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Ambiguity and Social Interaction

by

Juergen Eichberger, David Kelsey,
Burkhard C. Schipper

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Bonn Graduate School of Economics
Department of Economics
University of Bonn
Adenauerallee 24 - 42
D-53113 Bonn

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AMBIGUITY AND SOCIAL INTERACTION[1]

Jürgen Eichberger
Alfred Weber Institut,
Universität Heidelberg.

David Kelsey
Department of Economics,
The University of Birmingham.

Burkhard Schipper
Department of Economics,
University of Bonn.

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Abstract

We examine the impact of ambiguity on economic behaviour. We present a relatively non-technical account of ambiguity and show how it may be applied in economics. Optimistic and pessimistic responses to ambiguity are formally modelled. We show that pessimism has the effect of increasing (decreasing) equilibrium prices under Cournot (Bertrand) competition. We also examine the effects of ambiguity on peace processes. It is shown that ambiguity can act to select equilibria in coordination games with multiple equilibria. Some comparative statics results are derived for the impact of ambiguity in games with strategic complements.

Address for Correspondence David Kelsey, Department of Economics, The University of Birmingham, Birmingham, B15 2TT, ENGLAND

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1 Introduction

1.1 Motivation

Uncertainty has long been recognized being an important influence on economic behaviour. Knight (1921) made a distinction between *risk*, i.e., situations where the probabilities are known, and *uncertainty*, where probabilities are unknown or imperfectly known. In the subsequent literature, situations with unknown probabilities have been referred to as *ambiguity* to distinguish them from other kinds of uncertainty. Many current policy questions concern ambiguous risks, for instance, threats from terrorism and rogue states, the safety of the MMR (measles mumps and rubella) vaccine and the likely impact of new technologies. A key element of Knight's theory was that people differ in their attitudes to ambiguity. The majority of people tend to avoid ambiguous situations. However a minority of individuals actually appear to seek ambiguity. In his theory of profit and entrepreneurial activity, Knight argued that entrepreneurs tend to be individuals who are less ambiguity-averse.

Experimental evidence shows a similar pattern. A majority of experimental subjects behave more cautiously when probabilities are undefined, while a significant minority display the opposite attitude, (see for instance Camerer and Weber (1992)). Moreover the same individual may be pessimistic in one situation and optimistic in another. Henceforth we shall refer to such cautious behaviour in face of ambiguity as *pessimism.* Ambiguity seeking behaviour will be referred to as *optimism.* The evidence shows that ambiguity attitudes are distinct from risk attitudes, Cohen, Jaffray, and Said (1985). Individuals may be risk-averse and ambiguity-loving and vice-versa.

This paper aims to present a relatively accessible exposition of recent developments in the theory of uncertainty in the Knightian sense. In particular we study the impact of ambiguity in games. Players may react to ambiguity in a pessimistic way by putting more weight on the worst outcome of any possible course of action than an expected utility maximiser would. Alternatively, like Knight's entrepreneurs, they may react optimistically to ambiguity and over-weight the best outcome. The paper makes a number of new contributions. One of these is that the previous literature focused on ambiguity-aversion, while we allow for the possibility that some individuals may be ambiguity-seeking. In addition, the paper discusses new applications and presents some comparative static results on the impact of ambiguity. Specifically we consider the impact of ambiguity in some familiar models from industrial organisation. In a Cournot oligopoly with linear demand and constant marginal cost, the worst outcome would be perceived as a rival producing a large quantity. Under these assumptions, Cournot oligopoly is a game of strategic substitutes. Hence pessimism has the effect of reducing the perceived marginal benefit of producing more and so reduces the equilibrium output. This raises profits but reduces consumer surplus. In contrast, in Bertrand competition, a bad outcome would be perceived as rival firms charging a low price. Typically there is strategic complementarity in Bertrand models. In this case pessimism will reduce the incentive for any given firm to increase its price and hence will also reduce the equilibrium price. In both oligopoly models optimism has the opposite effect.

In these models, there is scope for strategic delegation. In both Bertrand and Cournot oligopoly we show that it is desirable for the owner of a firm to delegate decision-making to a manager who is more optimistic than (s)he is. This is in contrast to the so-called 'Dutch Book' arguments, see Green (1987) or Freedman and Purves (1969). In a typical version of the Dutch Book argument, there is a book-maker and an outsider. It is argued that the book-maker will lose money for certain if (s)he have does not have expected utility preferences. However these models are rather stylised. Often the book-maker is not allowed to consider the impact of his choices on the outsider's behaviour. In contrast, we show that optimistic behaviour may be an advantage in some standard economic models.

We believe that ambiguity will have an important impact in other social sciences as well as economics. For instance, environmental risks are often ambiguous due to limited knowledge of the relevant science and because outcomes will only be seen many decades from now. The effects of global warming and the environmental impact of GM crops are two examples. To illustrate the possibilities we consider an application to peace-making in section 5. We find that ambiguity-attitudes can be an important determinant of the success or failure of a peace process.

1.2 Background

For several decades, subjective expected utility (henceforth SEU) by Savage (1954) appeared to have rendered the distinction between risk and ambiguity obsolete. In this theory, individuals faced with uncertainty behave as if they held beliefs that can be represented by a subjective probability distribution. Hence, from an analytical point of view, there was little distinction between risk and ambiguity.

However early evidence by Ellsberg (1961) suggests that beliefs cannot be represented by conventional probabilities. Systematic laboratory experiments have confirmed Ellsberg's conjecture, Camerer and Weber (1992). Moreover, experimental work, e.g., by Gonzalez and Wu (1999) and Tversky and Kahneman (1992), suggests that decision-makers distort their beliefs in a predictable way. One of the best supported experimental results is that individuals react differently according to the source of uncertainty. In particular, they behave much more cautiously in circumstances, which are in some sense unfamiliar, Kilka and Weber (1998). The evidence appears to indicate that decision makers overweight unlikely events associated with bad outcomes.[1]

Despite the experimental evidence, SEU proved to be a successful modelling tool. Important insights were obtained from the distinction between risk preferences and

[1] Psychologists attribute such ambiguity-aversion to the categorical distinction individuals make between impossibility, possibility and certainty. Tversky and Wakker (1995) and Wakker (2001) provide further references for this evidence.

beliefs, which can be made in this approach. The economics of insurance and information could be developed in this context. It is desirable to develop a theory of ambiguity, which was equally suitable for application.

The inconsistencies between Savages's theory and empirically observed behaviour have stimulated efforts for alternative theories. In our opinion, one of the most promising of these is *Choquet expected utility (henceforth CEU)*, which involves representing individuals' beliefs by *non-additive probabilities* (or *capacities*), see Schmeidler (1989). In this theory, individuals maximise the expected value of a utility function with respect to a non-additive belief, (the expectation is expressed as a Choquet integral, Choquet (1953-4)). CEU is a generalization of subjective expected utility. It has the advantage that it maintains the separation of beliefs and outcome evaluation, which makes the theory easier to apply in economics.

We restrict attention to a special case, where preferences may be represented as a weighted average of the expected utility, the maximum utility and the minimum utility. This implies that preferences can be represented as choosing an act a to maximise:

$$\lambda M(a) + \gamma m(a) + (1 - \gamma - \lambda)\mathbf{E}_{\pi} u(a), \tag{1}$$

where $M(a)$ (resp. $m(a)$) denotes the maximum (resp. minimum) utility of act a and denotes $\mathbf{E}_{\pi} u(a)$ the expected utility of act a. In this case the decision-maker maximises a weighted average of the minimum, the maximum and the mean pay-offs.

1.3 Games with Ambiguity

So far ambiguity has been discussed mainly in regard to exogenous uncertainty. In economics, uncertainty often concerns the behaviour of others. This is usually modelled by Nash equilibrium. In this paper we use an alternative solution concept to model ambiguity in games. There are two main modifications to Nash equilibrium. Firstly players have CEU preferences rather than SEU preferences. Secondly a player may perceive some ambiguity about whether or not his/her opponents play best responses.

In addition to reducing mathematical complexity, the preferences described by equation (1) have the advantage of providing an intuitive representation of behaviour in the presence of ambiguity. This enables us to study how equilibrium behaviour varies with changes in ambiguity or attitudes to ambiguity. Working with these preferences is almost as easy as with expected utility theory. The theory is intuitive and can explain some puzzles of the traditional theory. Most importantly, it allows us to study the impact of ambiguity in economic models, an analysis which is impossible with expected utility theory. Despite such advantages, we believe that the success of the model will ultimately depend on its ability to provide new explanations for economic phenomena.

Organisation of the Paper Section 2 describes how to model individual behaviour in the presence of ambiguity. Then in section 3 we discuss a solution concept for games, which allows for the possibility that the behaviour of other players may be perceived to be ambiguous. In section 4 we demonstrate possible economic applications by applying these ideas to oligopoly models. Non-economic applications are demonstrated by the model of peace-making in section 5. Some more general results concerning the comparative statics of ambiguity are presented in section 6 and section 7 contains our conclusions. The appendix contains proofs of those results not proved in the text.

2 Modelling Ambiguity

In this section we explain how ambiguity can be modelled by non-additive beliefs. We present the concepts of a *neo-capacity* (non-additive belief) and of the *Choquet integral*. This enables to represent preferences as an ‘expectation’ with respect to a possibly non-additive belief.

2.1 Games

In this paper we shall consider the impact of ambiguity on behaviour in games. Here, ambiguity concerns the possible play of one's opponents. Consider a game $\Gamma = (I, (S_i, u_i)_{i \in I})$ with two players $I = \{1, 2\}$, where each player's *strategy set* $S_i \subseteq \mathbb{R}$ is a closed and bounded interval. Economic examples often deal with quantities and prices. Hence, for most purposes, it is sufficient to assume that each agent chooses a real number. (Extensions to more complex strategy sets are straightforward.) The pay-off function of individual i denoted by $u_i(s_i, s_{-i})$ is assumed to be concave and twice continuously differentiable.[2] The following notational conventions will be maintained throughout this paper. The set of strategy combinations will be denoted by $S = S_1 \times S_2$. A typical strategy combination $s \in S$ can be decomposed into the strategy s_i of player i and the strategy combination of the opposing player s_{-i}, $s = (s_i, s_{-i})$. The set of strategy combinations of player i's opponent is denoted by S_{-i}.

2.2 Non-Additive Beliefs and Expectations

Consider an economic agent whose profit may depend in part on the behaviour of rivals. We shall represent individuals' beliefs by capacities. A *capacity* plays a similar role to a subjective probability in SEU. For most of this paper we shall confine attention to neo-capacities, defined below.

Definition 2.1 *Let γ, λ be real numbers such that $0 < \gamma \leqslant 1, 0 < \lambda \leqslant 1 - \gamma$, define a neo-capacity ν by $\nu(A) = \lambda + (1 - \lambda - \gamma)\pi(A), \emptyset \subsetneqq A \subsetneqq S_{-i}; \nu(\varnothing) = 0, \nu(S_{-i}) = 1.$*[3]

[2]In most economic applications these assumptions are satisfied or can be relaxed in obvious ways. For instance, if strategies are multi-dimensional, it is sufficient to require that the strategy set is a closed and bounded subset of $\mathbb{R}^n$. This ensures that maxima or minima exist. Extensions to more than two players are possible but would introduce technical complications concerning the product capacity, see Eichberger and Kelsey (2000) and Eichberger and Kelsey (2002).

[3]Neo is an abbreviation for Non-extremal outcome additive. Neo-capacities are axiomatised in

Neo-capacities provide a useful example. We say that a neo-capacity is *optimistic* if $\gamma = 0$, *pessimistic* if $\lambda = 0$. If $\gamma = \lambda = 0$, a neo-capacity is *additive,* which implies that for all events A, B, $A \cap B = \emptyset$, $\nu(A \cup B) = \nu(A) + \nu(B)$. An additive capacity is a conventional probability distribution. Probability distributions, represent very precise beliefs.

As in standard decision theory, one wishes to assign an expected value to acts, which a decision-maker may choose. Let u, be a utility function which represents the decision-makers' pay-offs as a function of the acts of his/her opponents. An expectation of u with respect to a capacity ν, can be defined by the *Choquet integral,* which allows optimistic and/or pessimistic responses to uncertainty by over-weighting good and/or bad outcomes.[4]

Definition 2.2 *The Choquet expected value of the utility function,* $u_i : S_{-i} \to \mathbb{R}$ *with respect to the neo-capacity* $\nu = \lambda + (1 - \lambda - \gamma)\,\pi$ *is given by:*

$$\int u_i\left(s_i, s_{-i}\right) d\nu = \lambda M_i\left(s_i\right) + \gamma m_i\left(s_i\right) + (1 - \gamma - \lambda) \cdot \mathbf{E}_\pi u_i\left(s_i, s_{-i}\right), \tag{2}$$

where $\mathbf{E}_\pi u_i\left(s_i, \cdot\right)$ *denotes the expected utility of* u_i *with respect to the probability distribution* π *on* S_{-i} *and* $M_i\left(s_i\right) = \max_{s \in S_{-i}} u_i\left(s_i, s_{-i}\right)$ *and* $m_i\left(s_i\right) = \min_{s \in S_{-i}} u_i\left(s_i, s_{-i}\right).$

In this case, the Choquet integral is a weighted average of the minimum the maximum and the mean pay-offs. There is experimental evidence that preferences have this form see Lopes (1987).[5] Intuitively a neo capacity describes a situation in which the individual believes the likelihood of events is described by the additive probability distribution π. However (s)he lacks confidence in this belief. In part (s)he reacts to

Chateauneuf, Eichberger, and Grant (2002).

[4] Gilboa (1987), Schmeidler (1989) and Sarin and Wakker (1992) provide axiomatisations for CEU preferences. Ghirardato and Marinacci (2002) and Wakker (2001) characterise capacities representing ambiguity-averse or pessimistic attitudes of a decision maker. There is also a closely related literature which represents beliefs as sets of conventional probability distributions, see Bewley (1986), Gilboa and Schmeidler (1989), Kelsey (1994).

[5] Such preferences have been axiomatised in the context of risk by Cohen (1992).

this in an optimistic way measured by λ and in part the reaction is pessimistic, measured by γ. We shall assume that all individuals have CEU preferences and beliefs, which can be represented by a neo capacity. The following examples relate CEU to some more familiar decision rules:

1. If $\lambda = 0$, preferences have the maximin form and are extremely pessimistic;
2. If $\gamma = 0$, preferences exhibit the maximal degree of optimism;
3. If $\gamma + \lambda = 1$ these preferences coincide with the Hurwicz criterion, (see Hurwicz (1951)).

The Choquet integral is similar to a conventional expectation since it is a weighted average of utilities and the weights sum to 1. However the weights are not probabilities but decision weights. Neo-capacities allow us to model optimistic or pessimistic individuals according to the decision weights which they apply to outcomes. The best (resp. worst) outcome, M (resp. m) gets weight $\lambda + (1 - \gamma - \lambda)\,\pi(M)$ (resp. $\gamma + (1 - \gamma - \lambda)\,\pi(m)$). For any other outcome x the decision weight is $(1 - \gamma - \lambda)\,\pi(x)$. The Choquet integral is simply the sum over all outcomes of the act weighted by these decision weights. For additive capacities, the Choquet integral is the usual expected value of the act. If good (resp. bad) outcomes are over-weighted, we may interpret this as optimistic (resp. pessimistic) attitudes towards ambiguity.[6]

Definition 2.3 *Let $\nu = \lambda + (1 - \lambda - \gamma)\,\pi$ be a neo-capacity. We define the degree of optimism (resp. pessimism) of ν by $\lambda(\nu) = \lambda$, (resp. $\gamma(\nu) = \gamma$).*

One can interpret the additive part of a neo-capacity π as the decision-maker's belief and $(1-\lambda-\gamma)$ as the degree of confidence in that belief. In the light of equation

[6] Sarin and Wakker (1998) provide a detailed discussion of the relationship between decision weights and capacities. Wakker (2001) provides precise definitions of optimism and pessimism in CEU models.

(2), we refer to the parameter λ (resp. γ) as *degree of optimism (resp. pessimism).*[7] If beliefs are represented by conventional probabilities, it is not possible to model decision-makers who lack confidence in their beliefs. The ability to make this distinction offers opportunities to analyse the impact of ambiguity and optimism/pessimism in economic models. For a pessimistic decision maker, the lack of confidence in the equilibrium prediction is reflected by the weight given to the worst outcome. With neo-capacities, the comparative statics of ambiguity are relatively easy.

If $\lambda = 0$, then preferences may be represented on the form $a \succcurlyeq b \Leftrightarrow \min_{p \in C} \mathbf{E}_p u(a) \geqslant \min_{p \in C} \mathbf{E}_p u(b)$, where C is a set of conventional additive probabilities. We believe this formula is intuitive. When a decision-maker does not know the true probability (s)he considers a set of probabilities to be possible. He/she behaves cautiously and evaluates any course of action by the least favourable probability distribution. This small deviation from subjective probabilities allows us to capture the certainty effect, which is consistently observed in experimental work (Gonzalez and Wu (1999) and Kilka and Weber (1998)).

Definition 2.4 *The* support *of the neo-capacity* $\nu(A) = \lambda + (1 - \lambda - \gamma)\pi(A)$, *is defined by* $\operatorname{supp}\nu = \operatorname{supp}\pi$.[8]

The support represents the set of states which the decision-maker 'believes' in. We interpret the neo-capacity ν as representing a situation where the decision-maker 'believes' in the additive probability distribution π but lacks confidence in this belief. It thus seems intuitive that the support of ν should coincide with the support of π.

[7] Epstein (1999) and Ghirardato and Marinacci (2002) provide alternative concepts of ambiguity aversion. In the case of CEU, Ghirardato and Marinacci take additive capacities as the benchmark case of no ambiguity-aversion, while Epstein argues for capacities which are monotone transformations of an additive probability (*probabilistic sophistication*) as the relevant benchmark. This implies that a convex and increasing transformation of an additive probability is interpreted as *ambiguity-aversion* by Ghirardato and Marinacci and as *probabilistic risk-aversion* by Epstein. We cannot resolve this controversy in this paper. We mainly consider pure equilibria of games, in which case the two definitions agree.

[8] There have been a number of definitions of support proposed for convex capacities. See Ryan (1997) for a full discussion. If $\lambda = 0$ the neo-capacity is convex. This definition of support coincides with most of the proposed definitions in this case.

This is equivalent to the usual definition of a support, if the capacity is additive. However, this does not rule out that there may be strategy combinations outside the support, which influence a player's choice. If beliefs are represented by a neo-capacity then the best and worst outcome will influence choice in addition to members of the support. The support of a neo-capacity is itself an ambiguous event, since $\nu(\operatorname{supp}\nu) + \nu(\neg\operatorname{supp}\nu) = 1 + \lambda - \gamma$, which is not in general equal to 1. This is an important part of our model of ambiguity.

2.3 More General Models of Ambiguity

Representing preferences by a Choquet integral with respect to a neo-capacity is a special case of a decision theory due to Schmeidler (1989). In this theory beliefs are represented by a more general class of capacities. This implies that, a decision-maker will over-weight a number of good and bad outcomes compared to an expected utility maximiser. The effect of restricting attention to neo-capacities is that only the best and worst outcomes are over-weighted.

There are two features of the general CEU model, which make it difficult to apply. Firstly it can be mathematically complex. Secondly there are too many free parameters. A capacity on a set with n elements involves 2^n parameters, while $n-1$ parameters will describe a probability distribution on the same set. For this reason we have chosen to focus on the case where beliefs may be represented by neo-capacities which can be described by $n+1$ numbers.

3 Strategic Games with Ambiguity

3.1 Equilibrium

In this section we present a model of the impact of ambiguity in games. In this context, uncertainty concerns the possible play of one's opponents. Consider an arbitrary player i who is uncertain about his/her opponent's choice of strategy. If player i's beliefs are modelled by a neo-capacity ν_i on S_{-i} with an additive probability distribution π_i, degree of optimism λ_i and degree of pessimism γ_i, then the payoff function is the Choquet integral,

$$V_i(s_i; \pi_i, \lambda_i, \gamma_i) = \lambda_i m_i(s_i) + \gamma_i M_i(s_i) + (1 - \lambda_i - \gamma_i) \cdot \int u_i(s_i, s_{-i})\, d\pi_i(s_{-i}). \quad (1)$$

Definition 3.1 *A pair of neo-capacities* $\langle \nu_1^*, \nu_2^* \rangle$ *is an* Equilibrium Under Ambiguity (EUA) *if:* $\operatorname{supp} \nu_i^* \subseteqq \operatorname{argmax}_{s_{-i} \in S_{-i}} V_{-i}\left(s_{-i}; \pi_{-i}, \lambda_{-i}, \gamma_{-i}\right)$, *for* $i = 1, 2$.

In equilibrium, each individual plays a best response given his/her beliefs. (Since utility is assumed to be concave the first order condition is sufficient for s_i^* to be a best response.) Players 'believe' that their opponent will play best responses. However they lack confidence in this belief. This lack of confidence is reflected in the fact that the support of the capacity is an ambiguous event. They may respond to this lack of confidence in an optimistic way by over-weighting the best outcome, or in a pessimistic way by over-weighting the worst outcome. This notion of equilibrium is in the spirit of that suggested by Dow and Werlang (1994).[9]

The similarity of an EUA with a standard Nash equilibrium is obvious. Indeed, for no ambiguity, $\lambda_i = \gamma_i = 0$ for all $i \in I$, this solution concept would coincide with Nash equilibrium. If beliefs are strictly non-additive then, whether or not opponents play best responses, is itself an ambiguous event. Thus the possibility that particularly good or bad strategies may be played, could affect the strategy choice of a player.

In economics, equilibria are usually interpreted as situations where, given their beliefs, agents have no incentive to change their strategy and these beliefs are consistent with observations about the opponents' behaviour. The standard solution concept, Nash equilibrium has been characterised by two assumptions:

[9] It is more general, since unlike the earlier paper we allow for optimistic as well as pessimistic preferences. Dow and Werlang (1994) was also more general since they allowed equilibrium beliefs to be an arbitrary convex capacity.

- players are assumed to be rational in the sense of maximizing their objective functions by independently choosing a strategy given their beliefs about the opponents' behaviour,

- beliefs about behaviour have to be consistent with opponents' actual behaviour.

Mostly one assumes that actual behaviour can be observed and in the case of mixed strategies, situations are repetitive enough so that one can infer the mixed strategies played from the frequency of actual choices. The assumption of consistency of beliefs and observations is embodied in the equilibrium requirement that beliefs coincide with actual behaviour.

If beliefs are strictly non-additive, then behaviour, whether in pure or mixed strategies, simply cannot coincide with the strategies played, since there are no non-additive randomising devices. Thus a weaker notion of consistency is desirable. We will assume that, in equilibrium, players will focus their beliefs on strategies which are best responses. The support of a capacity, $\operatorname{supp}\nu_i$, is the set of strategies on which beliefs are focused. Our equilibrium concept adopts a notion of consistency suitable for modelling ambiguity.[10]

We define an equilibrium in terms of non-additive beliefs (ν_1^*, ν_2^*). Equilibrium strategies are given by the supports of the capacities, which are required to be best responses. If these are unique, we have a pure equilibrium. If there are several strategies, which a player considers as equal best, then any combination of these is possible in equilibrium. For example in *Matching Pennies*, any combination of "heads" and "tails" will represent equilibrium behaviour as long as both players do not believe that the opponent would favour a particular choice. If there is *no ambiguity*, then the equilibrium definition (3.1) specifies a set of independent additive

[10] Lo (1996) and Marinacci (2000) present alternative equilibrium definitions based on different notions of support. This has the effect of imposing different consistency requirements. In Lo's definition of equilibrium players do not perceive ambiguity about whether their opponents play best responses. This results in a solution concept which does not differ substantially from Nash equilibrium.

probability distributions, which are the mixed strategies of a Nash equilibrium.

In economic applications, players' strategy sets are mostly continuous variables, such as prices, quantities and investment expenditures. In such situations, pure equilibria exist. Even when players' beliefs are represented by conventional subjective probabilities there are problems with the interpretation of mixed equilibria.[11] Hence, we shall not consider mixed strategies in the present paper. The following result demonstrates the existence of pure equilibria.

Proposition 3.1 *If, for all players $i \in I$, the strategy sets S_i are* closed, bounded *and* convex, *and if the payoff functions $u_i(s_i, s_{-i})$ are* continuous *in s and* quasi-concave *in each player's own strategy s_i, then there exists an Equilibrium under Ambiguity (EUA) in pure strategies.*

In the present paper, we restrict attention to neo capacities $\nu_i(A) = \lambda_i + (1 - \lambda_i - \gamma_i)\,\pi_i(A)$, which implies, $\operatorname{supp}\nu_i^* = \operatorname{supp}\pi_i$. Hence, the support contains all strategy combinations which are in the support of its additive part, π_i, which we interpret as beliefs. In games, one can determine π_i, endogenously as the prediction of the players from the knowledge of the game structure and the preferences of others. In contrast. we treat the degrees of optimism, λ_i and pessimism, γ_i as exogenous. Unlike standard Nash equilibrium however, players may not have complete trust in their predictions. This is illustrated by the following example .

Example 1 *There are two players, an incumbent monopolist, M, and an entrant, E. If the entrant chooses not to enter, ne, (s)he will receive payoff 0 and the incumbent will receive the monopoly profits M. If the entrant enters the market, e, the monopolist has the choice of accommodating entry, a or fighting a price war, f. If the incumbent accommodates entry, both firms receive the duopoly profit d. Fighting entry causes both firms to sustain losses $-L$. The interaction between the monopolist and the entrant*

[11] For a discussion of the relevance of mixed strategy equilibrium the reader is referred to Osborne and Rubinstein (1994) .

may be represented as the following normal form game:

		Monopolist	
		a	f
Entrant	e	d, d	$-L, -L$
	ne	$0, m$	$0, m$

where $m > d > 0$ *and* $L > 0$.

There are two Nash equilibria (without ambiguity), $\langle a, e\rangle$, and $\langle f, ne\rangle$. In the first, the monopolist accommodates and the entrant enters, while in the second the monopolist fights and the entrant stays out. It is common to regard the latter equilibrium as less plausible. Once the entrant is in the industry, the monopolist will make lower profits by fighting than by accommodating. This equilibrium is eliminated by any standard refinement such as subgame perfection.

Now we shall consider how ambiguity affects this example. We shall assume that agents are purely pessimistic, i.e. $\gamma > 0$, $\lambda = 0$. When there is ambiguity, we find a new type of equilibrium. In this, the monopolist does not fight. However entry does not occur because the entrant is pessimistic and perceives considerable ambiguity about the incumbent's behaviour. Consider the following beliefs: $\nu^E(a) = \alpha$, $\nu^E(f) = 0, \frac{L}{(d+L)} > \alpha > 0, \nu^M(e) = 0$, $\nu^M(ne) = \beta, 1 \geqslant \beta > 0$. These beliefs show a high degree of pessimism for the entrant. With these beliefs the (Choquet) expected payoff of the monopolist is given by:

$$V^M(a) = m\beta + d(1-\beta), \qquad V^M(f) = m\beta - L(1-\beta). \tag{3}$$

Hence, a is a best response for the monopolist. If $\beta < 1$, then f is not a best response for the monopolist. Since this holds for all $\beta < 1$, even small amounts of ambiguity-aversion are capable of eliminating non-credible threats.[12] The (Choquet) expected utility of the entrant is given by, $V^E(e) = d\alpha - L(1-\alpha)$, $V^E(ne) = 0$. Thus ne is a best response for the entrant if and only if,

$$\alpha \leqslant \frac{L}{(d+L)}. \tag{4}$$

We interpret this as saying the entrant will not enter if (s)he is sufficiently ambiguity-averse. Equation (4) says that entry is more likely, the higher are the profits from successful entry d and the lower are the losses from a price war, L. In Nash equilibrium, entry is independent of these factors provided d and L are both positive. In our opinion, it is not implausible that these factors would affect the outcome. ■

The case of large ambiguity-aversion in Example 1, shows how non-Nash behaviour can arise in EUA. The entrant considers it more likely that the incumbent will accommodate entry and this belief is sustained in equilibrium. It is possible that such a decision might be affected by ambiguity, since a firm will usually have much less information about an industry in which it does not already have a presence. In practice, entry is likely to entail considerable expenditure before any returns are received. By definition the entrant is not already in the industry. Thus (s)he may face some considerable ambiguity about relevant variables, in particular the behaviour of the incumbent. It is not implausible that entrants might react by behaving cautiously and not entering even if they do not expect the incumbent to fight a price war.

In standard Nash theory, the assumption that the other player will be more likely to play a, implies that it is optimal to play e, which yields the higher payoff of d. This need not be the case in an EUA, if the entrant is sufficiently ambiguity-averse. Clearly, the possibility that the monopolist might fight entry, an event which is not in the support of the belief, influences the equilibrium outcome. Such behaviour is not implausible when players perceive ambiguity. If the entrant thinks that the

[12] This is true more generally see, Eichberger and Kelsey (2000), Proposition 5.1.

monopolist will be cautious and accommodate, (s)he may still not be bold enough, to enter, since a misjudgment will earn him/her an outcome of $-L$.

Our intuition suggests that (ne, a) is not an implausible way to behave. We suspect, however, that the degree of ambiguity-aversion depends upon observations. As evidence builds up that the opponent plays a (or f respectively), confidence may grow and choosing e (ne) may become more likely. With a support notion, which insists that strategy combinations outside the support do not affect behaviour, (ne, a) can never be an equilibrium. It seems to us an advantage of EUA, that it opens the possibility to model such testable hypothesis.

4 Oligopoly Models

In this section, we shall present some examples of how the techniques described in previous sections, can be used to examine the effect of ambiguity on economic behaviour. These examples will illustrate that the consequences of ambiguity can be examined without technical sophistication.

4.1 Cournot Oligopoly

4.1.1 Equilibrium under Ambiguity

First we consider a symmetric Cournot duopoly, where firms produce homogenous products and choose quantities as their strategic variable. We will show that, in this case, *optimism increases competition* because it induces more aggressive behaviour. Pessimism will, in general, have the opposite effect.

There are 2 firms, $i = 1, 2$, which compete in quantities. Assume that firm i faces the linear inverse demand curve $P_i(x_i, x_j) = \max\{1 - x_i - x_j, 0\}$. We shall assume that each firm can produce at constant marginal cost equal to c. Firm i chooses the quantity it wants to supply, x_i, from the interval [0, 1]. If beliefs are represented by neo-capacities, a firm over-weights the best and worst outcomes. We assume that firm i perceives the worst scenario to be a situation, where its rival dumps a large quantity on the market, driving the price down to zero. The firm's perceived best outcome is assumed to be where the rival produces zero output and the firm is a monopolist. Under these assumptions firm i's (Choquet) expected profit is:

$$\Pi_i = \lambda x_i (1 - x_i) + (1 - \gamma - \lambda) x_i [1 - x_j - x_i] - cx_i. \tag{5}$$

A possible criticism of this model is that the choice of the best and the worst outcome is arbitrary. However our results do not depend crucially on these assumptions. All that is required is that the best (resp. worst) outcome be below (resp. above) the Nash equilibrium output.[13]

The first order condition for maximising firm 1's profit is, $\frac{d\Pi_1}{dx_1} = \lambda_1 (1 - 2x_1) + (1 - \gamma_1 - \lambda_1)(1 - 2x_1 - x_2) = c$. Hence the reaction function of firm 1 is given by,

$$R^1 (x_2) = \frac{(1 - \gamma_1) - (1 - \gamma_1 - \lambda_1) x_2 - c}{2(1 - \gamma_1)}. \tag{6}$$

Proposition 4.1 *In a symmetric equilibrium where $\lambda_1 = \lambda_2 = \lambda$ and $\gamma_1 = \gamma_2 = \gamma$, the equilibrium output and price are given by*

$$\bar{x} = \frac{1 - \gamma - c}{3 - 3\gamma - \lambda}, \quad \bar{p} = \frac{1 - \gamma - \lambda + 2c}{3 - 3\gamma - \lambda}.$$

Proof. From equation (6), the equilibrium output $\bar{x}$ is given by $\bar{x} = \frac{(1-\gamma)-(1-\gamma-\lambda)\bar{x}-c}{2(1-\gamma)}$, which implies $\left[1 + \frac{(1-\gamma-\lambda)}{2(1-\gamma)}\right] \bar{x} = \frac{(1-\gamma)-c}{2(1-\gamma)}$ hence $\bar{x} = \frac{(1-\gamma)-c}{2(1-\gamma)+(1-\gamma-\lambda)} = \frac{1-\gamma-c}{3-3\gamma-\lambda}$. The equilibrium price is given by $\bar{p} = 1 - \frac{2-2\gamma-2c}{3-3\gamma-\lambda} = \frac{1-\gamma-\lambda+2c}{3-3\gamma-\lambda}$. ■

The next result shows that if firms become more optimistic, (i.e. λ increases) then equilibrium output will rise. An increase in optimism will increase the weight

[13]See Eichberger and Kelsey (2002) Proposition 3.1, for a related result which does not depend on assumptions about the perceived best and worst outcomes.

the firm puts on rivals producing a low output. This increases the marginal benefit of producing more and hence results in an increase in equilibrium output.

Proposition 4.2 *The effects of changes in ambiguity attitude on equilibrium in Cournot oligopoly are as follows:*

1. *An increase in optimism increases output and decreases prices in equilibrium;*

2. *If $\lambda < 3c$, an increase in pessimism reduces equilibrium output and increases prices.*

Proof. By inspection, $\bar{x}$ is an increasing function of λ. Since $\frac{dp}{d\bar{x}} = -2$, the equilibrium price is a decreasing function of λ. The effect of an increase in pessimism on output is given by, $\frac{d\bar{x}}{d\gamma} = \frac{3(1-\gamma-c)-(3-3\gamma-\lambda)}{(3-3\gamma-\lambda)^2} = \frac{\lambda-3c}{(3-3\gamma-\lambda)^2}$, which is negative provided $\lambda < 3c$. As before, $\frac{dp}{d\gamma} = -2\frac{d\bar{x}}{d\gamma}$. ■

Intuitively more optimism causes a firm to place more weight on the possibility that its rival will produce a low output. This increases the marginal profitability of extra output. Thus the given firm will produce more and hence the equilibrium output will rise. This reasoning is not restricted to the specific demand and cost functions but will apply whenever Cournot oligopoly is a game of strategic substitutes. In general one would not expect the effect of optimism to be large, hence it seems reasonable to assume $\lambda < 3c$, in which case an increase in pessimism would decrease equilibrium output. Likewise an increase in pessimism causes firms to place more weight on the possibility that a rival will produce a high output. This reduces the marginal benefit of producing more and hence tends to decrease equilibrium strategies.[14]

Corollary 4.1 *Assume $\bar{x} > \frac{1}{4}(1-c)$, then:*

1. *An increase in optimism decreases equilibrium profits,*

[14] However in this case there is an opposing effect. Assume cost is zero, by equation (5) the objective function of firm i is cardinally equivalent to $\tilde{\Pi}_i = \frac{\lambda}{1-\gamma} x_i (1-x_i) + \frac{(1-\gamma-\lambda)}{1-\gamma} x_i [1 - x_j - x_i]$. Increasing γ increases the weight placed on the monopoly profits in this expression and hence increases the output. A similar effect applies when cost is low but not zero i.e. $3c < \lambda$.

2. If $\lambda < 3c$, an increase in pessimism increases equilibrium profits.

Proof. Equilibrium profits are given by, $\pi = (p(x(\lambda)) - c)\, x(\lambda)$, hence
$\frac{\partial \Pi}{\partial \lambda} = (p(\bar{x}(\lambda)) - c) \frac{\partial \bar{x}}{\partial \lambda} + \frac{dp}{d\bar{x}} \frac{\partial \bar{x}}{\partial \lambda} \bar{x}(\lambda) = \frac{\partial \bar{x}}{\partial \lambda} \left[p(\bar{x}(\lambda)) - c + \frac{dp}{d\bar{x}} \bar{x}(\lambda) \right] = \frac{\partial \bar{x}}{\partial \lambda} [1 - c - 4\bar{x}(\lambda)]$.
Thus provided $\bar{x} > \frac{1}{4}(1-c)$ an increase in λ decreases profit. The proof of part 2 follows by similar reasoning. ■

The condition $\bar{x} > \frac{1}{4}(1-c)$ says that the effects of ambiguity are relatively small, in the sense that they do not induce firms to produce less than the collusive output. We would view this as the normal case.

To illustrate these results consider the case where there is no optimism $\lambda = 0$. By equation (6) the symmetric equilibrium is characterised by

$$\frac{1}{2} - \frac{3}{2}\bar{x} = \frac{1}{2(1-\gamma)} c. \tag{7}$$

Assume that there is an increase in pessimism, γ rises. Then the rhs. of equation (7) increases. Since the lhs. of equation (7) is decreasing in $\bar{x}(\gamma)$, $\bar{x}$ must be a decreasing function of γ. An increase in pessimism will decrease the quantities in a symmetric Cournot equilibrium as depicted in Figure 1. As firms are symmetric, EUA are intersections of the best response function with the 45-degree line.

Pessimism reduces the amount brought to market. Intuitively, ambiguity makes a decision-maker cautious about the behaviour of the opponent. By dumping output onto the market, the rival can drive down the price. If firms become more concerned about this possibility, they will reduce output in order to avoid the losses that would arise in such a case.

Due to the informational requirements, it may be difficult to identify the Nash equilibrium in an actual market. Hence, deviations may be hard to measure. Indirect evidence however may be gleaned from some experimental studies. Though the experiment was designed to study learning behaviour, Huck, Normann, and Oechseler

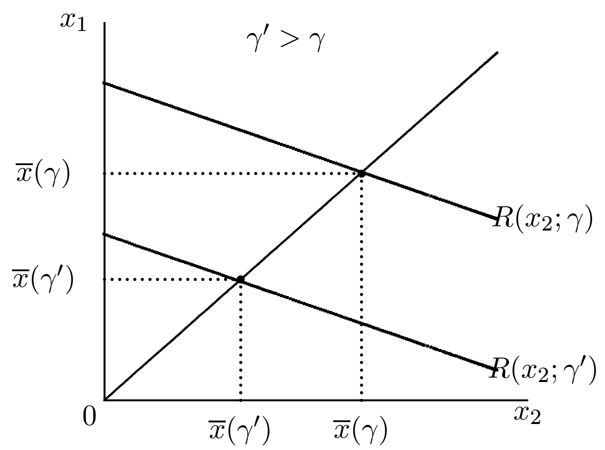

Figure 1: Cournot equilibrium and ambiguity

(1999)) found that "more information about behaviour and profits of others yields more competitive outcomes" (Result 2, p. C89). If ambiguity-aversion reflects a lack of confidence in one's information about the opponent, this may provide indirect evidence for ambiguity-aversion as a reason for reduced competition.

Traditionally, it has been suspected that oligopolies are prone to informal collusive arrangements. Scherer (1970) provides many examples from anti-trust cases. The presumption of regulators that oligopolists collude, suggests that output is, at least sometimes, below the Cournot level without clear evidence of collusion. Ambiguity may offer an alternative and as yet unexplored explanation for why competition may be less fierce in Cournot-style oligopoly than predicted by Nash equilibrium.

4.1.2 Strategic Delegation

In this section we show that it may be profitable to delegate decision-making to a manager who is more optimistic than the owner of the firm. This allows the owner to commit to producing a larger output, which is advantageous in a game of strategic substitutes.

Assume that firm 1 has a profit maximising owner who is ambiguity neutral, i.e. has additive beliefs. The owner hires a manager to operate the firm on his/her behalf. The owner pays him a wage, which is fraction α of firm 1's profit. The manager has CEU preferences and has beliefs represented by a neo-capacity. The owner chooses the manager to maximise his/her profit. Firm 2 is a conventional profit maximising firm. The following result finds the levels of optimism, λ_1 and pessimism, γ_1, which are optimal for the owner of firm 1.

Proposition 4.3 *The profit maximising levels of λ_1 and γ_1 satisfy*

$$\lambda_1 = 1 - 3\gamma_1 + 2\gamma_1 \frac{(1+c)}{(1-c)}. \tag{8}$$

Proof. Profit is maximised where the equilibrium output of firm 1 is equal to that of a Stackleberg leader which is $\frac{1}{2}(1-c)$. Thus by Lemma A.1, $\frac{(1-\gamma_1+\lambda_1)-(1+\gamma_1+\lambda_1)c}{3(1-\gamma_1)+\lambda_1} = \frac{1}{2}(1-c)$. Cross multiplying we obtain, $2(1-\gamma_1+\lambda_1) - 2(1+\gamma_1+\lambda_1)c$ $= (1-c)(3(1-\gamma_1)+\lambda_1)$, or $\lambda_1(2-2c) - 2\gamma_1 - 2\gamma_1 c + 2(1-c) = 3(1-\gamma_1)(1-c) + \lambda_1(1-c)$. Hence $\lambda_1(1-c) = 3(1-\gamma_1)(1-c) - 2(1-c) + 2\gamma_1(1+c)$, from which the result follows. ■

To understand this result it is useful to consider the special case where $\gamma_1 = 0$. Then equation (8) says $\lambda_1 = 1$, which implies that the manager will assign weight one to the possibility that the opponents will produce zero output and will himself produce the monopoly output. This is desirable, since in this example, the monopoly output coincides with the output of a Stackleberg leader, which is the most profitable output. Of course it is very unlikely that a manager would assign decision-weight one to the possibility that the opponents will produce zero output. However it remains true that profit can be raised by delegating to a manager who is more optimistic than the owner.[15] This suggests one reason why we might expect to see ambiguity-loving individuals. Such attitudes may increase an individual's income. In strategic interactions it may well not be in an individual's interest to follow the Savage axioms.

4.2 Bertrand Oligopoly

We shall now consider price (Bertrand) competition. In this case, ambiguity attitudes have the opposite effect. More pessimism increases competition by inducing firms to charge lower prices.

4.2.1 Equilibria without Ambiguity

Consider 2 firms producing heterogeneous goods which are close (but not perfect) substitutes. Firm i can produce at constant marginal and average cost, $k > 0$. Firm i charges price p_i for its output. We assume that firm i faces a linear demand curve: $D_i(p_i, p_j) = \max\{0, a + bp_j - cp_i\}, \quad a, b, c > 0, a > k, c > b$. The following result describes the equilibrium when firms choose prices simultaneously.

Proposition 4.4 *In Nash equilibrium of the Bertrand model,*

1. *the reaction function of firm i is given by*

$$p_i\left(p_j\right) = \frac{a + bp_j + ck}{2c}, \tag{9}$$

2. *both firms charge a price equal to, $\hat{p} = \frac{a+ck}{2c-b}$.*

[15] This result has found the value of λ_1, which will maximises the profit of the firm. However from the point of view of the owner, there is an additional advantage of hiring an optimistic owner. The more optimistic the owner the lower the value of α needed to induce the manager to work. This second effect also implies that it is advantageous to hire an optimistic manager.

Proof. The profits of firm i are given by, $\Pi_i = (p_i - k)(a + bp_j - cp_i)$. The first order condition for profit maximisation is: $\frac{\partial \Pi_i}{\partial p_i} = (a + bp_j - cp_i) - c(p_i - k) = 0$, from which part (1) follows. Let $\hat{p}$ denote the level of price charged by both firms in a symmetric Nash equilibrium then $\hat{p} = \frac{a+b\hat{p}+ck}{2c}$, which implies part (2). ■

Now suppose that firm 1 is a price leader and must set price first. Firm 2 then observes the price set by firm 1 and chooses its own price to maximise profit.

Proposition 4.5 *If firm 1 is a price leader its optimal price is given by:*

$$p_1 = \frac{2ac + b(a + ck)}{2(2c^2 - b^2)} + \frac{k}{2}.$$

With price leadership, firm 1 sets a price above the Bertrand equilibrium level since it takes into account the fact that when it raises price, this causes firm 2 to raise price as well. In this game there is a second mover advantage. Firm 2 makes higher profits since it can slightly undercut firm 1's price.

4.2.2 Equilibrium under Ambiguity

Now assume that each firm perceives its rival's behaviour as ambiguous and has beliefs represented by neo-capacities. We assume that each firm perceives the worst case to be where its rival reduces price to marginal cost. Hence we assume, a firm's strategy set is the interval $[k, K]$ for some sufficiently high K. We shall assume the best case is perceived to be where the rival firm sets price equal to K. Given this interpretation, it seems reasonable to require that K be above the Nash equilibrium price, i.e. $K > \frac{(a+ck)}{(2c-b)}$. We require $a + bk - cK > 0$ to ensure that demand is positive at all quantities in the firms' strategy sets. With these assumptions the (Choquet) expected profit of firm i becomes: $\Pi_i = (1 - \gamma_i - \lambda_i)(p_i - k)(a + bp_j - cp_i) + \gamma_i (p_i - k)(a + bk - cp_i) + \lambda_i (p_i - k)(a + bK - cp_i)$. Simplifying
$\Pi_i = (p_i - k)(a - cp_i) + (p_i - k) b [(1 - \gamma_i - \lambda_i) p_j + \gamma_i k + \lambda_i K]$. The first-order con-

dition for profit maximisation is: $\frac{\partial \Pi_i}{\partial p_i} = a - cp_i - c(p_i - k) + b[(1 - \gamma_i - \lambda_i) p_j + \gamma_i k + \lambda_i K] = 0$. Thus firm i's reaction function is given by

$$\rho^i(p_j) = \frac{a + b[(1 - \gamma_i - \lambda_i) p_j + \gamma_i k + \lambda_i K] + ck}{2c}. \tag{10}$$

Equation (10) defines a non-singular system of linear equations, which implies that the Bertrand equilibrium is unique. Since $K \geqslant p_j \geqslant k$, an increase in λ_i (resp. γ_i) will shift firm i's reaction curve up (resp. down) and hence increase (resp. decrease) the equilibrium price. The price of firm j will also increase, since reaction curves slope upwards. Consider firm 1. An increase in optimism causes it to place more weight on good outcomes. In this context, a good outcome would be firm 2 charging a high price. Since the model exhibits strategic complementarity this gives firm 1 an incentive to increase its price. This discussion is summarised in the following proposition.

Proposition 4.6 *In Bertrand oligopoly an increase in optimism (resp. pessimism) of firm i causes both firms to set higher (resp. lower) prices in equilibrium.*

The case of two firms with symmetric linear demand functions is illustrated in figure 2. An increase in pessimism causes the reaction curve to shift down and the slope to decrease. Firms have their own markets in which to react to the other's price. Uncertainty about the other price is equivalent to uncertainty about a firm's own demand. The lower a given firm sets the price, the smaller the market the opponents will face. Firms' concern about low demand in their respective market, provides an incentive for charging lower prices than in a conventional (Bertrand) equilibrium. Hence, pessimism tends to increase the competitiveness of Bertrand markets.

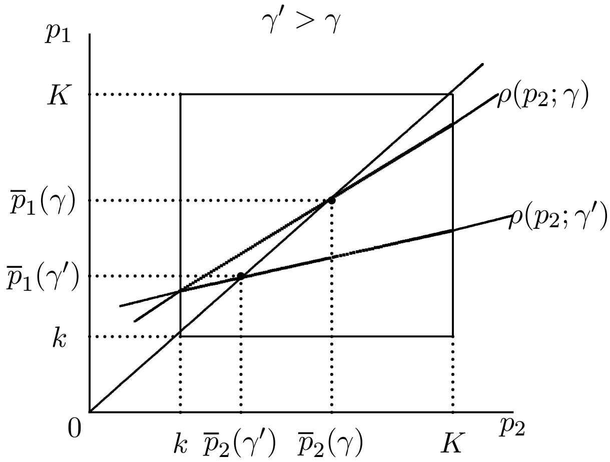

Figure 2: Bertrand equilibrium and uncertainty

4.2.3 Strategic Delegation

Equation (10) shows that an increase in λ_i will increase the equilibrium prices of both firms. In Bertrand oligopoly, a given firm can gain a strategic advantage by committing to price above the equilibrium level, see Fershtman and Judd (1987). This causes rivals to raise their prices, which gives the first firm an indirect benefit since its profits are higher the greater the prices of its rivals. Our results show that, appointing an optimistic manager would be a way to commit to a price above the Nash equilibrium level. Hence an optimistic manager will make more profit than an expected utility maximiser.

To illustrate the possibilities for strategic delegation we consider an specific form for the demand function.

Example 2 *Assume* $a = c = 2$ *and* $b = 1$, *i.e.* $D(p_i, p_j) = 2 + p_j - 2p_i$, *for* $i = 1, 2$. *By Propositions 4.4 and 4.5, the Nash equilibrium price is,* $\hat{p} = \frac{2}{3} + \frac{2}{3}k$, *while a price leader would charge* $\tilde{p}_1 = \frac{2ac + b(a+ck) + k(2c^2 - b^2)}{2(2c^2 - b^2)} = \frac{8 + (2+2k) + 7k}{14} = \frac{10+9k}{14}$. *We assume that firm 2 is a conventional profit maximising firm. Consider the case where firm 1 has ambiguity neutral owners, who delegate decision making to a manager, whose beliefs are represented by a neo-capacity. In this case we can show that* $\gamma_1 = 0, \lambda_1 = \frac{10-5k}{56K-38-37k}$, *are profit-maximising levels of* λ_1 *and* γ. [16]

The discussion of strategic delegation assumes that the ambiguity attitude of the manager is observable. Managers do make public speeches and reports, which could be used to reveal their ambiguity attitude. Many managers do appear to cultivate an optimistic view of their firm's performance. This could, in part, be motivated by strategic considerations.

In both Bertrand and Cournot oligopoly profits can be increased above the Nash equilibrium level by employing an optimistic manager. This is unusual as most comparative static results are reversed when one moves from Cournot to Bertrand competition.

5 Peace-making

We believe that formal modelling of ambiguity will aid the understanding of a number of problems in the social sciences. To illustrate this point, we next consider a model of a peace-making.

5.1 Motivation for Ambiguity in Conflict Situations

Consider the following stylised facts about peace-making between Israel and the Palestinians.

[16] This claim is proved in Lemma A.2 in the appendix.

1. “It takes two parties for peace but only one for war.”

2. “We don’t know what to believe. We offered them so much in the negotiations but they (Palestinians) said ‘no’ and started this terror (Intifada) on us. They even use the arms we provided to the police force of the Palestinian authority to shoot us.”

3. “We want peace but...”

Whereas (1) suggests that a player’s marginal benefit of contributing to peace jumps in the opponent’s contribution to peace, (2) indicates uncertainty about the opponent’s strategy. Fact (2) may indicate also that peace-making has positive externalities, i.e. an increase of one’s peace effort increases the opponent’s benefit. In the light of fact (3) it seems reasonable to assume that there is a dilemma. The Palestinian position appears to be not too dissimilar.

War is an inherently ambiguous situation. Even experienced generals admit to being often surprised by developments. In conflict situations such as Northern Ireland where do probability judgements come from? History provides only limited knowledge of similar conflicts, hence probabilities could not be interpreted as frequencies. It seems that a Bayesian model would not capture the nature of the conflict (stylized fact 2. “We don’t know what to believe.”).

5.2 A Peace-making Game

Consider two players $i = 1, 2$ interpreted as the parties involved in the conflict. Each player i chooses a strategy $s_i \in S_i = [0, 1]$. We interpret $s_i = 0$ as no effort, whereas $s_i = 1$ is full effort to peace-making. Higher values of s_i correspond to greater peace-making efforts by individual i. Let $s = (s_1, s_2) \in [0, 1] \times [0, 1]$. We assume that benefits from peace-making have the following form:

$$b(s_1, s_2) = \begin{cases} \varepsilon (s_1 + s_2)^2 \text{ if } s_1 < 1 \text{ or } s_2 < 1, 0 < \varepsilon < \frac{1}{4}, \\ 1, \text{ if } s_1 = s_2 = 1. \end{cases}$$

The benefits from peace making are increasing in the efforts of both parties, hence there are positive externalities. Avoiding a single terror act by the Palestinians or the killing one innocent school girl by the Israeli forces, brings an increase in benefit. The benefit function is convex, which implies there is strategic complementarity in peace-making. The more effort supplied by the Palestinians the greater the marginal benefit of peace-making by the Israelis. The discontinuity at $\langle 1, 1 \rangle$ indicates that there is a qualitative difference between peace and a war of very low intensity.

Peace-making can be costly, e.g. in Israel the right wing protests if the government does not respond forcefully enough to Palestinian terror. Increasing peace-making efforts may bear the risk of losing some right wing coalition partners and votes. There is likely to be similar pressures on other parties such as the Palestinian leadership or the IRA. For simplicity we assume some linear costs $cs_i, c > 0$. The payoff function u of either party $i = 1, 2$ is written

$$u_i(s_i, s_j) = ab(s_i, s_j) - cs_i,$$

with $a > 0$, being a real valued parameter weighting the importance of peace in the payoff function. The game is given by $\Gamma^P = \langle (S_i)_{i=1,2}, (u_i)_{i=1,2} \rangle$.[17] Thus peace can be viewed as a public good produced with increasing returns to scale. The following result characterises the Nash equilibria of the peace game.

Proposition 5.1 *Solutions without ambiguity of the peace-making game Γ^P, are characterised as follows:*

[17]This is a symmetric game. One may argue that many conflict situations are not symmetric, e.g. the IRA were fighting both the British army and the protestant paramilitaries. However we do not believe this simplification affects our conclusions qualitatively.

1. *if $a\varepsilon > c$ then full peace-making effort is the strictly dominant strategy for $i = 1, 2$;*

2. *if $a(1-\varepsilon) \geqslant c \geqslant a\varepsilon$ then there exist two Nash equilibria in pure strategies one with full peace-making effort and one where no effort is supplied by either party;*[18]

3. *if $c > a(1-\varepsilon)$ then no effort is the strictly dominant strategy for $i = 1, 2$.*

Case (3) describes a situation where each side views the benefits of peace as being less than the costs of peace-making, regardless of what the other party does. Consequently peace is not established. Such a situation would arise if benefits from peace are small compared to costs of peace-making efforts. This does not seem a realistic representation of situations such as Israel or Northern Ireland, where it seems most people perceive peace as worth achieving if possible.

Case (1) is the non-problematic case. Benefits from peace are strictly larger than the costs. Hence both parties provide full effort and peace is established. Again this does not appear to be a reasonable model of the world's conflict situations.

Case (2) is the interesting intermediate case. There are substantial benefits from peace but benefits from intermediate peace-making efforts are not enough to justify the costs. This seems to fit the circumstances in many potential conflict situations. It corresponds to stylized fact (3). Two Nash equilibria in pure strategies arise, one in which both parties engage in full peace-making efforts establishing peace and one in which no effort is made and peace is not achieved.

5.3 Peace-making under Ambiguity

Now we study the impact of ambiguity on the peace-making game. The following proposition shows more optimism makes a successful peace process more likely. In the

[18] There exists also a Nash equilibrium in mixed strategies where parties mix between zero and full peace-making effort.

case where Nash equilibrium is not unique, ambiguity can play a role in equilibrium selection. If the degree of optimism is sufficiently high, there is a unique equilibrium in which the peace process succeeds. Pessimism has the opposite effect. If there is enough pessimism, we can be sure that peace will not be established.

Proposition 5.2 *The impact of ambiguity in the peace-making game, Γ^P, is as follows:*

1. *If $a\varepsilon \geqslant c$: any equilibrium under ambiguity involves only the Nash equilibrium strategy under certainty, i.e. $s_i = 1$, $i = 1, 2$.*

2. *If $a(1-\varepsilon) > c > a\varepsilon$: any equilibrium under ambiguity involves only strategies $s_i = 1$ or $s_i = 0$, $i = 1, 2$. Moreover, there exists $\bar{\lambda}$ (resp. $\bar{\gamma}$) such that if $\lambda \geqslant \bar{\lambda}$ (resp. $\gamma \geqslant \bar{\gamma}$) then $s_i = 1$ (resp. $s_i = 0$) is the unique equilibrium strategy for $i = 1, 2$.*

3. *If $c \geqslant a(1-\varepsilon)$: any equilibrium under ambiguity involves only the Nash equilibrium strategy under certainty, i.e. $s_i = 0$, $i = 1, 2$.*

We believe the important part is case (2), which we argued is likely to be the relevant case when peace-making poses a serious political problem. It is in this case, that ambiguity makes a difference. A high degree of pessimism causes peace-making efforts to break down. On the other hand optimism (or more confidence in the actions of the other side) can cause the peace-making process to be successful.

This model is far from being the last word on peace-making. There are many features such as repeated interaction between the parties, that are completely omitted. Moreover, it is easy to observe that the process is more complicated since it involves complex political interactions within each side, not modelled here. This simple model cannot give precise advice to politicians but perhaps it is helpful for the peace-process to reduce ambiguity about each others actions and to encourage confidence in the peace-making activities of others. Differences in perceived ambiguity may explain why, arguably, Northern Ireland has a successful peace process while Palestine does not. The process of European integration may have served to reduce ambiguity in Northern Ireland, there seems to be no corresponding influence on Palestine.[19]

6 General Results

In this section we present some more general results on the comparative statics of changing ambiguity attitudes in 2-player games with strategic complements. We also show that ambiguity can act to select equilibria in coordination games. In a game with strategic complements and multiple Nash equilibria, if the level of pessimism is sufficiently high we can show that there is a unique equilibrium with ambiguity. The equilibrium strategies will be less than those in the Pareto inferior equilibrium without ambiguity. Optimism has the opposite effect. We consider a game with 2-players, $i = 1, 2$, where the strategy sets are subsets of the real line.[20] In particular let $S_i = [m_i, M_i]$ for $i = 1, 2$ and $S = S_1 \times S_2$. Player i has utility function $u^i(s_1, s_2)$ and has beliefs on S_{-i} represented by a neo-capacity $\nu_i = \lambda_i + (1 - \lambda_i - \gamma_i)\pi_i$. The following assumption is maintained throughout this section.

Assumption 6.1 (Strict Concavity) *There exists δ, such that for all $s_1, s_2 \in S, u_{11}(s_1, s_2) < \delta < 0$.*

Proposition 6.1 *If the amount of ambiguity perceived by player i, $\lambda_i + \gamma_i$ is sufficiently large the equilibrium with ambiguity is unique.*

This result may be explained as follows. If player i believes player j's behaviour to be more ambiguous, player i's behaviour becomes less responsive to changes in

[19] European integration rules out both the extreme Protestant claim that Ulster should have no links whatsoever with Dublin and the extreme Nationalist case for a united Ireland with in which the Great Britain plays no role. By focusing political debate on the middle ground it may help to reduce ambiguity.

[20] This can be generalised to the case where the strategy sets are other ordered spaces, see Milgrom and Roberts (1990).

j's strategy. Thus the reaction curves become steeper, which results in a unique equilibrium.

Next we investigate the comparative statics of changing ambiguity attitudes. To get unambiguous comparative static results we need to assume strategic complementarity.

Assumption 6.2 *There are* positive externalities, *i.e.* $u^i(s_i, s_{-i})$ *is increasing in* s_{-i}, *for* $i = 1, 2$.

Assumption 6.3 (Strategic Complementarity) $u^i_{12}(s_1, s_2) > 0$, *for* $i = 1, 2$.[21]

Strategic complementarity says that if player j increases his/her strategy this raises the marginal benefit to i of increasing his/her own strategy. If player i becomes more optimistic (s)he will place higher weight on good outcomes. If there are positive externalities a good outcome will be interpreted as the opponent playing a high strategy. In the presence of strategic complementarity this gives i an incentive to increase his/her strategy. If equilibrium is unique, an increase in optimism will increase equilibrium strategies of both players. If equilibrium is not unique we get a similar result. The *set* of equilibria increases, in the sense that the strategies played in the highest and lowest equilibria increase.

Proposition 6.2 *Under Assumptions 6.2 and 6.3 the strategies of both players in the highest and lowest equilibria are increasing (resp. decreasing) functions of* λ_1 *and* λ_2 *(resp.* γ_1 *and* γ_2*).*

Corollary 6.1 *If* λ_1 *(resp.* γ_1*) is sufficiently large (resp. small), equilibrium is unique and is larger (resp. smaller) than the largest (resp. smallest) equilibrium without ambiguity.*

Proof. Follows from Propositions 6.1 and 6.2. ■

[21] As usual u_{12} denotes $\frac{\partial^2 u}{\partial s_1 \partial s_2}$.

Thus ambiguity can act as an equilibrium selection device. If agents are sufficiently optimistic, all will focus on an equilibrium in which high strategies are played. The assumption of positive externalities and strategic complementarity implies that the highest equilibrium is Pareto superior. In this case, optimism would select the equilibrium with the highest level of economic activity. As usual, pessimism has the opposite effect.[22]

7 Concluding Remarks

It is possible to generalise these applications in many ways. The applications here were chosen to represent cases of strategic substitutes (Cournot equilibrium), strategic complements with a unique equilibrium (Bertrand equilibrium) and strategic complements with multiple equilibria (peace-making).

In Eichberger and Kelsey (2002) we study the comparative statics of uncertainty in games with strategic substitutes or complements and finite strategy sets. In that paper our results are proved for general pessimistic CEU preferences. This demonstrates that the restriction to neo-capacities is not crucial for our results. That paper, provides some further applications. In a model of voluntary contributions to public goods we show that uncertainty increases the provision of public goods. This potentially explains why donations often appear to be well above Nash equilibrium levels. This frequently occurs in situations where there is much uncertainty, such as an appeal to raise funds for disaster relief.

In a macroeconomic coordination model, we show that increasing ambiguity has the effect of reducing economic activity. If there are multiple equilibria, optimism

[22]Some related results can be found in Eichberger and Kelsey (2002). The present paper extends those results since they apply to optimistic as well as pessimistic attitudes to ambiguity. In addition the present paper has continuous rather than discrete strategy spaces. Eichberger and Kelsey (2002) confined attention to symmetric equilibria of symmetric games, assumptions not used in this section. Moreover the earlier paper was only established uniqueness of equilibrium with a high degree of ambiguity when a restrictive assumption was satisfied (Assumption 3.2 of Eichberger and Kelsey (2002)).

(resp. pessimism) can cause the economy to move to a higher (resp. lower) equilibrium. This example could potentially be developed to provide macroeconomic models with a Keynesian flavour. It bears an intriguing resemblance to Keynes' argument that macroeconomic activity is at times influenced by waves of optimism and pessimism.

The economic applications, presented in this paper, can only serve to illustrate the type of results which one can expect to obtain by including ambiguity in economic analysis. Our intuition suggests that the conclusions obtained are not unreasonable. So far, there exists experimental evidence only for the impact of ambiguity aversion on individual decision making, Kilka and Weber (1998). Experimental results on ambiguity in games, are to our knowledge not yet available. Indirect evidence, as in Huck, Normann, and Oechseler (1999) for the Cournot case, provides some support for our model.

This paper suggest a number of directions for future research. One, which we are actively pursuing is experimental testing of the impact of ambiguity in games. The theory in the present paper provides some testable hypotheses. One of these is that ambiguity has the opposite effect in games of strategic complements and substitutes. This result should, in principle, be experimentally testable. An experiment could be set up with matched pairs of games, which are as similar as possible, except than one is a game of strategic complements and the other is a game of strategic substitutes. If ambiguity is introduced into the two games then our theory predicts that it will have the opposite effect on equilibrium strategies.

A Appendix

This appendix contains the proofs of those results not already proved in the text.

A.1 Games with Ambiguity

To see the effect of ambiguity on behaviour in a game, it helps to consider the following perturbation of the agents' preferences. Let $V_i\left(s_i, s_{-i}; \gamma_i, \lambda_i\right) = \lambda_i \max_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + \gamma_i \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) + (1 - \lambda_i - \gamma_i) u_i(s_i, s_{-i})$ for $i = 1, 2$. This perturbed payoff is just player i's Choquet expected payoff from the pure strategy profile (s_i, s_{-i}), when player i's beliefs may be represented by a neo-capacity with degrees of optimism λ_i and pessimism γ_i. Now consider the following perturbed game $\Gamma(\lambda, \gamma) = \left(\langle I, \left(S_i, V_i\left(\cdot; \lambda_i, \gamma_i\right)\right\rangle i = 1, 2\right)$. The following result shows that an equilibrium with uncertainty can be viewed as a conventional Nash equilibrium of this modified game.

Proposition A.1 *For any pure strategy Nash equilibrium of* $\Gamma(\lambda, \gamma)$, *there is a corresponding pure strategy* Equilibrium under Ambiguity (EUA) *of* Γ, *in which player* i *has degrees of optimism* λ_i *and pessimism* γ_i.[23]

Proof. Let $s^* = (s_1^*, s_2^*)$ be a Nash equilibrium strategy profile of the game $\Gamma(\lambda, \gamma) = \left\langle I, \left(S_i, V_i\left(\cdot; \lambda_i, \gamma_i\right), i = 1, 2\right\rangle\right.$. Define a neo-capacity, ν_i^*, for $i = 1, 2$, by, $\nu_i^* = \lambda_i + \left(1 - \lambda_i - \gamma_i\right) \pi_i$, where π_i is the probability distribution that assigns probability 1 to s_{-i}^*. The degree of uncertainty-aversion of ν_i^* is γ_i and $\operatorname{supp} \nu_i^* = \operatorname{supp} \pi_i$. Hence, $\operatorname{supp} \nu_i^* = \{s_{-i}^*\}$. We assert that the profile of neo capacities (ν_1^*, ν_2^*) is an *Equilibrium under Ambiguity (EUA)* of Γ. Since (s_1^*, s_2^*) is a Nash equilibrium of $\Gamma(\lambda, \gamma)$, $\operatorname{supp} \nu_i^* = \{s_{-i}^*\} \in R_{-i}(\nu_{-i}^*)$. The profile of beliefs (ν_1^*, ν_2^*) is therefore an EUA. ■

An Equilibrium under Ambiguity of the game $\Gamma(0, 0, 0, 0)$ is a Nash equilibrium of the game Γ. So a standard Nash equilibrium is the special case of an EUA where

[23] To clarify, by "corresponding" we mean that each player plays the same strategy in the two equilibria.

all players have additive beliefs. Moreover, EUA can be applied wherever Nash equilibrium can, as Proposition 3.1 shows.

Proof of Proposition 3.1: By Proposition A.1, it suffices to show that a Nash equilibrium of the game $\Gamma(\lambda, \gamma)$ exists. One can show $V_i(\cdot; \lambda_i, \gamma_i)$ is continuous and quasi-concave if $u_i(s_i, s_{-i})$ is continuous in s and quasi-concave in s_i. Existence of a pure strategy equilibrium for $\Gamma(\lambda, \gamma)$ then follows from standard arguments. ■

A.2 Oligopoly

Lemma A.1 *Assume that the manager of firm 2 is ambiguity neutral (i.e.* $\gamma_2 = \lambda_2 = 0$), *while the manager of firm 1 is not necessarily ambiguity neutral. Then under Cournot quantity competition the equilibrium output of firm 1 is given by:*

$$\bar{x}_1 = \frac{(1 - \gamma_1 + \lambda_1) - (1 + \gamma_1 + \lambda_1)\,c}{3\,(1 - \gamma_1) + \lambda_1}.$$

Proof. From equation (6), firm 1's reaction function is given by, $R^1(x_2) = \frac{(1-\gamma_1)-(1-\gamma_1-\lambda_1)x_2-c}{2(1-\gamma_1)}$. By similar reasoning firm 2's reaction function is, $R^2(x_1) = \frac{1-c-x_1}{2}$. Solving for equilibrium in the usual way we obtain: $x_1 = \frac{2(1-\gamma_1)-(1-\gamma_1-\lambda_1)(1-c-x_1)-2c}{4(1-\gamma_1)}$. Thus $\left[\frac{4(1-\gamma_1)-(1-\gamma_1-\lambda_1)}{4(1-\gamma_1)}\right] x_1 = \frac{(1-\gamma_1+\lambda_1)-(1+\gamma_1+\lambda_1)c}{4(1-\gamma_1)}$ from which the result follows. ■

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Proof of Proposition 4.5 By equation (9), firm 2's reaction function is given by $p_2 = \frac{a+bp_1+ck}{2c}$. Firm 1's profits are given by, $\Pi_1 = (p_1 - k)\left(a + b\frac{a+bp_1+ck}{2c} - cp_1\right)$. The first order condition for profit maximisation is: $\frac{\partial \Pi_1}{\partial p_1} = a + b\frac{a+ck}{2c} + \left(\frac{b^2-2c^2}{2c}\right)p_1 + (p_1 - k)\left(\frac{b^2-2c^2}{2c}\right) = 0$. Solving for p_1 we find, $2\left(\frac{2c^2-b^2}{2c}\right)p_1 = k\left(\frac{2c^2-b^2}{2c}\right) + a + b\frac{a+ck}{2c}$, from which the result follows. ■

Lemma A.2 *Provided* $K \geqslant \frac{10+9k}{14}$, *in example 2 the optimal value of* λ_1 *is*

$$\lambda_1 = \frac{10 - 5k}{56K - 38 - 37k} \geqslant 0. \tag{11}$$

Proof. By equation (10) firm 2's reaction function is given by $p_2 = \frac{2+p_1+2k}{4}$. Firm 1's reaction function is given by $p_1 = \frac{2+2k+\lambda_1 K+(1-\lambda)p_2}{4}$. Solving for equilibrium, $p_1 = \frac{8+8k+4\lambda_1 K+(1-\lambda)(2+2k)}{16} + (1-\lambda)\frac{p_1}{16}$. Hence $\left[\frac{16-(1-\lambda_1)}{16}\right] p_1 = \frac{8+8k+4\lambda_1 K+(1-\lambda_1)(2+2k)}{16}$ or $p_1 = \frac{8+8k+4\lambda_1 K+(1-\lambda_1)(2+2k)}{16-(1-\lambda_1)}$.

Profit is maximised where this is equal to the price which would be chosen by a price leader without ambiguity. Hence $\frac{8+8k+4\lambda_1 K+(1-\lambda_1)(2+2k)}{16-(1-\lambda_1)} = \frac{10+9k}{14}$. Solving for λ_1, $112 + 112k + 56\lambda_1 K + 28(1-\lambda_1)(1+k) = 160 - 10(1-\lambda_1) + 144k - 9k(1-\lambda_1), 56\lambda_1$,which implies $K + 38(1-\lambda_1) + 37(1-\lambda_1)k = 48 + 32k \Leftrightarrow 56\lambda K - 38\lambda_1 - 37\lambda_1 k = 10 - 5k$, from which equation (11) follows.

Note that $56K - 38 - 37k \geqslant 56\left(\frac{10+9k}{14}\right) - 38 - 37k = 40 + 36k - 37k = 2 - k \geqslant 0$, since $2 = a > k$. (Recall $\frac{10+9k}{14}$ is the output a price leader would choose.) ■

A.3 Peace Processes

Proof of Proposition 5.1 Since b is convex, any party's best response is either $s_i = 0$ or $s_i = 1$.

Case (1) If $s_2 \neq 1, u(1, s_2) - u(s_1, s_2) = a\varepsilon(1+s_2)^2 - c - \left[a\varepsilon(s_1+s_2)^2 - cs_1\right]$

$= a\varepsilon\left(1 - s_1^2 + 2s_2 - 2s_1 s_2\right) - c(1-s_1) = a\varepsilon((1+s_1)(1-s_1) + 2s_2(1-s_1)) - c(1-s_1) = (1-s_1)[a\varepsilon(1+s_1+2s_2) - c] > 0$, since, by assumption, $a\varepsilon > c$.

The case $s_2 = 1$ can be covered as follows: $u(1,1) - u(s_1, 1) = a - c - \left[a\varepsilon(s_1+1)^2 - cx_1\right]$ $= a(1-4\varepsilon) + 4a\varepsilon - a\varepsilon\left(s_1^2 + 2s_1 + 1\right) - c(1-s_1) = a(1-4\varepsilon) + 2a\varepsilon(1-s_1) + a\varepsilon\left(1-s_1^2\right) - c(1-s_1) = a(1-4\varepsilon) + (1-s_1)[2a\varepsilon + a\varepsilon(s_1+1) - c] > 0$ since $a\varepsilon > c$.

Case (2) To show that $s_1 = s_2 = 1$ is a Nash equilibrium, by convexity of b it is enough to show $u(1,1) \geqslant u(0,1)$. This holds since, $u(1,1) - u(0,1) = a - c - a\varepsilon = a(1-\varepsilon) - c \geqslant 0$. Now $u(0,0) = 0, u(1,0) = a\varepsilon - c \leqslant 0$, by assumption, which implies that $s_1 = s_2 = 0$ is also a Nash equilibrium.

Case (3) If $s_1 \neq 1$, $u(0, s_2) - u(s_1, s_2) = a\varepsilon s_2^2 - a\varepsilon(s_1+s_2)^2 + cs_1$

$= cs_1 - a\varepsilon\left(2s_1 s_2 + s_1^2\right) \geqslant c - 3a\varepsilon > 0$, since $c > a(1-\varepsilon)$ implies $c > \frac{3}{4}a \geqslant 3a\varepsilon$. The remaining case follows since $u(1,1) - u(0,1) = a - c - a\varepsilon = a(1-\varepsilon) - c < 0$. ■

Proof of Proposition 5.2 Cases (1) and (3) are proven by Proposition 5.1 and the observation that CEU preferences respect strict dominance.

Case (2) By convexity the only best responses can be 0 or 1. Without loss of generality consider player 1 Assume that his/her beliefs are represented by a neo-capacity $\nu = \lambda + (1-\gamma-\lambda)\pi$. Let $U(1)$ (resp. $U(0)$) denote his/her (Choquet) expected utility if (s)he chooses 1 (resp. 0). Then, $U(1) = \lambda[a-c] + \gamma[a\varepsilon - c] + (1-\gamma-\lambda)\left[a\varepsilon\mathbf{E}_\pi(1+s_2)^2 - c\right]$, $U(0) = \lambda a\varepsilon + (1-\gamma-\lambda)a\varepsilon\mathbf{E}_\pi s_2^2$, where $\mathbf{E}_\pi$ denotes expectation with respect to the additive probability π. Now $U(1) - U(0) = \lambda[a-c] + \gamma[a\varepsilon - c] + (1-\gamma-\lambda)\left[a\varepsilon\mathbf{E}_\pi(1+s_2)^2 - c\right] - \lambda a\varepsilon - (1-\gamma-\lambda)a\varepsilon\mathbf{E}_\pi s_2^2$
$= \lambda[a(1-\varepsilon) - c] + \gamma[a\varepsilon - c] + (1-\gamma-\lambda)[a\varepsilon\mathbf{E}_\pi(1+2s_2) - c]$. By assumption $[a(1-\varepsilon)-c] > 0$ and $a\varepsilon - c < 0$ hence if λ (resp. γ) is sufficiently large $U(1) > U(0)$ (resp. $U(1) < U(0)$), from which the result follows. ■

A.4 General Results

Lemma A.3 *The slope of the reaction functions is given by*

$$\begin{aligned}
R^{1\prime}(s_2) &= \frac{-(1-\lambda_1-\gamma_1)u_{12}^1\left(R^1(s_2), s_2\right)}{\lambda_1 u_{11}^1(R^1(s_2), M_2) + \gamma_1 u_{11}^1(R^1(s_2), m_2) + (1-\lambda_1-\gamma_1)u_{11}^1(R^1(s_2), s_2)} \\
R^{2\prime}(s_1) &= \frac{-(1-\lambda_2-\gamma_2)u_{12}^2\left(s_1, R^2(s_1)\right)}{\lambda_2 u_{22}^2(M_1, R^2(s_1)) + \gamma_2 u_{22}^2(m_1, R^2(s_1)) + (1-\lambda_2-\gamma_2)u_{22}^2(s_1, R^2(s_1))}.
\end{aligned}$$

Proof. Let R^i denote the reaction function of player i. Consider player 1, his/her Choquet expected utility is given by: $\lambda_1 u^1(s_1, M_2) + \gamma_1 u^1(s_1, m_2) + (1-\lambda_1-\gamma_1)u^1(s_1, s_2)$. By Assumption 6.1 his reaction function is defined by,

$$\lambda_1 u_1^1\left(R^1(s_2), M_2\right) + \gamma_1 u_1^1\left(R^1(s_2), m_2\right) + (1-\lambda_1-\gamma_1)u_1^1\left(R^1(s_2), s_2\right) = 0. \quad (12)$$

Differentiating (12) with respect to s_2 we obtain, $\lambda_1 u_{11}^1\left(R^1(s_2), M_2\right)R^{1\prime}(s_2)$
$+\gamma_1 u_{11}^1\left(R^1(s_2), m_2\right)R^{1\prime}(s_2) + (1-\lambda_1-\gamma_1)u_{11}^1\left(R^1(s_2), s_2\right)R^{1\prime}(s_2)$

$+ (1-\lambda_1-\gamma_1)\, u_{12}^1 \left(R^1(s_2), s_2\right) = 0$. From which the result follows. The slope of R^2 can be derived by similar reasoning. ■

Proof of Proposition 6.1 Consider the function, $g : S^1 \times S^2 \to S^1 \times S^2$, defined by $g(s_1, s_2) = \left\langle R^1(s_2) - s_1, R^2(s_1) - s_2 \right\rangle$. The partial derivatives of g are $\frac{\partial g^1}{\partial s_1} = -1$, $\frac{\partial g^1}{\partial s_2} = R^{1\prime}(s_2)$, $\frac{\partial g^2}{\partial s_1} = R^{2\prime}(s_1)$ and $\frac{\partial g^2}{\partial s_2} = -1$. Let J denote the Jacobian matrix of g. Then $J = \begin{pmatrix} -1 & R^{1\prime}(s_2) \\ R^{2\prime}(s_1) & -1 \end{pmatrix}$. The trace of J is -2. Thus if the determinant of J is positive, both eigenvalues must be negative and hence J is negative definite. The determinant of J is $1 - R^{1\prime}(s_2)\, R^{2\prime}(s_1) \geqslant 1 - \frac{(1-\lambda_2-\gamma_2)(1-\lambda_1-\gamma_1)Q^2}{\delta^2}$, since $R^{1\prime}(s_2) = \frac{-(1-\lambda_1-\gamma_1)u_{12}^1\left(R^1(s_2),s_2\right)}{\lambda_1 u_{11}^1(R^1(s_2),M_2)+\gamma_1 u_{11}^1(R^1(s_2),m_2)+(1-\lambda_1-\gamma_1)u_{11}^1(R^1(s_2),s_2)}$, $\left|R^{1\prime}(s_2)\right| \leqslant \frac{(1-\lambda_1-\gamma_1)Q}{\delta}$, where $Q = \max_{\langle s_1, s_2\rangle \in S} \left|u_{12}^1(s_1, s_2)\right|$. It follows that J is negative definite if $\lambda_1 + \gamma_1$ is sufficiently large. By Theorem 4.3 of Eichberger (1993), negative definiteness of J is implies that equilibrium is unique. ■

The next result characterises the highest and lowest equilibria in terms of the slope of the reaction functions.

Lemma A.4 *If the highest and lowest equilibria are interior equilibria, then* $R^{1\prime}(s_2)\, R^{2\prime}(s_1) \leqslant 1$ *at these equilibria.*

Proof. Define $\rho : [m_1, M_1] \to [m_1, M_1]$ by $\rho(s_1) = R^1\left(R^2(s_1)\right)$. By assumption there are no corner equilibria, hence $\rho(m_1) > m_1$ and $\rho(M_1) < M_1$. Let $\langle \hat{s}_1, \hat{s}_2 \rangle$ be an equilibrium such that $R^{1\prime}(s_2)\, R^{2\prime}(s_1) > 1$. Then for all sufficiently small $\delta > 0$, $\rho(\hat{s}_1 + \delta) > \hat{s}_1 + \delta$. Let $\phi(s_1) = \rho(s_1) - s_1$. Then $\phi(s_1 + \delta) > 0$ and $\phi(M_1) < 0$. By the intermediate value theorem there exists $\bar{s} \in (\hat{s}_1 + \delta, M_1)$ such that $\phi(\bar{s}_1) = \bar{s}_1$. Therefore $\langle \hat{s}_1, \hat{s}_2 \rangle$ is not the highest equilibrium. A similar argument applies to the lowest equilibrium. ■

Proof of Proposition 6.2 Let $\langle \hat{s}_1, \hat{s}_2 \rangle$ denote the highest equilibrium. Assume first that $\langle \hat{s}_1, \hat{s}_2 \rangle$ is an interior equilibrium. Since $\langle \hat{s}_1, \hat{s}_2 \rangle$ is an interior equilibrium it satisfies the first order conditions for best responses.

$$\lambda_1 u_1^1(s_1, M_2) + \gamma_1 u_1^1(s_1, m_2) + (1 - \lambda_1 - \gamma_1) u_1^1(s_1, s_2) = 0, \tag{13}$$

$$\lambda_2 u_2^2(M_1, s_2) + \gamma_2 u_2^2(m_1, s_2) + (1 - \lambda_2 - \gamma_2) u_2^2(s_1, s_2) = 0. \tag{14}$$

Differentiating (14) with respect to λ_1 we obtain:

$\left[\lambda_2 u_{22}^2(M_1, s_2) + \gamma_2 u_{22}^2(m_1, s_2) + (1 - \lambda_2 - \gamma_2) u_{22}^2(s_1, s_2)\right] \frac{\partial s_2}{\partial \lambda_1} + (1 - \lambda_2 - \gamma_2) u_{12}^2(s_1, s_2) \frac{\partial s_1}{\partial \lambda_1} = 0$. Hence

$$\frac{\partial s_2}{\partial \lambda_1} = R^{2\prime}(s_1) \frac{\partial s_1}{\partial \lambda_1} = 0. \tag{15}$$

Differentiating (13) with respect to λ_1 we obtain,

$\left[\lambda_1 u_{11}^1(s_1, M_2) + \gamma_1 u_{11}^1(s_1, m_2) + (1 - \lambda_1 - \gamma_1) u_{11}^1(s_1, s_2)\right] \frac{\partial s_1}{\partial \lambda_1} + (1 - \lambda_1 - \gamma_1) u_{12}^1(s_1, s_2) \frac{\partial s_2}{\partial \lambda_1}$

$= u_1^1(s_1, s_2) - u_1^1(s_1, M_2)$.

Hence $\frac{\partial s_1}{\partial \lambda_1} + \frac{(1-\lambda_1-\gamma_1)u_{12}^1(s_1,s_2)}{\left[\lambda_1 u_{11}^1(s_1,M_2)+\gamma_1 u_{11}^1(s_1,m_2)+(1-\lambda_1-\gamma_1)u_{11}^1(s_1,s_2)\right]} \frac{\partial s_2}{\partial \lambda_1}$

$= \frac{u_1^1(s_1,s_2)-u_1^1(s_1,M_2)}{\left[\lambda_1 u_{11}^1(s_1,M_2)+\gamma_1 u_{11}^1(s_1,m_2)+(1-\lambda_1-\gamma_1)u_{11}^1(s_1,s_2)\right]}$. Substituting from (15),

$\frac{\partial s_1}{\partial \lambda_1} - R^{1\prime}(s_2) R^{2\prime}(s_1) \frac{\partial s_1}{\partial \lambda_1} = \frac{u_1^1(s_1,s_2)-u_1^1(s_1,M_2)}{\left[\lambda_1 u_{11}^1(s_1,M_2)+\gamma_1 u_{11}^1(s_1,m_2)+(1-\lambda_1-\gamma_1)u_{11}^1(s_1,s_2)\right]}$. Hence $\frac{\partial s_1}{\partial \lambda_1} = \frac{u_1^1(s_1,s_2)-u_1^1(s_1,M_2)}{\left[\lambda_1 u_{11}^1(s_1,M_2)+\gamma_1 u_{11}^1(s_1,m_2)+(1-\lambda_1-\gamma_1)u_{11}^1(s_1,s_2)\right]} \left[1 - R^{1\prime}(s_2) R^{2\prime}(s_1)\right]^{-1}$. By Lemma A.4, $1 - R^{1\prime}(s_2) R^{2\prime}(s_1) \geqslant 0$, hence $\frac{\partial s_1}{\partial \lambda_1} \geqslant 0$. By equation (15) $\frac{\partial s_2}{\partial \lambda_1} \geqslant 0$. The results for the effect of changing the other parameters on the highest and lowest equilibria can be obtained by similar reasoning.

Now consider the case where the highest equilibrium is on the boundary of the strategy set. In particular suppose that when $\lambda_1 = \tilde{\lambda}_1$ that the highest equilibrium is $\langle M_1, M_2 \rangle$. Firstly it is trivially true that a decrease in λ_1 must (weakly) decrease the equilibrium strategies of both players. Now suppose λ_1 increases from $\tilde{\lambda}_1$ to $\hat{\lambda}_1$. The equilibrium at $\langle M_1, M_2 \rangle$ satisfies the Kuhn-Tucker conditions:

$$\tilde{\lambda}_1 u_1^1 (M_1, M_2) + \gamma_1 u_1^1 (M_1, m_2) + \left(1 - \tilde{\lambda}_1 - \gamma_1\right) u_1^1 (M_1, M_2) \geqslant 0, \quad (16)$$

$$\lambda_2 u_1^2 (M_1, M_2) + \gamma_2 u_1^2 (m_1, M_2) + (1 - \lambda_2 - \gamma_2) u_1^2 (M_1, M_2) \geqslant 0. \quad (17)$$

Since $\hat{\lambda}_1 u_1^1 (M_1, M_2) + \gamma_1 u_1^1 (M_1, m_2) + \left(1 - \hat{\lambda}_1 - \gamma_1\right) u_1^1 (M_1, M_2) = \tilde{\lambda}_1 u_1^1 (M_1, M_2) + \gamma_1 u_1^1 (M_1, m_2) + \left(1 - \tilde{\lambda}_1 - \gamma_1\right) u_1^1 (M_1, M_2)$, the Kuhn-Tucker conditions are still satisfied when $\lambda_1 = \hat{\lambda}_1$. By concavity, these conditions are sufficient hence $\langle M_1, M_2 \rangle$ remains the highest equilibrium when $\lambda_1 = \hat{\lambda}_1$. Similar reasoning applies to the lowest equilibrium. ■

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