asymptotic expansion scheme with the freezing techniques, that is, our pricing formula: Formula (74) with equations (97)-(106) is applied to CEV-Heston LMM, while the one with equations (112)-(123) is applied to Quadratic-Heston LMM. The value in the round bracket denotes the implied volatility corresponding to each swaption price. As explained in Remark 2, we have no problems in computation, which is very fast: It only takes less than 1/1000 seconds with core i7-870 processor to evaluate a 10x10 swaption, (although we partially rely on numerical integrations since we make the parameter $\nu(t)$ time-dependent for the reduction of the approximation errors as much as possible.)

Next, we note that the values in the lower layers of Table 4, 5, 6 and 7 denote the approximation errors caused by the freezing techniques and/or the asymptotic expansion.

It can be seen that significantly accurate prices are obtained by our swaption pricing formula under the CEV-Heston LMM in Table 4 and 5,

Table 4: 10y×10y Payers Swaption Prices under CEV-Heston LMM

Strike Rate (%)	3.00	4.00	5.00	6.00	7.00
(a) Full MC	0.1236	0.0779	0.0400	0.0157	0.0048
	(12.07)	(10.04)	(8.62)	(7.75)	(7.34)
(b) FT + MC	0.1239	0.0783	0.0403	0.0158	0.0048
	(12.51)	(10.26)	(8.72)	(7.77)	(7.31)
(c) FT + AE	0.1241	0.0784	0.0403	0.0156	0.0047
	(12.74)	(10.29)	(8.69)	(7.74)	(7.25)
(b) − (a)	0.0003	0.0004	0.0003	0.0001	-0.0001
	(0.44)	(0.22)	(0.10)	(0.02)	(-0.03)
(c) − (b)	0.0002	0.0001	-0.0001	-0.0001	-0.0001
	(0.24)	(0.03)	(-0.03)	(-0.03)	(-0.06)
(c) − (a)	0.0005	0.0005	0.0002	0.0000	-0.0002
	(0.68)	(0.25)	(0.07)	(-0.01)	(-0.09)

Table 5: 5y×5y Payers Swaption Prices under CEV-Heston LMM

Strike Rate (%)	3.00	4.00	5.00	6.00	7.00
(a) Full MC	0.0786	0.0467	0.0222	0.0080	0.0023
	(17.48)	(14.67)	(12.77)	(11.62)	(10.98)
(b) FT + MC	0.0787	0.0468	0.0222	0.0080	0.0022
	(17.72)	(14.75)	(12.78)	(11.57)	(10.90)
(c) FT + AE	0.0788	0.0467	0.0221	0.0079	0.0022
	(17.84)	(14.74)	(12.76)	(11.54)	(10.86)
(b) − (a)	0.0001	0.0001	0.0000	0.0000	-0.0001
	(0.24)	(0.09)	(0.00)	(-0.05)	(-0.07)
(c) − (b)	0.0001	-0.0001	-0.0001	-0.0001	0.0000
	(0.13)	(-0.01)	(-0.01)	(-0.03)	(-0.04)
(c) − (a)	0.0002	0.0001	0.0000	-0.0001	-0.0001
	(0.36)	(0.07)	(-0.01)	(-0.08)	(-0.12)

while the prices under the Quadratic-Heston LMM in Table 6 and 7 are less accurate than the ones under the CEV-Heston LMM. However, even when pricing deep in-the-money swaptions in Table 6 and 7, the approximation errors in terms of the implied volatilities are less than 1%. Therefore, the level of these errors seems acceptable in practice.

7 Conclusion

This paper proposed the LSV-LMM with affine-type stochastic volatility models combined with local volatility models, where all parameters in the LSV-LMM can be time-dependent. In particular, applying standard freezing techniques and an asymptotic expansion method, it provided a new analytic

Table 6: 10y×10y Payers Swaption Prices under Quadratic-Heston LMM

Strike Rate (%)	3.00	4.00	5.00	6.00	7.00
(a) Full MC	0.1258	0.0812	0.0437	0.0182	0.0058
	(14.47)	(11.70)	(9.80)	(8.51)	(7.78)
(b) FT + MC	0.1252	0.0803	0.0425	0.0169	0.0049
	(13.91)	(11.26)	(9.42)	(8.13)	(7.36)
(c) FT + AE	0.1249	0.0799	0.0422	0.0168	0.0046
	(13.55)	(11.06)	(9.32)	(8.09)	(7.20)
(b) − (a)	-0.0006	-0.0009	-0.0012	-0.0012	-0.0009
	(-0.56)	(-0.44)	(-0.38)	(-0.37)	(-0.42)
(c) − (b)	-0.0003	-0.0004	-0.0003	-0.0002	-0.0003
	(-0.35)	(-0.20)	(-0.10)	(-0.05)	(-0.16)
(c) − (a)	-0.0009	-0.0013	-0.0015	-0.0014	-0.0012
	(-0.91)	(-0.64)	(-0.48)	(-0.42)	(-0.58)

Table 7: 5y×5y Payers Swaption Prices under Quadratic-Heston LMM

Strike Rate (%)	3.00	4.00	5.00	6.00	7.00
(a) Full MC	0.0802	0.0489	0.0245	0.0096	0.0030
	(20.14)	(16.58)	(14.21)	(12.65)	(11.74)
(b) FT + MC	0.0798	0.0482	0.0236	0.0087	0.0024
	(19.50)	(15.99)	(13.63)	(12.05)	(11.08)
(c) FT + AE	0.0796	0.0482	0.0237	0.0089	0.0024
	(19.17)	(15.95)	(13.74)	(12.18)	(11.07)
(b) − (a)	-0.0004	-0.0007	-0.0010	-0.0009	-0.0006
	(-0.64)	(-0.59)	(-0.58)	(-0.60)	(-0.67)
(c) − (b)	-0.0002	-0.0001	0.0002	0.0002	-0.0000
	(-0.33)	(-0.04)	(0.11)	(0.12)	(-0.01)
(c) − (a)	-0.0006	-0.0008	-0.0008	-0.0007	-0.0006
	(-0.97)	(-0.63)	(-0.47)	(-0.48)	(-0.68)

approximation formula for pricing swaptions under the model. To demonstrate effectiveness of our approach, the paper took CEV-Heston LMM and Quadratic-Heston LMM as examples and confirmed sufficient accuracies of our approach for calibration to a caplet market and numerical evaluation of swaptions under the models.

Our future research topics are as follows: Fist, in order to improve the accuracy of our current approximation formula, the higher order computational scheme of the asymptotic expansion developed by Takahashi, et al. (2009) has to be applied. Alternatively or at the same time, the full application of the asymptotic expansion might be necessary without freezing techniques. Second, for more accurate calibration, we may need to implement a pricing formula for swaptions under the LSV-LMM with a multi-dimensional

stochastic volatility. Finally, in order to compute exotic interest rate derivatives and their Greeks, we have to develop efficient Monte Carlo simulation techniques.

A Formulas for the conditional expectations of the Wiener-Itô integrals

This appendix summarizes conditional expectation formulas useful for explicit computation of the asymptotic expansions. In the following, $q_i \in L^2[0,T]$, $i = 1, 2, ..., 5$. Also, $H_n(x;\Sigma)$ denotes the Hermite polynomial of degree n and $\Sigma = \int_0^T |q_{1t}|^2 dt$. For the derivation and more general results, see Section 3 in Takahashi, Takehara and Toda(2009).

1.

$$\mathbf{E}\left[\int_0^T q_{2t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] = \left(\int_0^T q_{2t}^{'} q_{1t} dt\right) \frac{H_1(x;\Sigma)}{\Sigma}$$

2.

$$\mathbf{E}\left[\int_0^T \int_0^t q_{2u}^{'} dW_u q_{3t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T \int_0^t q_{2u}^{'} q_{1u} du q_{3t}^{'} q_{1t} dt\right) \frac{H_2(x;\Sigma)}{\Sigma^2}$$

3.

$$\mathbf{E}\left[\left(\int_0^T q_{2u}^{'} dW_u\right)\left(\int_0^T q_{3s}^{'} dW_s\right) | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T q_{2u}^{'} q_{1u} du\right)\left(\int_0^T q_{3s}^{'} q_{1s} ds\right) \frac{H_2(x;\Sigma)}{\Sigma^2}$$
$$+ \int_0^T q_{2t}^{'} q_{3t} dt$$

4.

$$\mathbf{E}\left[\int_0^T \int_0^t \int_0^s q_{2u}^{'} dW_u q_{3s}^{'} dW_s q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T q_{4t}^{'} q_{1t} \int_0^t q_{3s}^{'} q_{1s} \int_0^s q_{2u}^{'} q_{1u} du ds dt\right) \frac{H_3(x;\Sigma)}{\Sigma^3}$$

5.

$$\mathbf{E}\left[\int_0^T \left(\int_0^t q_{2u}^{'} dW_u\right)\left(\int_0^t q_{3s}^{'} dW_s\right) q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left\{\int_0^T \left(\int_0^t q_{2u}^{'} q_{1u} du\right)\left(\int_0^t q_{3s}^{'} q_{1s} ds\right) q_{4t}^{'} q_{1t} dt\right\} \frac{H_3(x;\Sigma)}{\Sigma^3}$$
$$+ \left(\int_0^T \int_0^t q_{2u}^{'} q_{3u} du q_{4t}^{'} q_{1t} dt\right) \frac{H_1(x;\Sigma)}{\Sigma}$$

6.

$$\mathbf{E}\left[\left(\int_0^T\int_0^t q_{2s}^{'}dW_s q_{3t}^{'}dW_t\right)\left(\int_0^T\int_0^r q_{4u}^{'}dW_u q_{5r}^{'}dW_r\right)\mid\int_0^T q_{1v}^{'}dW_v=x\right]=$$

$$\left(\int_0^T q_{3t}^{'}q_{1t}\int_0^t q_{2s}^{'}q_{1s}dsdt\right)\left(\int_0^T q_{5r}^{'}q_{1r}\int_0^r q_{4u}^{'}q_{1u}dudr\right)\frac{H_4(x;\Sigma)}{\Sigma^4}$$

$$+\left\{\int_0^T q_{3t}^{'}q_{1t}\int_0^t q_{5r}^{'}q_{1r}\int_0^r q_{2u}^{'}q_{4u}dudrdt+\int_0^T q_{5t}^{'}q_{1t}\int_0^t q_{3r}^{'}q_{1r}\int_0^r q_{2u}^{'}q_{4u}dudrdt\right.$$

$$+\int_0^T q_{3t}^{'}q_{1t}\int_0^t q_{2r}^{'}q_{5r}\int_0^r q_{4u}^{'}q_{1u}dudrdt+\int_0^T q_{3t}^{'}q_{5t}\left(\int_0^t q_{2s}^{'}q_{1s}ds\right)\left(\int_0^t q_{4u}^{'}q_{1u}du\right)dt$$

$$\left.+\int_0^T q_{5r}^{'}q_{1r}\int_0^r q_{3u}^{'}q_{4u}\int_0^u q_{2s}^{'}q_{1s}dsdudr\right\}\frac{H_2(x;\Sigma)}{\Sigma^2}$$

$$+\int_0^T\int_0^t q_{2u}^{'}q_{4u}duq_{3t}^{'}q_{5t}dt$$