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Loss Analysis of a Life Insurance Company Applying Discrete-time Risk-minimizing Hedging Strategies

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LOSS ANALYSIS OF A LIFE INSURANCE COMPANY APPLYING DISCRETE-TIME RISK-MINIMIZING HEDGING STRATEGIES

AN CHEN*

Abstract. In this paper, we consider the net loss of a life insurance company issuing identical equity-linked pure endowment contracts in the case of periodic premiums. Under this construction, financial risks as well as the mortality risk are included. Based on Møller (1998), we particularly investigate the situation where the company applies a time-discretized risk-minimizing hedging strategy, i.e., a trading restriction is imposed on a continuous-time risk-minimizing strategy. Therefore, the considered model is incomplete where the incompleteness results not only from the mortality risk but also from the trading restrictions. Through an illustrative example, it is observed from the simulations that a substantial reduction in the ruin probability is achieved by using the time-discretized risk-minimizing strategy. However, as the hedging frequency is set higher, this advantage almost disappears, because a higher frequency leads to more hedging errors which constitute a vital part of the hedger's net loss. In order to improve the simulated results, another type of discrete-time risk-minimizing strategy is taken into consideration. It is obtained by discretizing the hedging model instead of the hedging strategy. For this purpose, Møller's (2001) discrete-time (binomial) risk-minimizing strategy is adopted. For both strategies, a number of sensitivity analyses are carried out, e.g. how the ruin probability changes with the fair combination of the minimum interest rate guarantee and the participation rate.

JEL: G10, G13, G22

Keywords: Net Loss, Discrete-time Risk-minimizing Hedging Strategies, Pure Endowment Equity-linked Life Insurance

1. Introduction

The topic of insolvency risk of life insurance companies has attracted a great deal of attention. Since the 1980s a long list of defaulted life insurance companies in Europe, Japan and USA has been reported. Here are two noticeable examples from the United States:

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First Executive Life Insurance Co. in 1991, the 12th largest bankruptcy in the United States in the period 1980 - 2005, and Conseco Inc. in 2002, the 3rd largest bankruptcy in the United States in the period 1980 - 2005[1]. In Japan, the following life insurance carriers defaulted: Nissan Mutual Life in 1997, Chiyoda Mutual Life Insurance Co. and Kyoei Life Insurance Co. in 2000 and Tokyo Mutual Life Insurance in 2001. In Europe, there were the following most noticeable insolvency cases: Garantie Mutuelle des Fonctionnaires in France in 1993, the world's oldest life insurer Equitable Life in the United Kingdom in 2000 and Mannheimer Leben in Germany in 2003. Therefore, the task of how to reduce the insolvency risk becomes a more and more important topic.

The insolvency risk of an insurance company can usually be reduced in two different ways: externally or internally. Concerning the external risk management, a regulator may be introduced who imposes an intervention rule in order to prevent the insurance company from insolvency. This is the approach taken e.g. by Grosen and Jørgensen (2002)[2]. In their model, the firm defaults and is liquidated if up to the maturity time the value of the total assets has not been sufficiently high to cover the nominal liability multiplied by some pre-specified constant parameter. The regulator controls the strictness of intervention by setting the size of this parameter. Concerning the internal risk management, the insurance company actively manages its exposure to insolvency by appropriately hedging the risks of the issued contracts. This approach has already been used e.g. in Mahayni and Schlögl (2003). They mainly investigate how to determine the contract parameters conservatively and implement robust risk management strategies. It is worth mentioning that different contract designs and different hedging criteria would lead to very different results by using this approach. In the present paper, we mainly study the case when the insurance company applies a risk-minimizing hedging strategy to an equity-linked pure endowment life insurance contract. Moreover, we go one step further and investigate the net loss of the contract-issuing company.

Equity-linked life insurance contracts are an example of the interplay between insurance and finance. In contrast to the financial risks[3], the insurance risk is not tradable in the

[1] Data taken from http://www.bankruptcydata.com.

[2] Bernard, Le Courtois and Quittard-Pinon (2005) extended Grosen and Jørgensen's (2002) model by incorporating the stochastic interest rate. Chen and Suchanecki (2005) extended Grosen and Jørgensen's (2002) model to study the effects of various bankruptcy procedures.

[3] Besides the financial risk related to the asset, there is also interest rate risk because life insurance policies are typically long term contracts and the time horizons are long enough to capture significant

financial market. Hence, there are different methods to deal with this untradable risk. Following Brennan and Schwartz (1979), most authors (e.g. Bacinello and Ortu (1993), Bacinello and Persson (2002), Grosen and Jørgensen (2000) and Miltersen and Persson (2000)) replace the uncertainty of the insured individuals' death/survival by the expected values according to the law of large numbers. So, the actual insurance claims including mortality risk as well as financial risk are replaced by modified claims, which only contain financial uncertainty. This allows the use of standard financial valuation and hedging techniques for complete markets. Although some other authors add mortality risk to their model, they neglect the hedging perspective and mainly deal with fair valuation of the equity-linked life insurance contracts, see e.g. Aase and Persson (1994), Ekern and Persson (1996) and Nielsen and Sandmann (1995, 1996, 2002). In contrast to all the authors mentioned above, Møller (1998) attempts to **hedge** the combined actuarial and financial risk. In his work, continuously adjustable risk-minimizing (in the sense of variance-minimizing) hedging strategies are determined for equity-linked life insurance contracts. In this paper, we use Møller's risk-minimizing strategy with a modification, namely a trading restriction is imposed on this continuous strategy, i.e., the hedging of the contingent claim occurs at discrete times only. Therefore, the considered model is incomplete in two aspects where the incompleteness results not only from the mortality risk but also from the trading restriction.

Through an illustrative simulation example, it is observed numerically that a substantial reduction in the ruin probability[4] is achieved by using the time-discretized risk-minimizing strategy, in comparison with the scenario, where the insurer invests the premiums in a risk free asset with a rate of return corresponding to the market interest rate. However, the extent of the reduction becomes less apparent and the advantage of using this strategy almost disappears when the trading frequency is increased. This is due to the fact that extra duplication errors are caused when the original mean-self-financing risk-minimizing hedging strategy is discretized with respect to time and that these errors increase with the frequency. In order to improve the numerical results, another type of discrete-time risk-minimizing strategy is taken into consideration. It is obtained by discretizing the hedging model instead of the hedging strategy. For this purpose, we consider the Cox, Ross and

variations in the interest rate. For the sake of clarity, a deterministic term structure is applied in the present paper, but it is not difficult to add a stochastic term structure to the model.

[4]Due to the specific modelling of the contract (pure endowment contracts), the ruin probability equals the relative frequency the simulated net loss of the insurer at the maturity of the contract is larger than zero.

Rubinstein (1979) model (CRR), which converges in the limit to the Black-Scholes (1973) model. In this discrete-time framework, Møller's (2001) binomial risk-minimizing strategy is adopted. When comparing the simulation results with the scenario where the strategy is discretized, we observe considerably smaller ruin probabilities, in particular, when the frequency is increased.

This paper is organized as follows: In Section 2, the net loss of an insurance company is defined and for two simple scenarios the loss is computed. Section 3 focuses on the net loss caused by using the time-discretized originally continuous risk-minimizing hedging strategies. Section 4 contains the demonstration of how to calculate the relevant discretized risk-minimizing strategy with the help of an example. In Section 5, we show by simulating the ruin probability caused by discretizing the hedging strategy that some of the numerical results are not very satisfactory. In Section 6, the hedging model is discretized instead of the hedging strategy and the numerical results are improved substantially. Section 7 concludes the paper.

2. Definition of net loss and two extreme scenarios

This section aims at defining the net loss of a life insurance company and at exhibiting two extreme cases. Suppose that at the beginning n identical customers of age x engage in the same pure endowment contract with the insurance company, which promises each of them a payment of $f(S)$ at the maturity date if they survive until this point in time. The function $f(S)$ describes the dependence of the final payment on the evolution of the stock price. It can be a function of the terminal stock price S_T only or of the whole path of the stock and possibly it contains embedded options[5]. In return, each customer pays a premium of K periodically, which is determined at the beginning of the contract and which will be kept constant through the duration of the contract. Let $Y_t^{(n)}$ denote the number of customers who survive time t. As most authors do, we also assume that the surviving times of each customer are independent. This leads to a binomial distribution of $Y_t^{(n)}$ with parameters $(n, {}_tp_x)$, where ${}_tp_x$ gives the probability that an x-aged insured survives time t. Furthermore, it is assumed that the discount factor δ is deterministic and that mortality and financial risks are independent. By this definition of the contract, we observe that both the payment of the insurance company and that of the customers depend on the mortality uncertainty, while the size of the payment of the insurer also hinges on the performance of the stock. Consequently, the net loss of the insurance company at the

[5]In Section 4, a specific payment $f(S)$ is illustrated.

maturity date of the contract is defined as the difference of its accumulated outflows and its accumulated inflows by that point in time.

$$
\begin{aligned}
&\text{Net loss of the insurer at time } T \\
&= \text{Payment of the insurer at time } T - \text{Accumulated premium incomes till time } T \\
&\quad -\text{Trading gains (losses) from investment strategies} \\
&= Y_T^{(n)} f(S) - \sum_{i=0}^{M-1} (Y_{t_i}^{(n)} - Y_{t_{i+1}}^{(n)}) \sum_{j=0}^{i} K e^{(T-t_j)\delta} - Y_T^{(n)} \sum_{j=0}^{M-1} K e^{(T-t_j)\delta} \\
&\quad -\text{Trading gains (losses) from investment strategies.}
\end{aligned}
$$

Those, who die during $]t_i, t_{i+1}]$ only pay the premiums till t_i and those who survive the end of the contract $T = t_M$ pay all of the premiums. Naturally, the trading gains (losses) depend on the hedging/investment strategies the insurer chooses.

2.1. **Net loss when investing the premiums in a risk free asset.** As a starting point, we consider the net loss of the company when the insurance company invests the premiums in a risk free asset with a rate of return δ. Hence, the net loss of the insurer at time T is simplified to:

$$
L_n = Y_T^{(n)} f(S) - \sum_{i=0}^{M-1} (Y_{t_i}^{(n)} - Y_{t_{i+1}}^{(n)}) \sum_{j=0}^{i} K e^{(T-t_j)\delta} - Y_T^{(n)} \sum_{j=0}^{M-1} K e^{(T-t_j)\delta}. \tag{1}
$$

The expected loss can be derived as follows:

$$
\mathrm{E}[L_n] = n \cdot {}_T p_x \cdot \mathrm{E}[f(S)] - n \sum_{i=0}^{M-1} \sum_{j=0}^{i} K e^{(T-t_j)\delta} ({}_{t_i}p_x - {}_{t_{i+1}} p_x) - n \cdot {}_T p_x \sum_{j=0}^{M-1} K e^{(T-t_j)\delta}. \tag{2}
$$

The independence assumption between financial and mortality risks and the equality $\mathrm{E}[Y_{t_i}^{(n)}] = n \cdot {}_{t_i}p_x$ are needed for the above derivation. It is observed that the expected loss is equal to 0, if and only if

$$
K^* = \frac{{}_T p_x \mathrm{E}[f(S)]}{\sum_{i=0}^{M-1} \sum_{j=0}^{i} e^{(T-t_j)\delta} ({}_{t_i}p_x - {}_{t_{i+1}} p_x) + {}_T p_x \sum_{j=0}^{M-1} e^{(T-t_j)\delta}}. \tag{3}
$$

It is observed that the optimal K^* does not depend on the number of the contracts the insurer issues. Only with this premium, $E[L_n]/n = 0$ holds. If the charged premium is larger than K^*, then $E[L_n]/n < 0$, i.e., $\lim_{n\to\infty} E[L_n] = -\infty$. This means that the company makes an infinitely large expected profit as the number of the contract-holders is increased to infinity. On the contrary, if the charged premium is smaller than K^*, this will result in an infinitely large expected loss for the company as the number of the contract-holders goes to infinity. In the numerical analysis, the equivalent martingale measure is used to calculate the optimal periodic premium payment K^*.

The variance of the net loss can be derived without major difficulty. It is noticed that asymptotically, i.e., as $n \to \infty$,

$$\mathrm{Var}\left[\frac{1}{n} L_n\right] \quad \to \quad {}_T p_x^2 \mathrm{Var}[f(S)].$$

As expected, by increasing the number of the insured, the insurer can eliminate all the mortality risk. This is the so-called diversification over sub-populations (law of large numbers). However, the financial uncertainty concerning the future evolution of the stock remains with the insurer, since all contracts are linked to the same stock.

2.2. **Net loss in the case of a static hedge.** In contrast to the above extreme scenario, we now assume that there are some static ("buy-and-hold") hedging strategies which completely duplicate the final payment $f(S)$, so that the insurer can eliminate the entire risk. Assume that the company applies the static strategy, i.e., it purchases $n \cdot {}_T p_x$ financial contracts at the beginning of the insurance contract and holds them until the maturity date of the insurance contract. Each of these financial contracts pays the amount $f(S)$ at time T. Let V_0 be today's price of such a financial contract. Hence, the loss is described as the difference of the loss in the first case and the profit/loss from trading:

$$L_n^s = L_n - \text{profit/loss from trading} = L_n - \left(n \cdot {}_T p_x \cdot f(S) - n \cdot {}_T p_x \cdot V_0 e^{\delta T}\right). \qquad (4)$$

Not surprisingly, it is observed here

$$\mathrm{Var}\left[\frac{L_n^s}{n}\right] \to 0$$

as $n \to \infty$, i.e., in this case, the total risk (mortality risk + financial risk) could be eliminated by increasing the number of policies in the portfolio and by buying the static hedging strategy on the stock. However, this static strategy is not realistic because the usual term of these insurance contracts is quite long, e.g, 12 to 30 years in Germany, while standard options are typically short-term transactions, say, less than one year. Hence, any realistic hedging strategy will leave the insurance company exposed to some risks which lie between the above two extreme scenarios. Due to this unrealistic restriction, this second scenario will not be considered later. As mentioned in the introduction, the paper focuses on the net loss analysis when the hedger adopts a time-discretized risk-minimizing hedging strategy. Hence, before coming to Møller's (1998) risk-minimizing hedging strategy, we review some fundamentals about cost processes and duplication errors caused by using time-discretized originally continuous hedging strategies.

3. Cost process and net loss when applying the discretized originally continuous risk -minimizing strategy

Due to two reasons, namely, high transaction costs and the fact that security markets do not operate but are closed at nights, at weekends and on holidays, it is impossible for a hedger to make continuous adjustments to his hedging portfolio. In this context, discrete-time strategies receive a wide application. There are different ways to generate a discrete-time strategy. In the following, we consider a discrete-time hedging strategy which is generated from discretizing a continuous-time hedging strategy with respect to time. That is, the underlying price dynamics is a continuous-time stochastic process so that a continuous-time hedging strategy is received at first[6]. Later, in Section 5, another type of discrete-time trading strategy is generated directly from assuming that the relevant price dynamics is driven by a binomial model. Before the loss analysis is taken into consideration, the corresponding cost process and the duplication error resulting from the use of a time-discretized hedging strategy are studied.

3.1. **Cost process and duplication error.** Assume, the set of trading dates is characterized by a sequence of refinements τ^Q of the interval $[0,T]$, namely,

$$\tau^Q = \{0 = t_0 < t_1 < \cdots < t_Q = T\}$$

with $|t_{k+1} - t_k| \to 0$ for $Q \to \infty$. For simplification, Q is assumed to be a multiple of M. Transactions are carried out immediately after the prices are announced at a certain discrete point in time and are kept constant throughout the time period until the next trading decision takes place. On the one hand, $\phi^Q = (\xi^Q, \eta^Q)$ denotes the discrete-time trading strategy with respect to the refinement τ^Q, where ξ^Q gives the number of stocks S and η^Q the number of bonds B. It is defined as follows

$$\phi_t^Q := \phi_{t_k}^Q, \quad t \in]t_k, t_{k+1}], \quad k \leq Q-1.$$

On the other hand, $\phi^c = (\xi^c, \eta^c)$ denotes the corresponding continuous-time trading strategy. Both ϕ^Q and ϕ^c depend on the payoff $f(S)$. The equality

$$\phi_{t_k}^Q = \phi_{t_k}^c, \quad k = 0, 1, \cdots, Q-1$$

does not necessarily hold in general. Its validity crucially depends on the specification of the model and the trading strategy. For instance, $\phi_{t_k}^Q = \phi_{t_k}^c$ holds if ϕ^c is the Black-Scholes or risk-minimizing hedging strategy and ϕ^Q corresponds to the time-discretized version of

[6]For example, we obtain a continuous-time hedging strategy by assuming that the asset dynamics follows a geometric Brownian motion as in the Black-Scholes (1973) model.

these strategies. However, the equality is not valid any more when the discrete-time strategy is obtained in a binomial model, while ϕ^c is any continuous strategy. Since our main interest lies in deriving the cost process of the time-discretized risk-minimizing strategy, we are in the situation of $\phi^Q_{t_k} = \phi^c_{t_k}$.

According to the relation between gain and cost processes, the net loss can be rephrased as follows:

$$
\begin{aligned}
\text{Net loss} &= Y_T^{(n)} f(S) - \text{trading gains/losses} - \text{Premium incomes} \\
&= Y_T^{(n)} f(S) - [V_T(\phi^Q) - V_0(\phi^Q)e^{\delta T} - L_T(\phi^Q)] - \text{Premium incomes} \\
&= V_0(\phi^Q)e^{\delta T} + L_T(\phi^Q) + [V_T(\phi^c) - V_T(\phi^Q)] - \text{Premium incomes},
\end{aligned}
$$

where $L_T(\phi^Q)$ is the cost until the maturity resulting from ϕ^Q, and $V_T(\phi^c) - V_T(\phi^Q)$ the duplication error[7] caused by using the time-discretized hedging strategy. For the above derivation the equality $Y_T^{(n)} f(S) = V_T(\phi^c)$ is used, i.e it is assumed that the contingent claim $Y_T^{(n)} f(S)$ is perfectly duplicated by the final payment of the continuous hedging strategy. This assumption is satisfied when continuous risk-minimizing hedging strategies[8] are taken into account. Since the premium incomes are known and since the initial value of ϕ^Q equals the initial value of ϕ^c, the net loss can be readily obtained as soon as the cost and the duplication error with respect to ϕ^Q are calculated.

The corresponding cost process $L(\phi^Q)$ associated with ϕ^Q is defined as follows:

$$
\begin{aligned}
L_t(\phi^Q) &= V_t(\phi^Q) - V_0(\phi^Q) - \sum_{j=0}^{k-1} \Bigg[\xi^c_{t_j}(S_{t_{j+1}} - S_{t_j}) + \eta^c_{t_j}(B_{t_{j+1}} - B_{t_j}) \\
&\quad + \xi^c_{t_k}(S_t - S_{t_k}) + \eta^c_{t_k}(B_t - B_{t_k}) \Bigg] \\
&= \xi^c_{t_k} S_{t_k} + \eta^c_{t_k} B_{t_k} - \left(V_0(\phi^c) + \sum_{j=0}^{k-1} \xi^c_{t_j}(S_{t_{j+1}} - S_{t_j}) + \eta^c_{t_j}(B_{t_{j+1}} - B_{t_j}) \right), \; t \in]t_k, t_{k+1}].
\end{aligned}
$$

[7]It is well-known that the discrete-time version of Gaussian hedging strategies could lead to an extra duplication bias, even when there are no model or parameter misspecifications, see e.g. Mahayni (2003).

[8]This is a dynamic hedging approach which relies on the condition that contingent claims can be duplicated by the final value of the hedging portfolio and basically amounts to minimizing the variance of the hedger's future costs. However, this approach has the undesirable property that minimization of the variance (or the expected value of the square of the future costs) implies that relative losses and relative gains are treated equally. C.f. Föllmer and Sondermann (1986) and Föllmer and Schweizer (1988).

It is noted that ϕ^Q is not necessarily self-financing or mean-self-financing even if this holds for ϕ^c.[9] Moreover, the value of the discrete-time version ϕ^Q differs from that of the continuous strategy ϕ^c by an amount, which is given by:

$$V_t(\phi^c) - V_t(\phi^Q) = (\xi_t^c - \xi_{t_k}^c)S_t + (\eta_t^c - \eta_{t_k}^c)B_t,\ t \in]t_k, t_{k+1}].$$

That means, if the contingent claim is duplicated by the value of ϕ_T^c, in general it cannot be duplicated by the value of the time-discretized strategy for maturity date T simultaneously, because it takes the value of $\phi_{t_{Q-1}}^c$. In the following we denote by $L_T^C(\phi^Q)$ the accumulated hedging error of the insurer, which is defined as the sum of the cost until time T and the generated duplication error, i.e.,

$$\begin{aligned}
L_T^C(\phi^Q) &= L_T(\phi^Q) + V_T(\phi^c) - V_T(\phi^Q) \\
&= \xi_{t_{Q-1}}^c S_{t_{Q-1}} + \eta_{t_{Q-1}}^c B_{t_{Q-1}} - \left(\xi_{t_0}^c S_{t_0} + \eta_{t_0}^c B_{t_0} + \sum_{j=0}^{Q-2} \xi_{t_j}^c (S_{t_{j+1}} - S_{t_j}) + \eta_{t_j}^c (B_{t_{j+1}} - B_{t_j})\right) \\
&\quad + (\xi_Q^c - \xi_{t_{Q-1}}^c) S_{t_Q} + (\eta_Q^c - \eta_{t_{Q-1}}^c) B_{t_Q} \\
&= \sum_{j=1}^{Q} (\xi_{t_j}^c - \xi_{t_{j-1}}^c) S_{t_j} + (\eta_{t_j}^c - \eta_{t_{j-1}}^c) B_{t_j}.
\end{aligned}$$

Up to now we have only considered the accumulated hedging error caused by a time-discretized continuous hedging portfolio. Below we will specify this continuous strategy, and have a look at Møller's risk-minimizing hedging strategy.

Møller (1998)[10] considers discounted processes. By using the above definitions, Møller's dynamic risk-minimizing hedging strategy in the pure endowment insurance turns now into

$$\xi_t^Q := \xi_{t_k}^c = Y_{t_k}^{(n)}{}_{T-t_k}p_{x+t_k} f_s(t_k, S_{t_k}) \qquad t \in]t_k, t_{k+1}], \tag{5}$$

$$\eta_t^Q := \eta_{t_k}^c = V_{t_k}^* - \xi_{t_k}^c S_{t_k}^* \qquad t \in]t_k, t_{k+1}], \tag{6}$$

[9] Assume ϕ^c is self-financing, then

$$\begin{aligned}
L_t(\phi^Q) &= V_0(\phi^c) + \int_0^{t_k} \xi_u^c dS_u + \int_0^{t_k} \eta_u^c dB_u - V_0(\phi^c) - \sum_{j=0}^{k-1} \xi_{t_j}^c (S_{t_{j+1}} - S_{t_j}) + \eta_{t_j}^c (B_{t_{j+1}} - B_{t_j}) \\
&= \sum_{j=0}^{k-1} \left(\int_{t_j}^{t_{j+1}} (\xi_u^c - \xi_{t_j}^c) dS_u + \int_{t_j}^{t_{j+1}} (\eta_u^c - \eta_{t_j}^c) dB_u \right).
\end{aligned}$$

From this, without extra conditions, even $E[L_t(\phi^Q)]$ is not equal zero, i.e., ϕ^Q is not mean-self-financing.

[10] Møller (1998) derives the risk-minimizing strategies along the lines of Föllmer and Sondermann (1986) for different equity-linked life insurance contracts under the assumption that the asset price processes are martingales under the objective probability measure.

where $f(t, S_t)$ represents the value of the contingent claim at time t and $f_s(t, S_t)$ the corresponding derivative of $f(t, S_t)$ with respect to the stock price S_t. S_t^* is the discounted stock price at time t and V_t^* gives the discounted value of the hedging portfolio at time t. The hedge ratio (the number of stocks the insurer should hold) at time $t \in]t_k, t_{k+1}]$ is described as the product of the hedge ratio in the case of financial risk only and the average number of customers who survive the contract's maturity time T given that they have survived time t_k. The number of bonds is determined as the difference between the discounted value of the portfolio and the amount invested in the stock[11]. After some transformations, the discounted accumulated hedging error in this specific case has the form of

$$\begin{aligned} L_0^C(\phi^Q) &= \sum_{j=1}^{Q} (\xi_{t_j}^c - \xi_{t_{j-1}}^c) S_{t_j}^* + \eta_{t_Q}^c - \eta_{t_0}^c \\ &= \sum_{j=1}^{Q} \left(Y_{t_j}^{(n)} {}_{T-t_j}p_{x+t_j} f_s(t_j, S_{t_j}) - Y_{t_{j-1}}^{(n)} {}_{T-t_{j-1}}p_{x+t_{j-1}} f_s(t_{j-1}, S_{t_{j-1}}) \right) S_{t_j}^* \\ &\quad + Y_{t_Q}^{(n)} f(S) e^{-\delta T} - Y_{t_Q}^{(n)} f_s(t_Q, S_{t_Q}) S_T^* - V_0(\phi^c) + n_T p_x f_s(t_0, S_{t_0}) S_{t_0} \end{aligned} \tag{7}$$

According to Møller (1998), the hedger could eliminate all the financial risk by using continuously adjustable risk-minimizing hedging strategies, i.e., the hedging errors left to the hedger completely result from the mortality risk. However, this argument loses its validity if the continuous risk-minimizing strategy is applied discretely. In a word, the discrete version of a continuous risk-minimizing hedging strategy cannot be variance-minimizing. It is observed from Equation (7) that the accumulated hedging error hinges not only on the mortality risk, but also on the financial risk.

3.2. **Net loss.** By applying the continuous risk-minimizing strategy in discrete time, not all the financial risks are eliminated. In fact, the financial risk could even make the hedger worse off in the sense that more losses are caused. The net loss of the insurer using the discretized risk-minimizing hedging strategy consists of the initial investment plus the

[11] The bond value $B_{t_k}^*$ does not appear in Equation (6) because discounted assets are considered and hence the value of $B_{t_k}^*$ is identical to 1.

accumulated hedging error with respect to ϕ^Q less the premium inflows of the hedger:

$$\begin{aligned}
L_T^{r1} &= V_0(\phi^Q)e^{\delta T} + L_0^C(\phi^Q)e^{\delta T} - \sum_{i=0}^{M-1}(Y_{t_i}^{(n)} - Y_{t_{i+1}}^{(n)})\sum_{j=0}^{i} Ke^{(T-t_j)\delta} - Y_T^{(n)}\sum_{j=0}^{M-1} Ke^{(T-t_j)\delta} \\
&= \sum_{j=1}^{Q}\left[Y_{t_{jz}}^{(n)}\,{}_{T-t_{jz}}p_{x+t_{jz}} f_s(t_{jz}, S_{t_{jz}}) - Y_{t_{(j-1)z}}^{(n)}\,{}_{T-t_{(j-1)z}}p_{x+t_{(j-1)z}} f_s(t_{(j-1)z}, S_{t_{(j-1)z}})\right] S_{t_{jz}} \\
&\quad \cdot e^{(T-t_{jz})\delta} + Y_T^{(n)} f(S) - Y_{t_{Qz}}^{(n)} f_s(t_{Qz}, S_{t_{Qz}})S_T + n_T p_x f_s(t_0, S_{t_0})S_{t_0}e^{\delta T} \\
&\quad - \sum_{i=0}^{M-1}(Y_{t_i}^{(n)} - Y_{t_{i+1}}^{(n)})\sum_{j=0}^{i} Ke^{(T-t_j)\delta} - Y_T^{(n)}\sum_{j=0}^{M-1} Ke^{(T-t_j)\delta}. \qquad (8)
\end{aligned}$$

Here the new notation $z := \frac{M}{Q}$ is introduced in order to make the time index of hedging conform to premium payment time points. As we are interested in the net loss of the hedger at the maturity time, the first two terms in Equation (8) are accrued till the maturity date with accumulation factor δ. Later Equation (8) is used in order to simulate the ruin probability of the hedger in this case.

4. An illustrative example

Because equity-linked products with an asset value guarantee have become very popular in Germany both as pure investment contracts and in the context of life insurance policies since 1996, a specific guaranteed equity-linked insurance contract is also considered as an illustrative example. Our goal is not only to price the issued contract, but to derive the discretized originally continuous risk-minimizing strategy, to study the cost process, and further to investigate the hedger's net loss.

We consider a specific guaranteed equity-linked pure endowment life insurance contract, which provides the buyer of such a contract the payoff

$$f(S) = \sum_{i=0}^{M-1} Ke^{gt_{i+1}} + \alpha\sum_{i=0}^{M-1}(i+1)K\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+, \qquad (9)$$

if she/he survives the maturity of the contract. In this specific case, the final payment is dependent on the minimum guaranteed interest rate g, the participation rate in the surpluses α, the duration of the contract M and more importantly the whole stock prices. Specified at the beginning of the contract, the premium K (e.g. $K = K^*$) is paid periodically by the insured till the maturity of the contract or the death of the insured, whichever comes first. If the insured survives the maturity of the contract, she/he obtains the guaranteed amount and the accumulated boni (participation in the surplus of the company), which are represented by a sequence of European call options with strike $e^{g(t_{i+1}-t_i)}$.

After plugging the $f(S)$-value into Equation (1), we easily obtain the loss of the company for the first situation, where the insurer invest all the premiums in the risk free asset with a rate of return δ.

$$\begin{aligned} L_n &= Y_T^{(n)}\left(\sum_{i=0}^{M-1} K e^{g t_{i+1}} + \alpha \sum_{i=0}^{M-1}(i+1)K\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\right) \\ &\quad - \sum_{i=0}^{M-1}(Y_{t_i}^{(n)} - Y_{t_{i+1}}^{(n)})\sum_{j=0}^{i} K e^{(T-t_j)\delta} - Y_T^{(n)}\sum_{j=0}^{M-1} K e^{(T-t_j)\delta}. \end{aligned} \tag{10}$$

Due to the unrealistic constraint of the second extreme case, we skip this case and jump to the third case, where the insurer hedges her/his risk by using the risk-minimizing strategy. Above all, the discretized risk-minimizing strategy for this specific contract is to be derived in order to be able to computer the loss of the insurer.

4.1. **The case of time-discretizing the continuous risk-minimizing strategy.** Following Equation (5), we need to calculate $f_s(t, S_t)$ for this specific equity-linked life insurance contract in order to obtain the discrete-time version of the continuous risk-minimizing strategy. It is well-known that the price of a contingent claim at time t equals the expected discounted value of the terminal payoff conditional on the information structure till time t, $t \in [0, T]$, under the equivalent martingale measure, that is,

$$\begin{aligned} f(t_{jz}, S_{t_{jz}}) &= E^*[e^{-\delta(T-t_{jz})} f(S_T)|\mathcal{F}_{t_{jz}}] \\ &= e^{-\delta(T-t_{jz})}\sum_{i=0}^{M-1} K e^{g t_{i+1}} + \alpha K \sum_{i=0}^{M-1}(i+1)\Bigg\{\mathbf{1}_{\{t_{jz}>t_{i+1}\}} e^{-\delta(T-t_{jz})}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+ \\ &\quad + \mathbf{1}_{\{t_i<t_{jz}\le t_{i+1}\}} e^{-\delta(T-t_{i+1})}\left(\frac{S_{t_{jz}}}{S_{t_i}} N(d_1^{(jz,i)}) - e^{g(t_{i+1}-t_i)} e^{-\delta(t_{i+1}-t_{jz})} N(d_2^{(jz,i)})\right) \\ &\quad + \mathbf{1}_{\{t_{jz}\le t_i\}} e^{-\delta(t_{M-1}-t_{jz})}\left(N(\tilde{d}_1) - e^{(g-\delta)(t_{i+1}-t_i)} N(\tilde{d}_2)\right)\Bigg\}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} d_{1/2}^{(jz,i)} &= \frac{\ln S_{t_{jz}}/S_{t_i} - g(t_{i+1}-t_i) + (\delta \pm \frac{1}{2}\sigma^2)(t_{i+1}-t_{jz})}{\sigma\sqrt{t_{i+1}-t_{jz}}} \\ \tilde{d}_{1/2} &= \frac{-g(t_{i+1}-t_i) + (\delta \pm \frac{1}{2}\sigma^2)(t_{i+1}-t_i)}{\sigma\sqrt{(t_{i+1}-t_i)}}, \end{aligned}$$

where $N(\cdot)$ is the cumulative standard normal distribution function. As above z is used for conformity and $\mathbf{1}$ denotes the indicator function. The detailed derivation of $f(t, S_t)$ is given in Appendix A. From the derived price of the contingent claim we take the derivative with respect to $S_{t_{jz}}$ and obtain:

$$f_s(t_{jz}, S_{t_{jz}}) = \frac{\partial f(t_{jz}, S_{t_{jz}})}{\partial S_{t_{jz}}} = \alpha K \sum_{i=0}^{M-1} (i+1) e^{-\delta(T-t_{i+1})} \mathbf{1}_{\{t_i < t_{jz} \leq t_{i+1}\}} \frac{1}{S_{t_i}} N(d_1^{(jz,i)}). \qquad (12)$$

Since the main interest of this paper lies in studying the loss distribution of a life insurance company, we are more concerned with the hedging error caused by using risk-minimizing strategies, which constitutes the main part of the insurer's loss. Plugging Equation (12) in Equation (8), we come to the net loss of the insurer:

$$\begin{aligned} L_T^{r1} &= \sum_{j=1}^{Q} \Bigg[Y_{t_{jz}}^{(n)} {}_{T-t_{jz}}p_{x+t_{jz}} \alpha K \sum_{i=0}^{M-1} (i+1) e^{-\delta(T-t_{i+1})} \mathbf{1}_{\{t_i < t_{jz} \leq t_{i+1}\}} \frac{1}{S_{t_i}} N(d_1^{(jz,i)}) \\ &\quad - Y_{t_{(j-1)z}}^{(n)} {}_{T-t_{(j-1)z}}p_{x+t_{(j-1)z}} \alpha K \sum_{i=0}^{M-1} (i+1) e^{-\delta(T-t_{i+1})} \mathbf{1}_{\{t_i < t_{(j-1)z} \leq t_{i+1}\}} \frac{1}{S_{t_i}} N(d_1^{((j-1)z,i)}) \Bigg] \\ &\quad \cdot S_{t_{jz}} e^{(T-t_{jz})\delta} + Y_T^{(n)} f(S) - Y_{t_{Qz}}^{(n)} \frac{\alpha M K}{S_{t_{M-1}}} N(d_1^{(Qz,M-1)}) S_T + n_T p_x e^{-\delta T} \alpha K N(d_1^{(z,0)}) e^{\delta T} \\ &\quad - \sum_{i=0}^{M-1} (Y_{t_i}^{(n)} - Y_{t_{i+1}}^{(n)}) \sum_{j=0}^{i} K e^{(T-t_j)\delta} - Y_T^{(n)} \sum_{j=0}^{M-1} K e^{(T-t_j)\delta}. \qquad (13) \end{aligned}$$

In the following Equations (10) and (13) are used in a simulation in order to analyze the net loss of the insurer. If we repeat the simulation many times, we can conclude which strategy is more beneficial to the insurance company by comparing the simulated technical ruin probabilities. Usually, ruin is defined as a "first passage" event, but due to our contract specification (pure endowment contracts), ruin is defined as the event that the net loss of the insurance company at the maturity date T is larger than zero. Hence, the ruin probability is given as the frequency of the net loss of the insurer is larger than zero. The bigger the ruin probability, the more unstable the insurance company. Hence, an insurance company aims at reaching a ruin probability which is as small as possible.

5. Numerical results

This section targets at simulating the insurer's losses for different cases:

1) the insurer invests the premiums in the risk free asset at a fixed rate of interest δ (Equation (10));
2) the hedger uses a time-discretized risk-minimizing hedging strategy (Equation (13)).

Scenario 2) will still be sub-categorized into two situations: the insurer adjusts his strategy as often as the premium payment dates occur, namely once a year ($M = Q$) and the insurer adjusts his portfolio once a month, while the premium payment occurs once a year ($Q = 12M$). This is done in order to find out whether the hedger is able to reach a smaller ruin probability by increasing the trading frequency.

Due to the independence assumption between the mortality risk and the financial risk, in principle the simulation of the losses reduces to simulating: a) the survival process $\{Y_t^{(n)}\}_{t\in[0,T]}$ and b) the payoff of the pure endowment insurance contract $f(S)$ (or the corresponding derivative of $f(S)$ with respect to the stock) respectively. In order to simulate the survival process, we just need to know the survival probability $\{{}_tp_x\}_{t\in[0,T]}$, which can be calculated by a hazard rate function. For the numerical calculation, the Gompertz-Makeham hazard rate function from Møller (1998) is adopted, i.e:

$$\mu_{x+t} = 0.0005 + 0.000075858 \cdot 1.09144^{x+t} \qquad t \geq 0.$$

This function was used in the Danish 1982 technical basis for men. Consequently, the survival probability of an x-aged life is given by

$${}_tp_x = \exp\left\{-\int_0^t (0.0005 + 0.000075858 \cdot 1.09144^{x+u})du\right\}.$$

Another parameter which should be considered before starting a simulation is the fair premium K^*. According to the analyses in Section 2, non-optimal K-values could cause infinite losses or profits to the hedger asymptotically. However, in this specific example, the fair premium[12] cannot be determined explicitly, because the final payment of the contract depends on the periodic premiums. Substituting this final payment in the expression of the optimal premium equation (Equation (3)), the K-terms would be left out in the calculation. Hence the optimal K^* can only be determined implicitly through the fair relationship between the participation rate α and the minimum guaranteed interest rate g. That is, for a given g, we obtain a corresponding participation α^*, which makes the contract fair. Under the equivalent martingale measure, α as a function of g is given by:

$$\alpha^*(g) = \frac{\sum_{i=0}^{M-1}\sum_{j=0}^{i} e^{-\delta t_j}({}_{t_i}p_x - {}_{t_{i+1}}p_x) + {}_Tp_x \sum_{j=0}^{M-1} e^{-\delta t_j} - {}_Tp_x e^{-\delta T}\sum_{i=0}^{M-1} e^{g t_{i+1}}}{\sum_{i=0}^{M-1}(i+1)e^{-\delta t_{M-1}}(N(\tilde{d}_1) - e^{(g-\delta)(t_{i+1}-t_i)}N(\tilde{d}_2))}. \tag{14}$$

In Table 1, some exemplary fair values are listed. Obviously, there exists a negative relationship between fair α's and g's. Furthermore, the fair α^* rises substantially as the

[12] Taking mortality risk into consideration, a premium is called fair, if the expected discounted accumulated premium income equals the expected discounted accumulated payoff of the contract under the equivalent martingale measure.

Duration M	Minimum Guarantee g	Fair Participation Rate α^*
$M = 12$	$g = 0.0275$	0.37587
$M = 12$	$g = 0.0325$	0.31939
$M = 12$	$g = 0.0375$	0.25634
$M = 20$	$g = 0.0275$	0.49067
$M = 30$	$g = 0.0275$	0.70779

TABLE 1. Fair participation rates α's with following parameters: $\delta = 0.05$, $x = 35$, $\sigma = 0.2$.

duration of the contract increases. This is due to the fact that the periodic boni in the issued contract are held by the insurer till the maturity date, without giving any compensations to the customer. A long duration of the contract implies that the insurer keeps more boni of his customers for a longer time, which hampers the insured to reinvest the periodic boni to a large extent. According to the principle of equivalence, a larger α-value becomes necessary to make the contract fair. These values for the fair participation rate α^* combined with the corresponding g's and M's are used in simulating the ruin probabilities. Of course the fair participation rate also depends on some other parameters like σ and the survival probabilities. However, these dependencies are not of interest here.

Simulating the loss distribution of the first case, where the company invests the premium incomes in a risk free asset, is relatively simple. Simulate the price processes $\frac{S(t_{i+1})}{S(t_i)}$, $i = 0, \cdots M-1$ under the market measure and substitute them into the $f(S)$ expression, then one sample of the claim $f(S)$ is obtained. Combined with the simulated $Y_1^{(n)}, \cdots, Y_T^{(n)}$, one path of the loss is generated. If the whole simulation is repeated m times, the ruin probability of the insurance company is approximated as the ratio:

$$\frac{\text{the number of the paths where the simulated loss is above 0}}{m}.$$

The ruin probabilities for the risk-minimizing strategies are achieved similarly according to Equation (13). Following the procedure we introduced above, the ruin probabilities for Cases 1) and 2) (two subcategories) are obtained after simulating the losses 100000 times.

Table 2 exhibits how the ruin probability depends on the market performance of the stock, which is described by the rate of return μ. Three different μ values, $\mu < \delta$, $\mu = \delta$, and $\mu > \delta$ are used. The percentage numbers in the last column of the table give the ratio of the ruin probability in the case of $Q = M$ and $Q = 12M$ to the ruin probability in Case

Case 1	
μ	Ruin Prob.
0.04	0.45291
0.05	0.47996
0.06	0.53353

Case 2: $Q = M$		
μ	Ruin Prob.	%
0.04	0.13914	30.72%
0.05	0.13213	27.53%
0.06	0.11762	22.05%

Case 2: $Q = 12M$		
μ	Ruin Prob.	%
0.04	0.10861	23.98%
0.05	0.12262	25.55%
0.06	0.12412	23.26%

TABLE 2. Ruin probabilities for different μ's with parameters: $n = 100$, $\alpha = 0.37587$, $g = 0.0275$, $M = 12$, $\delta = 0.05$, $x = 35$, $\sigma = 0.2$.

1 respectively. First of all, it is observed that the ruin probability in the case of discretized risk-minimizing hedging is considerably smaller than in the first case. In the situation $Q = M$, the ruin probabilities are reduced by 69.28%, 62.47% and 77.95% respectively for $\mu = 0.04$, $\mu = 0.05$ and $\mu = 0.06$. The same phenomenon is observed for the situation of $Q = 12M$ with the percentage numbers 76.02%, 74.45% and 77.74%. Second, a common observation for the first case and the case $Q = 12M$ is that the ruin probability increases with the value of μ. This is due to the fact that a better performance of the stock leads to a higher liability of the insurer. However, this relationship between μ and the ruin probability in the discretized risk-minimizing hedge ($Q = 12M$) is not so noticeable as in Case 1. And in case $Q = M$ this relationship ceases to be valid, i.e. the relationship between the ruin probability and μ is quite ambiguous (see also Tables 3-5). Theoretically, it is valid that the more frequently the insurer updates his risk-minimizing hedging strategies, the more the financial risks are reduced. Furthermore, the insurer can eliminate all the financial risks if he could hedge continuously. However, the accumulated hedging error caused by discretizing the continuous risk-minimizing hedging strategy destroyed this argument. This is why it is observed that not all the ruin probabilities in the case $Q = 12M$ are smaller than in the case $M = Q$.

The relation between the ruin probability and the duration of the contract is illustrated in Tables 3, 4 and 5 for different μ-values. Above all, M plays a very important role in determining the fair participation rate α (c.f Table 1). For different g's and M's different fair α's are obtained. Also in these cases the ruin probabilities are reduced substantially, with the use of discretized risk-minimizing strategies. Almost overall a positive relationship between the ruin probability and M is observed. In the first case, obviously the effect of M on the insurer's liability dominates that of M on his accumulated premium incomes. Ruin appears more likely as M increases. In the second case, on the one hand, it is known that some discretization and duplication errors exist when the discretized risk-minimizing

Case 1	
M	Ruin Prob.
12	0.45291
20	0.47796
30	0.55110

Case 2: $Q = M$		
M	Ruin Prob.	%
12	0.13914	30.72%
20	0.12112	25.34%
30	0.19770	35.87%

Case 2: $Q = 12M$		
M	Ruin Prob.	%
12	0.10861	23.98%
20	0.15732	32.91%
30	0.27158	49.28%

TABLE 3. Ruin probabilities for different M with parameters: $n = 100$, $\alpha = 0.37587\,(M = 12)$, $\alpha = 0.49067\,(M = 20)$, $\alpha = 0.70779\,(M = 30)$, $g = 0.0275$, $\mu = 0.04$, $\delta = 0.05$, $x = 35$, $\sigma = 0.2$.

Case 1	
M	Ruin Prob.
12	0.47996
20	0.51102
30	0.57715

Case 2: $Q = M$		
M	Ruin Prob.	%
12	0.13213	27.53%
20	0.14114	27.62%
30	0.20320	35.21%

Case 2: $Q = 12M$		
M	Ruin Prob.	%
12	0.12262	25.55%
20	0.19740	38.63%
30	0.31127	53.93%

TABLE 4. Ruin probabilities for different M with parameters: $n = 100$, $\alpha = 0.37587\,(M = 12)$, $\alpha = 0.49067\,(M = 20)$, $\alpha = 0.70779\,(M = 30)$, $g = 0.0275$, $\mu = 0.05$, $\delta = 0.05$, $x = 35$, $\sigma = 0.2$.

hedging strategy is used and that they are an essential part of the hedger's loss. As time goes by, the hedge errors accumulate (negative effect). On the other hand, a longer duration of the contract leads to higher premium inflows. Consequently, in the long run this reduces the insurer's loss to a certain extent (positive effect). Here the negative effect dominates the positive effect overall. This negative impact is so distinct that quite big ruin probabilities have resulted for $M = 30$ for the case of $Q = 12M$. In this subcategory, the insurer adjusts his portfolio much more frequently than the premium payment dates occur. The more often the hedger updates his strategy, the more duplication and discretization errors arise. Consequently, relatively high ruin probabilities are caused as the duration of the contract increases.

Table 6 demonstrates how the ruin probability changes with the fair combination of α and g. Overall, the effect of the minimum guarantee g dominates that of α. This is due to the fact that the resulting α's are relatively small, and consequently the boni part of the payment does not play a role as important as the minimum guarantee parameter g. Hence, a higher minimum interest rate guarantee leads to a higher ruin probability. Conversely,

Case 1	
M	Ruin Prob.
12	0.53353
20	0.58912
30	0.62525

Case 2: $Q = M$		
M	Ruin Prob.	%
12	0.11762	22.05%
20	0.14314	24.30%
30	0.23073	36.90%

Case 2: $Q = 12M$		
M	Ruin Prob.	%
12	0.12412	23.26%
20	0.20641	35.03%
30	0.38382	61.39%

TABLE 5. Ruin probabilities for different M with parameters: $n = 100$, $\alpha = 0.37587\,(M = 12)$, $\alpha = 0.49067\,(M = 20)$, $\alpha = 0.70779\,(M = 30)$, $g = 0.0275$, $\mu = 0.06$, $\delta = 0.05$, $x = 35$, $\sigma = 0.2$.

g, α	Case 1	$Q = M$	%	$Q = 12M$	%
$g = 0.0275$, $\alpha = 0.37587$	0.53353	0.11762	22.05%	0.13527	25.35%
$g = 0.0325$, $\alpha = 0.31939$	0.53607	0.12112	22.59%	0.14214	26.52%
$g = 0.0375$, $\alpha = 0.25634$	0.54609	0.13563	24.84%	0.15231	27.89%

TABLE 6. Ruin probabilities for different combinations of α and g with parameters: $n = 100$, $\mu = 0.06$, $M = 12$, $\delta = 0.05$, $x = 35$, $\sigma = 0.2$.

it is expected that the effect of the α's will dominate that of the g's for relatively small minimum interest rate guarantees g, say near 0, and relatively high participation rates.

6. Loss analysis in a discrete-time hedging model

Some of the numerical results obtained in the last section are not very satisfactory. The reduction in the ruin probabilities is relatively small when a high rebalancing frequency is combined with a long duration. Naturally, the question will be asked whether discretizing the hedge model instead of discretizing the strategy would improve the results. According to Mahayni (2003), discretizing the hedging model (CRR-based hedging model) yields a more favorable result for the hedger than discretizing the continuous hedging strategy, in the sense that the binomial hedge with a suitably adjusted drift component is mean-self-financing, while the discretized Gaussian hedge sub-replicates the convex payoff for both a positive or a negative drift component. For the discrete-time setup, we consider Møller's (2001) risk-minimizing strategy for equity-linked life insurance contracts derived in the

CRR model. There, the trading strategy has the form of

$$\xi_t^B = Y_{t-1}^{(n)}{}_{T-(t-1)}p_{x+t-1}\alpha_t^f \qquad t = t_1, \cdots, t_Q = T \tag{15}$$

$$\eta_t^B = Y_t^{(n)}{}_{T-t}p_{x+t}f(t, S_t) - Y_{t-1}^{(n)}{}_{T-(t-1)}p_{x+t-1}\alpha_t^f S_t^* \qquad t = t_1, \cdots, t_Q = T \tag{16}$$

where $f(t, S_t)$ gives the value of the contingent claim at time t and α_t^f stands for the hedging strategy calculated in the binomial model without mortality risk. In addition, the binomial model contains Q periods. The discounted accumulated hedging error from using the risk-minimizing strategy at time $t_Q = T$ has the form of

$$L_0^C(\phi^B) = \sum_{j=1}^{Q} e^{-\delta t_j} f(t_j, S_{t_j})_{T-t_j}p_{x+t_j}(Y_{t_j}^{(n)} - Y_{t_{j-1}}^{(n)}\, p_{x+t_{j-1}}). \tag{17}$$

The last term of the above equation $(Y_{t_j}^{(n)} - Y_{t_{j-1}}^{(n)}\, p_{x+t_{j-1}})$ indicates that this unhedgeable risk results exactly from the difference between the actual number of survivors at time t_j and the conditional expected number of survivors at time t_j calculated at time t_{j-1}. In this case all the hedge errors are caused by mortality risk and the expected hedge errors are zero under both the subjective and the martingale measure, i.e., the discrete risk-minimizing strategy is mean-self-financing.

Similarly, the net loss of the insurance company is decomposed into three parts: the initial investment plus the hedging errors and minus the premium incomes.

$$\begin{aligned} L_T^{r2} &= V_0(\phi^B)e^{\delta T} + L_0^C(\phi^B)e^{\delta T} - \sum_{i=0}^{M-1}(Y_{t_i}^{(n)} - Y_{t_{i+1}}^{(n)})\sum_{j=0}^{i} Ke^{(T-t_j)\delta} - Y_T^{(n)}\sum_{j=0}^{M-1} Ke^{(T-t_j)\delta} \\ &= V_0(\phi^B)e^{\delta T} + \sum_{j=1}^{Q} e^{\delta(T-t_{jz})} f\left(t_{jz}, S_{t_{jz}}\right)_{T-t_{jz}}p_{x+t_{jz}}\left(Y_{t_{jz}}^{(n)} - Y_{t_{(j-1)z}}^{(n)}\, {}_z p_{x+t_{(j-1)z}}\right) \\ &\quad - \sum_{i=0}^{M-1}(Y_{t_i}^{(n)} - Y_{t_{i+1}}^{(n)})\sum_{j=0}^{i} Ke^{(T-t_j)\delta} - Y_T^{(n)}\sum_{j=0}^{M-1} Ke^{(T-t_j)\delta}. \end{aligned} \tag{18}$$

Also here z is used for conformity reasons and all the terms are accumulated to the maturity date with the conform interest rate δ. In accordance with the net loss expression when the hedging model is discretized (Equation (18)), only the values of the contingent claims at certain discrete trading times $f\left(t_{jz}, S_{t_{jz}}\right)$ are relevant for the examination of the net loss of the hedger. Let $f\left(t_{jz}, S_{t_{jz}}\right)$ denote the time t_{jz}-value of the contract's payoff in the binomial model. In the following, again the specific contract construction introduced in Section 4 is used to obtain some numerical results in the binomial model.

μ	up		down		w	
	$M=Q$	$Q=12M$	$M=Q$	$Q=12M$	$M=Q$	$Q=12M$
0.04	1.2214	1.05943	0.818731	0.9439	0.600	0.529
0.05	1.2214	1.05943	0.818731	0.9439	0.625	0.536
0.06	1.2214	1.05943	0.818731	0.9439	0.650	0.543

TABLE 7. up, down and w-values with $\sigma = 0.2$.

In the binomial model, the market rate of return μ can be expressed as a function of the weighted sum of up and down values as follows:

$$\mu\, M = \mathrm{E}\left[\ln\left(\frac{S(T)}{S(t_0)}\right)\Big|\mathcal{F}_{t_0}\right] = Q(w\, \ln \mathrm{up} + (1-w)\, \ln \mathrm{down}), \tag{19}$$

where w gives the probability that the stock moves upwards under the market measure and E denotes the corresponding expected value under this measure. In order to make this case comparable to the discretized originally continuous risk-minimizing hedging strategy, the up, the down movement and the interest rate per period are set as follows:

$$\mathrm{up} = \exp\left\{\sigma\sqrt{M/Q}\right\}, \quad \mathrm{down} = \exp\left\{-\sigma\sqrt{M/Q}\right\}, \quad r(Q) = \exp\left\{\delta\frac{M}{Q}\right\} - 1. \tag{20}$$

Plugging Equation (20) in (19), the market performance can also be characterized consequently by w:

$$w = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{M/Q}.$$

Although μ/w is irrelevant in determining the hedging strategy in the binomial model, it does decide how the market performs and with which probability that the underlying asset reaches a certain knot under the market measure. Table 7 demonstrates several values of up, down and w, which are used later for the calculation of the ruin probability. In order to determine the loss of the insurer (Equation (14)), only the values of the contingent claims at $t_i, i = 0,\, z,\, \cdots,\, (Q-1)z$ together with the survival probabilities and processes matter. Since in the binomial model the calculations of these values and of the risk-minimizing strategy are quite simple, we directly jump to the results, which are demonstrated in Tables 8 and 9.

Table 8 illustrates how the ruin probability depends on the market performance of the stock for two subcases $Q = M$ and $Q = 12M$. First, an increase in the ruin probability is observed as μ goes up for $M = Q$, but this effect is not so obvious as in the first case.

Binomial Hedge: $Q = M$		Binomial Hedge: $Q = 12M$	
μ	Ruin Prob.	μ	Ruin Prob.
0.04	0.33283	0.04	0.04372
0.05	0.34284	0.05	0.06553
0.06	0.34689	0.06	0.03924

TABLE 8. Ruin probabilities with a binomial hedge with parameters: $n = 100$, $x = 35$, $\sigma = 0.2, M = 12$, $g = 0.0275$, $\alpha = 0.203596$.

Binomial Hedge: $Q = M$		Binomial Hedge: $Q = M$	
M	Ruin Prob.	g, α	Ruin Prob.
12	0.34689	$g = 0.0275, \alpha = 0.37587$	0.34689
20	0.27327	$g = 0.0325, \alpha = 0.31939$	0.35986
30	0.14014	$g = 0.0375, \alpha = 0.25654$	0.42693

TABLE 9. Ruin probabilities with a binomial hedge with parameters: $n = 100$, $x = 35$, $\sigma = 0.2, \mu = 0.06$ Left: $g = 0.0275$; Right: $M = 12$.

Furthermore, it ceases to be valid as the trading frequency increases to $Q = 12M$. Second, with a more frequent rebalancing of the portfolio ($Q = M \to Q = 12M$) the ruin probability becomes very small. That is, almost all the financial risks are eliminated when the trading occurs 12 times as often as the premium payment. Here the advantages of the binomial hedge are completely displayed. Because the considered binomial hedging strategy is indeed risk-minimizing, no duplication errors are experienced. Instead, the above considered discretized originally risk-minimizing hedging strategy actually loses its "risk-minimizing" nature and duplication errors are encountered with each adjustment of the portfolio. As the adjustment frequency rises, the advantages from this rise can be largely ruined by these duplication errors and consequently higher ruin probabilities are caused (c.f. Tables 2-5).

Table 9 is generated for the case $M = Q$ and shows the dependence of the ruin probabilities on the duration of the contract M (left table) and on the different α-g-combinations (right table). In contrast to Case 1 and the case of the originally continuous risk-minimizing strategy, the ruin probability does not go up with the duration of the contract M. It is known that only some intrinsic hedging errors will result from the use of this binomial hedging strategy, which are completely caused by the mortality risk. The size of these intrinsic hedging errors is small in comparison with the premium inflows of the insurer. Therefore, a quite small ruin probability is observed, e.g. 0.14014 for $M = 30$. It could easily be shown that almost no ruin probability will result if a long duration of the contract is combined with a high adjustment frequency. Hence, a binomial hedge improves the stability of those insurers, who mainly deal with long-term contracts or/and adjust their trading portfolio very frequently. The effect of the combination of α and g on the ruin probability remains unchanged (the effect of g dominates α). Rather, larger values of the ruin probability are observed compared to the originally continuous risk-minimizing strategy. This is due to the fact that both the duration of the contract ($M = 12$) and the frequency of adjusting the trading portfolio are chosen quite low ($Q = 12$). Consequently, the advantages from the binomial hedge are not so pronounced.

7. Conclusion

This paper represents a simulation study to investigate the net loss of a life insurance company issuing identical pure endowment contracts to n identical customers. It is observed that a considerable decrease in the ruin probability is achieved when the hedger uses a time-discretized risk-minimizing strategy. Nevertheless, the magnitude of the reduction becomes quite small and the advantage of using this time-discretized strategy almost disappears as the hedging frequency is increased. This is due to the fact that by discretization the originally mean-self-financing continuous risk-minimizing hedging strategy is not mean-self-financing any more. Furthermore, it causes some extra duplication errors, which increase the insurer's net loss to a big extent. It is shown that the simulation results are greatly improved when the hedging model instead of the hedging strategy is discretized. The effect is particularly distinct when long-term contracts are taken into consideration or when the hedging strategy is adjusted quite frequently.

In this paper, the simulation errors are not taken into consideration. However, since the results for these two discrete-time hedging strategies differ much from each other, analogous results could be expected after the simulation errors are taken into account. Furthermore, the result in this paper is contract- and model-dependent, i.e., another specification of the contract or another dynamics of the underlying asset could lead to different results.

The contract considered in the present paper is a pure endowment contract. It will be a natural extension to analyze an endowment contract, in which the insured will get paid both on an early death and on survival of the maturity date. Furthermore, all customers and all the issued insurance contracts are assumed to be identical in this paper. It would be interesting to study the net loss and the corresponding ruin probability when different customers, e.g., customers with different entering or/and exiting times are considered.

Appendix A

The fair price $f(t, S_t)$ of the contingent claim $f(S)$ at time t is derived as follows:

$$
\begin{aligned}
f(t,S_t) &= E^*[e^{-\delta(T-t)}f(S_T)|\mathcal{F}_t] \\
&= E^*\left[e^{-\delta(T-t)}\left(\sum_{i=0}^{M-1} Ke^{gt_{i+1}} + \alpha\sum_{i=0}^{M-1}(i+1)K\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\right)\Big|\mathcal{F}_t\right] \\
&= e^{-\delta(T-t)}\sum_{i=0}^{M-1} Ke^{gt_{i+1}} + \alpha(i+1)K\sum_{i=0}^{M-1} E^*\left[e^{-\delta(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\Big|\mathcal{F}_t\right] \\
&= e^{-\delta(T-t)}\sum_{i=0}^{M-1} Ke^{gt_{i+1}} + \alpha K\sum_{i=0}^{M-1}(i+1)\Bigg(E^*\left[e^{-\delta(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\mathbf{1}_{\{t>t_{i+1}\}}\Big|\mathcal{F}_t\right] \\
&\quad +E^*\left[e^{-\delta(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\mathbf{1}_{\{t_i<t\le t_{i+1}\}}\Big|\mathcal{F}_t\right] \\
&\quad +E^*\left[e^{-\delta(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\mathbf{1}_{\{t\le t_i\}}\Big|\mathcal{F}_t\right]\Bigg) \\
&= e^{-\delta(T-t)}\sum_{i=0}^{M-1} Ke^{gt_{i+1}} + \alpha K\sum_{i=0}^{M-1}(i+1)\Bigg(e^{-\delta(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\mathbf{1}_{\{t>t_{i+1}\}} \\
&\quad +e^{-\delta(T-t)}e^{\delta(t_{i+1}-t)}E^*\left[e^{-\delta(t_{i+1}-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\mathbf{1}_{\{t_i<t\le t_{i+1}\}}\Big|\mathcal{F}_t\right] \\
&\quad +e^{-\delta(T-t)}e^{\delta(t_{i+1}-t_i)}E^*\left[E^*\left[e^{-\delta(t_{i+1}-t_i)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\mathbf{1}_{\{t\le t_i\}}\Big|\mathcal{F}_{t_i}\right]\Big|\mathcal{F}_t\right]\Bigg) \\
&= e^{-\delta(T-t)}\sum_{i=0}^{M-1} Ke^{gt_{i+1}} + \alpha K\sum_{i=0}^{M-1}(i+1)\Bigg(e^{-\delta(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)}\right]^+\mathbf{1}_{\{t>t_{i+1}\}} \\
&+ e^{-\delta(T-t_{i+1})}\left(\frac{S_t}{S_{t_i}}N(d_1^{(t,i)}) - e^{g(t_{i+1}-t_i)}e^{-\delta(t_{i+1}-t)}N(d_2^{(t,i)})\right)\mathbf{1}_{\{t_i<t\le t_{i+1}\}} \\
&+ e^{-\delta(T-t)}e^{\delta(t_{i+1}-t_i)}\left(N(\tilde{d}_1) - e^{(g-\delta)(t_{i+1}-t_i)}N(\tilde{d}_2)\right)\mathbf{1}_{\{t\le t_i\}}\Bigg)
\end{aligned}
$$

with where

$$
\begin{aligned}
d_{1/2}^{(t,i)} &= \frac{\ln S_t/S_{t_i} - g(t_{i+1}-t_i) + (\delta \pm \frac{1}{2}\sigma^2)(t_{i+1}-t)}{\sigma\sqrt{t_{i+1}-t}} \\
\tilde{d}_{1/2} &= \frac{-g(t_{i+1}-t_i) + (\delta \pm \frac{1}{2}\sigma^2)(t_{i+1}-t_i)}{\sigma\sqrt{(t_{i+1}-t_i)}}.
\end{aligned}
$$

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