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A non-standard approach to a market with boundedly rational consumers and strategic firms. Part I: A microfoundation for the evolution of sales

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In our model, individual consumers follow simple behavioral decision rules based on imitation and habit as suggested in consumer research, social learning, and related fields. Demand can be viewed as the outcome of a population game whose revision protocol is determined by the consumers' behavioral rules. The consumer dynamics are then analyzed in order to explore the demand side and first implications for a strategic supply side.

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1. Introduction

Optimal economic choices are often very hard to obtain for consumers. Especially if a product is new on the market, its quality can only be anticipated. Consider for example an innovative and new liquid crystal display (LCD) TV shortly after its product introduction. A potential consumer does not know the true product quality (cf. Smallwood and Conlisk (1979), p. 2), though she can have some (not necessarily true) idea about it. In such a situation, as supported by studies in social psychology and consumer research (e. g. Assael (1984), p. 371ff; Venkatesan (1966)), consumers often base their product choices on *imitation* of others who already own the product. Imitation can take place either in an implicit or explicit manner. For instance, when a person encounters someone owning the LCD TV, the familiarity with this product increases, implicitly rising the probability to buy it. On the other hand, explicit or deliberate imitation may occur if a consumer understands the product's popularity as a hint to its past performance (cf. Ellison and Fudenberg (1993)).

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Imitation represents a widely used concept in areas such as evolutionary game theory or social learning. For example, Schlag (1998) shows that a proportional imitation rule is better than any other (well-performing or so-called improving) behavioral rule in a multi-armed bandit environment. Ellison and Fudenberg (1993) employ imitation through the concept of popularity weighting. For a high degree of popularity weighting the agents choose the most popular choice, independent of its payoff. An extreme result of such imitative behavior is herding. In this context Bikhchandani, Hirshleifer, and Welch (1992) explain how social conventions, norms, fashion or new behavior can arise through informational cascades. Another approach to modeling conventions, where the own choice depends also on past actions of others, is due to Young (1993). In his model n agents, drawn randomly from a population, play an n-person game. They act as in fictitious play, except that their information about the predecessors is incomplete and that the agents are not free from errors. Banerjee (1992) shows that imitative behavior yields inefficient equilibria: Herding can emerge since the agents rather imitate others than following the information contained in their own signal. Word-of-mouth learning is another field closely related to imitation and the herding literature and is said to be an important mechanism for consumers to choose a product brand. Banerjee and Fudenberg (2004) for example examine rational word-of-mouth learning. They search for aspects that affect the characteristics of word-of-mouth learning in the long run by investigating several sampling rules for imitation. Ellison and Fudenberg (1995) represents a nice example where boundedly rational word-of-mouth learning from a few other agents may lead to socially efficient results. Similar forerunners of word-of-mouth learning are for instance Kirman (1993), where a vivid example of herding (ants eating only from one of two exactly identical food sources) is explained within a recruiting model, and the seminal work of Smallwood and Conlisk (1979). In order to identify market (share) equilibria, they investigate a model in which consumers respond to breakdowns of their products. As in our model, they consider consumers who are uncertain about the product quality. Their consumers buy a new product each period—either of the same brand as before or of another one, depending on whether a breakdown happened. After breakdown, the consumer makes a decision taking the population shares into account, i. e. how many agents bought each brand in the last period. The analysis is then divided into that of weakly dissatisfied and strongly dissatisfied consumers, who will not buy the product again. Furthermore, Smallwood and Conlisk (1979) explore the equilibria for different values of a parameter representing the consumers' degree of confidence in the significance of brand popularity. Only the best quality brands are used in equilibrium (only weakly dissatisfied consumers) or additionally some non-best (only strongly dissatisfied consumers) if there are moderate consumers' beliefs about brand popularity. In a second step, Smallwood and Conlisk consider the supply side of the market and assume that the firms set their products' breakdown probabilities each period in order to maximize their profit. Due to the high mathematical complexity they only obtain approximate firm strategies. For the demand side we use an approach quite similar to

Smallwood's and Conlisk's paper, extending and refining some ideas, for example including a weighting of imitation according to (anticipated) product quality as well as introducing a habit mechanism.

Experiments in consumer research have revealed further important mechanisms of decision making besides imitation. In particular, consumers obviously also take their own experience and satisfaction into account, resulting in a habitual behavior (Assael (1984)). Let us return to the TV example: A consumer who owned an LCD TV for a while and is satisfied with it rather sticks to LCD TVs instead of following the majority of other agents buying a plasma TV. Consequently, we introduce *habit* as a second mechanism.

An indication of habitual purchases was found in several studies, summarized in Assael (1984, p. 53). Implicitly, the amount of habit induced by a product is also modeled in Smallwood and Conlisk (1979) via its breakdown probability or so-called "dissatisfaction probability". The consumers buy the same brand—hence follow their habit—until they encounter a problem. Expressed in terms of our model, if the consumers' habit is disturbed, the consumers imitate the buying decision of other consumers. A habit-related aspect has also already been advanced in the early papers of bounded rationality (for instance Simon (1955)), where satisficing as opposed to optimizing is motivated. Satisficing describes the consumer practice to already accept the first satisfying choice instead of extensively searching for the optimal choice. Similarly, in our model we will assume that consumers who own a satisfying good will continue to buy it.

Generally, our approach can be classified as belonging to the bounded rationality field as it is surveyed in at least three excellent reviews, Ellison (2006), Sobel (2000), and Conlisk (1996), with focuses on industrial organization, learning, and general motivations, respectively. There seems to exist quite some work taking a basically similar perspective as our approach. Early precursors of bounded rationality in industrial organization focus on firms' rather than consumers' bounded rationality, for instance Rothschild (1947) and Cyert and March (1956). Another interesting early paper by Heiner (1983) hypothesizes that predictable behavior in economics results mostly from rules of thumb behavior of the agents and not from optimizing agents, since maximizing agents would lead to irregularities on the market. More recently, Thadden (1992) introduced a model in which the firms act strategically, whereas the consumers do not, which today represents the most common deviation from rationality.

In order to describe markets, we will define a two-level game, of which the lower level is the content of the present paper (part I). This lower level uses a continuous time consumer model based on behavioral rules which are reasonable according to psychologic and experimental studies (Assael (1984), Venkatesan (1966)). The model is stated in form of a population game as defined in Sandholm (2005) and Sandholm (2006), who makes use of the fact that for a large population size, the stochastic process generated by the evolutionary process can be approximated by solutions to ordinary differential equations (Benaïm and Weibull (2003)). The upper level, which will be the topic of a second paper (part II: Product pricing when demand follows a rule of thumb), describes the strategic behavior of firms, which base their decisions on the results of the lower level model. Altogether, it will be seen that two simple rule of thumb ingredients, imitation and habit, are sufficient to generate typical patterns observed in consumer markets, such as product life cycles (as described in Kluyver (1977), Brockhoff (1967), Polli and Cook (1969)). Concerning the methodology, we employ a new approach to describe interacting consumers in continuous time as well as a new method to describe supply and demand side of a market via a two-level game.

Although a complete market description has to take into account supply and demand side, the model presented in this first paper stands on its own. It is for example in itself applicable in cases where the supply side is (temporarily) inactive, as in marginal cost pricing (Bertrand, perfect competition), supply sides with strong inertia, or more general decision processes like choosing when to take one's holiday. Furthermore, some implications on product feasibility can already be obtained, if prices and production costs are implicitly encoded in the model parameters.

The outline of the paper is as follows. We start with the introduction of the used methodology. The general methodology is then applied to a market where demand follows a simple rule of thumb. Subsequently, we deduce the important sales equation, motivate an appropriate revision protocol (similar to a transition probability), and find the corresponding mean dynamic which determines the demand evolution dependent on two parameters per product, a convincement factor and a habit coefficient. Finally, the demand evolution is analyzed for varying numbers of different products (or brands).

2. Model and microfoundation of sales

This section proposes a mathematical model for the evolution of sales based on a microfoundation. The microfoundation constitutes a means of macroscopically describing a system of a large number of interacting agents taking part in a so-called modified population game. The agents behave non-rationally and myopically change their activities. The mean over all agents then provides the governing differential equations to describe the overall system behavior.

After briefly introducing the methodology of a microfoundation following Sandholm (2005) and (2006) it is applied to a goods market, yielding a first simple result about the relation between the evolution of product distribution and the sales. Finally, we suggest an exemplary model of consumer behavior and derive the resultant sales evolution via the methods above.

2.1. Methodology

A typical normal form game consists of a population of agents, their actions, and the payoff function which rewards or penalizes the agents' actions depending on the other players' behavior. Since the agents are usually assumed to be rational, their behavior directly follows from maximizing the payoff. However, in the case of non-rational agents the payoff function is insufficient to determine their actions; a revision protocol, for example similar to Sandholm (2006), seems more adequate to describe the agents' behavior.

Definition 2.1 (Revision protocol)**.** *A* revision protocol ρ *is a map*

$$\rho : X \times \mathbb{R} \to \mathbb{R}_+^{(n+1)\times(n+1)}, \tag{1}$$

where $X = \left\{x = (x_0, \dots, x_n) \in [0,1]^{n+1} \,|\sum_{i=0}^n x_i = 1\right\}$ *is called the* state space*. The scalar* $\rho_{ij}(x,t)$ *is called the* conditional switch rate *from activity* i *to activity* j *at time* t *and state* x*. The sum*

$$R_i = \sum_{j=0}^{n} \rho_{ij}(x,t) \tag{2}$$

is called the alarm clock rate *of subpopulation* i *and the scalar* $p_{ij} = \frac{\rho_{ij}}{R_i}$ *the* switching probability*.*

Before motivating this definition we shall first use the revision protocol to specify the model framework.

Definition 2.2 (Modified population game)**.** *A* modified population game *is defined by the triple* $\mathcal{G} = (N, \mathcal{A}, \rho)$*, where* N *is the agent* population size*,* $\mathcal{A} = \{0, \dots, n\}$ *is the* activity set*, and* $\rho \in \mathbb{R}_+^{(n+1)\times(n+1)}$ *is a* revision protocol*.*

The motivation is as follows: In a modified population game a situation with N agents is considered, each of whom plays precisely one activity $i \in \mathcal{A}$ at a time. We use the term 'activity' as opposed to the game theoretic term 'action' in order to emphasize the continuity of the activity in time, whereas actions take place at discrete times (e. g. in repeated games). The subpopulation of those agents playing activity i amounts to the time-varying fraction x_i of the total population so that the set of all possible states is given by $X = \{x \in [0,1]^{n+1} | \sum_{i=0}^n x_i = 1\}$. Each agent playing activity i is equipped with an independent *Poisson alarm clock* of rate R_i, i. e. an alarm clock which rings after an exponentially distributed time with expected value R_i^{-1}. Each time the alarm goes off the agent revises her activity and switches to activity $j \in \mathcal{A}$ with probability p_{ij}.

The objective of studying the macroscopic behavior of a system requires averaging over all agents. In order to limit complexity of analysis we shall make use of the continuum hypothesis so that differential equations can be applied. Hence we assume N very large with the effect that the state space X can be approximated as continuous and that the mean agent behavior can be approximated by expected behavior according to the law of large numbers. Under these conditions Sandholm (2006) derives the macroscopic system behavior:

Definition 2.3 (Mean dynamic)**.** *Let $\mathcal{G}$ be a modified population game, and let ρ be its revision protocol. The* mean dynamic *corresponding to $\mathcal{G}$ is*

$$\dot{x}_i = \sum_{j=0}^{n} x_j \rho_{ji}(x,t) - x_i \sum_{j=0}^{n} \rho_{ij}(x,t), \quad i = 0, \ldots, n, \tag{3}$$

where $\dot{x}_i$ denotes the time derivative of x_i.

2.2. Application to a consumer market

Let us consider a market consisting of N consumers and n different products produced by some firms. As opposed to some standard industrial organization models the focus is here laid on the consumer side of the market, i. e. given the firms' behavior in the sense of product properties[1] (e. g. price and quality) the evolution of the market is governed by the consumers' purchase decisions. Though in principle the application of modified population games allows for time-dependent product properties (i. e. time-varying firm strategies) and consumer characteristics, let us concentrate on a market with static parameters.

The consumers, who take the role of the agents in the modified population game, own at most one good each, where the different products are substitutes but not perfect substitutes to them. Their activity set comprises the activities "not owning any product", denoted 0, "owning good 1", denoted 1, and so forth. The alarm clock of each consumer announces the time when she can decide to buy a new product and hence to switch her activity. Reasons can be a breakage or a defect when owning a product or a suddenly arising interest of the consumer in a product when currently being without any good. Due to the memorylessness of such incidents, the use of a Poisson alarm clock seems most adequate to model their occurrence. Since the old products are broken, they are disposed and therefore reselling is assumed to be impossible. Finally, let the firms have—as is standard practice in industrial organization—perfect knowledge about the consumer behavior depending on the product properties, and hence they produce exactly as many goods as are demanded at any time.

2.3. The sales equation

We define $S_i(t)$ as the number of units of product $i \in \{1, \ldots, n\}$, sold per time at time t. S_i shall be called the *sales of product i*.

The sales of product i correspond to the rate of consumers switching to or rather

[1] Exogenously given prices can for instance be applied in a Bertrand competition for oligopolies.

buying this good,

$$S_i(t) \quad = \quad N\left[x_i(t)\rho_{ii}(t) + \sum_{\substack{j=0\\ j\neq i}}^{n} x_j(t)\rho_{ji}(t)\right], \quad i = 0,\dots,n. \tag{4}$$

Using the mean dynamic (3), equation (4) can be transformed into the *sales equation*, which describes the relationship between sales and consumer subpopulations,

$$\frac{S_i}{N} = \dot{x}_i + x_i R_i, \quad i = 0,\dots,n. \tag{5}$$

To enhance intuition, the rate of change $\dot{x}_i$ can be interpreted as the difference between consumers adopting and abandoning good i, while the second summand represents all those owners of good i who currently reorientate themselves and either abandon or stick with[2] product i. As the leaving consumers cancel, the net effect consists of those consumers buying product i.

Let us provide a brief analysis of the sales equation: As indicated above, the equation decomposes into a pair of additive terms, leading to two cases, each corresponding to one term outweighing the other. In the case of small x_i, e. g. during product launch, sales precisely reflect the increasing distribution among the population, e. g. due to growing product awareness. On the other hand, for negligible $\dot{x}_i$ (after reaching a comparatively stable situation) sales of good i solely originate from replacements of both defect units of i and different products. The rate of replacements is equivalent to the rate of consumer reorientation or revision, $x_i R_i$.

2.4. A consumer revision protocol and the resulting mean dynamic

The specific revision protocol which we employ borrows well-established ideas from social learning (Smallwood and Conlisk 1979, Ellison and Fudenberg 1995), psychological, experimental, and consumer research literature (Assael 1984). On a goods market there are mainly two factors influencing the buying behavior of a consumer, the goods' properties and their perceived distribution among other consumers. It is for example beyond question that any purchase decision depends on the anticipated product quality, including functionality, reliability, value for money, and many further properties. Also, the life span of a product plays a role as it affects the frequency of purchases. On the other hand, the product distribution among other consumers may have an effect by simply determining the level of product awareness or inducing a fashion or even networking. The revision protocol ought to reflect these influences on the consumer behavior.

A revision protocol $\rho(x,t)$ is uniquely defined by the product-dependent alarm clock rates R_i and the switching probabilities $p_{ij}(x,t)$, for which we shall suggest a specification in the following.

[2]Note that these consumers replace the broken unit they own and buy a new one.

The alarm clock rate represents the average frequency at which consumers think about buying a specific product and hence replacing their old one—unless they do not yet possess any good. Since this frequency depends on the durability of the currently owned product, alarm clock rates in general differ from product to product. For simplicity, we shall assume the rates R_i to be given for each product i and to be invariant with time.

Of those people, who do not own any product, the fraction of consumers deciding to buy product i can be described by the switching probability p_{0i}. When consumers start thinking about buying a specific product they scan the market and become susceptible to various types of information about possible alternatives. In the accompanied decision process passive and active decision mechanisms can be distinguished: When consumers passively encounter a product, its level of familiarity rises, thus increasing the possibility for this product to be bought. On the other hand, consumers may actively imitate others in buying the same good since the popularity of a product might give information about the product's past performance (cf. Ellison and Fudenberg (1993)), so that according to Smallwood and Conlisk (1979) the consumers' choices are sensitive to market shares or popularities of the products. Additionally, studies in social psychology support the individual's conformity to group norms, i. e. that consumers imitate group behavior (Assael (1984), p. 371ff).

Purely active, purely passive, and intermediate decision mechanisms are categorized in the following. They imply a proportionality between the switching probability p_{0i} and the fraction x_i of consumers currently owning product i. The mechanisms are ordered from rather passive to rather active.

- In daily life consumers encounter product i at a frequency proportional to its distribution x_i among the population. Hence, familiarity with the good and proneness to buy it rise accordingly.
- A consumer's idea of an ideal product is partly shaped by the surroundings. Product characteristics often observed are commonly desired. The intensity of influence by good i and thus proneness to buy it may be assumed proportional to its frequency x_i.
- Information about products can be obtained from various media (including the Internet), e. g. experience is exchanged on product evaluation websites. The media in total roughly reflect the real world including the market; the more widespread a product is, the more frequently the media report about it, thereby increasing product awareness proportionally to x_i.
- The expected quality of a product is often inferred from the number of sales, assuming that superior products always find a ready market. The prevalently perceived indicator is thus the observed distribution x_i.
- Consumers also actively ask around their friends and let their purchase decision be influenced by the found product distribution, which on average

matches the x_i. For certain goods, networking may play a role as well, intrinsically implying the positive effect of high x_i on the sales of product i.

- Finally, a fashion might be induced by a high distribution of a specific product, leading to even higher sales of that product.

In summary, we may assume a linear relation between switching probability p_{0i} and x_i as a first approximation, that is $p_{0i} \sim x_i$. The proportionality factor, denoted φ_i, still remains to be determined. Generally, it differs from product to product (Assael (1984), p. 432, 414) and can even be time dependent. We need $\varphi_i \in [0,1]$, since

$$1 = \sum_{i=0}^{n} p_{0i} = p_{00} + \sum_{i=1}^{n} \varphi_i x_i,$$

must also hold for any state $x_i = 1,\ x_{j\neq i} = 0$. φ_i constitutes the accumulated influence of product frequency on the consumers' purchase decision via all different mechanisms. It can be interpreted as the intensity by which a consumer is convinced during an encounter with the product and is therefore termed *convincement factor* in the following. It is similar to an anticipated product quality. Of course, φ_i does depend on the good properties as there are the price, the (expected) quality, the strength of networking, and fashion effects for that product etc. To summarize,

$$p_{0i} = \varphi_i x_i,\ i \neq 0. \tag{6}$$

Let us now turn to those people owning product i. Someone who is content with that good, tends to buy a new unit of that good when the alarm clock rings, even though a better product might exist. Surveying the market is time-consuming, and furthermore, consumers usually act conservatively and avoid changes, so that the same product i is bought. Assael (1984, p. 53) summarizes several studies on the topic and comes to the conclusion that a form of habit evolves, leading to repeat purchases of a product without further information search or evaluating brand alternatives.

We may deduce that the fraction p_{ii} of consumers sticking to product i only depends on the habit induced by this good or consumer sluggishness and is independent of any other factors. Also, we will assume this habit level to represent a characteristic of the product, as justified by the fact that a consumer usually gets the more discontent with a product the more often it breaks down or the less satisfactory its functionality seems. In our model, a fixed, product-specific percentage of consumers will develop a buying habit, so that finally

$$p_{ii} = s_i \in [0,1],\ i \neq 0. \tag{7}$$

Obviously, the fraction of switching consumers $(1-p_{ii})$ divides up into the fractions p_{ij} of people switching to product $j \neq i$. Naturally, they behave just like those consumers not yet owning any good, except that they do not purchase product i

again. Therefore,

$$\begin{aligned} p_{ij} &= (1-p_{ii})p_{0j} \\ &= (1-s_i)\varphi_j x_j,\ i \neq 0 \wedge j \neq 0, i. \end{aligned} \tag{8}$$

The switching probabilities p_{i0} and p_{00} are now uniquely determined by the constraints $\sum_{j=0}^{n} p_{ij} = 1$, $i = 0, \ldots, n$,

$$p_{00} = 1 - \sum_{j=1}^{n} p_{0j} = 1 - \sum_{j=1}^{n} \varphi_j x_j, \tag{9}$$

$$p_{i0} = 1 - \sum_{j=1}^{n} p_{ij} = (1-p_{ii})(p_{00}+p_{0i}) = (1-s_i)\left(1 - \sum_{\substack{j=1 \\ j\neq i}}^{n} \varphi_j x_j\right). \tag{10}$$

For $i = 1, \ldots, n$, the mean dynamic (3) eventually takes the form

$$\begin{aligned} \dot{x}_i &= \rho_{0i} + \sum_{j=1}^{n} x_j(\rho_{ji} - \rho_{0i}) - x_i R_i \\ &= x_i \left(\varphi_i R_0 - (1-s_i)R_i - \varphi_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[R_0 - (1-s_j)R_j\right] x_j - \varphi_i R_0 x_i \right). \end{aligned} \tag{11}$$

This specific mean dynamic has Lotka-Volterra form $\dot{x}_i = x_i \left(a_i - \sum_{j=1}^{n} b_{ij} x_j\right)$ of competitive species with coefficients $a_i = \varphi_i R_0 - (1-s_i)R_i$, $b_{ii} = \varphi_i R_0$, and for $j \neq i$, $b_{ij} = \varphi_i \left[R_0 - (1-s_j)R_j\right]$.

All constants R_i, φ_i, and s_i may in principle be time-dependent so that product modifications or fashion trends can be modeled. However, for a theoretical analysis, we shall first assume the coefficients to be constant. Non-constant coefficients will be dealt with in a second paper (Part II: Product pricing when demand follows a rule of thumb). In particular, we will show how to interpret the convincement factor φ_i as a demand function of the product price.

φ_i and s_i should be seen as the mean parameters over the whole population of heterogeneous, boundedly rational agents. The probability φ_i, for example, may be seen as that fraction of the heterogeneous population which is convinced by product i. Even more, φ_i may also be seen as the probability of an individual to be convinced, i. e. due to the individual's bounded rationality her decision is not deterministic. Hence, the parameters incorporate both an individual variation and a global variation over the population.

Modified population games with the consumer revision protocol motivated above will frequently be used in the later model analysis. For simplicity we shall therefore define the following:

Definition 2.4 (Habitual imitative consumers)**.** *Agents who follow the above revision protocol (6) to (10) are called* habitual imitative consumers*. A modified population game with such agents is called the demand side of a* market with habitual imitative consumers*.*

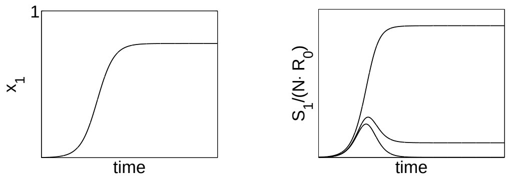

Figure 1: Sketch of a typical behavior of x_1 (left) and $\frac{S_1}{N \cdot R_0}$ (right) in time.

3. Model analysis

3.1. Single good market

For the single good case the mean dynamic has the following form

$$\dot{x}_1 = x_1 \varphi_1 R_0 (\Psi - x_1) \tag{12}$$

with $\Psi = 1 - \frac{R_1}{R_0} \frac{1-s_1}{\varphi_1}$. This ordinary differential equation can be solved analytically. Using partial fraction expansion and separation of variables the solution reads

$$x_1(\tau) = \frac{\Psi}{1 + \left(\frac{\Psi}{x_1(0)} - 1\right) \exp\left[-\tau \Psi \varphi_1\right]} \tag{13}$$

with dimensionless time $\tau = tR_0$. The evolution of the population x_1 either has a sigmoidal shape with initial exponential growth and saturation value Ψ (figure 1) or—in case of an undesirable product—immediately decays to zero.

Definition 3.1 (Feasibility)**.** *We call a good* feasible *if a positive number of consumers owns the product in the steady state.*

In this case, the number of sales is positive at least within a certain time period. Mathematically, feasibility of a good is determined by the parameter Ψ, which can readily be shown by investigating the stability of the (unique) steady states $x_1 = 0$ and $x_1 = \Psi$ respectively. Figure 2 shows a transcritical bifurcation to occur at $\Psi = 0$ which renders the steady state $x_1 = 0$ instable and $x_1 = \Psi$ stable. Hence we obtain:

Proposition 3.1. *The single product on a market with habitual imitative consumers is feasible iff* $\Psi > 0$. □

The sales equation for the single good market with a feasible good takes the explicit form

$$\frac{S_1(\tau)}{N \cdot R_0} = \frac{\Psi}{1 + \left(\frac{\Psi}{x_1(0)} - 1\right) \exp\left[-\tau \Psi \varphi_1\right]} \left[\frac{R_1}{R_0} + \varphi_1 \Psi - \frac{\varphi_1 \Psi}{1 + \left(\frac{\Psi}{x_1(0)} - 1\right) \exp[-\tau \Psi \varphi_1]}\right] \tag{14}$$

with saturation value

$$\lim_{\tau \to \infty} \frac{S_1(\tau)}{N \cdot R_0} = \Psi \frac{R_1}{R_0}. \tag{15}$$

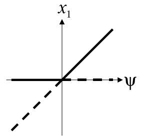

Figure 2: Transcritical bifurcation with stable (solid line) and unstable (dashed line) steady state values of x_1.

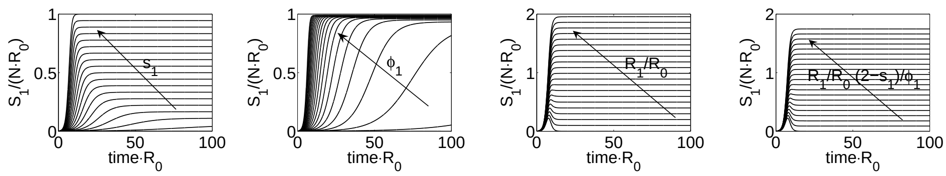

Figure 3: Comparative statics of $\frac{S_1}{N \cdot R_0}$ with respect to the parameters s_1, φ_1, $\frac{R_1}{R_0}$, and $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$. In each graph, the fixed parameters have the values $\frac{R_1}{R_0} = 1$, $x_1(0) = 0.001$, $\varphi_1 = 0.9$, $s_1 = 0.99$. The arrows indicate the variation of the curve for an increasing parameter value, where the parameter values of s_1 and φ_1 are varied from zero to one in steps of 0.05 and the values of $\frac{R_1}{R_0}$ and $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$ are varied from zero to two in steps of 0.1.

We may assume that the initial condition $x_1(0) > 0$ is near zero. Depending on $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$, the sales can take qualitatively different shapes (figure 1 right) corresponding to different patterns of monopoly product life cycles:

For $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1} < 1$ we find an initial exponential growth, a maximum, and then a decay with saturation. Intuitively, the stronger the effect of habituation s_1 or imitation φ_1 (i. e. the more convincing the product is), the stronger is the increase of the subpopulation. After a maximum is reached however, the effect of product durability gains in importance, since the most purchases are replacements of broken products. Hence, the sales decrease and approach a saturation value. In the special case $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1} = 0$ of an infinitely durable good ($R_1 = 0$), the saturation value is zero since once the good is bought, the consumer owns it forever and does not purchase another unit of this product.

For $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1} \geq 1$ the curve is sigmoidal. In this case habit and imitation cause an initial increase, then the good has to be replaced quite often and a saturation value is approached.

S_1 is uniquely determined by three parameters, s_1, φ_1, and $\frac{R_1}{R_0}$, which together with $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$ lend themselves for a comparative statics analysis (figure 3). Generally, the time scale inherent in the sales evolution decreases for rising φ_1, s_1, $\frac{R_1}{R_0}$, $\frac{R_1}{R_0} \cdot \frac{2-s_1}{\varphi_1}$ (i. e. the system becomes faster), and the saturation value increases, which can also be directly inferred from equation (15) and the mean dynamic (12). The effect is especially sensitive to changes in φ_1. These results are intuitive, since rising habit, imitation, and replacement rates are indeed expected to speed up the system and to cause more frequent purchases.

A single product market can be a model of different firm constellations: Either a monopoly produces the good or an oligopoly, where all firms produce exactly the same good and the brands do not influence the consumers' decision (the consumers do not even differentiate between them by any means).

3.2. Two goods market

In the two goods market the mean dynamic has the form

$$\dot{x}_i = x_i \varphi_i R_0 \left[\Psi_i - x_i - \Phi_j x_j\right], \; i = 1, 2, \; j \neq i, \tag{16}$$

where the $\Phi_j = 1 - \frac{R_j}{R_0}(1 - s_j)$ are the slopes of the nullclines in the corresponding phase plane (i. e. the lines along which $\dot{x}_i = 0$, see figure 4) and the $\Psi_i = 1 - \frac{R_i}{R_0}\frac{1-s_i}{\varphi_i}$ are the intersections of nullclines with the axes and thus represent the respective saturation values of the single good case. Obviously, the Ψ_i have to be positive for feasible goods, which we will assume in the following.

Phase planes are a very elegant representation of the dynamic system, in which the nullclines divide the state space into regions with $\dot{x}_i > 0$ and $\dot{x}_i < 0$. Steady states are given by the intersection points $x = (x_1, x_2)$ of nullclines $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Obviously, in the two goods case there are four possible distinct steady states,

$$\mathbf{x}^A = \left(\frac{\Psi_1 - \Phi_2\Psi_2}{1 - \Phi_1\Phi_2}, \frac{\Psi_2 - \Phi_1\Psi_1}{1 - \Phi_1\Phi_2}\right), \;\; \mathbf{x}^B = (0, \Psi_2), \;\; \mathbf{x}^C = (\Psi_1, 0), \;\; \mathbf{x}^D = (0, 0).$$

$\mathbf{x}^D$, the trivial steady state, is instable. For a stability analysis of the other steady states two cases have to be distinguished, $\mathbf{x}^A \in \mathbb{R}^2_{\geq 0}$ and $\mathbf{x}^A \notin \mathbb{R}^2_{\geq 0}$ respectively (cf. figure 4). For the description of the time evolution of populations x_1 and x_2 and the according product life cycles we may again assume $x_1(0), x_2(0) > 0$ to be near zero.

1. $\mathbf{x}^A \in \mathbb{R}^2_{\geq 0}$. This is possible if either the denominators and numerators of x_1^A and x_2^A are all positive (1a) or all zero (1b), in which case the nullclines lie on top of each other.

    a) $1 > \Phi_1\Phi_2$, $\Psi_1 > \Phi_2\Psi_2$, $\Psi_2 > \Phi_1\Psi_1$ (figure 4 left). The only stable steady state is $\mathbf{x}^A$. Hence the products coexist lastingly on the market. Depending on the initial condition, x_1 or x_2 may have a maximum before approaching the saturation value; the form of the trajectory strongly depends on its starting point.

       For the product life cycles—similar to the single good case—a maximum may or may not exist depending on the parameter values. The number of products sold initially and overall strongly depends on the initial condition.

    b) $1 = \Phi_1\Phi_2$, $\Psi_1 = \Phi_2\Psi_2$, $\Psi_2 = \Phi_1\Psi_1$ (figure 4 middle). This case occurs for either $R_i = 0$ or $s_i = 1$ for $i = 1, 2$, i. e. for infinitely durable goods or

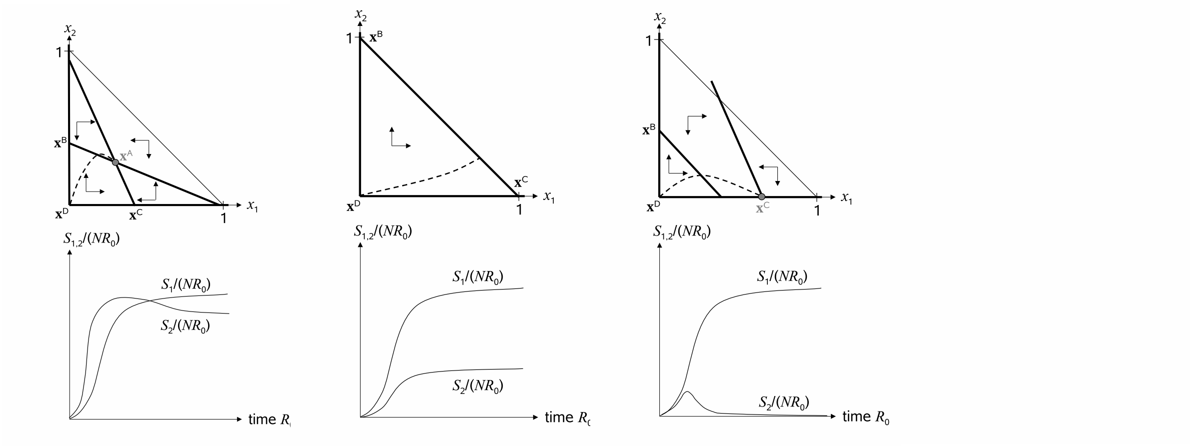

Figure 4: Phase planes for all three possible cases 1a, 1b, 2 (from left to right), with nullclines (solid lines) and one trajectory (dotted line) starting near zero each. The (asymptotically) stable steady states are indicated by filled circles. Resulting product life cycles are displayed beneath each diagram.

perfect habituation. $\mathbf{x}^A$ degenerates to a continuous line of (not asymptotically) stable steady states. $x_1(t)$ and $x_2(t)$ are monotonous, while the sales evolution may exhibit a maximum depending on the R_i. For infinitely durable goods ($R_i = 0$, $i = 1, 2$) the sales eventually approach zero.

2. $\mathbf{x}^A \notin \mathbb{R}^2_{\geq 0}$ (figure 4 right). Without loss of generality let $\Psi_1 > \Phi_2\Psi_2$ (otherwise simply renumber the species; $\Psi_1 < \Phi_2\Psi_2$ and $\Psi_2 < \Phi_1\Psi_1$ is not possible for $\Phi_i \leq 1$). Since either x_1^A or x_2^A has to be negative to prevent an intersection of nullclines in $\mathbb{R}^2_{\geq 0}$, this implies $1 = \Phi_1\Phi_2$ or $\Psi_2 < \Phi_1\Psi_1$. $\mathbf{x}^C$ is stable and $\mathbf{x}^B$ is instable (reverse for $\Psi_1 < \Phi_2\Psi_2$ and $\Psi_2 > \Phi_1\Psi_1$). Hence, one product (here product 2) dies out and the other product survives on the market. For the vanishing product, both the subpopulation and sales reach a maximum and then approach zero. The corresponding curves of the surviving product qualitatively behave as in the single good case. The total sales of the inferior product strongly depend on the initial condition.

The bifurcation diagram in figure 5 illustrates the parameter regions for the different cases, in particular the region of coexistence (1a,1b) and the region of exclusion of a product (2). The following proposition can immediately be inferred:

Proposition 3.2. *Product 1 is feasible on a two goods market with habitual imitative consumers, iff either* $\Psi_1 > 0$ *and* $\Psi_2 \leq 0$ *or* $\Psi_1 > \Phi_2\Psi_2$*. The analogous result holds for product 2.* □

Since Ψ_i encodes the "quality" of the good (Ψ_i is large for high durability, convincement, and habituation factor), the previous proposition basically performs a comparison of product qualities. However, due to the consumers' heterogeneity and

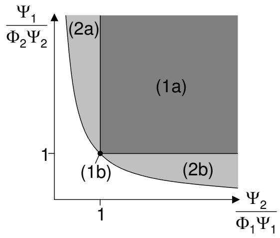

Figure 5: Bifurcation diagram. (1a) represents the coexistence region of both products, good 2 dies out in (2a) and good 1 in (2b). The region below $\Phi_1\Phi_2 = 1$ cannot be reached.

bounded rationality, the "quality" of one good need only be larger than the "quality" of the other downscaled by a factor Φ_j, which is monotonously increasing with the habit induction by the competing product.

A further, quite intuitive result follows directly from the phase planes in figure 4:

Proposition 3.3. *If a competitor enters a single good market with habitual imitative consumers, the steady state market share of the incumbent decreases, while the joint market share of entrant and incumbent increases. Formally,*

$$x_i^* \leq \Psi_i = x_i^\star \leq x_1^* + x_2^*, \quad i = 1, 2, \tag{17}$$

where x_i^* is the (or a) stable steady state value of x_i in the two goods market and $x_i^\star$ the steady state value in the single good market. □

As in the single good case, a two product market can model different firm constellations: There might be a monopoly producing both goods, or the products are offered by a duopoly or even oligopoly, but with only two distinguishable goods. Equation (17) suggests that it might be beneficial for a monopoly to offer a variety of goods instead of just one (due to their bounded rationality this even holds for completely homogeneous consumers, in which case a single, optimally fitted product would seem more profitable at first glance).

3.3. n goods market

Extension to multi-goods markets, where the mean dynamic takes the form

$$\dot{x}_i \quad = \quad x_i \varphi_i R_0 \left[\Psi_i - x_i - \sum_{j=1,\, j\neq i}^{n} \Phi_j x_j \right], \; i = 1, \ldots, n, \tag{18}$$

is obvious and will therefore be kept brief here. We will only derive the feasibility condition for the nth good (and hence for any good after renumbering) and take a look at the special case of a symmetric market. In the following we will abbreviate vectors of scalars according to $(\sigma_i)_{i=1,\ldots,m} = \vec{\sigma}$. Also, we will need the following lemma, whose proof is given in the appendix:

Lemma 3.4. *Let $0 < \Phi_i < 1$ and let A_n for $n \in \mathbb{N}$ be the matrix defined as*

$$(A_n)_{ij} = \begin{cases} 1, & i = j, \\ \Phi_j, & i \neq j, \end{cases} \quad i, j = 1 \dots n.$$

Then $\det(A_n) > 0$.

Now we can characterize the feasibility of a good:

Proposition 3.5. *Consider an n-product market with habitual imitative consumers on which the products i, $i = 1, \dots, n-1$, coexist with $0 < \Phi_i < 1$. Then product n is feasible iff*

$$\Psi_n > A_{n-1}^{-1} \vec{\Psi} \cdot \vec{\Phi} = \vec{\tilde{x}} \cdot \vec{\Phi} = \sum_{i=1}^{n-1} \Phi_i \tilde{x}_i \quad \text{for} \quad (A_{n-1})_{ij} = \begin{cases} 1, & i = j, \\ \Phi_j, & i \neq j, \end{cases}$$

where $\vec{\tilde{x}}$ is the vector of market shares on the $(n-1)$-goods market (i. e. without product n).

In this proposition, the identity $A_{n-1}^{-1}(\Psi_1, \dots, \Psi_{n-1})^{\mathrm{T}} = (\tilde{x}_1, \dots, \tilde{x}_{n-1})^{\mathrm{T}}$ results from the steady state equations of the mean dynamic (18) for the $(n-1)$-goods market if all market shares are non-zero. Hence, intuitively, the above proposition implies that the (hypothetic) monopoly market share Ψ_n has to be larger than the weighted sum of market shares of products 1 to $n-1$, where the weights Φ_i increase with habit induction. The proof is deferred to the appendix.

For a sensible market model, we postulate that it be consistent with our economic intuition. The next few lines are devoted to such a proof of consistency: We aim to show the intuitive fact that competition becomes harder when another competitor enters the market. For this purpose, let $\bar{\Psi}_{n-1}$ be the feasibility boundary from proposition 3.5 for an nth product to enter a market with $n-1$ existing products and with habitual imitative consumers. Furthermore, let $\bar{\Psi}_{n-2,j}$ be the same feasibility boundary, however, only for entering a market with just $n-2$ existing products, namely all products from the $(n-1)$-goods market except for product j. Then the following lemma can be proven:

Lemma 3.6. *Under the conditions of the previous proposition and with $\bar{\Psi}_{n-1}$ and $\bar{\Psi}_{n-2,j}$ as just defined,*

$$\bar{\Psi}_{n-1} - \bar{\Psi}_{n-2,j} = \prod_{i=1; i \neq j}^{n-1} (1 - \Phi_i)(\Psi_j - \bar{\Psi}_{n-2,j}) \frac{\Phi_j}{\det(A_{n-1})}.$$

□

We shall not give the tedious proof here, however, the reader may easily verify the relation by trying out different n. This lemma now yields the desired result:

Proposition 3.7. *In a market with habitual imitative consumers, competition gets harder the more competitors enter the market, i. e.*

$$\bar{\Psi}_{n-1} > \bar{\Psi}_{n-2,j} \quad \text{for all } j = 1, \dots n-1.$$

Hence, it is harder to enter the $(n-1)$-goods market than it is to enter the same market, but with any one of the products removed.

Proof. $(1-\Phi_i)$ is positive due to $0 < \Phi_i < 1$. $(\Psi_j - \bar{\Psi}_{n-2,j})$ is positive since otherwise product j would not be feasible and would hence not exist. Thus the result follows from lemma 3.6 together with lemma 3.4. □

We shall now briefly provide a further interpretation of the feasibility condition in proposition 3.5: Analogously to $\bar{\Psi}_{n-2,j}$, let the matrix $A_{n-2,j}$ be defined by omitting the jth row and the jth column of matrix A_{n-1}. Furthermore, let matrix $(A_n)_{i\to\vec{\Psi}}$ be equal to A_n with column i replaced by $\vec{\Psi}$, let the vector of ones be denoted by $\vec{1}$ and let $M_{\not{i},\not{j}}$ denote a matrix M with the ith row and jth column removed. Then

$$\begin{aligned}
\det(A_{n-1})\bar{\Psi}_{n-1} &= \sum_{i,j=1;i\neq j} (-1)^{i+j}\Phi_i\Psi_i\Phi_j \det(((A_{n-1})_{j\to\vec{1}})_{\not{j},\not{i}}) + \sum_{i=1}^{n-1}\Phi_i\Psi_i\det((A_{n-1})_{\not{i},\not{i}}) \\
&= \sum_{j=1}^{n-1}\left[-\sum_{i=1;i\neq j}^{n-1}(-1)^{i+j}\Phi_i\Psi_i\Phi_j\left((-1)^{i+j}\det((A_{n-2,j})_{i\to\vec{1}})\right) + \Phi_j\Psi_j\det(A_{n-2,j})\right] \\
&= \sum_{j=1}^{n-1}\left[-\bar{\Psi}_{n-2,j}\Phi_j\det(A_{n-2,j}) + \Phi_j\Psi_j\det(A_{n-2,j})\right] \\
&= \sum_{j=1}^{n-1}\Phi_j\det(A_{n-2,j})\left[\Psi_j - \bar{\Psi}_{n-2,j}\right]
\end{aligned}$$

Hence, the lower feasibility bound $\bar{\Psi}_{n-1}$ can be interpreted as the weighted sum of the feasibility $(\Psi_j - \bar{\Psi}_{n-2,j})$ of all existing products. $(\Psi_j - \bar{\Psi}_{n-2,j})$ indeed expresses how much more feasible good j is in comparison with the minimum to exist on the market.

In the special case of a symmetric market with $n-1$ identical products, A_{n-1}^{-1} has a simple form, which allows to look at proposition 3.5 from a slightly different viewpoint. The following corollary compares the quality of the entrant with the quality of the incumbents and enables us to examine this relation for growing numbers of products.

Corollary 3.8. *Consider a symmetric $(n-1)$-product market with identical firms ($\Phi_i = \Phi$, $\Psi_i = \Psi$) and habitual imitative consumers on which the products coexist with $0 < \Phi < 1$. Then a new (not necessarily identical) nth product is feasible iff*

$$\Psi_n > \frac{(n-1)\Phi\Psi}{1+(n-2)\Phi}.$$

Proof. By calculating $A_{n-1}^{-1}A_{n-1}$ we can readily verify

$$(A_{n-1}^{-1})_{ij} = \begin{cases} \frac{(n-3)\Phi+1}{(1-\Phi)[1+(n-2)\Phi]}, & i = j, \\ \frac{-\Phi}{(1-\Phi)[1+(n-2)\Phi]}, & i \neq j. \end{cases}$$

Hence, by proposition 3.5,

$$\Psi_n > A_{n-1}^{-1} \vec{\Psi} \cdot \vec{\Phi} = \frac{(n-1)\Phi\Psi}{1+(n-2)\Phi}.$$

□

Obviously, the "quality" Ψ_n of the new nth product may always be smaller than the "quality" Ψ of the $n-1$ incumbents. However, the factor $\frac{(n-1)\Phi\Psi}{1+(n-2)\Phi}$ monotonously increases with the number of products n, and in the limit

$$\Psi_n > \frac{(n-1)\Phi\Psi}{1+(n-2)\Phi} \overset{n\to\infty}{\longrightarrow} \Psi$$

so that the entrant's "quality" has to approach the incumbents' "quality" Ψ.

4. Discussion

So far we have worked out how the sales dynamics of products or brands evolve in an environment with habitual imitative consumers and how this evolution is influenced by the product dependent habit and imitation parameters. In this section we shall briefly mention further implications for firms deduced from the model.

For non-varying products and thus fixed habit and imitation parameter values, the final steady state cannot be influenced by the firms (unless the parameter values are fixed functions $\varphi_i(x_1, \ldots, x_n)$, $s_i(x_1, \ldots, x_n)$ of the state so that the mean dynamic (11) no longer has Lotka-Volterra form). Nevertheless, a moderately large initial share $x_i(t = 0)$ generally is profitable for firms (i. e. to give away a number of product units for free initially) in order to sell more product units on the whole. The effect arises, because a larger initial product share leads to a faster approach of the steady state. Similarly it pays off for a firm to launch a product early in comparison to the competitors in order to have a large market share when the other products enter the market. This causes sales advantages such as the initial hump in figure 4 (left).

Of course any real market which is to be modeled needs careful choice of habit, imitation, and durability parameters. Such parameters can be obtained from matching empirical curves with simulated sales development, for instance using the observed time scales and steady state market shares. As an example consider the evolution of consumption of filter cigarettes as found in Polli and Cook (1969, p. 389). Here we will regard filter cigarettes as a product class within which brand differences (at least initially) do not matter much so that a monopoly approximation (i. e. there are two options: buying or not buying filter cigarettes) may be appropriate. The fact that habit plays a strong role here is unquestionable (and rumor has it that imitation takes place as well). From the data we roughly estimated a high (packet) consumption rate of $R_1 = 200$ per year and a rather low consideration frequency of $R_0 = 20$ per year as well as a low imitation of $\varphi_1 = 0.01$. Habit on the other hand is

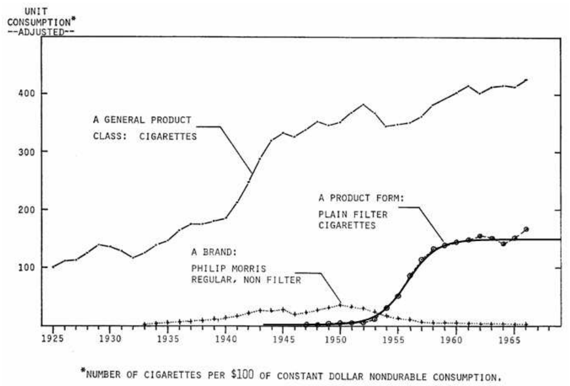

Figure 6: Figure from Polli and Cook (1969) with simulated sales evolution of filter cigarettes in the USA overlaid (solid, smooth line).

close to one with $s_1 = 0.9992$. This results in $x_1 = 20\%$ of the population smoking filter cigarettes in the steady state, which seems a reasonable estimate considering that according to Polli and Cook (1969) in the 1960s filter cigarettes consumption amounted to roughly 40% of total cigarette consumption (and nearly half the US population were smokers). The curves are given in figure 6.

A subsequent work (Part II: Product pricing when demand follows a rule of thumb) will additionally model the supply side of the market and analyze firm's pricing or advertising strategies as well as the welfare in such a market. Moreover, we show how product life cycles of the typically empirically supported form can be generated in a market with habitual imitative consumers.

5. Conclusion

We examined the evolution of a consumer market, where boundedly rational consumers follow rules of thumb, basing their decisions on imitation and habit. To achieve this goal we set up a modified population game with a corresponding revision protocol as framework. We then analyzed the resulting sales evolution of products and investigated product feasibility in a market with habitual imitative consumers. The behavioral parameters can be adapted to match observed sales evolutions of products or product classes, and the introduced methodology allows for broad applications and qualitative theoretical analysis. In particular, demand by habitual imitative consumers will serve as the basis for modeling a strategic supply side in subsequent work.

One of the main achievements of this paper consists in having cast psychological and experimental results into a mathematical model with boundedly rational and habitual imitative consumers, as well as investigation of the consequences. For example, such a market model is shown to be consistent with standard economic intuition in that it becomes more difficult for products to survive on the market the more competitors enter the market.

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A. Appendix

A.1. Proof of lemma 3.4

Proof. Consider the matrix function $M(t) := I + t(A_n - I)$ such that $M(0) = I$ and $M(1) = A_n$, where I denotes the identity matrix. Since $\det(M)$ is continuous in M, it is also continuous in t. If $\det(M(1))$ were non-positive, then according to Rolle's theorem there would exist some $t \in (0, 1]$ with $\det(M(t)) = 0$. However, $\det(M(t))$ cannot be zero due to the following reasoning: For a contradiction, assume the columns of A_n to be linearly dependent, i. e.

$$\vec{0} = \sum_{i=1}^{n} \alpha_i \left(\Phi_i, \ldots, 1, \Phi_i, \ldots\right)^{\mathrm{T}}.$$

This can be rewritten as

$$\vec{0} = \begin{pmatrix} \alpha_1(1-\Phi_1) \\ \vdots \\ \alpha_n(1-\Phi_n) \end{pmatrix} + \sum_{i=1}^{n} \alpha_i \begin{pmatrix} \Phi_i \\ \vdots \\ \Phi_i \end{pmatrix},$$

which together with $\Phi_i < 1$ implies $\alpha_i = \frac{k}{1-\Phi_i}$, for some number k. Plugging this back into the equation we obtain

$$0 = k\left(1 + \sum_{i=1}^{n} \frac{\Phi_i}{1-\Phi_i}\right),$$

which is impossible for $\frac{\Phi_i}{1-\Phi_i} > 0$. $\square$

A.2. Proof of proposition 3.5

Proof. For the computation, assume that also product n exists in the steady state.

1. *Linear system of equations:* Since by assumption $x_i > 0$ in the steady state for $i = 1, \ldots, n$ we obtain a linear system of steady state equations from the mean dynamic (18):

$$A_n \vec{x} = \vec{\Psi}$$

2. *Cramer's rule:* Due to the previous step, A_n is invertible. According to Cramer's rule the solution to $A_n\vec{x} = \vec{\Psi}$ is given by

$$x_i = \frac{\det\left((A_n)_{i\to\vec{\Psi}}\right)}{\det(A_n)}$$

where matrix $(A_n)_{i\to\vec{\Psi}}$ equals A_n with column i replaced by $\vec{\Psi}$.

3. *Cramer's rule backwards:* Let us denote the vector of ones by $\vec{1}$ and let $M_{\cancel{i},\cancel{j}}$ be matrix M with the ith row and jth column removed. Using Laplace expansion for the last column,

$$\begin{aligned}
\det\left((A_n)_{n\to\vec{\Psi}}\right) &= \sum_{i=1}^{n}(-1)^{i+n}\Psi_i \det((A_n)_{\cancel{i},\cancel{n}}) \\
&= \sum_{i=1}^{n-1}(-1)^{i+n}\Phi_i\Psi_i \det(((A_n)_{\cancel{i},\cancel{n}})_{i\to\vec{1}}) + \Psi_n \det(A_{n-1}) \\
&= \sum_{i=1}^{n-1}(-1)\Phi_i\Psi_i \det((A_{n-1})_{i\to\vec{1}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{i,j=1}^{n-1}(-1)^{i+j}\Phi_i\Psi_i \det((A_{n-1})_{\cancel{j},\cancel{i}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}(-1)^{i+j}\Phi_i\Psi_i\Phi_j \det(((A_{n-1})_{j\to\vec{1}})_{\cancel{j},\cancel{i}}) - \sum_{i=1}^{n-1}\Phi_i\Psi_i \det((A_{n-1})_{\cancel{i},\cancel{i}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}(-1)^{i+j}\Phi_i\Psi_i\Phi_j \det(((A_{n-1})_{i\to\vec{1}})_{\cancel{i},\cancel{j}}) - \sum_{i=1}^{n-1}\Phi_i\Psi_i \det((A_{n-1})_{\cancel{i},\cancel{i}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{i,j=1}^{n-1}(-1)^{i+j}\Phi_j\Psi_i \det((A_{n-1})_{\cancel{i},\cancel{j}}) + \Psi_n \det(A_{n-1}) \\
&= -\sum_{j=1}^{n-1}\Phi_j \det((A_{n-1})_{j\to\vec{\Psi}}) + \Psi_n \det(A_{n-1}) \\
&= \det(A_{n-1})\left(\Psi_n - \sum_{i=1}^{n-1}\Phi_i\left(A_{n-1}^{-1}\vec{\Psi}\right)_i\right),
\end{aligned}$$

where the last step follows from Cramer's rule and Laplace expansion has been applied various times.

Overall, we obtain

$$\begin{aligned}
\text{product } n \text{ feasible} &\Leftrightarrow x_n > 0 \\
&\overset{\text{lemma3.4}}{\Leftrightarrow} x_n \det(A_n) > 0 \\
&\overset{\text{step2}}{\Leftrightarrow} \det((A_n)_{n\to\vec{\Psi}}) > 0 \\
&\overset{\text{step3}}{\Leftrightarrow} \det(A_{n-1})\left(\Psi_n - \sum_{i=1}^{n-1}\Phi_i\left(A_{n-1}^{-1}\vec{\Psi}\right)_i\right) > 0 \\
&\overset{\text{lemma3.4}}{\Leftrightarrow} \Psi_n > \sum_{i=1}^{n-1}\Phi_i\left(A_{n-1}^{-1}\vec{\Psi}\right)_i
\end{aligned}$$

□

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