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The Effect of Policyholders' Rationality on Unit-linked Life Insurance Contracts with Surrender Guarantees

by

Jing Li and Alexander Szimayer

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Bonn Graduate School of Economics
Department of Economics
University of Bonn
Kaiserstrasse 1
D-53113 Bonn

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Jing Li
Bonn Graduate School of Economics
University of Bonn
Adenauerallee 24-42
D-53111 Bonn, Germany
email: lijing@uni-bonn.de

Alexander Szimayer
School of Economics
University of Bonn
Adenauerallee 24-42
D-53111 Bonn, Germany
email: szimayer@uni-bonn.de

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Abstract

We study the valuation of unit-linked life insurance contracts with surrender guarantees. Instead of solving an optimal stopping problem, we propose a more realistic approach accounting for policyholders' rationality in exercising their surrender option. The valuation is conducted at the portfolio level by assuming the surrender intensity to be bounded from below and from above. The lower bound corresponds to purely exogenous surrender and the upper bound represents the limited rationality of the policyholders. The valuation problem is formulated by a valuation PDE and solved with the finite difference method. We show that the rationality of the policyholders has a significant effect on average contract value and hence on the fair contract design. We also present the separating boundary between purely exogenous surrender and endogenous surrender. This provides implications on the predicted surrender activity of the policyholders.

Keywords: unit-linked life insurance contracts, surrender guarantee, limited rationality, fair contract analysis

JEL: G13, G22, C65

1 Introduction

Most unit-linked life insurance contracts entitle the policyholders to terminate the contract before the maturity date and receive a certain cash refund called the surrender value. In the literature, at least four approaches are found to evaluate such contracts. The first approach is to consider the surrender decision as caused by exogenous reasons and a surrender table can be constructed to capture the statistics on surrenders, see Bacinello [3]. The second approach is to work within the contingent-claim framework and consider the surrender option as an American-style contingent claim to be exercised rationally. This approach is favored by most literature in recent years. Examples are Grosen and Jørgensen [12][13], Bacinello [2][3], and Bacinello et al. [4], to just name a few. The argument is that the policyholder should not complain about the contract depreciation caused by his own non-optimal surrender, even due to exogenous reasons like financial difficulties, when he does have the right to do it optimally. The third approach takes suboptimal surrender into consideration. This is suggested by Bernard and Lemieux [5]. They consider a single policyholder's decision behavior, which is characterized by a decision parameter. The policyholder is assumed to exercise the surrender option only when the ratio between the surrender value and the continuation value exceed the decision parameter. The fourth approach is carried out on the portfolio level. It is first proposed by Albizzati and Geman [1] who incorporate both the exogenous and the endogenous surrender reasons into the valuation problem. They assume that the proportion of surrender among the active contracts is an increasing function of the ratio of the surrender value and the value when holding the contract until maturity. In case the ratio is below one, the surrender rate is set to its minimum reflecting base level surrender due to exogenous reasons. The surrender rate is then linear increasing with increasing ratio until a fixed upper bound is reached. The upper bound represents the maximal surrender rate. Recently, similar idea was implemented by Giovanni [11] to model the policyholders' rationality in contract surrender.

We consider the approaches of Albizzati and Geman [1] as well as Giovanni [11] as more realistic than the other three approaches. The first two approaches only address part of the story. Surrender decisions are not only triggered by exogenous reasons but also by endogenous reasons. The empirical study conducted by Kuo, Tsai and Chen [14] shows that not only the unemployment rate (which corresponds to the exogenous surrender reason) but also the interest rate (which corresponds to the endogenous surrender reason) has impact on surrender behavior. Without treating the endogenous surrender risk properly, the policy issuer will suffer an underestimated loss when disadvantageous financial market movement brings about more surrender cases than that have been summarized by the surrender table. However, it has never been observed that all the policyholders simultaneously take the same surrender action when it is optimal to do so. Treating the surrender action merely as an optimal stopping problem will overestimate the funds needed to manage the contracts. Overall, it is difficult to identify each policyholder's decision rule and to figure out the proportion of policyholders who are characterized by the same decision parameter. Since the policy issuers cannot identify the rationality of the policyholders separately, all the policyholders should be charged the same at the beginning. The premiums charged by considering both the exogenous and the endogenous surrender reasons can be argued to be reasonable on the portfolio level.[1]

Although we tend to follow Albizzati and Geman [1], we also bear in mind that there are some limitations in their approaches that we try to avoid. In Albizzati and Geman [1], mortality is considered as one of the surrender events. However, in most cases death benefit and surrender benefit are not equal to each other. Surrender is usually accompanied by a penalty in payment which does not apply to death benefit. Hence, the distinction between the death event and the surrender event should be considered. In addition, Albizzati and Geman [1] assume that a policyholder surrenders the contract by comparing the surrender value and the value of initiating a new contract which he holds till the maturity. A closed-form solution is obtained by assuming independence between the surrender probabilities at different time points. However, usually a new contract also allows for surrender. In this case, a surrender probability in the future also has influence on the surrender probability at present. This effect should be taken into consideration when evaluating a contract with surrender guarantees. If the assumption about the independence between the surrender probabilities is suspended, the Monte Carlo simulation method is suggested by them to solve the valuation problem which is very time consuming.

In this paper we propose the intensity-based valuation of unit-linked life insurance contracts with surrender guarantees. Surrender is not modeled as a binary event but randomized where the surrender intensity reflects the local likelihood of surrender. The intensity based approach was first used in credit risk modeling to describe the arrival of the credit event. Recently, a similar approach has been adopted in other areas. For example, the mortality risk embedded in insurance contracts is characterized by the mortality intensity, (e.g. Milevsky et al. [17], Dahl [7], Dahl and Møller [8]) and the prepayment risk embedded in mortgage loans is captured by the prepayment intensity (e.g. Stanton [19], Dai et al. [9]). In our paper, we describe the arrival of the surrender event also by an intensity-based approach and solve the valuation problem for a representative policyholder. We assume that the surrender intensity of the policyholder is bounded from below and from above. As in Albizzati and Geman [1] and Giovanni [11] the lower bound represents the surrender base level due to exogenous reasons. And the upper bound represents the maximal surrender

[1] For those competent policyholders who are able to exercise their surrender option optimally, less premiums are charged than those are needed to support the contracts. It is the irrational policyholders who have born the extra costs.

rate that is attributed to exercise of the surrender option when it is financially optimal to do so. Since the optimal decision will not be made by all the policyholders simultaneously and equivalently not by the representative policyholder, both the lower and the upper bound of the surrender intensity are finite numbers between zero and infinity.[2] They can be easily backed out from the relevant statistics in the past. By capturing the surrender risk with the surrender intensity, and similarly, the mortality risk with the mortality intensity, we are able to establish a partial differential equation whose solution is the contract value we are looking for. The finite difference method is then applied to solve the problem. In this sense, our approach is quite similar to Giovanni [11] but is also different from him in two aspects. We have incorporated the mortality risk in our model which is but ignored by Giovanni [11]. In addition, we emphasize the fair contract design in our paper.

To formalize the problem, we introduce the model setup in Section 2. The valuation of the contracts is carried out in Section 3. In Section 4 we study the impact of the policyholders' rationality on the contract value through numerical examples. Moreover, the relationship of the parameters in the contract will be analyzed. Section 5 concludes.

2 Setup

Unit-linked life insurance contracts link the financial market and the insurance market together. On the financial market, we assume that there is a non-dividend paying risky asset with price process S and a riskless money market account with price process B. Under the real world measure $\mathbb{P}$, the two asset price processes are governed respectively by the stochastic differential equations

$$\mathrm{d}S_t = a(t, S_t)\, S_t\, \mathrm{d}t + \sigma(t, S_t)\, S_t\, \mathrm{d}W_t\,, \tag{1}$$

and

$$\mathrm{d}B_t = r(t)\, B_t\, \mathrm{d}t\,, \tag{2}$$

for $0 \leq t \leq T$, where a is the local mean rate of return of the risky asset and σ is the volatility of the risky asset. Both of them are Markovian. The risk-free interest rate r is assumed to be deterministic. Moreover, W refers to the 1-dimensional Brownian motion under $\mathbb{P}$ and generates the financial market filtration $\mathcal{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. The financial market is complete and arbitrage free, which is equivalent to the existence of a risk-neutral martingale measure $\mathbb{Q}$ so that the price process S is described as

$$\mathrm{d}S_t = r(t)\, S_t\, \mathrm{d}t + \sigma(t, S_t)\, S_t\, \mathrm{d}\hat{W}_t,\ 0 \leq t \leq T, \tag{3}$$

[2]If the surrender option is exercised optimally, the surrender intensity switches between zero and infinity.

where $\hat{W}$ is a Brownian motion under $\mathbb{Q}$ which satisfies $\mathrm{d}\hat{W}_t = \mathrm{d}W_t + \frac{a-r}{\sigma}\mathrm{d}t$.

The insurance market is modeled by the random time τ denoting the death time of an individual aged x at the starting time 0. The jump process associated with it is H with $H_t = 1_{\{\tau \leq t\}}$, for $t \in [0,T]$, and generates the filtration $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t \leq T}$. Furthermore, the hazard rate of the random time τ (or the mortality intensity) is denoted by μ. In recent literature, the mortality intensity is often assumed to be stochastic based on the observation of the systematic longevity risk in recent decades. However, in another paper [15] we find that the stochastic feature of the mortality intensity is of minor impact on unit-linked life insurance contracts when the risk profiles at death and at maturity are not dramatically different. We assume here, therefore, that the mortality intensity is described by a deterministic function $\mu(t)$, for $t \in [0,T]$. In fact, the mortality risk is then unsystematic and can be diversified away over a large pool of policyholders. Accordingly, we can work under the risk-neutral measure $\mathbb{Q}$ extended to the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ such that $\hat{W}$ is a $(\mathbb{Q}, \mathbb{G})$-Brownian motion and μ is the $(\mathbb{Q}, \mathbb{G})$-intensity of H. See Bielecki and Rutkowski [6] for details.

3 Contract Valuation

In this section we introduce the contract and derive the valuation equation. The contract is comprised of a survival benefit, a death benefit and a surrender benefit. Survival benefit and death benefit both offer a guaranteed rate and the possibility to participate in a potentially profitable development of the risky asset. The surrender benefit depends on time only, effectively representing a put option, see Bernard and Lemieux [5] for a similar approach. The contract value is derived using the balance law of financial economics, see Dai et al. [9].

We assume that the policyholder pays at the beginning time 0 the single premium P for the contract with the maturity date T. The payoff of the contract is linked to the underlying asset S. When the policyholder survives time T, the payment to him is

$$\Phi(S_T) = P \max\left(\alpha\,(1+g)^T, \left(\frac{S_T}{S_0}\right)^k\right), \tag{4}$$

where α refers to the percentage of the initial premium which is provided with the minimum guaranteed rate g and k refers to the policyholder's participation rate in the performance of the underlying asset. When the policyholder dies at time $\tau < T$, the death benefit is

$$\Psi(\tau, S_\tau) = P \max\left(\alpha\,(1+g_d)^\tau, \left(\frac{S_\tau}{S_0}\right)^{k_d}\right), \tag{5}$$

where the parameters g_d and k_d refer respectively to the minimum guaranteed rate and the participation rate in the asset performance upon the occurrence of the death event. They need not be identical with g and k. However, in practice, death as a natural event is neither penalized nor rewarded, so that $g = g_d$ as well as $k = k_d$ is very common. Furthermore, the surrender benefit is introduced into the contract. Following Bernard and Lemieux [5] we set the surrender benefit L to be independent of the asset performance. If the policyholder surrenders the contract at time λ, he obtains

$$L(\lambda) = (1 - \beta_\lambda)\, P\, (1 + h)^\lambda\,, \tag{6}$$

where β_λ is a penalty charge for the surrender action at time λ and h refers to the minimum guaranteed rate for the surrender benefit. The penalty β is typically constant over one calender year and a decreasing function of time such that early surrender is more penalized.[3] In practice the minimum guaranteed rate h is not allowed to fall below the minimum guaranteed rate g for the survival benefit, see Bernard and Lemieux [5].

Following our rationale in Section 1 we describe the arrival of the surrender action at a random time λ by a generalized Poisson process with stochastic intensity γ. The intensity γ depends on the relationship between the surrender benefit L and the present value of the contract V. When the surrender benefit is smaller than the contract value, the surrender intensity takes a lower value $\underline{\rho}$. On the contrary, a higher value $\bar{\rho}$ is taken. Formally, γ can be expressed as

$$\gamma_t = \begin{cases} \underline{\rho}\,, & \text{for } L(t) < V_t\,, \\ \bar{\rho}\,, & \text{for } L(t) \geq V_t\,. \end{cases} \tag{7}$$

This formulation is inspired by Dai et al. [9] and can be traced back to Stanton [19] who deals with the prepayment terms in mortgage loans. In this way, we are not explicitly solving an optimal stopping problem but a randomized version of it. However, in the limiting case, when $\underline{\rho} \searrow 0$ and $\bar{\rho} \nearrow \infty$, we obtain the solution to the accompanying optimal stopping problem. Accordingly, our approach includes in the limit the aforementioned American-style contingent claim analysis of Grosen and Jørgensen [12][13], Bacinello [2][3] and Bacinello, et al. [4].

The next step is to establish the contract value V. There are at least two ways to derive it. One is to evaluate the expectation under the risk-neutral measure. We follow an alternative approach and derive the contract value by the PDE characterization using the balance law, see Dai et al. [9]. However, in the appendix we provide a detailed derivation of the main result in Proposition 1 below using the risk-neutral expectation. Of course, in our Markovian setting both methods are connected via the Feynman-Kac theorems.

[3]For examples of penalty functions please refer to Palmer [18].

The balance law is based on the no-arbitrage condition

$$r(t)V_t\mathrm{d}t \;\; = \;\; \mathbb{E}_{\mathbb{Q}}\left[\mathrm{d}V_t|\mathcal{G}_t\right], \quad 0 \le t \le T\,. \tag{8}$$

Provided that the policyholder is still alive at time t and has not surrendered the contract yet, we consider the following cases under the assumption that the two stopping times τ and λ are conditionally independent of each other:

1) The conditional probability that death occurs over $(t, t + \mathrm{d}t)$ while the surrender does not is $\mu_t\mathrm{d}t(1 - \gamma_t\mathrm{d}t) = \mu_t\mathrm{d}t$.

2) The conditional probability that surrender occurs over $(t, t + \mathrm{d}t)$ while the death event has not happened is $\gamma_t\mathrm{d}t(1 - \mu_t\mathrm{d}t) = \gamma_t\mathrm{d}t$.

3) The conditional probability that both the surrender and the death events occur over $(t, t + \mathrm{d}t)$ is 0.

Suppose that the contract value at time t is of the form $V_t = 1_{\{\lambda>t,\tau>t\}}v(t, S_t)$ for a suitably differentiable function $v : [0, T] \times \mathbb{R} \to \mathbb{R}, (t, s) \mapsto v(t, s)$. Thus we can also express γ_t as a function of the state variables, i.e. $\gamma_t = \gamma(t, S_t)$ where $\gamma : [0, T] \times \mathbb{R} \to \mathbb{R}, (t, s) \mapsto \gamma(t, s)$. Upon the occurrence of the death there is a change in the payment liability of the amount $\Psi(t, s) - v(t, s)$ and upon the occurrence of the surrender the change in the payment liability is $L(t) - v(t, s)$. Hence, we can rewrite (8) as

$$r(t)v(t, S_t)\mathrm{d}t \;\; = \;\; \mathbb{E}_{\mathbb{Q}}[\mathrm{d}v(t, S_t)|\mathcal{F}_t] + (\Psi(t, S_t) - v(t, S_t))\mu_t\mathrm{d}t + (L(t) - v(t, S_t))\gamma(t, S_t)\mathrm{d}t.$$

Applying the Ito's Lemma to $\mathrm{d}v(t, S_t)$ and assuming sufficient integrability we obtain

$$\mathbb{E}_{\mathbb{Q}}[\mathrm{d}v(t, S_t)|\mathcal{F}_t] \;\; = \;\; \mathbb{E}_{\mathbb{Q}}[\mathcal{L}v(t, S_t)\,\mathrm{d}t + \sigma(t, S_t)S_t\frac{\partial v}{\partial s}v(t, S_t)\,\mathrm{d}\hat{W}_t|\mathcal{F}_t] \;\; = \;\; \mathcal{L}v(t, S_t)\,\mathrm{d}t\,,$$

where $\mathcal{L}$ is the differential operator comprised of the partial derivative with respect to time and the generator of the process S defined in (3), i.e.

$$\mathcal{L}f(t, s) \;\; = \;\; \frac{\partial f}{\partial t}(t, s) + r(t)s\frac{\partial f}{\partial s}(t, s) + \frac{1}{2}\sigma^2(t, s)s^2\frac{\partial^2 f}{\partial s^2}(t, s)\,.$$

Then we obtain

$$\mathcal{L}v(t, s) + \mu(t)\Psi(t, s) + \gamma(t, s)L(t) - (r(t) + \mu(t) + \gamma(t, s))\,v(t, s) \;\; = \;\; 0\,.$$

By no-arbitrage, we must also have $v(T, s) = \Phi(s)$, for all $s > 0$. We have just derived the pricing PDE summarized in the following proposition.

Proposition 1. *The contract value is given by* $V_t = 1_{\{\lambda>t,\tau>t\}} v(t,S_t)$ *where the price function* v *is the solution of the partial differential equation*

$$\mathcal{L}v(t,s) + \mu(t)\Psi(t,s) + \gamma(t,s)L(t) - (r(t)+\mu(t)+\gamma(t,s))\, v(t,s) \;=\; 0\,, \tag{9}$$

for $(t,s) \in [0,T) \times \mathbb{R}^+$ *with terminal condition* $v(T,s) = \Phi(s)$, *for* $s \in \mathbb{R}^+$.

By the Feynman-Kac theorem we obtain the immediate corollary.

Corollary 1. *The value of the contract* V_t *can be represented by*

$$\begin{aligned} V_t \;=\; & 1_{\{\tau>t\}} 1_{\{\lambda>t\}} \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^T (r(u)+\mu(u)+\gamma_u)\mathrm{d}u} \Phi(S_T) \Big| \mathcal{F}_t \right] \\ & + 1_{\{\tau>t\}} 1_{\{\lambda>t\}} \mathbb{E}_{\mathbb{Q}} \left[\int_t^T e^{-\int_t^u (r(s)+\mu(s)+\gamma_s)\mathrm{d}s} \mu(u)\Psi(u,S_u)\mathrm{d}u \Big| \mathcal{F}_t \right] \\ & + 1_{\{\tau>t\}} 1_{\{\lambda>t\}} \mathbb{E}_{\mathbb{Q}} \left[\int_t^T e^{-\int_t^u (r(s)+\mu(s)+\gamma_s)\mathrm{d}s} \gamma_u L(u)\mathrm{d}u \Big| \mathcal{F}_t \right], \quad 0 \le t \le T\,. \end{aligned} \tag{10}$$

The contract value can be viewed as the discounted value of a fictitious security with the dividend payment at any time t being $\mu(t)\Psi(t,S_t) + \gamma_t L(t)$ and the final payment at T being $\Phi(S_T)$. The discount factor in this fictitious world is $e^{-\int_0^\cdot (r(u)+\mu(u)+\gamma_u)\mathrm{d}u}$.

Remark 1. *The results derived in Proposition 1 and Corollary 1 can be generalized. We have assumed that the bounds of the surrender intensity,* $\underline{\rho}$ *and* $\bar{\rho}$, *respectively, are constant. In fact, we can allow the bounds being driven by the financial market and other non-financial state variables* X, *i.e.* $\underline{\rho}_t = \underline{\rho}(t,S_t,X_t)$ *and* $\bar{\rho}_t = \bar{\rho}(t,S_t,X_t)$. *Further, we can include stochastic interest rates and stochastic volatility in our model. Under this extended setup the valuation PDE in* (9) *and the value in* (10) *carry over.*

The contract value V is influenced by the bounds $\underline{\rho}$ and $\bar{\rho}$. Intuitively it is clear that a lower value for $\underline{\rho}$ leads to less frequent surrender due to exogenous reasons and accordingly increases the contract value. Likewise, a higher value for $\bar{\rho}$ allows a higher surrender activity when it is financially profitable to do so and therefore increases the contract value. The following proposition states this fact precisely. The proof can be found in the appendix.

Proposition 2. *Suppose that* v *is the value function of the contract with bounds* $\underline{\rho}$ *and* $\bar{\rho}$, *and that* w *is the value function of the contract with bounds* $\underline{\zeta}$ *and* $\bar{\zeta}$. *Assume that* $\underline{\zeta} \le \underline{\rho}$ *and* $\bar{\rho} \le \bar{\zeta}$. *Then we have* $w(t,s) \ge v(t,s)$, *for* $(t,s) \in [0,T] \times \mathbb{R}^+$.

Corollary 2. *In the setting of Proposition 2 define the sets where exclusively exogenous surrender occurs by* $C^v = \{(t,s) \in [0,T] \times \mathbb{R}^+ : L(t) < v(t,s)\}$ *and* $C^w = \{(t,s) \in [0,T] \times \mathbb{R}^+ : L(t) < w(t,s)\}$, *respectively. Then* $C^v \subseteq C^w$.

Proof. This is an immediate consequence of Proposition 2. ∎

4 Numerical Analysis

In this section we study the life insurance contract we have specified above closely through numerical analysis. We assume that the underlying of the contract is the S&P 500 with volatility $\sigma(t, s) = 0.2$. The market interest rate is constant at $r = 0.04$. At the beginning $P = \$100$ is paid. The contract life time is 10 years. For the moment we assume that the participation rate into the minimum guaranteed amount is $\alpha = 0.85$. The minimum guaranteed rates at survival, at death and at surrender satisfy $g = g_d = h = 0.02$. The participation coefficient at survival and at death satisfy $k = k_d = 0.9$. The penalty rates are $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$ and $\beta_t = 0$ for $t \geq 5$. We further assume that the mortality intensity follows the deterministic process $\mu(t) = A + Bc^{x+t}$ for the policyholder aged x at time $t = 0$ with $A = 5.0758 \times 10^{-4}$, $B = 3.9342 \times 10^{-5}$, $c = 1.1029$. The pool of policyholders are assumed to be 40-aged at the moment they enter into the contract, see Li and Szimayer (2010) for a similar setup.

4.1 Rationality and Contract Price

We first study the effect of the policyholders' rationality on the contract price. Table 1 displays contract values V_0 for various rationalities of the policyholders that are parametrized by the lower and upper bound of the surrender intensity γ. The lower bound $\underline{\rho}$ is the base level surrender intensity representing surrender due to exogenous reasons, and takes the values 0, 0.03, 0.3. The upper bound $\bar{\rho}$ limits the local exercise probability in case exercising the surrender option is financially advantageous, see (7). It takes the values 0, 0.03, 0.3, and ∞. We may say that a policyholders acts financially more rational the lower the lower bound $\underline{\rho}$ and the higher the upper bound $\bar{\rho}$. It is clear that a higher degree of rationality leads to a higher contract price, see Proposition 2.

	$\bar{\rho}$				
$\underline{\rho}$	0.00	0.03	0.30	3.00	∞
0	102.7630	103.9335	108.2971	110.6107	110.9602
0.03	-	99.4447	103.5910	105.5440	105.8250
0.3	-	-	92.7071	94.4926	94.9999

Table 1: Contract value V_0 for various bounds $\underline{\rho}$ and $\bar{\rho}$ for the surrender intensity.

For $\underline{\rho} = \bar{\rho} = 0.00$ the surrender option is never exercised. Therefore we obtain a European-style contract with value 102.7630. Keeping $\underline{\rho} = 0.00$ and increasing the upper bound $\bar{\rho}$ to the limit ∞ results in a contract where the surrender option is exercised optimally. The value of the American-style contract is 110.9602, and is about 8% higher than the value of the corresponding European-style contract. In general we can observe that the contract values are increasing with increasing $\bar{\rho}$ as stated in Proposition 2. Purely exogenous surrender can be presented by assuming that the upper and lower bound are identical, i.e. $\underline{\rho} = \bar{\rho}$. The values on the diagonal of Table 1 are decreasing with increasing surrender rate. This is not a general effect but due to the fact that for this contract the surrender value L is on average lower than the value V of a contract that is still alive. Fixing the upper bound and varying the lower bound representing the exogenous surrender the contract values are increasing with decreasing lower bound $\underline{\rho}$ what is in line with Proposition 2.

Let us now focus on the benchmark parameters for the subsequent fair contract analysis in Section 4.3, i.e. set $\underline{\rho} = 0.03$ and $\bar{\rho} = 0.30$. The resulting contract value is 103.5910. To obtain the corresponding purely exogenous surrender situation the upper bound is set to $\bar{\rho} = 0.03$ and the value decreases to 99.4447. In contrast, for optimal exercise of the surrender option the upper bound is set to ∞. The corresponding contract value increases to 105.8250. We can interpret the benchmark value of 103.5910 as a weighted average of the purely exogenous surrender value and the value obtained when the surrender option is optimally exercised, with weights 35% and 65%, respectively.

4.2 The Separating Boundary

For the insurance company writing the contract it is instructive to identify the actual surrender intensity γ for any given time t and asset value $S_t = s$. According to (7) γ is determined by the current contract value and surrender benefit. Once the value function v is obtained by solving the pricing PDE in Propostion 1 we can identify the region C where purely exogenous surrender occurs, $\gamma(t,s) = \underline{\rho}$, i.e. $C = \{(t,s) \in [0,T] \times \mathbb{R}^+ : v(t,s) > L(t)\}$, and its complement C^c where surrender occurs at the maximal intensity, $\gamma(t,s) = \bar{\rho}$, i.e. $C^c = \{(t,s) \in [0,T] \times \mathbb{R}^+ : v(t,s) \leq L(t)\}$. The separating boundary is then the set $\partial C = \{(t,s) \in [0,T] \times \mathbb{R}^+ : v(t,s) = L(t)\}$. Moreover, Corollary 2 characterizes the relationship of C when the bounds of the surrender intensity $\underline{\rho}$ and $\bar{\rho}$ are varied. Decreasing $\underline{\rho}$ or, alternatively, increasing $\bar{\rho}$ expands the set C where purely exogenous surrender occurs.

Figure 1 displays the separating boundary for the benchmark parameters on the left and for the case when the upper bound of the surrender intensity is set to ∞ on the right. For both figures we observe that a higher underlying price makes the participation in it more attractive, and hence indicates a lower surrender rate in this region. While a lower underlying price suggests that it is not promising to benefit from the growth of the underlying price. In addition, three factors affect the separating boundary. One is the interest rate effect. In our example, the minimum guaranteed rates at death, at survival and at surrender are all smaller than the interest rate on the market. An early surrender enables the policyholders to invest their money into a riskless asset with a higher rate of return than the minimum guaranteed rate and is hence preferred. The incentive to surrender

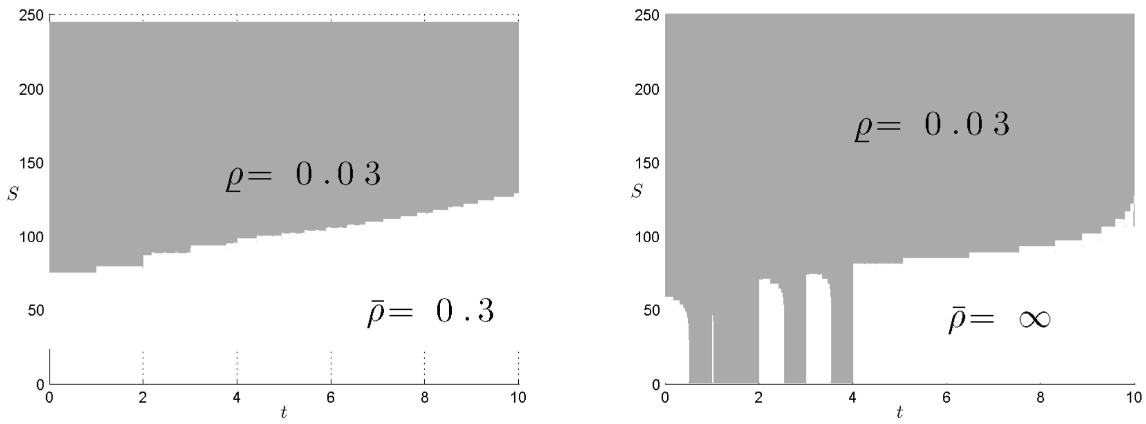

Figure 1: The separating boundary ∂C for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$ and $\beta_t = 0$ for $t \geq 5$.

the contract earlier can be reduced if the asset price is high enough so that the probability of receiving a higher payoff increases which offsets the interest rate effect. The second one is the time effect. For the same asset price level, the earlier it is, the more higher is the possibility that the asset price at a certain time point in the future will rise to a higher level, and hence, the higher is the continuation value of the contract. Thus, a lower asset price at the early stage can be more tolerated and the separating boundary can be lower at this stage due to the time effect. The third one is the penalty effect. In our example, there is $\alpha P(1+g)^t < (1-\beta_t)P(1+h)^t$, for all $t \geq 0$. Besides, $(1-\beta_t)P(1+h)^t \leq (1-\beta_{t'})P(1+h)^{t'}$ for $t \leq t'$. This indicates that for S small enough, the surrender value is always higher than the minimum guarantee. As time increases, the dominance of the surrender value is more obvious, and hence, the asset price must be higher to compensate the disadvantage of the relatively lower guaranteed amount. Figure 1 results from the three effects mentioned. Within one year, the interest rate effect dominates, while between the different years, the other two effects dominate. Consequently, the separating boundary is not smooth in the first 4 years and it is smooth and monotonically increasing afterwards. Comparing the benchmark case (left in Figure 1) with the case where the upper bound $\bar{\rho}$ is set to ∞ (right in Figure 1) we observe that the set indicating purely exogenous surrender C expands. This is expected due to Corollary 2.

Now, the penalty term is eliminated by setting $\beta_t = 0$ for all t. Then we obtain a separating boundary as displayed in Figure 2. The penalty and the time effect dominate the interest rate effect. Hence, we observe the monotonic increase of the separating boundary over the life time of the contract. Moreover, the boundary is now smooth, since the penalty parameters for different years are identical. Again, the set C where purely exogenous surrender occurs expands when the upper bound $\bar{\rho}$ is increased from 0.30 (left) to ∞ (right).

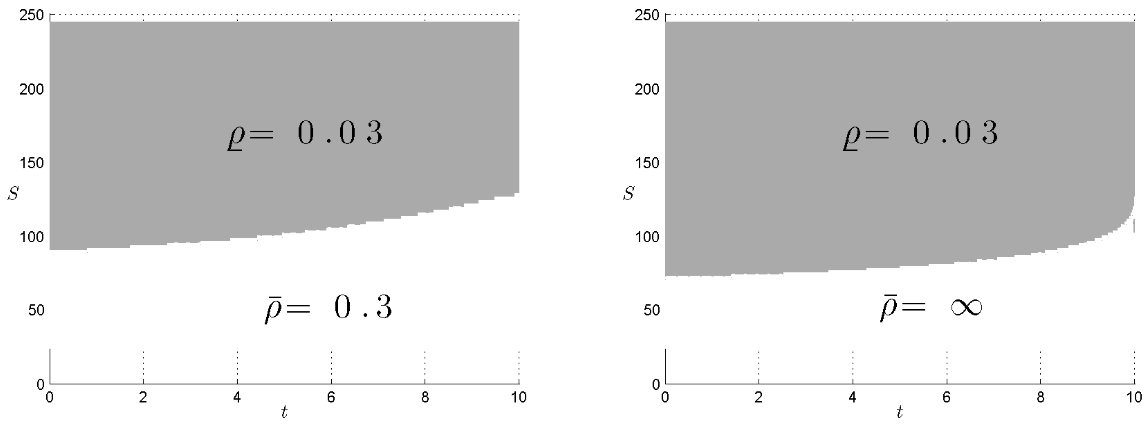

Figure 2: The separating boundary of ∂C for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_t = 0$ for $t \geq 0$.

4.3 Fair Contract Analysis

In this section we study how the parameters should be specified to ensure a fair contract, i.e. $V_0 = P = 100$. Since the contract price depends on the assumption about the rationality of the policyholders in our model, our fair contract analysis is conducted in a narrow sense by fixing the rationality of the policyholders. The price obtained is the amount that should be charged on average based on this assumption. We assume in this part that $\underline{\rho} = 0.03$ and $\bar{\rho} = 0.30$. Furthermore, we also compare the result with the case $\bar{\rho} = 0.03$ and $\bar{\rho} = \infty$. For our original parameters chosen in Section 4.1 the contract value is 103.5910 and is therefore over par. To reduce to the contract value, there are potentially three ways. The first way is to reduce the minimum guarantee at survival or at death or in both cases. The second way is to enhance the penalty in the early surrender case. The third way is to reduce the participation in the performance of the underlying asset.

We investigate the effect of a reduction of the minimum guarantee on the contract value. The reduction of minimum guarantee can be achieved either by reducing the participation rate α, the minimum guarantee rate g_1, or g_2. Since their effects are similar, we only focus on the participation rate α. In Figure 3 we present the contract values with different choices of α while other parameters are kept the same as we chose at the beginning. We notice from Figure 3 that the effect of the minimum guarantee on the contract value depends on the rationality of the policyholders. For the completely rational policyholders (i.e., $\bar{\rho} = \infty$), the minimum guarantee hardly has any effect on the contract value. When the policyholders are on average more rational than those who only surrender for exogneous reasons, the effect of the minimum guarantee is also minor. This is because a reasonable surrender guarantee is supplied in the contract. If it is unprofitable to go on holding the contract, the policyholders

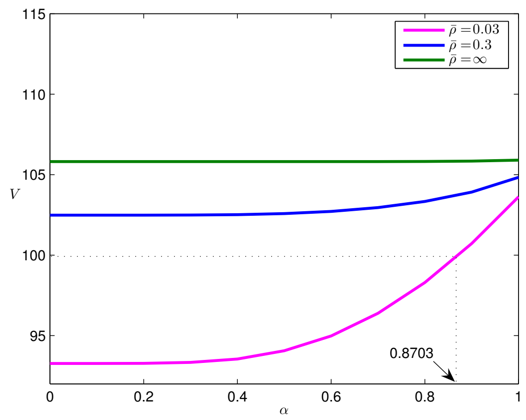

Figure 3: The contract value V_0 depending on the participation rate in the minimum guarantee α for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

can simply terminate the contract and obtain the guaranteed surrender value which may be higher than the minimum guarantee. On the contrary, for irrational policyholders (i.e., when $\bar{\rho} = 0.03$), their surrender decisions do not depend on the surrender guarantee. The effect of the minimum guarantee on the contract value is hence much higher. We can verify this rationale by setting the surrender to value to zero.

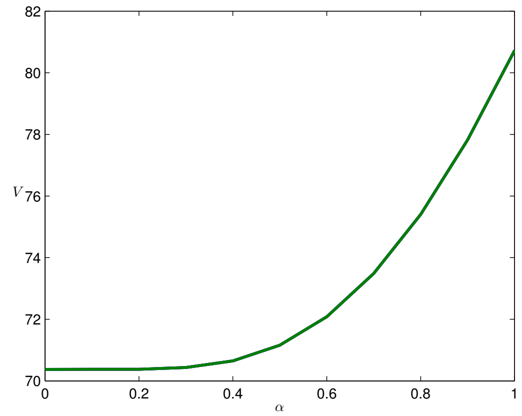

Figure 4: The contract value V_0 depending on the participation rate in the minimum guarantee α for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_t = 1$, for $t \geq 0$.

The contract values for $\beta = 1$, and hence $L = 0$, and various values for α are displayed in Figure 4. We see that the contract value in this case is actually independent of the rationality. This is because the surrender value is zero so that always the lowest surrender intensity $\underline{\rho}$ applies which is identical for the different choices for $\bar{\rho}$. We also see that when the surrender guarantee is small the participation rate α plays a more important role in determining the contract value. The contract values for $\alpha = 0$ and $\alpha = 1$ differ by 10.3529 whereas in the previous setting the difference was just 2.3552, both for $\bar{\rho} = 0.30$. The pattern is similar for $\bar{\rho} = 0.03$ and $\bar{\rho} = \infty$. On the other hand we can interpret from Figure 4 that to ensure the contract to be issued at par the policyholders should not be overpenalized. In Bernard and Lemieux [5], the participation rate α is included both in the minimum guarantee and in the asset performance. Hence, the variation of the parameter works simultaneously on both parts which may display a more significant effect. However, when we observe these two parts separately we are more clear about the specific effect of each parameter and gain insight into the design of effective contracts. According to the contract that we have designed we can simply keep $\alpha = 1$ so that the contract looks more attractive to the policyholders. While other parameters should be adjusted more carefully.

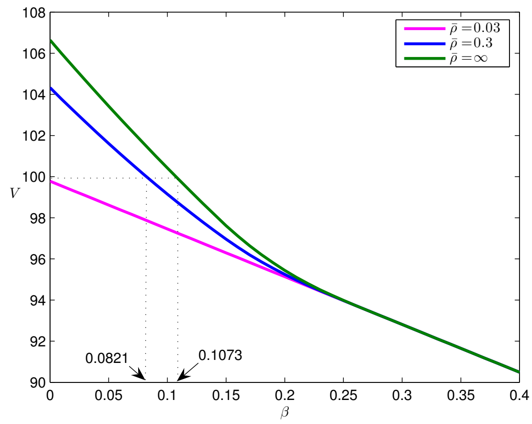

Figure 5: The contract value V_0 depending on the penalty parameter β for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = h = 0.02$, $k = k_d = 0.9$.

Next, we investigate the relationship between the penalty parameter and the contract value. In Figure 5 we display the contract value V_0 as a function of penalty parameter β graphically for different degrees of rationality, $\bar{\rho} = 0.03, 0.30, \infty$. The contract value is monotonically decreasing in the penalty parameter. For the contract to be fairly issued the penalty parameter should be 0.0821 for $\bar{\rho} = 0.30$. In case of rational surrender, i.e. $\bar{\rho} = \infty$, in presence of exogenous surrender with $\underline{\rho} = 0.03$ the penalty parameter has to be increased to 0.1073 for the contract to be fair. While for purely exogenous surrender, i.e. $\bar{\rho} = 0.03$, the contract value is always under par in our example. This means that other parameters must be adjusted so as to take the policyholders' irrationality into account properly.

Finally, we analyze the effect of the participation rate in the asset performance on the contract value. For simplicity we assume the participation rates for both, the survival and

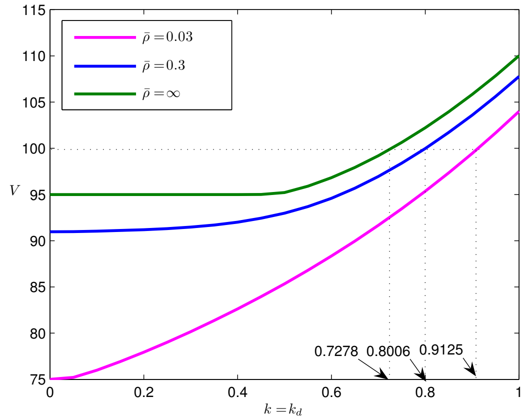

Figure 6: The contract value V_0 depending on the participation rates $k = k_d$ for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

the death events, to be the same namely, $k = k_d$. Other parameters are consistent with the values detailed at the beginning of Section 4. In Figure 6 we display the relationship of the participation rates in the asset performance with the contract value graphically for $\bar{\rho} = 0.03, 0.30, \infty$. We see that the contract value increases monotonically with the participation rates. For $\bar{\rho} = 0.30$ and ∞ the increase is not that large for small values of the participation rates $k = k_d$. This is because in these cases holding the contract generally brings lower benefit to the policyholders than surrendering the contract prematurely. The surrender benefit thus plays a dominant role in determining the contract value. Since the surrender guarantee is independent of k and k_d in our numerical example, the contract value does not vary too for small values of $k = k_d$. Also notice that that in this case the contract value is under par. On the contrary, when k and k_d are large the survival benefit and the death benefit dominate the contract value, the contract value increases is more sensitive to changes of $k = k_d$. However, when the policyholders surrender due to exogenous reasons (indicated by $\bar{\rho} = 0.03$) the survival and death benefit are driving the contract value. Hence the effect of an increase in $k = k_d$ on the contract value is nearly linear. To obtain a fair contract the participation rates $k = k_d$ should be set to 0.8006 for $\bar{\rho} = 0.30$. Increasing $\bar{\rho}$ to its limit ∞ requires a lower participation of $k = k_d = 0.7278$ for $\bar{\rho} = 0.03$ for the contract to be fair. In contrast, for the cases of purely exogenous surrender, i.e. $\bar{\rho} = 0.03$, the participation rate has to increase to 0.9125 to constitute a fair contract.

In the remainder of this section we focus on the design of a fair contract and investigate the interaction of various parameters. First, we study the relationship between participation rate in the minimum guarantee α and the minimum guaranteed rate at survival and at death $g = g_d$. To produce realistic results we alter the benchmark parameters by setting

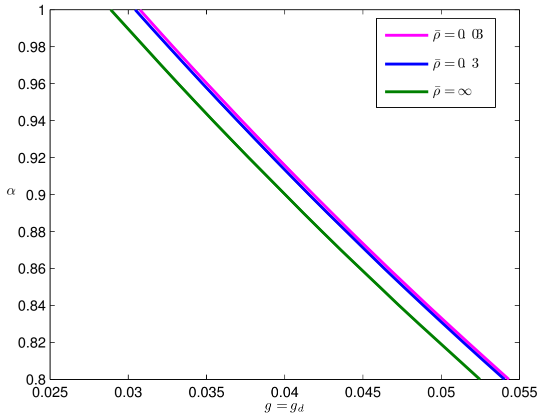

Figure 7: Parameter combinations of the participation rate in the minimum guarantee α and the minimum guaranteed rates at survival and at death $g = g_d$ ensuring a fair contract, for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $g = g_d = h = 0.02$, $k = k_d = 0.7$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

$k = k_d = 0.7$ to ensure the existence of a fair contract. We present the relationship between α and $g = g_d$ in Figure 7. We see that α is decreasing in $g = g_d$. For α below 0.9 the minimum guaranteed rate of return at survival and at death must be higher than the market interest rate for the contract value to be higher. Further note that the higher the rationality of policyholders is, the lower is the $\alpha - g$ level in Figure 7. Since the more rational policyholders can judge the situation more correctly and make the better out of it, they need less compensation offered by the minimum guarantee.

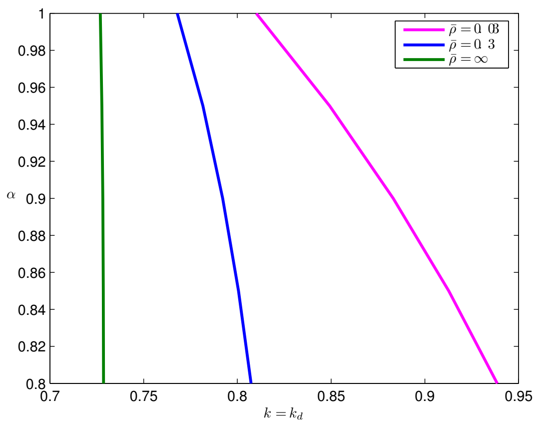

Figure 8: Parameter combinations of the participation rate in the minimum guarantee α and participation rates in the asset performance at survival and at death $k = k_d$ ensuring a fair contract, for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $g = g_d = h = 0.02$, $k = k_d = 0.9$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

Next we study pairs of the participation rate in the minimum guarantee α and the participation parameters in the asset performance k and k_d such that a fair contract is obtained. The other parameters are kept as in the benchmark case. A graphical illustration for this setting is given in Figure 8. We observe that for the same level of α, a lower (higher) $k = k_d$ is required to account for the higher (lower) rationality of the pool of policyholders. Moreover, when the policyholders act more rational, the sensitivity of α with respect to $k = k_d$ is higher, or in other words, the sensitivity of $k = k_d$ with respect to α is lower.

We have mentioned in Section 3 that the growth rate h for the surrender case is, in practice, not allowed to fall below the minimum guaranteed rate g for the survival benefit. For our numerical analysis, however, we loose this restriction and study the relationship of h with other parameters. As an example, we present in Figure 9 the relationship between h and $k = k_d$. It is obvious that for given $k = k_d$, h must be set lower (higher) to account for the higher (lower) rationality of the policyholders. For the policyholders with low rationality, a fair contract may not even exist if we keep h at the same level as g, and at the same time, only allow the policyholders to participate in the asset performance less than proportionally.

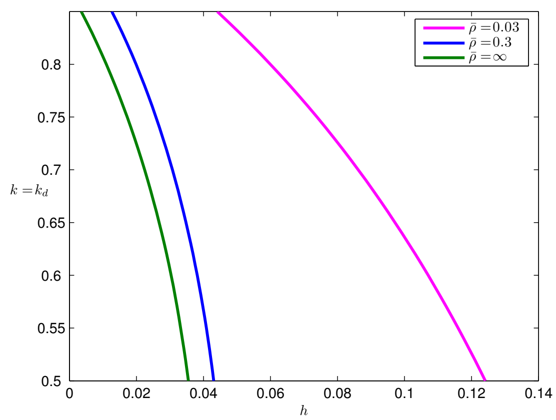

Figure 9: Parameter combinations of the minimum guaranteed rate h for the surrender benefit and the participation rates in the asset performance at survival and at death $k = k_d$ ensuring a fair contract, for $\underline{\rho} = 0.03$, $\bar{\rho} \in \{0.03, 0.3, \infty\}$, $\alpha = 0.85$, $g = g_d = 0.02$, $\beta_1 = 0.05$, $\beta_2 = 0.04$, $\beta_3 = 0.02$, $\beta_4 = 0.01$, and $\beta_t = 0$, for $t \geq 5$.

The above fair contract analysis tells us that the rationality of the policyholders should also be taken into account when designing a fair contract.

5 Conclusion

In this paper we have studied the valuation of unit-linked life insurance contracts with surrender guarantees. Instead of solving an optimal stopping problem, we have proposed a more realistic approach accounting for policyholders' rationality in exercising their surrender option. The valuation is conducted at the portfolio level by assuming the surrender intensity to be bounded from below and from above. The lower bound corresponds to purely exogenous surrender and the upper bound represents the limited rationality of the policyholders. In practice, the lower and the upper bounds can be obtained from historical data. We have shown that for different degrees of rationality indicated by the difference between the lower and the upper bounds of the surrender intensity, the average contract values are significantly different. Hence, it is important to judge the rationality of the potential policyholders realistically. Based on the realistic estimation of their rationality, the contract can be designed more reasonably and an average overvaluation can be avoided. We provide the separating boundary between purely exogenous surrender and surrender due to financial reasons. This may help insurance companies to better understand the surrender activity of their policyholders affecting also the companies' hedge programs. In addition, our fair contract analysis has revealed specific contract designs that are fairly robust with respect to the degree of rationality of the policyholders.

This paper can be extended in several ways. As indicated in Remark 1 we can extend the model to allow for stochastic interest rates and stochastic volatility. Further, the bounds $\underline{\rho}$ and $\bar{\rho}$ need not be constant but can be driven by market variables and non-financial factors. The general results in Proposition 1 and Proposition 2 and the respective corollaries are likely to carry over. However, in a multi-factor model solving the valuation PDE can easily become a high dimensional problem. In this case, least-squared Monte Carlo simulation following Longstaff and Schwartz [16] can be adapted. This issue will be addressed in future research. A further interesting perspective is to incorporate a secondary market where the policyholder are given the additional option to sell their contract to a third party. In an extended version of our framework we plan to study the impact of a secondary market on contract value and fair contract design.

Appendix A: Proof of Proposition 2

Proof. The function v is the solution of the PDE (9) with terminal condition $v(T,s) = \Phi(s)$ and bounds $\underline{\rho}$ and $\bar{\rho}$. The function w is the solution of the same PDE (9) with identical terminal condition $w(T,s) = \Phi(s)$ but different bounds $\underline{\zeta}$ and $\bar{\zeta}$. Assume that $\underline{\zeta} \leq \underline{\rho}$ and $\bar{\rho} \leq \bar{\zeta}$. Now define $z = w - v$. It follows directly that $z(T,s) = w(T,s) - v(t,s) = \Phi(s) - \Phi(s) = 0$. To obtain the dynamics of z take the difference of the PDEs describing w

and v, i.e.:

$$\begin{aligned} 0 &= \mathcal{L}w(t,s) + \mu(t)\Psi(t,s) + \gamma^{\zeta}(t,s)L(t) - (r(t)+\mu(t)+\gamma^{\zeta}(t,s))\,w(t,s) \\ &\quad - (\mathcal{L}v(t,s) + \mu(t)\Psi(t,s) + \gamma^{\rho}(t,s)L(t) - (r(t)+\mu(t)+\gamma^{\rho}(t,s))\,v(t,s)) \\ &= \mathcal{L}z(t,s) + (\gamma^{w}(t,s) - \gamma^{v}(t,s))\,(L(t) - w(t,s)) - (r(t)+\mu(t)+\gamma^{v}(t,s))\,z(t,s)\,, \end{aligned}$$

were γ^v and γ^w, respectively, are given by (7) using the appropriate bounds. By Feynman-Kac we obtain the stochastic representation of z as follows

$$z(t,s) = \mathbb{E}^{t,s}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u (r(x)+\mu(x)+\gamma^v(x,S_x))\,\mathrm{d}x}(\gamma^w(u,S_u) - \gamma^v(u,S_u))\,(L(u) - w(u,S_u))\mathrm{d}u\right],$$

where $\mathbb{E}^{t,s}_{\mathbb{Q}}$ denotes the expectation conditioned on $S_t = s$. From the definition of γ^w in (7) and the assumption $\bar{\zeta} \geq \bar{\rho}$ we see that if $(L - w) \geq 0$ we have $\gamma^w = \bar{\zeta} \geq \bar{\rho} \geq \gamma^v$ and thus $(\gamma^w - \gamma^v) \geq 0$. On the other hand, if $(L - w) < 0$ then $\gamma^w = \underline{\zeta}$. By assumption we have $\underline{\zeta} \leq \underline{\rho}$ and thus $\gamma^w \leq \underline{\rho} \leq \gamma^v$, or, $(\gamma^w - \gamma^v) \leq 0$. Thus, we see that the integrand in the above equation is nonnegative and therefore $z \geq 0$. Since $z = w - v$ we obtain $w \geq v$. □

Appendix B: Alternative Derivation of Proposition 1

To apply the martingale approach, we first describe the arrival of the surrender event with the jump process J satisfying $J_t = 1_{\{\lambda \leq t\}}$ for $t \in [0,T]$. The jump process generates the filtration $\mathbb{J} = (\mathcal{J}_t)_{0 \leq t \leq T}$. We denote the joint filtration of $\mathbb{F}$, $\mathbb{H}$ and $\mathbb{J}$ as $\mathbb{G}$. We know from the specification of the intensity of the jump process J that it is $\mathbb{F}$-measurable. However, it only determines the probable but not the real occurrence of λ. Mathematically it means that the σ-fields $\mathcal{F}_T$ and $\mathcal{J}_t$ are independent given $\mathcal{F}_t$ under the real world probability measure $\mathbb{P}$. Hence, λ is a $\mathbb{G}$- but not an $\mathbb{F}$-stopping time. The contract value at time t can be expressed as

$$\begin{aligned} V_t &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}1_{\{\tau>T,\lambda>T\}}\Phi(S_T)\Big|\mathcal{G}_t\right] && \text{(Part 1: survival benefit value)} \\ &\quad + \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^\tau r(u)du}1_{\{t<\tau\leq T,\tau<\lambda\}}\Psi(\tau,S_\tau)\Big|\mathcal{G}_t\right] && \text{(Part 2: death benefit value)} \\ &\quad + \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^\lambda r(u)du}1_{\{t<\lambda\leq T,\lambda<\tau\}}L(\lambda)\Big|\mathcal{G}_t\right] && \text{(Part 3: surrender benefit value)} \end{aligned} \quad (11)$$

Following Duffie et al. [10] we can express V_t with a so-called reduced form which we summarize in Proposition 3.

Proposition 3. *Suppose the setup detailed in Section 2 and Section 3, the value process V_t*

in (11) *has the representation*

$$\begin{aligned} V_t &= 1_{\{\tau>t\}}1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T(r(u)+\mu(u)+\gamma_u)du}\Phi(S_T)\Big|\mathcal{F}_t\right] \\ &+1_{\{\tau>t\}}1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u(r(s)+\mu(s)+\gamma_s)ds}\mu(u)\Psi(u,S_u)du\Big|\mathcal{F}_t\right] \\ &+1_{\{\tau>t\}}1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u(r(s)+\mu(s)+\gamma_s)ds}\gamma_u L(u)du\Big|\mathcal{F}_t\right], \quad 0\le t\le T. \end{aligned} \tag{12}$$

Proof. We prove the above equation part by part.

$$\begin{aligned} \text{Part 1} &= \mathbb{E}_{\mathbb{Q}}\left[1_{\{\tau>T\}}1_{\{\lambda>T\}}\Phi(S_T)e^{-\int_t^T r(u)du}\Big|\mathcal{F}_t\vee\mathcal{H}_t\vee\mathcal{J}_t\right] \\ &= 1_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T(r(u)+\mu(u))du}1_{\{\lambda>T\}}\Phi(S_T)\Big|\mathcal{F}_t\vee\mathcal{J}_t\right] \\ &= 1_{\{\tau>t\}}1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T(r(u)+\mu(u)+\gamma_u)du}\Phi(S_T)\Big|\mathcal{F}_t\right]. \end{aligned} \tag{13}$$

The second equation follows from Bielecki et al. [6] Corollary 5.1.1. Notice that $1_{\{\lambda>T\}}\Phi(S_T)$ is $(\mathcal{F}_T\vee\mathcal{J}_T)$-measurable. Similarly, the same argument leads to the third equation relying on the $\mathcal{F}_T$-measurability of $\Phi(S_T)$.

$$\begin{aligned} \text{Part 2} &= \mathbb{E}_{\mathbb{Q}}\left[1_{\{t<\tau\le T\}}1_{\{\tau<\lambda\}}\Psi(\tau,S_\tau)e^{-\int_t^\tau r(u)du}\Big|\mathcal{F}_t\vee\mathcal{H}_t\vee\mathcal{J}_t\right] \\ &= 1_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[1_{\{t<\tau\le T\}}1_{\{\tau<\lambda\}}e^{\int_0^t\mu(u)du}\Psi(\tau,S_\tau)e^{-\int_t^\tau r(u)du}\Big|\mathcal{F}_t\vee\mathcal{J}_t\right] \\ &= 1_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[\int_t^T 1_{\{\lambda>u\}}e^{\int_0^t\mu(u)du}\Psi(u,S_u)e^{-\int_t^u r(s)ds}\mathrm{d}\mathbb{Q}(\tau\le u|\mathcal{F}_T)\Big|\mathcal{F}_t\vee\mathcal{J}_t\right] \\ &= 1_{\{\tau>t\}}\int_t^T\mathbb{E}_{\mathbb{Q}}\left[1_{\{\lambda>u\}}e^{\int_0^t\mu(u)du}\Psi(u,S_u)e^{-\int_t^u r(s)ds}\mu(u)e^{-\int_0^u\mu(s)ds}du\Big|\mathcal{F}_t\vee\mathcal{J}_t\right]\mathrm{d}u \\ &= 1_{\{\tau>t\}}\int_t^T\mathbb{E}_{\mathbb{Q}}\left[1_{\{\lambda>u\}}e^{-\int_t^u(r(s)+\mu(s))ds}\mu(u)\Psi(u,S_u)\Big|\mathcal{F}_t\vee\mathcal{J}_t\right]du \\ &= 1_{\{\tau>t\}}\int_t^T 1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^u(r(s)+\mu(s)+\gamma_s)ds}\mu(u)\Psi(u,S_u)\Big|\mathcal{F}_t\right]du \\ &= 1_{\{\tau>t\}}1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\int_t^T\left[e^{-\int_t^u(r(s)+\mu(s)+\gamma_s)ds}\mu(u)\Psi(u,S_u)du\Big|\mathcal{F}_t\right]. \end{aligned} \tag{14}$$

And finally

$$\begin{aligned}
\text{Part 3} &= \mathbb{E}_{\mathbb{Q}}\left[1_{\{t<\lambda\leq T\}}1_{\{\lambda<\tau\}}L(\lambda)e^{-\int_t^\lambda r(u)du}\middle|\mathcal{F}_t\vee\mathcal{H}_t\vee\mathcal{J}_t\right] \\
&= 1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\left[1_{\{t<\lambda\leq T\}}e^{\int_0^t\gamma_u du}1_{\{\lambda<\tau\}}L(\lambda)e^{-\int_t^\lambda r(u)du}\middle|\mathcal{F}_t\vee\mathcal{H}_t\right] \\
&= 1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{\int_0^t\gamma_s ds}1_{\{\tau>u\}}L(u)e^{-\int_t^u r(s)ds}d\mathbb{Q}(\lambda\leq u|\mathcal{F}_T)|\mathcal{F}_t\vee\mathcal{H}_t\right] \\
&= 1_{\{\lambda>t\}}\int_t^T\mathbb{E}_{\mathbb{Q}}\left[e^{\int_0^t\gamma_s ds}1_{\{\tau>u\}}L(u)e^{-\int_t^u r(s)ds}\gamma_u e^{-\int_0^u\gamma_s ds}\middle|\mathcal{F}_t\vee\mathcal{H}_t\right]du \\
&= 1_{\{\lambda>t\}}\int_t^T 1_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^u(r(s)+\mu(s)+\gamma_s)ds}\gamma_u L(u)\middle|\mathcal{F}_t\right]du \\
&= 1_{\{\tau>t\}}1_{\{\lambda>t\}}\mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u(r(s)+\mu(s)+\gamma_s)ds}\gamma_u L(u)du\middle|\mathcal{F}_t\right]. \qquad (15)
\end{aligned}$$

Summing up (13), (14) and (15) we obtain (12). □

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