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Matching Heterogeneous Agents with a Linear Search Technology

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Abstract: Steady state equilibria in heterogeneous agent matching models with search frictions have been shown to exist in Shimer and Smith (2000) under the assumption of a quadratic search technology. We extend their analysis to the commonly investigated linear search technology.

Keywords: Search, Matching, Steady State Equilibrium.

***JEL* Classification Numbers:** C78, D83.

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1 Introduction

In an important contribution Shimer and Smith (2000) establish the existence of a steady state equilibrium in a heterogeneous agent matching model with search frictions, characterize the structure of the resulting equilibrium matches, and obtain conditions under which assortative matching (Becker, 1973) arises in equilibrium. Smith (2006) extends these results, derived under the assumption of transferable utilities, to a setting with non-transferable utilities. While Shimer and Smith (2000) and Smith (2006) derive their results in a setting with one population of agents and continuous time, their results also apply when there are two distinct populations (women and men or buyers and sellers) or time is discrete. Similarly, while Shimer and Smith (2000) and Smith (2006) assume that there is a fixed population of agents with entry of unmatched agents resulting from the exogenous destruction of matches, as noted in Eeckhout (1999) this is equivalent to the alternative formulation in which there is an exogenous entry flow of agents and unmatched agents exit from the market at a fixed, strictly positive rate.[1]

In one respect, though, the analysis in Shimer and Smith (2000) and Smith (2006) hinges on a particular assumption employed in both papers, namely that matches are generated according to a quadratic search technology, which "seems a poor approximation" unless one deals with "an economy with a low density of potential partners" (Diamond and Maskin, 1979, p. 283). To prove existence of a steady state equilibrium, both Shimer and Smith (2000) and Smith (2006) rely on Lemma 4 in Shimer and Smith (2000). Following Smith (2006) we will refer to this result as the fundamental matching lemma. The fundamental matching lemma asserts that the steady state condition, which requires that the flow creation and flow destruction of matches for every type of agent balances, implicitly defines the density of unmatched agents as a continuous function of agents' acceptance decisions. As noted in Shimer and Smith (2000) their proof of this result makes essential use of the assumption of a quadratic search technology. In particular, their proof does not work for the commonly studied linear search technology,[2] precluding the application of their existence result to heterogeneous agent matching models positing such a search technology.

The purpose of this note is to show that the fundamental matching lemma holds for a linear search technology in a setting which is otherwise identical to the one considered in Shimer and Smith (2000) and Smith (2006). Because the proof of the fundamental matching lemma is the only result in these papers which relies on the quadratic search technology, our result implies that not only the characterization but also the existence results from these papers hold for models with a linear search

[1] The results from Shimer and Smith (2000) and Smith (2006) also apply if one assumes that agents who match and leave the market are immediately replaced by identical clones. Even though many papers in the literature (McNamara and Collins, 1990; Morgan, 1998; Burdett and Wright, 1998; Bloch and Ryder, 2000; Chade, 2001; Adachi, 2003; Chen, 2005) consider such cloning models, it is difficult to think of any good economic motivation for such an assumption.

[2] In a linear search technology the mass of matches is proportional to the mass of agents searching for a partner. Linear search technologies thus possess constant returns to scale.

technology. We view this result as significant because – with the exception of Burdett and Coles (1997, Theorem 1), who obtain existence under a log-concavity assumption – the existence of steady state equilibria in heterogeneous agent matching models with a linear search technology has so far only been established for some simple examples with discrete type distributions,[3] whereas we consider continuous type distributions. In particular, our version of the fundamental matching lemma does implies existence of steady state equilibria for the models in Lu and McAfee (1996) and Eeckhout (1999) and dispenses with the log-concavity assumption in Burdett and Coles (1997).

2 Flow Balance and the Fundamental Matching Lemma

As suggested in Burdett and Coles (1999) it is useful to think of the equilibrium problem in heterogeneous agent matching models in terms of two smaller problems. First, for a given steady state population of unmatched agents in the market one may consider partial equilibrium (Burdett and Coles, 1997), requiring that all unmatched agents use optimal strategies (specifying, in particular, their decision whether or not to match with a particular type of agent they may encounter) given the strategies of everyone else. Second, given a steady state population of unmatched agents and their strategies one may calculate the mass and distribution of types exiting the market per unit time. In a steady state the corresponding exit flow for each type should be equal to the entry flow of that type. We will refer to this condition as the flow balance condition. A steady state equilibrium obtains if a population of unmatched agents and a strategy profile satisfy both the partial equilibrium and the flow balance condition.

Our interest in the following is in the flow balance condition. Consequently, there is no need to spell out the strategic aspects relevant for the partial equilibrium analysis. In Shimer and Smith (2000) and Smith (2006) flow balance requires

$$\delta\left(\ell(x) - u(x)\right) = \rho u(x) \int_0^1 \alpha(x,y) u(y) dy \tag{1}$$

for all $x \in [0,1]$, where x denotes the type of an agent. In this equation the Borel measurable function $\ell : [0,1] \to (0,\infty)$ is the type density, satisfying $\underline{\ell} \stackrel{\text{def}}{=} \inf_x \ell(x) > 0$ and $\overline{\ell} \stackrel{\text{def}}{=} \sup_x \ell(x) < \infty$, and the function $u : [0,1] \to (0,\infty)$, satisfying $u(x) \leq \ell(x)$ for all x, is the endogenous unmatched density. Let

$$\mathcal{U} \quad = \quad \{u \in L_1([0,1]) \mid u(x) > 0 \ \forall x, \quad u \text{ essentially bounded}\}.$$

The density of matched agents is $\ell(x) - u(x)$. These agents' matches are destroyed at an exogenous rate $\delta > 0$, so that the left side of (1) represents the entry flow

[3] In unpublished work Chen (2002) proves existence in a marriage model featuring a linear search technology. His argument makes heavy use of the simplicity afforded by his assumption that there are only two types of agents. Burdett and Coles (1997) consider a similar example.

of agents of type x into the pool of unmatched agents. The right side represents the corresponding exit flow and embodies the assumption of a quadratic search technology: anyone unmatched encounters unmatched agents in $Y \subseteq [0,1]$ at rate $\rho \int_Y u(y)dy$, where $\rho > 0$. Hence, the total mass of encounters between unmatched agents per unit time is $\rho(\int_0^1 u(y)dy)^2$ and thus quadratic in the mass of unmatched agents. Not all encounters result in a match. This is captured by the inclusion of the match indicator function $\alpha : [0,1]^2 \to [0,1]$ in the right hand side of (1): the term $\alpha(x,y) = \alpha(y,x)$ indicates the proportion of encounters between agents of type x and y resulting in a match. Let

$$\mathcal{A} \quad = \quad \{\alpha \in L_1([0,1]^2) \mid 0 \leq \alpha(x,y) = \alpha(y,x) \leq 1 \ \forall x,y\}.$$

In their fundamental matching lemma Shimer and Smith (2000, Lemma 4) assert that (1) has a unique solution $u_\alpha \in \mathcal{U}$ for all $\alpha \in \mathcal{A}$ and that the map $\alpha \to u_\alpha$ from $\mathcal{A}$ to $\mathcal{U}$ is continuous with respect to the $||\cdot||_1$ -norm. The counterpart to (1) for a *linear search technology*, in which it is assumed that any unmatched agent encounters unmatched agents with types in $Y \subseteq [0,1]$ at rate $\rho \int_Y u(y)dy / \int_0^1 u(y)dy$, is

$$\delta\left(\ell(x) - u(x)\right) = \rho u(x) \frac{\int_0^1 \alpha(x,y)u(y)dy}{\int_0^1 u(y)dy}. \tag{2}$$

Fundamental Matching Lemma. *For all match indicator functions $\alpha \in \mathcal{A}$ there is a unique unmatched density $u_\alpha \in \mathcal{U}$ such that the flow balance condition (2) holds. The map $\alpha \mapsto u_\alpha$ from $\mathcal{A}$ to $\mathcal{U}$ is continuous with respect to the $||\cdot||_1$ -norm.*

To obtain this result we proceed in three steps, establishing existence, uniqueness, resp. continuity of the population solving the flow balance condition. Throughout details are relegated to the Appendix. In Subsection 3.4 we explain how and why the structure of our proof differs from the proof in Shimer and Smith (2000).

3 Proof of the Fundamental Matching Lemma

It is convenient to define $\hat{\rho} = \rho/\delta$, so that the flow balance condition (2) can be rewritten as

$$\ell(x) = u(x)\left[1 + \hat{\rho}\frac{\int_0^1 \alpha(x,y)u(y)dy}{\int_0^1 u(y)dy}\right]. \tag{3}$$

Furthermore, let $\underline{u} = \underline{\ell}/(1+\hat{\rho})$, $\overline{u} = \overline{\ell}$, and

$$\mathcal{C} = \{u \in L_1([0,1]) \mid \underline{u} \leq u(x) \leq \overline{u} \ \forall x\}.$$

3.1 Step 1: Existence

Here we show that an unmatched density $u \in \mathcal{U}$ satisfying the flow balance condition (3) exists for any match indicator function $\alpha \in \mathcal{A}$. We proceed by reformulating (3) as a continuous fixed point equation in the Lebesgue space L_1 and verifying that our fixed point mapping satisfies the assumptions of Schauder's fixed point theorem.

For all $\alpha \in \mathcal{A}$, $u \in \mathcal{U}$ solves (3) if and only if it is a fixed point of $\phi_\alpha : \mathcal{U} \to \mathcal{U}$, where

$$\phi_\alpha(u)(x) \quad = \quad \frac{\ell(x)}{1+\hat{\rho}\frac{\int u(y)\alpha(x,y)\mathrm{d}y}{\int u(y)\mathrm{d}y}}. \tag{4}$$

Any fixed point belongs to $\mathcal{C}$ because

$$\phi_\alpha(\mathcal{U}) \subseteq \mathcal{C}. \tag{5}$$

The restricted map $\phi_\alpha : \mathcal{C} \to \mathcal{C}$ has a fixed point because it satisfies the assumptions of Schauder's fixed point theorem (see Granas and Dugundji (2003, p. 119) for the version of Schauder's theorem we are using). First, ϕ_α is continuous with respect to the $||\cdot||_1$ -norm (see Lemma 2 in the Appendix). Second, $\mathcal{C}$ is convex. Third (due to the presence of the integral operator in the definition of ϕ_α), the set $\phi_\alpha(\mathcal{C})$ is relatively compact; i.e., the $||\cdot||_1$ -closure $\overline{\phi_\alpha(\mathcal{C})}$ is compact (Lemma 3). Forth, because any sequence that converges in the $||\cdot||_1$ -norm has a subsequence that converges almost everywhere (Aliprantis and Burkinshaw, 1998, Lemma 31.6), $\mathcal{C}$ is $||\cdot||_1$ -closed, implying $\overline{\phi_\alpha(\mathcal{C})} \subseteq C$.

3.2 Step 2: Uniqueness

By appropriately rescaling the unmatched density we reformulate the flow balance condition as an algebraically convenient quadratic equation: Lemma 4 shows that $u \in \mathcal{U}$ solves (3) if and only if $\nu \in \mathcal{U}$ defined by

$$\nu \quad = \quad \frac{u}{\sqrt{\int u(y)\mathrm{d}y}} \tag{6}$$

solves

$$\ell = G(\alpha, \nu), \tag{7}$$

where $G : \mathcal{A} \times L_2([0,1]) \to L_2([0,1])$ is defined by

$$G(\alpha,\nu)(x) \quad = \quad \nu(x)\int_0^1 \nu(y)(1+\hat{\rho}\alpha(x,y))\mathrm{d}y. \tag{8}$$

To obtain uniqueness, it is sufficient to show that G (when viewed as a function of ν) is injective on the set

$$\mathcal{D} \quad = \quad \{\nu \in L_1([0,1]) \mid \underline{\nu} \le \nu(x) \le \overline{\nu} \ \forall x\},$$

where $\underline{\nu} = \underline{u}/\sqrt{\overline{u}}$ and $\overline{\nu} = \overline{u}/\sqrt{\underline{u}}$: because $G(\alpha,\nu) = \ell$ implies $\nu \in \mathcal{D}$ (Lemma 5), it follows that (7) has at most one solution for given $\alpha \in \mathcal{A}$. As the transformation

$\nu = u/\sqrt{\int u(y)\mathrm{d}y}$ is invertible on $\mathcal{U}$ (Lemma 1), it follows that for all $\alpha \in \mathcal{A}$ there exists at most one $u \in \mathcal{U}$ solving (3).

To show that G is injective we use the observation that a quadratic function is injective if its Frechet derivative, viewed as a linear operator on the space $L_2([0,1])$, is injective (Rall, 1969). Let $\|\cdot\|_2$ denote the standard norm in the Hilbert space $L_2([0,1])$. For all $\alpha \in \mathcal{A}$, the Frechet derivative of $G(\alpha,\cdot)$ at $\nu \in L_2([0,1])$ with respect to the $\|\cdot\|_2$-norm is the continuous linear operator $J(\alpha,\nu) : L_2([0,1]) \to L_2([0,1])$ given by

$$J(\alpha,\nu)(g)(x) = \int (g(x)\nu(y) + \nu(x)g(y))(1+\hat{\rho}\alpha(x,y))\mathrm{d}y.$$

Lemma 6 establishes that for all $\nu \in \mathcal{D}$, $\alpha \in \mathcal{A}$, and $g \in L_2([0,1])$, $g \neq 0$, we have $J(\alpha,\nu)(g) \neq 0$. Observing that

$$G(\alpha,\nu) - G(\alpha,\nu') = J(\alpha, \frac{\nu+\nu'}{2})(\nu-\nu')$$

yields the desired conclusion.

3.3 Step 3: Continuity

For any $\alpha \in \mathcal{A}$, the quadratic equation $G(\alpha,\nu) = \ell$ has a solution $\nu \in \mathcal{U}$ (this follows from Step 1 and Lemma 4). From Step 2, the solution is unique and in $\mathcal{D}$; denote it by ν_α. It is sufficient to show that the function $\alpha \mapsto \nu_\alpha$ is continuous on $\mathcal{A}$ with respect to the $\|\cdot\|_1$-norm. (Using Lemma 1, it follows that $\alpha \mapsto u_\alpha$ is continuous with respect to the $\|\cdot\|_1$-norm.)

Consider any sequence $(\alpha_n) \subseteq \mathcal{A}$ and $\alpha \in \mathcal{A}$ such that $\|\alpha_n - \alpha\|_1 \to 0$. Because $|\alpha_n(x) - \alpha(x)| \leq 2$ for all x, $(\|\alpha_n - \alpha\|_2)^2 \leq 2\|\alpha_n - \alpha\|_1$, implying $\|\alpha_n - \alpha\|_2 \to 0$. In Lemma 7 we establish that $\alpha \mapsto \nu_\alpha$ is (Lipschitz) continuous with respect to the $\|\cdot\|_2$-norm. Hence, $\|\nu_{\alpha_n} - \nu_\alpha\|_2 \to 0$. From the Hölder inequality, $\|\nu_{\alpha_n} - \nu_\alpha\|_1 \leq \|\nu_{\alpha_n} - \nu_\alpha\|_2$. Hence, $\|\nu_{\alpha_n} - \nu_\alpha\|_1 \to 0$, as was to be shown.

3.4 Linear vs. Quadratic Search Technology

The proof of the fundamental matching lemma for the quadratic search technology in Shimer and Smith (2000) employs a different method from the one we use above to obtain existence and uniqueness of a population solving the flow balance condition. Their counterpart (Shimer and Smith, 2000, Appendix B) to equation (3) is

$$\ell(x) = u(x)\left[1 + \hat{\rho}\int_0^1 \alpha(x,y)u(y)dy\right], \tag{9}$$

which gives rise to the fixed point equation

$$u(x) = \frac{\ell(x)}{1 + \hat{\rho}\int_0^1 \alpha(x,y)u(y)dy}. \tag{10}$$

Their critical idea is to apply a log transformation of u to rewrite (10) as

$$\nu(x) = \log\left(\frac{\ell(x)}{1+\hat{\rho}\int_0^1 \alpha(x,y)e^{\nu(y)}dy}\right) \tag{11}$$

and to show that the mapping obtained from the right side of this equation is a contraction mapping (with respect to the sup-norm) from an appropriately chosen subset of L_∞ into itself, so that existence and uniqueness of a solution follows from Banach's fixed point theorem. There is no straightforward extension of this approach to the case of the linear search technology. Of course, starting from either (3) or (7) a log transformation may be applied to obtain a fixed point equation similar to (11) for the linear case. The difficulty is that the arguments from Shimer and Smith (2000) no longer apply to show that the corresponding fixed point mappings is contractive.

While Step 1 of our proof has no counterpart in Shimer and Smith (2000),[4] Steps 2 and 3 of our proof build on arguments provided by Shimer and Smith to prove continuity of the unmatched density in the match indicator function. Besides our demonstration that these arguments may be used to infer uniqueness, there are two novel insights in our analysis. First, because (9) is already quadratic in the unmatched density u, no rescaling as in (6) is required when using the arguments from Step 2 and 3 to infer uniqueness and continuity for the quadratic search technology. Second, while the right side of (9) contains a linear term such a term is absent in the corresponding quadratic function (8) we use in the linear case. The presence of such a linear term simplifies the proofs of the counterparts to our Lemmas 6 and 7 obtained in Shimer and Smith (2000, Appendix B).

4 Appendix

The following lemmas are used in the proof of the fundamental matching lemma.

Lemma 1. *For all $r \in I\!R$, the map $\tau_r : \mathcal{U} \to \mathcal{U}$ defined by*

$$\tau_r(u)(x) = \frac{u(x)}{(\int u(y)dy)^r}$$

is continuous with respect to the $||\cdot||_1$ -norm. If $r \neq 1$, then $\tau_{r/(r-1)}$ is the inverse of τ_r.

Proof. To show that τ_r is continuous, consider $u_n \to u$. Then $\int u_n \to \int u$, hence, $(\int u_n)^{-r} \to (\int u)^{-r}$. Using

$$\begin{aligned}\tau_r(u_n)(x) - \tau_r(u)(x) &= u_n(x)(\int u_n)^{-r} - u_n(x)(\int u)^{-r} \\ &\quad + u_n(x)(\int u)^{-r} - u(x)(\int u)^{-r},\end{aligned}$$

[4]It is easy to see, though, that the same method as in Step 1 of our proof can be used to establish the existence of a solution to (10).

one sees that

$$\begin{aligned} \| \tau_r(u_n) - \tau_r(u) \|_1 \quad &\leq \quad \| u_n \|_1 \left| (\int u_n)^{-r} - (\int u)^{-r} \right| \\ &\quad + \| u_n - u \|_1 \ (\int u)^{-r}, \end{aligned}$$

showing that $\tau_r(u_n) \to \tau_r(u)$.

For any s, we have $\tau_r \tau_s = \tau_{r+s-rs} = \tau_s \tau_r$. Choosing $s = r/(r-1)$ if $r \neq 1$, we have $\tau_{r+s-rs} = \tau_0$, showing that τ_s is the inverse of τ_r. □

Lemma 2. *Let $\alpha \in \mathcal{A}$. The map ϕ_α is continuous with respect to the $\| \cdot \|_1$ -norm.*

Proof. Let $\overline{\mathcal{U}} = \{u \in L_1([0,1]) \mid u(x) \geq 0 \ \forall x\}$. Observe that $\phi_\alpha = \kappa \psi_\alpha \tau_1$, where $\psi_\alpha : \mathcal{U} \to \overline{\mathcal{U}}$ is defined by

$$\psi_\alpha(u)(x) = \int u(y) \alpha(x,y) \mathrm{d}y, \tag{12}$$

and $\kappa : \overline{\mathcal{U}} \to \mathcal{U}$ is given by

$$\kappa(u)(x) = \frac{\ell(x)}{1 + \hat{\rho} u(x)}.$$

From Lemma 1, τ_1 is continuous. To show that ψ_α is continuous, let $u, v \in \mathcal{U}$; then

$$\| \psi_\alpha(u) - \psi_\alpha(v) \|_1 \ \leq \int \int \mid u(y) - v(y) \mid \alpha(x,y) \mathrm{d}y \mathrm{d}x \leq \ \| u - v \|_1 .$$

To show that κ is continuous, let $u, v \in \overline{\mathcal{U}}$; then

$$\kappa(u)(x) - \kappa(v)(x) \quad = \quad \ell(x) \hat{\rho} \frac{v(x) - u(x)}{(1 + \hat{\rho} u(x))(1 + \hat{\rho} v(x))},$$

implying $\| \kappa(u) - \kappa(v) \|_1 \ \leq \overline{u} \hat{\rho} \ \| u - v \|_1$. □

Lemma 3. *Let $\alpha \in \mathcal{A}$. The set $\phi_\alpha(\mathcal{C})$ is relatively compact.*

Proof. Using the decomposition $\phi_\alpha = \kappa \psi_\alpha \tau_1$ from the proof of Lemma 2, it is sufficient to show that the set $\psi_\alpha(\tau_1(\mathcal{C}))$ is relatively compact.

It is convenient to extend any $\alpha \in \mathcal{A}$ to $\mathbb{R}^2$ via $\alpha(x,y) = 0$ for all $(x,y) \notin [0,1]^2$, and extend any $\psi_\alpha(u)$ to $\mathbb{R}$ via $\psi_\alpha(u)(x) = 0$ for all $x \notin [0,1]$. Then (12), with integration area $\mathbb{R}$, holds for all $x \in \mathbb{R}$. Let $\epsilon > 0$. From Adams (1975, p. 31), $\psi_\alpha \tau_1(\mathcal{C})$ is relatively compact if (i) it is $\| \cdot \|_1$ -bounded and (ii) there exists $\delta' > 0$ such that

$$\int_{\mathbb{R}} \mid \psi_\alpha(u)(x+h) - \psi_\alpha(u)(x) \mid \mathrm{d}x \quad < \quad \epsilon \qquad \text{for all } u \in \tau_1(\mathcal{C}) \text{ and } |h| < \delta'.$$

Because

$$u(x) \leq \frac{\overline{u}}{\underline{u}} \quad \text{for all} \;\; x \in [0,1],\; u \in \tau_1(\mathcal{C}), \tag{13}$$

we have $\|\, \psi_\alpha(u) \,\|_1 \; \leq \overline{u}/\underline{u}$ for all $u \in \tau_1(\mathcal{C})$, implying (i).

For all $h \in \mathbb{R}$, define the translation operator $T_h : L_1(\mathbb{R}^2) \to L_1(\mathbb{R}^2)$ by $T_h(\alpha)(x,y) = \alpha(x+h,y)$. As remarked by Adams (1975, p. 32), we have the continuity property

$$\|\, T_h(\alpha) - \alpha \,\|_1 \;\to 0 \;\text{ as }\; h \to 0. \tag{14}$$

For all $u \in \tau_1(\mathcal{C})$ and $x, h \in \mathbb{R}$,

$$|\; \psi_\alpha(u)(x+h) - \psi_\alpha(u)(x) \;| \quad \overset{(13)}{\leq} \quad \frac{\overline{u}}{\underline{u}} \int \;|\; T_h(\alpha)(x,y) - \alpha(x,y) \;|\; \mathrm{d}y. \tag{15}$$

From (14), there exists $\delta' > 0$ such that $\|\, T_h(\alpha) - \alpha \,\|_1 \; < \epsilon \underline{u}/\overline{u}$ if $|h| < \delta'$. Thus, for all $u \in \tau_1(\mathcal{C})$ and $|h| < \delta'$,

$$\begin{aligned} &\int_{\mathbb{R}} \;|\; \psi_\alpha(u)(x+h) - \psi_\alpha(u)(x) \;|\; \mathrm{d}x \\ &\overset{(15)}{\leq} \frac{\overline{u}}{\underline{u}} \int_{\mathbb{R}^2} |\; T_h(\alpha)(x,y) - \alpha(x,y) \;|\; \mathrm{d}y \; \mathrm{d}x \;=\; \frac{\overline{u}}{\underline{u}} \;\|\, T_h(\alpha) - \alpha \,\|_1 \;\; < \epsilon, \end{aligned}$$

implying (ii). □

Lemma 4. *For all $\alpha \in \mathcal{A}$ and $u, \nu \in \mathcal{U}$: $\phi_\alpha(u) = u \Leftrightarrow \big(G(\alpha,\nu) = \ell$ and $\nu = \tau_{1/2} u\big)$.*

Proof. Using Lemma 1 and the definition of ϕ_α,

$$\begin{aligned} \phi_\alpha(u) = u \quad &\Leftrightarrow \quad (\phi_\alpha \tau_{-1}(\nu) = \tau_{-1}(\nu) \text{ and } u = \tau_{-1}\nu) \\ &\Leftrightarrow \quad \big(\phi_\alpha(\nu) = \tau_{-1}(\nu) \text{ and } \nu = \tau_{1/2} u\big)\,. \end{aligned}$$

A straightforward computation shows that the equation $\phi_\alpha(\nu) = \tau_{-1}(\nu)$ is satisfied if and only if $G(\alpha,\nu) = \ell$. □

Lemma 5. *For all $\alpha \in \mathcal{A}$ and $\nu \in \mathcal{U}$, if $G(\alpha,\nu) = \ell$ then $\nu \in \mathcal{D}$.*

Proof. From Lemma 1, the inverse of $\tau_{1/2}$ is τ_{-1}. Hence, $G(\alpha,\nu) = \ell$ implies $\phi_\alpha(\tau_{-1}\nu) = \tau_{-1}\nu$ by Lemma 4. Hence, $\tau_{-1}\nu \in \mathcal{C}$ by (5), implying $\nu \in \tau_{1/2}(\mathcal{C}) \subseteq \mathcal{D}$ by definition of $\mathcal{D}$. □

Lemma 6. *There exists $\eta > 0$ such that, for all $\nu \in \mathcal{D}$, $\alpha \in \mathcal{A}$, and $g \in L_2([0,1])$,*

$$\|\, J(\alpha,\nu)(g) \,\|_2 \quad \geq \quad \eta \,\|\, g \,\|_2 \;.$$

Proof. Fix some ν. Using the norm $\|\cdot\|_{2'}$ and the inner product $\langle\cdot,\cdot\rangle$ in the Hilbert space $L_2\{[0,1], 1/\nu\}$,

$$\begin{aligned}
&2 \,||\, g \,||_{2'} \;\; ||\, J(\alpha,\nu)(g) \,||_{2'} \\
&\quad \geq 2\langle g, J(\alpha,\mu)(g)\rangle \\
&\quad = 2\int g(x)J(\alpha,\nu)(g)(x)\frac{1}{\nu(x)}\mathrm{d}x \\
&\quad = \int\int \left[2g(x)^2\frac{\nu(y)}{\nu(x)} + 2g(x)g(y)\right](1+\hat\rho\alpha(x,y))\mathrm{d}x\mathrm{d}y \\
&\overset{\alpha \text{ symmetric}}{=} \int\int \left[g(x)^2\frac{\nu(y)}{\nu(x)} + g(y)^2\frac{\nu(x)}{\nu(y)} + 2g(x)g(y)\right](1+\hat\rho\alpha(x,y))\mathrm{d}x\mathrm{d}y \\
&\quad = \int\int \left[g(x)\sqrt{\nu(y)/\nu(x)} + g(y)\sqrt{\nu(x)/\nu(y)}\right]^2 (1+\hat\rho\alpha(x,y))\mathrm{d}x\mathrm{d}y \\
&\quad \geq \int\int \left[g(x)\sqrt{\nu(y)/\nu(x)} + g(y)\sqrt{\nu(x)/\nu(y)}\right]^2 \mathrm{d}x\mathrm{d}y \\
&\quad \geq \frac{\underline{\nu}}{\overline{\nu}}\int g(x)^2\mathrm{d}x + \frac{\underline{\nu}}{\overline{\nu}}\int g(y)^2\mathrm{d}y + \int\int 2g(x)g(y)\mathrm{d}x\mathrm{d}y \\
&\quad \geq 2\frac{\underline{\nu}^2}{\overline{\nu}}\int g(x)^2\frac{1}{\nu(x)}\mathrm{d}x + 2\left(\int g(x)\right)^2 \\
&\quad \geq 2\frac{\underline{\nu}^2}{\overline{\nu}}\int g(x)^2\frac{1}{\nu(x)}\mathrm{d}x \\
&\quad = 2\frac{\underline{\nu}^2}{\overline{\nu}}(\,||\, g \,||_{2'}\,)^2.
\end{aligned}$$

Dividing by $||\, g \,||_{2'}$ yields

$$||\, J(\alpha,\nu)(g) \,||_{2'} \quad \geq \quad \frac{\underline{\nu}^2}{\overline{\nu}} \,||\, g \,||_{2'} \;.$$

We obtain the desired formula with $\eta = \underline{\nu}^{5/2}/\overline{\nu}^{3/2}$. □

Lemma 7. *There exists $\zeta > 0$ such that,*
if

$$\alpha,\alpha' \in \mathcal{A}, \quad \nu,\nu' \in D, \quad G(\alpha,\nu) = \ell, \quad G(\alpha',\nu') = \ell,$$

then

$$||\, \nu - \nu' \,||_2 \quad \leq \quad \zeta \,||\, \alpha - \alpha' \,||_2 \;.$$

Proof. Let $h = \nu - \nu'$. Straightforward calculation shows

$$G(\alpha',\nu) - \ell \;=\; G(\alpha',\nu'+h) - G(\alpha',\nu') \;=\; J(\alpha',\nu'+\frac{h}{2})(h). \tag{16}$$

Moreover,

$$G(\alpha',\nu)(x) - G(\alpha,\nu)(x) \quad = \quad \hat\rho\nu(x)\int \nu(y)(\alpha'(x,y) - \alpha(x,y))\mathrm{d}y.$$

Hence,

$$\begin{aligned}
&\int \mid G(\alpha',\nu)(x) - G(\alpha,\nu)(x) \mid^2 \mathrm{d}x \\
&\leq \hat{\rho}^2 \overline{\nu}^4 \int \left(\int \mid \alpha'(x,y) - \alpha(x,y) \mid \mathrm{d}y \right)^2 \mathrm{d}x \\
&\overset{\text{by Hölder's inequality}}{\leq} \hat{\rho}^2 \overline{\nu}^4 \int \int (\alpha'(x,y) - \alpha(x,y))^2 \mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Hence,

$$\| G(\alpha',\nu) - \underbrace{G(\alpha,\nu)}_{=\ell} \|_2 \quad \leq \quad \hat{\rho}\overline{\nu}^2 \| \alpha - \alpha' \|_2 \ . \tag{17}$$

Hence,

$$\| \nu - \nu' \|_2 \overset{\text{Lemma 6}}{\leq} \frac{1}{\eta} \| J(\alpha', \mu' + \frac{h}{2})(h) \|_2 \overset{(16),(17)}{\leq} \frac{\rho\overline{\nu}^2}{\eta} \| \alpha - \alpha' \|_2 \ ,$$

so that the desired claim holds with $\zeta = \rho\overline{\nu}^2/\eta$. □

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