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Evolution and Correlated Equilibrium

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Evolution and Correlated Equilibrium

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Abstract

We show that a set of outcomes outside the convex hull of Nash equilibria can be asymptotically stable with respect to convex monotonic evolutionary dynamics. Boundedly rational agents receive signals and condition the choice of strategies on the signals. A set of conditional strategies is asymptotically stable only if it represents a strict (correlated-)equilibrium set. There are correlated equilibria that cannot be represented by an asymptotically stable signal contingent strategy. For generic games it is shown that if signals are endogenous but no player has an incentive to manipulate the signal generating process and if the signal contingent strategy is asymptotically stable, then and only then, the outcome must be a strict Nash equilibrium.

Keywords: Dynamic Stability, Noncooperative Games, Correlated Equilibrium, Evolution

JEL codes: C72, D80

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1 Introduction and Related Literature

Consider a situation of strategic interaction in which agents perceive signals before they choose their strategy. Restricting the general setup of Aumann (1974), we demand that all agents share common prior on the distribution of the signals. Given the own signal and given the conditional distribution of the opponents' signals, each agent optimally chooses a strategy. Finally, suppose that there is common knowledge of rationality. According to Aumann (1987), a resulting outcome must be a correlated equilibrium. Due to the potential correlation between signals, a correlated equilibrium does not need to be a Nash equilibrium. Indeed, a situation of strategic interaction without signals seems artificial – signals are all around us in the real world, we can hardly avoid perceiving them and then condition our behavior on them in many situations. For example, in a financial market agents may receive signals on the value of some asset that are correlated. Several firms competing on a market for some consumption good may receive correlated information on the parametrization of the demand function. Consumers observe signals displaying information on the quality of some good when planning their consumption. Football fans perceive signals concerning the success of their favorite team within some tournament and condition their betting behavior on this information.

Rationality in the sense of Aumann (1987) requires that agents understand the underlying probability space and that this is commonly known. Here, the concept of correlated equilibrium is supported from the perspective of bounded rationality. We assume evolutionary dynamics on the game in which agents receive signals and show that states persisting over time in the presence of small mutations are correlated equilibria – and therefore may be non-Nash outcomes. Before the model is described in detail in the next section, I discuss the concept of evolution. In his survey on adaptive heuristics, Hart (2005) describes evolutionary dynamics as one extreme of bounded rationality: individuals' behavior is completely deterministic. The concept of evolutionary game theory originates from biology; see Dawkins (1990) or Björnerstedt and Weibull (1996) for socio-economic interpretations. Rationality is imposed on an aggregate level: strategies with higher relative success spread faster. Evolutionary game theory contributes by showing that *even if* agents are boundedly rational, certain outcomes predicted by concepts requiring rationality persist over time.

This paper characterizes the set of correlated equilibria that persist over time, given boundedly rational agents. The first part of the chapter assumes an exogenous and stationary process of signal generation. A set of signal contingent strategies is asymptotically stable with respect to convex monotonic dynamics[1], if it is a strict equilibrium set[2] of the game with signals. Given this selection, I consider endogenous signals. A signal generating process is robust, if no population has an incentive to manipulate the process, given equilibrium choice of the signal contingent strategies. I show for generic games that a signal contingent strategy is asymptotically stable and the signal generating processs is robust, if and only if the induced outcome is a strict Nash equilibrium. For the special case of the traditional example that has an equilibrium outcome with payoffs outside the convex hull of Nash-payoffs, the Chicken game, I show that a correlated equilibrium has robust signals if and only if it induces payoffs that lie *inside* the convex hull of Nash-payoffs.

The remainder of this section classifies this paper to the literature. It is well understood that the aggregate can display some rationality. Ritzberger and Weibull (1995) show that only strict Nash equilibria are asymptotically stable in the multipopulation replicator dynamics. For asymmetric games (animal conflicts), Selten (1980) shows that evolutionary stable strategies must be strict Nash equilibria. I make use of a concept introduced by Balkenborg (1994), *strict equilibrium set*. Each element of a strict equilibrium set

[1] Hofbauer and Weibull (1996)

[2] Balkenborg (1994)

is a Nash equilibrium, the set is closed under mixed best replies. Balkenborg and Schlag (2007) show asymptotic stability of restpoints within this set for general asymmetric games.[3] I rely on the concept of strict equilibrium set to characterize sets of correlated equilibria that are asymptotically stable. Lenzo and Sarver (2006) build up a model of subpopulations in which agents are matched according to a distribution over the set of subpopulations. They show that every interior[4] Lyapunov stable state is equivalent to a correlated equilibrium. Their model is inspired by the work of Mailath et al. (1997) who show that equilibria in a static model of local interactions coincide with correlated equilibria in the original game. In both models the correlation device is a "matching technology" with which agents of different populations are matched non-uniformly. I show that Lenzo and Sarver (2006) is a special case of the general model considered here, if one chooses a particular signal generating process. Cripps (1991) analyzes a two player model in which in a first stage nature randomly allocates row or column to the players and in a second step assigns one role of a finite set of roles to each player. He shows that an ESS in the symmetric game yields a distribution over the set of outcomes that is a strict correlated equilibrium. I abstain from analyzing the symmetrization and extend his model to dynamic analysis. Kim and Wong (2007) define evolutionary stable correlation for symmetric 2×2-games. They apply a special signal space, I discuss this matter after introducing the static model. Finally, we consider endogenous signals. I imagine situations, in which some agents exercise control over the generation of signals. Attention is not restricted to cheap talk games, situations in which a signal consists of a message of each player. In such a case, the player can manipulate a part of the signal. I consider players who can replace a signal entirely and

[3]Other setwise concepts defined for symmetric one population games are introduced by Balkenborg and Schlag (2001) and Thomas (1985). Cressman (2003) also elaborates on the strict equilibrium set.

[4]Interiority in the subpopulation model means that the state is interior for each subpopulation. It is more stringent than interiority in our case.

model this by considering the choice of probability distributions over the set of signal generating elementary events. I characterize the set of strategies in the original game for evolutionary dynamics of signalcontingent strategies, if no population has an incentive to manipulate the signal generating process.

This paper is structured as follows: section 2 sets up the model, section 3 lists propositions which are already available in the literature and which We transfer to this model to characterize asymptotically stable sets of correlated equilibria. Section 4 gives some examples. Section 5 shows the generalization of the subpopulation model of Lenzo and Sarver (2006), section 6 characterizes the set of stable outcomes that have a robust signal generating process and the appendix collects the remaining proofs.

2 Model

2.1 Static Model

We give a brief description of the model before we proceed to define it formally. At each point of time, nature randomly and independently draws a tuple of agents from a fixed set of infinite populations. A signal generating process reveals information to each of the active agents, this information may be correlated. Each agent chooses a strategy to interact with the other agents in a normal form game. Each agent is characterized by a *rule* that prescribes the strategic choice given the received signal. The resulting payoff determines wether the applied rule spreads in the population.

Let $\Gamma = \{\mathcal{N}, S, f\}$ be a finite game in normal form where $\mathcal{N} = \{1, \ldots, N\}$ is the set of population, $S = \times_{i \in \mathcal{N}} S^i$ and $S^i = \{s^i_1, \ldots, s^i_{m_i}\}$ is population i's finite set of pure strategies and $f : S \rightarrow \mathbb{R}^N$ is a utility or fitness function. Let $\Sigma^i = \Delta(S^i)$ be the set of probability measures on S^i and let $\hat{\Sigma}^i$ be a finite subset of Σ^i that contains the vertices of Σ^i. Let $\Sigma = \times_{i \in \mathcal{I}} \Sigma^i$ be the set of product measures on S, define $\hat{\Sigma} = \times_{i \in \mathcal{I}} \hat{\Sigma}^i$ accordingly. $\Delta = \Delta(S)$ is the set

of all probability measures on S. Denote by $s^{-i} = (s^1, \dots, s^{i-1}, s^{i+1}, \dots, s^N)$ a vector of strategies without the one of population i and by $S^{-i} = \times_{j \in \mathcal{N} \setminus \{i\}} S^j$ the Cartesian product of all but i's strategy spaces. Define $\Sigma^{-i} = \times_{j \in \mathcal{N} \setminus \{i\}} \Sigma^j$ and $\Delta^{-i} = \Delta\left(\times_{j \in \mathcal{N} \setminus \{i\}} S^j\right)$. I extend f to the space of mixed strategies, $f : \Sigma \to \mathbb{R}$, defined by $f^i(\sigma^i, \sigma^{-i}) = \sum_{s \in S} f^i(s) \prod_{j \in \mathcal{N}} \sigma^j(s^j) \ \forall\ i \in \mathcal{N}$.
A Strategy $s^i \in S^i$ is *dominated* if there exists some mixed strategy $\sigma^i \in \Sigma^i$ such that $f^i(s^i, \sigma^{-i}) \leq f^i(\sigma^i, \sigma^{-i}) \ \forall\ \sigma^{-i} \in \Sigma^{-i}$, with strict inequality for at least one σ^{-i}. If the inequality is strict for all σ^{-i}, s^i is *strictly dominated*. It is immediate to show that if s^i is dominated then there is a dominating strategy σ^i with $\sigma^i(s^i) = 0$.

A strategy tuple $\sigma = (\sigma^i, \sigma^{-i})$ is a Nash Equilibrium (NE) in Γ, if $\forall\ i \in \mathcal{N},\ f^i(\sigma^i, \sigma^{-i}) - f^i(s^i_h, \sigma^{-i}) \geq 0 \ \forall\ s^i_h \ \in S^i$.

Following Aumann (1987), I define a probability space $\langle \Omega, \mathcal{A}, P \rangle$ which generates signals (that are potentially correlated) on which agents can condition their strategic choices. Both, the original game Γ and the probability space constitute the primitives of my model. Assume Ω to be a nonempty and finite set of generic elements ω. Let $\mathcal{A}$ be the powerset of Ω and let $\{\mathcal{A}^i\}_{i \in \mathcal{N}}$ be a collection of partitions of Ω. $\mathcal{A}^i$ represents an information structure for population i; if nature draws an elementary event $\omega \in \Omega$, population i knows $A^i \in \mathcal{A}^i$ if and only if $\omega \in A^i$. Since for each population i there may be events ω, ω' that i cannot distinguish, the agents are not able to 'learn' P. Therefore, I need to assume P to be a *common prior* on $(\Omega, \mathcal{A})$. I regard P as an objective statistic environment. Without loss of generality, I assume that $P(\omega) > 0 \ \forall\ \omega \in \Omega$. All subjectivity enters the model via the set of partitions $\{\mathcal{A}^i\}_{i \in \mathcal{N}}$. I define the signaling structure $\mathcal{I} = \left\{ \langle \Omega, \mathcal{A}, P \rangle, \{\mathcal{A}^i\}_{i \in \mathcal{N}} \right\}$. I refer to an element $\omega \in \Omega$ as a complete description of a state of the world while an element $A^i \in \mathcal{A}^i$ is called a signal for the true state of the world. I assume that each agent has access to some private randomization device that allows for independent mixing, such that any mixed strategy in $\hat{\Sigma}^i$ is available. Wherever necessary, I assume that $\hat{\Sigma}^i$ is rich enough. Define $A^i(\omega) = \{A^i \in \mathcal{A}^i \mid \omega \in A^i\}$ the information set available to an agent in population i if nature draws ω. Throughout the model I make the assumption that the populations' fitnesses (represented by $f : S \Rightarrow \mathbb{R}^N$) do not depend on any ω. This is because I want to show that *even if information is payoff-irrelevant*, outcomes that are no Nash-equilibra of Γ can be stable under boundedly rational behavior, if agents perceive correlated signals.

Let a *rule* be a mapping from the set of states to strategies, $r^i : \Omega \to \hat{\Sigma}^i$. I assume for all i that r^i is $\mathcal{A}^i$-measurable, that is if for some ω, $r^i(\omega) = \sigma^i$ then $r^i(\omega') = \sigma^i \ \forall \ \omega' \in A^i(\omega)$. In words, agents cannot distinguish states that are in the same information set A. Define as $r^i_{s^i}(\omega)$ the probability with which an agent who uses rule r^i chooses strategy s^i given event ω, that is $r^i_{s^i}(\omega) = \sigma^i(s^i)$, where $\sigma^i = r^i(\omega)$. Denote the finite set of all rule-profiles by $\mathcal{R}$. I denote the share of agents in population i applying rule r^i by $\rho^i(r^i)$, the set of all shares in population i, ρ^i by $\Delta(\mathcal{R}^i)$, the set of all population shares ρ by $\Delta_{\mathcal{R}} = \underset{i \in \mathcal{N}}{\times} \Delta(\mathcal{R}^i)$. As before, I denote by r^{-i} the vector r without the element r^i, and by ρ^{-i} the vector ρ without the element ρ^i. Denote by $\mathcal{F} : \Delta_{\mathcal{R}} \to \mathbb{R}^N$ the expected fitness from the choice of the rules, where the components are defined as follows: $\mathcal{F}^i(\rho) = \sum_{\omega \in \Omega} P(\omega) \sum_{r \in \mathcal{R}} f^i(r(w)) \prod_{j \in \mathcal{N}} \rho^j(r^j)$. Given the signaling structure $\mathcal{I}$ and the normal form game Γ, I call $\mathcal{G}_{(\mathcal{I},\Gamma)} = \{\mathcal{N}, \mathcal{R}, \mathcal{F}\}$ the *expansion* of Γ.

A rule $r^i \in \mathcal{R}^i$ is strictly dominated if there exists some population share $\rho^i \in \Delta(\mathcal{R}^i)$ such that $\mathcal{F}^i(r^i, \rho^{-i}) < \mathcal{F}^i(\rho^i, \rho^{-i}) \ \forall \ \rho^{-i} \in \Delta(\mathcal{R}^{-i})$.

To get a flavor of the model, I begin the analysis with a very straight forward result that is helpful to show the extinction of dominated strategies.

Lemma 1

If strategy s^i is strictly dominated in Γ by some mixed strategy $\hat{\sigma}^i \in \hat{\Sigma}^i$, any rule r^i with $r^i_{s^i}(\omega) > 0$ for some ω is strictly dominated in the game $\mathcal{G}_{(\mathcal{I},\Gamma)}$, if $\hat{\Sigma}^i$ is rich enough.

Proof

Assume without loss of generality that $\hat{\sigma}^i(s^i) = 0$. Define for each $\omega \in \Omega$ the new rule $\hat{r}^i_{\tilde{s}^i}(\omega) = r^i_{\tilde{s}^i}(\omega) + r^i_{s^i}(\omega) \cdot \hat{\sigma}^i(\tilde{s}^i)\ \forall\ \tilde{s}^i \neq s^i$ and $\hat{r}^i_{s^i}(\omega) = 0$. It is easy to verify that $\hat{r}^i(\omega) \in \Sigma^i\ \forall\ \omega$, however I need to assume that $\hat{\Sigma}^i$ is rich enough such that $\hat{r}^i(\omega) \in \hat{\Sigma}^i\ \forall\, \omega$. For convenience I define $f^i(r^i(\omega), \rho^{-i}(\omega)) = \sum_{r^{-i}} f^i(r^i(\omega), r^{-i}(\omega)) \prod_{j \neq i} \rho^j(r^j)$. We then have $\forall\ \rho^{-i} \in \Delta_{\mathcal{R}^{-i}}$:

$$\begin{aligned}\mathcal{F}^i(\hat{r}^i, \rho^{-i}) \;&=\; \mathcal{F}^i(r^i, \rho^{-i}) + \\ &\underbrace{\sum_{\substack{\omega \in \Omega \\ r^i_{s^i}(\omega) > 0}} P(\omega) r^i_{s^i} \left[f^i(\hat{\sigma}^i, \rho^{-i}(\omega)) - f^i(s^i, \rho^{-i}(\omega)) \right]}_{>0}\end{aligned}$$

□

A strategy s^i is iteratively strictly dominated in Γ if there exists a sequence $\{s^{i_t}, \Gamma_t\}_{t=0}^n$ such that s^{i_t} is strictly dominated in Γ_t, where Γ_t is obtained from Γ_{t-1} by removing $s^{i_{t-1}}$ from i_{t-1}'s set of pure strategies in Γ_{t-1}, $\Gamma = \Gamma_0$ and $s^i = s^{i_0}$. The same definition applies for a rule r^i in the game $\mathcal{G}$.

As a consequence of Lemma 1 one can state an analogous statement for iteratively strictly dominated rules:

Lemma 2

If strategy s^i is iteratively strictly dominated in Γ by some mixed strategy $\hat{\sigma}^i \in \hat{\Sigma}^i$, any rule r^i with $r^i_{s^i}(\omega) > 0$ for some ω is iteratively strictly dominated in the game $\mathcal{G}_{(\mathcal{I},\Gamma)}$, if $\hat{\Sigma}^i$ is rich enough.

Definition CORRELATED EQUILIBRIUM (*c.e.*)
Given $\mathcal{I}$, a *correlated equilibrium* in Γ is a mixed rule $\rho \in \Delta_{\mathcal{R}}$ such that for all i, $\mathcal{F}^i(\rho) \geq \mathcal{F}^i(\tilde{\rho}^i, \rho^{-i})\ \forall\ \tilde{\rho}^i \in \Delta(\mathcal{R}^i)$. A *c.e.* is *strict*, if inequalities hold strictly for all $\tilde{\rho}^i \neq \rho^i$ and $i \in \mathcal{N}$.

Here, an equilibrium is a point in the set of mixed rules.

Definition INDUCED DISTRIBUTION
Let $\rho \in \Delta_{\mathcal{R}}$ be some distribution over the set of rules. Then $\mathcal{I}$ and ρ induce a distribution over the set of outcomes. Define $\forall\ s \in S$:

$$\lambda(s) = \sum_{\omega} P(\omega) \prod_{i \in \mathcal{N}} \sum_{r^i \in \mathcal{R}^i} \rho^i(r^i) \cdot r^i_{s^i}(\omega)$$

Definition CORRELATED EQUILIBRIUM DISTRIBUTION (*c.e.d.*)
A distribution $\lambda \in \Delta$ induced by $\mathcal{I}$ and a *c.e.* ρ is a *correlated equilibrium distribution.*

Given some expanded game $\mathcal{G}_{(\mathcal{I},\Gamma)}$, there may exist multiple *c.e.* ρ, some being strict and some other being non-strict. See Example 4.1 .

Fix some signal generating process $\mathcal{I}$. Then, a mixed rule $\rho \in \Delta_{\mathcal{R}}$ is a *c.e.* in Γ, if and only if ρ is a Nash equilibrium of expanded game $\mathcal{G}_{(\mathcal{I},\Gamma)}$. Note the generality of the signal space. Consider instead the special case $\Omega = S$ and $\mathcal{A}^i = \{\{s^i \times S^{-i}\}_{s^i}\}$, that is each population gets a recommendation to play a particular strategy. Kim and Wong (2007) use this signal space. With these direct signals, it is optimal to follow the recommendation if the signals are distributed according to a *c.e.d.* . However, two problems come with this approach: firstly, even if the distribution of signals P is a *c.e.d.*, it might still be an equilibrium if the agents deviate from the recommendation (see example 4.1). Secondly, if one pins down a special signal generating process, one can always construct a meta game in which agents can condition their choice of rules on some extra signals they might receive. The general formulation of the signal space includes such extra signals.[5]

Definition Evolutionary Stability (Swinkels (1992))
$\rho \in \Delta_{\mathcal{R}}$ is evolutionary stable in $\mathcal{G}_{(\mathcal{I},\Gamma)}$, if $\exists\, \epsilon' > 0 :\ \forall\, \epsilon \in (0, \epsilon')$ and $\tilde{\rho} \in \Delta_{\mathcal{R}}$

$$\mathcal{F}^i(\tilde{\rho}_i, (1-\epsilon)\rho_{-i} + \epsilon\tilde{\rho}_{-i}) \geq \mathcal{F}^i(\rho_i, (1-\epsilon)\rho_{-i} + \epsilon\tilde{\rho}_{-i}) \Rightarrow \tilde{\rho} = \rho \ .$$

It follows immediately that a rule is evolutionary stable if and only if it is a strict Nash equilibrium of $\mathcal{G}$.[6] Note that the above definition is for multi-population (and asymmetric) games.

The definitions of evolutionary stable *sets* by Thomas (1985), Balkenborg and Schlag (2001) and Cressman (2003) are all specified for symmetric one population games. Therefore I do not list them but state a concept for general asymmetric games:

Definition Strict Equilibrium set (SEset) (Balkenborg (1994))
A nonempty set $F \subset \Delta_{\mathcal{R}}$ is a *strict equilibrium set* if it is a set of Nash equilibria of $\mathcal{G}$ that is closed under mixed-rule best replies by each population i, i.e. if for some $\rho \in F$, $(\tilde{\rho}^i, \rho^{-i}) \in F$ whenever $\mathcal{F}^i(\tilde{\rho}^i, \rho^{-i}) = \mathcal{F}^i(\rho)$ for each population i.

Such a set does not need to exist, see Example 5.4.

[5] I am grateful to Andreu Mas-Colell to scrutinize the special signal generating process in an earlier version of this paper.

[6] See Swinkels (1992), Theorem 2.

Theorem (Balkenborg and Schlag (2007)[7], Cressman (2003)[8])
If F is an SESet of $\mathcal{G}$, then F is a finite union of faces of $\Delta_{\mathcal{R}}$. In particular, F is closed and contains at least one pure rule $r \in \mathcal{R}$.

2.2 Dynamic Model

We assume that at each point in time, agents update their behavior such that the population shares $\rho = (\rho^1, \ldots, \rho^n)$ change according to the regular differential equation

$$\dot{\rho}^i(r^i) = g^i(r^i, \rho) \cdot \rho^i(r^i),\ \forall\ r^i \in \mathcal{R}^i,\ \forall\ i \in \mathcal{N}, \tag{1}$$

where regularity presumes that $g = \times_{i\in\mathcal{N}} g^i$ is (i) Lipschitz continuous on $\Delta_{\mathcal{R}} = \times_{i\in\mathcal{N}} \Delta(\mathcal{R}^i)$ and (ii) $g^i(\cdot, \rho) \cdot \rho^i = 0\ \forall \rho \in \Delta_{\mathcal{R}}$. By the Picard-Lindelöf Theorem[9], there exists a unique solution $\hat{\rho}(\cdot, \rho)$ for each initial condition $\rho \in \Delta_{\mathcal{R}}$.

The following definition is taken from Hofbauer and Weibull (1996):[10]
(1) is *convex monotonic* (CM), if it satisfies

$$\mathcal{F}^i(r^i, \rho^{-i}) < \mathcal{F}^i(\rho^i_k, \rho^{-i}) \Rightarrow g^i(r^i, \rho) < g^i(\cdot, \rho) \cdot \rho^i_k\ \forall\ i \in \mathcal{N}\ .$$

(1) is the *replicator dynamics*, if

$$g^i(r^i, \rho) = \mathcal{F}^i(r^i, \rho^{-i}) - \mathcal{F}^i(\rho)\ \forall\ r^i \in \mathcal{R}^i \text{ and } i \in \mathcal{N}.$$

Clearly, the replicator dynamic is convex monotonic.
Define $\rho^+ = \{\rho' \in \Delta_{\mathcal{R}} \mid \exists\ t\ \in \mathbb{R}_+, \rho' = \hat{\rho}(t, \rho)\}$, as the subset of $\Delta_{\mathcal{R}}$ that is

[7] Proposition 2, p.299
[8] Theorem 3.1.2, p.71
[9] A function $\phi : X \to \mathbb{R}^k$, where $X \subset \mathbb{R}^k$, is (locally) Lipschitz continuous if for every compact subset $C \subset X$ there exists some real number λ such that it holds for all $x, y \in C$: $||\phi(x) - \phi(y)|| \leq \lambda||x - y||$. If $X \subset \mathbb{R}^k$ is open and the vector field $\phi : X \to \mathbb{R}^k$ is Lipschitz continuous, then the system $\dot{x} = \phi(x)$ has a unique solution $\hat{x}(\cdot, x^0) : T \to X$ through every state $x^0 \in X$. Moreover, $\hat{x}(t, x^0)$ is continuous in $t \in T$ and $x^0 \in X$. (Weibull (1995) pp.232)
[10] Convex monotonicity is implied by aggregate monotonicity, it is not implied by and does not imply monotonicity (both Samuelson and Zhang (1992), Definition 3, p.369)

reached if the dynamics start at ρ.

Definition STABILITY

A closed set $\Lambda \subseteq \Delta_{\mathcal{R}}$ is *Lyapunov stable* if for every neighborhood $\mathcal{U}'$ of Λ there exists a neighborhood $\mathcal{U}''$ such that $\rho^+ \subset \mathcal{U}' \; \forall \; \rho \in \mathcal{U}'' \cap \Delta_{\mathcal{R}}$.

A closed set $A \subseteq \Delta_{\mathcal{R}}$ is *asymptotically stable* if it is Lyapunov stable and if there exists a neighborhood $\mathcal{U}$ of A such that $\hat{\rho}(t,\rho) \underset{t\to\infty}{\longrightarrow} A$ for all $\rho \in \mathcal{U} \cap \Delta_{\mathcal{R}}$.

3 Propositions

This section collects the propositions.

Proposition 1

Let g be convex monotonic. If $F \subset \Delta_{\mathcal{R}}$ is a Lyapunov stable set of rest points, then each $\rho \in F$ is a *c.e.* .

Proof: see Appendix.

The converse of Proposition 1 is not true in general:
Fix some $\mathcal{I}$ and *c.e.* ρ in which for a population i, the rule $r^i \in \mathcal{R}^i : \rho^i(r^i) > 0$ is weakly dominated by some mixed rule $\tilde{\rho}^i$. Then there exists a neighborhood $\mathcal{U}$ of ρ such that $\forall \; \tilde{\tilde{\rho}} = (\rho^i, \tilde{\tilde{\rho}}^{-i}) \in \mathcal{U}, \tilde{\tilde{\rho}}^{-i} \in \mathrm{int}(\Delta_{\mathcal{R}^{-i}})$ it holds that $\mathcal{F}^i(r^i, \tilde{\tilde{\rho}}^{-i}) < \mathcal{F}^i(\tilde{\rho}^i, \tilde{\tilde{\rho}}^{-i})$. Therefore, for some $\tilde{\tilde{\rho}}^{-i}$ there exists some $r^i_h \in \mathcal{R}^i$ with $\tilde{\rho}^i(r^i_h) > 0$ such that $\mathcal{F}^i(r^i_h, \tilde{\tilde{\rho}}^{-i}) > \mathcal{F}^i(r^i, \tilde{\tilde{\rho}}^{-i})$. Since (1) is convex monotonic, $g^i(r^i_h, \tilde{\tilde{\rho}}) > g^i(r^i, \tilde{\tilde{\rho}})$, contradicting Lyapunov stability.

The next propositions specify the relationship of asymptotic stability and correlated equilibrium:

Proposition 2 (cf Balkenborg and Schlag (2007), Theorem 6 and Cressman (2003), Theorem 4.5.3)
If a non-empty set $F \subset \Delta(\mathcal{R})$ of rules ρ is an asymptotically stable set of rest points under the standard replicator dynamic, F is a SEset.

Balkenborg and Schlag (2007) and Cressman (2003) actually show equivalence, if (1) is the replicator dynamic. Balkenborg and Schlag (2007) also show the reverse for a wide class of other dynamics. I show the reverse for the distinct class of convex monotone dynamics.[11]

Proposition 3
Let (1) be convex monotonic. If a set F is a SEset, then F is an asymptotically stable set of rest points.

Proof: see Appendix A.

If the process does not start in the interior of $\Delta_{\mathcal{R}}$, there may exist some $\rho_0 \in \Delta_{\mathcal{R}}$ such that $\lambda(\hat{\rho}(t, \rho_0))$ is not a *c.e.d.* for all $t > 0$, even if an asymptotically stable set exists.

Proposition 4 (Hofbauer and Weibull (1996) Theorem 1)
If a rule $r^i \in \mathcal{R}^i$ is iteratively strictly dominated and the process starts in the interior of the rulespace and if the selection dynamics (1) is convex monotonic, r^i gets eliminated.

[11] In Proposition 13 Balkenborg and Schlag (2007) demand (A) that $g^i(r^i, \rho) \geq 0$ whenever r^i is a best response to ρ^{-i}, (B) that $g^i(r^i, \rho) > 0$ whenever r^i is a best response to ρ^{-i} but ρ^i is not and (C) that $g^i(r^i, \rho) < 0$ whenever ρ^i is a best response to ρ^{-i} but r^i is not. Neither does convex monotonicity imply (A),(B) and (C) nor vice versa. Consider some ρ, $\tilde{\rho}^i$ and r^i such that $\mathcal{F}^i(\tilde{\rho}^i, \rho^{-i}) > \mathcal{F}^i(\rho) > \mathcal{F}^i(r^i, \rho^{-i})$. (A),(B),(C) imply that $g^i(r^i, \rho^{-i}) = 0$. From regularity I have $\rho^i \cdot g^i(\cdot, \rho) = 0$, hence g cannot be convex monotonic.

We do not give a statement whether the induced distribution over outcomes converges. Viossat (2004) shows for symmetric 3×3-games that the multipopulation replicator dynamics eliminates all strategies not used in a correlated equilibrium (with interior initial conditions), however Viossat (2007) gives an example of a class of symmetric 4×4 games for which the replicator dynamics eliminates *all* strategies used in correlated equilibrium along interior solutions. Lemma 2 and Proposition 4 allow me to pin down a weaker result, namely to rule out iteratively strictly dominated outcomes in the induced distribution in the long run.

Corollary

If the process starts in the interior of the rulespace and if the selection dynamics (1) is convex monotonic, then $\lim_{t \to \infty} \lambda(t)$ attaches zero probability to outcomes s that involve strategies that are iteratively strictly dominated, if $\hat{\Sigma}$ is rich enough.

4 Examples

This section demonstrates how the model can be applied to various examples. The examples are complementary to the propositions of the preceeding section.

4.1 A Coordination Game

This example illustrates that one signal generating process $\mathcal{I}$ allows for multiple stable rules r, r' that do not induce the same distribution λ over outcomes S. Even if the signal generating process $\mathcal{I}$ itself is a distribution over S and is regarded as a 'recommendation', other strategy choices can well be stable.

Let the game Γ be defined by $\mathcal{N} = \{1, 2\}$, $S = \{u, d\} \times \{l, r\}$ and

	l	r
$f = u$	(1,1)	(0,0)
d	(0,0)	(1,1)

.

We specify $\mathcal{I}$ with $\Omega = S$, $\mathcal{A}^1 = \big\{\{ul, ur\}, \{dl, dr\}\big\}$, $\mathcal{A}^2 = \big\{\{lu, ld\}, \{ru, rd\}\big\}$. A rule for population 1 (row) assigns a strategy for the first and the second element of $\mathcal{A}^1$ respectively. UD means "choose u if $\omega \in \{ul, ur\}$ and choose d if $\omega \in \{dl, dr\}$". I analogously denote the rules of population 2.

$\mathcal{F}_i =$

	LL	LR	RL	RR
UU	1	$P(\{ul, dl\})$	$P(\{ur, dr\})$	0
UD	$P(\{ul, ur\})$	$P(\{ul, dr\})$	$P(\{ur, dl\})$	$P(\{dl, dr\})$
DU	$P(\{dl, dr\})$	$P(\{dl, ur\})$	$P(\{ul, dr\})$	$P(\{ul, ur\})$
DD	0	$P(\{ur, dr\})$	$P(\{ul, dl\})$	1

for $i = 1, 2$

The rules (UU, LL) and (DD, RR) are the strict correlated equilibria that correspond to the pure Nash equilibria of the original game for any P with full support. Consider P to be a uniform measure over Ω. Then, (UD, LR) is a non-strict *c.e.*. For general P, the induced distribution λ does not need to coincide with P, although it still may be a *c.e.d* . Suppose $P(\{ul\}) = p$ and $P(\{dr\}) = 1 - p$. The pair (DU, RL) is a strict *c.e.* and induces the following distribution over the set of outcomes: $\lambda(ul) = 1 - p$ and $\lambda(dr) = p$.

4.2 Chicken

A non-Nash outcome may be asymptotically stable.

Consider the "chicken game" originally presented in Aumann (1974):

	l	r
$\Gamma = u$	(6,6)	(2,7)
d	(7,2)	(0,0)

Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let $\mathcal{A}^1 = \big\{\{\omega_1, \omega_2\}, \{\omega_3\}\big\}$ and $\mathcal{A}^2 = \big\{\{\omega_1\}, \{\omega_2, \omega_3\}\big\}$, let $P(\omega) \equiv \frac{1}{3}$. Given this $\mathcal{I}$, the resulting expanded game is

$\mathcal{G}_{(\Gamma,\mathcal{I})} =$

	LL	RL	LR	RR
UU	(6,6)	$(4\frac{2}{3}, 6\frac{1}{3})$	$(3\frac{1}{3}, 6\frac{2}{3})$	(2,7)
UD	$(6\frac{1}{3}, 4\frac{2}{3})$	(5,5)	$(2\frac{2}{3}, 4\frac{1}{3})$	$(1\frac{1}{3}, 4\frac{2}{3})$
DU	$(6\frac{2}{3}, 3\frac{1}{3})$	$(4\frac{1}{3}, 2\frac{2}{3})$	(3,3)	$(\frac{2}{3}, 2\frac{1}{3})$
DD	(7,2)	$(4\frac{2}{3}, 1\frac{1}{3})$	$(2\frac{1}{3}, \frac{2}{3})$	(0,0)

(UD, LR) is a strict *c.e.*, hence it is a singleton evolutionary stable rule and therefore asymptotically stable in any convex monotonic dynamic. As is well known, the payoffs generated by (UD, LR) lie outside the convex hull of the Nash equilibria of the original game Γ.

4.3 A SEset of correlated equilibria

The chicken example above shows that a single outcome can be asymptotically stable producing payoffs that lie outside the convex hull of the Nash equilibrium outcomes. This example does the same for a *set* of outcomes. Consider the following game

$\Gamma =$

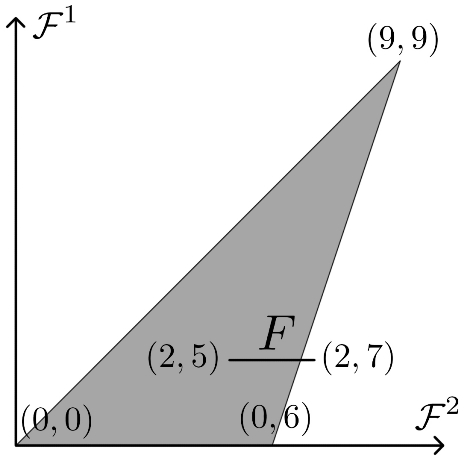
	l	r
u	(0,0)	(0,6)
m	(3,-6)	(0,0)
d	(9,9)	(-3,6)

The pure Nash equilibria are (u, r), (m, r) and (d, l), the unique mixed Nash equilibrium is $(\sigma^1(m) = \frac{1}{3}, \sigma^1(d) = \frac{2}{3}, \sigma^2(l) = \frac{1}{3})$. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{A}^1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \mathcal{A}^2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, $P(\omega) \equiv 3$. Each population has two signals, the row population therefore has 9 rules, column has 4 rules. The payoff matrix of the expanded game is given by

$\mathcal{F} =$

	LL	LR	RL	RR
UU	(0,0)	(0,4)	(0,2)	(0,6)
UM	(1,-2)	(0,2)	(1,0)	(0,4)
UD	(3,3)	(-1,4)	(3,5)	(-1,6)
MU	(2,-4)	(1,0)	(0,2)	(0,2)
MM	(3,-6)	(1,-2)	(2,-4)	(0,0)
MD	(5,-1)	(0,0)	(4,1)	(-1,2)
DU	(6,6)	$(\mathbf{2}, \mathbf{7})$	(2,5)	(-2,6)
DM	(7,4)	$(\mathbf{2}, \mathbf{5})$	(3,3)	(-2,4)
DD	(9,9)	(1,7)	(5,8)	(-3,6)

.



In the figure, the shaded triangle is the convex hull of the Nash equilibrium payoffs of Γ, the thick line connecting the points $(2, 5)$ and $(2, 7)$ represents the SEset $F = \{\rho \in \Delta(\mathcal{R}) \mid \rho^1(DM) = 1 - \rho^1(DU), \rho^2(LR) = 1\}$, which is not fully contained in the convex hull.

4.4 Matching Pennies

Non-existence of SEset.

Consider the original two population game with strategies $\{h,t\}^2$ and payoff matrix

		h	t
Γ =	h	(1,-1)	(-1,1)
	t	(-1,1)	(1,-1)

Let the information structure be given by a singleton $\Omega = \{\omega\}$, in other words let there be no signals. Therefore the rules coincide with the strategies. The set of Nash equilibria of $\mathcal{G}$ has only one element which is not strict (and hence is not closed under mixed-rule best replies). In fact, any information structure $\mathcal{I} = \left\{ \langle \Omega, \mathcal{A}, P \rangle, \{\mathcal{A}^i\}_{i\in\mathcal{N}} \right\}$ that has a common prior induces an expanded game $\mathcal{G}_{\{\mathcal{I},\Gamma\}}$ which has no SEset. If instead of P there would exist some subjective priors $\{P^i\}_{i\in\mathcal{I}}$ with $P^i : \mathcal{A}^i \to \mathbb{R}$ violating the common prior assumption, it would be straightforward to construct an expansion of Γ with strict equilibria, see Aumann and Dreze (2005), example 6.5 .

5 Subpopulations

In this section, I illustrate that the model of Lenzo and Sarver (2006) can be expressed as a special case of the general formulation of the model presented in this paper. I give a special interpretation of the signals: a signal assigns one of finitely many subpopulations to each agent. Let each population i have a set of subpopulations $M^i = \{m^i_1, \ldots, m^i_{|M^i|}\}$, defining $M = \underset{i\in\mathcal{N}}{\times} M^i$. Denote by $x^{m^i}_{s^i}$ the share of agents in subpopulation m^i that choose strategy s^i. Let $\eta \in \Delta(M)$ be a probability distribution over M, with $\eta(m^i, \cdot) > 0\ \forall m^i \in M^i$ and $i \in \mathcal{N}$. Note that this distribution may be correlated and that there may be matches $m \in M$ that receive zero-probability.

We show that given a game Γ, for any M, η with state x, there is an $\mathcal{I}$ and a state ρ such that the induced distributions are the same. One can represent any state x of the subpopulations model by a state ρ of our model if one gives a particular specification of the signalling structure. Furthermore we show that ρ needs not to be unique and that the dynamic properties of x and ρ need not be the same.

Let $\Omega = M$, $\mathcal{A}^i = \left\{\{m^i \times M^{-i}\}_{m^i \in M^i}\right\}$, $P = \eta$ and $\rho^i(r^i) = \prod_{m^i \in M^i} x^{m^i}_{r^i(m^i)}$.[12] Firstly we show that $\sum_{r^i \in \mathcal{R}^i} \rho^i(r^i) = 1$. Note that $\sum_{\substack{r^i \in \mathcal{R}^i \\ r^i(m^i_1)=s^i}} \prod_{\substack{m^i \in M^i \\ m^i \neq m^i_1}} x^{m^i}_{r^i(m^i)} = \sum_{\substack{r^i \in \mathcal{R}^i \\ r^i(m^i_1)=s^i_h}} \prod_{\substack{m^i \in M^i \\ m^i \neq m^i_1}} x^{m^i}_{r^i(m^i)} \ \forall\ s^i, s^i_h \in S^i$.

$$\begin{aligned}
\sum_{r^i \in \mathcal{R}^i} \rho^i(r^i) &= \sum_{r^i \in \mathcal{R}^i} \prod_{m^i \in M^i} x^{m^i}_{r^i(m^i)} \\
&= \sum_{s^i \in S^i} \sum_{\substack{r^i \in \mathcal{R}^i \\ r^i(m^i_1)=s^i}} x^{m^i_1}_{s^i} \prod_{\substack{m^i \in M^i \\ m^i \neq m^i_1}} x^{m^i}_{r^i(m^i)} \\
&= \underbrace{\left(\sum_{s^i \in S^i} x^{m^i_1}_{s^i}\right)}_{=1} \sum_{\substack{r^i \in \mathcal{R}^i \\ r^i(m^i_1)=s^i_h}} \prod_{\substack{m^i \in M^i \\ m^i \neq m^i_1}} x^{m^i}_{r^i(m^i)} \\
&\ \vdots \\
&= \sum_{s^i \in S^i} \sum_{\substack{r^i \in \mathcal{R}^i \\ r^i(m^i_1) = s^i_h \\ \vdots \\ r^i(m^i_{|M^i|-1}) = s^i_h \\ r^i(m^i_{|M^i|}) = s^i}} \prod_{\substack{m^i \in M^i \\ m^i \neq m^i_1 \\ \vdots \\ m^i \neq m^i_{|M^i|-1} \\ m^i = m^i_{|M^i|}}} x^{m^i_{|M^i|}}_{s^i} \\
&= \sum_{s^i \in S^i} x^{m^i_{|M^i|}}_{s^i} \cdot 1
\end{aligned}$$

[12]More precisely: $r^i(m^i) = r^i(m^i, m^{-i})$ for some m^{-i} ($r^i(m^i, m^{-i})$ has the same value $\forall\ m^{-i}$).

To calculate $\lambda(s)$ for some $s \in S$:

$$\begin{aligned}
\lambda(s) &= \sum_{\omega\in\Omega} P(\omega) \prod_{i\in\mathcal{N}} \sum_{\substack{r^i\in\mathcal{R}^i\\ r^i(\omega)=s^i}} \rho^i(r^i) \\
&= \sum_{m\in M} \eta(m) \prod_{i\in\mathcal{N}} \sum_{\substack{r^i\in\mathcal{R}^i\\ r^i(m^i)=s^i}} \prod_{m_k^i\in M^i} x_{r^i(m_k^i)}^{m_k^i} \\
&= \sum_{m\in M} \eta(m) \prod_{i\in\mathcal{N}} x_{s^i}^{m^i} \sum_{\substack{r^i\in\mathcal{R}^i\\ r^i(m^i)=s^i}} \prod_{\substack{m_k^i\in M^i\\ m_k^i\neq m^i}} x_{r^i(m_k^i)}^{m_k^i}
\end{aligned}$$

From the third line of the calculation of $\sum\limits_{r^i\in\mathcal{R}^i} \rho^i(r^i)$ we know that $\sum\limits_{\substack{r^i\in\mathcal{R}^i\\ r^i(m^i)=s^i}} \prod\limits_{\substack{m_k^i\in M^i\\ m_k^i\neq m^i}} x_{r^i(m_k^i)}^{m_k^i} = 1$ and have the desired result that the distributions over outcomes are the same. However, there is no one-to-one mapping from one model to the other model. Consider the following simple example with $M = M^1 \times M^2 = \{m_1^1, m_2^1\} \times \{m_1^2, m_2^2\}$, $S = \{s_1^1, s_2^1\} \times \{s_1^2, s_2^2\}$, $\eta(m) = \frac{1}{4}\ \forall\, m \in M$, $x_{s_1^i}^{m_1^i} = x_{s_2^i}^{m_2^i} = 1$, in words: for each population i all agents of subpopulation 1 choose their strategy 1 and all agents from subpopulation 2 choose their strategy 2. If $\Omega = M$, $\mathcal{A}^1 = \big\{\{(m_1^1, m_1^2), (m_1^1, m_2^2)\}, \{(m_2^1, m_1^2), (m_2^1, m_2^2)\}\big\}$ (and $\mathcal{A}^2$ analogous), $P = \eta$ and ρ as constructed above, there is probability mass one on the rule $r^i : r^i(m_1^i) = s_1^i, r^i(m_2^i) = s_2^i$. Alternatively, but for the same Ω, $\mathcal{A}^i$, P, one could assign $\tilde{\rho}^i(r^i) = \frac{1}{4}\ \forall\ r^i \in \mathcal{R}^i$. Both ρ and $\tilde{\rho}$ induce the same distribution λ but while ρ is pure, $\tilde{\rho}$ is completely mixed and therefore ρ and $\tilde{\rho}$ have different dynamic properties.

6 Robust Signals

Until now, it was assumed that the signal generating process is stationary. This is plausible, if the signals originate from an object that is completely exogenous, i.e. if they are independent from interaction – a somehow polar case. The other polar case would be that the agents themselves can choose messages that serve as signals. I regard situations in which one population i can alter the complete signal and consider the case in which population i can choose a particular probability distribution P. I offer the following interpretation: suppose some institution determines P. Every population knows the design of the institution and therefore has access to the information how the institution determines P. Population i can influence the institution, because – for example – some key positions within the institution are held by members of population i. In this section we derive conditions such that population i does not have an incentive to change P in a stable state ρ. Suppose nature draws a certain elementary event $\omega \in \Omega$. Then, for a given distribution of rules $\rho = \{\rho^i\}_i$, population i's expost payoff is $f^i(\rho(\omega))$. Population i has an incentive to change P if there is some other event ω' with $f^i(\rho(\omega)) < f^i(\rho(\omega'))$. This leads to the following definition:

Definition *Robust to Manipulation*
Given ρ, $P \in \Delta(\Omega)$ is *robust to manipulation* if for all populations i

$$P(\omega) > 0 \Rightarrow f^i(\rho(\omega)) \geq f^i(\rho(\omega')) \ \forall \omega' \in \Omega \ .$$

If a distribution P is robust to manipulation given ρ, no population (regardless wether it has the capability to change P or not) has an incentive to manipulate P. We do not demand that any population *can* change P. We characterize those pairs (P, ρ) such that no population *wants* to change P given ρ. Nevertheless, we have implicitly assumed some constrained reasoning. Suppose there is some mapping $g : \Delta(\Omega) \rightrightarrows \Delta(\mathcal{R})$ such that given distribution P, agents play an equilibrium $\rho \in g(P)$. In the approach above, agents believe g to be singlevalued and constant and agents compare the ex-post payoffs. Alternatively, one could argue that population i does not have an incentive to change P to P' if $\mathcal{F}^i_P(\rho) \geq \mathcal{F}^i_{P'}(\rho') \ \forall \ \rho' \in g(P')$. That is, no population has an incentive to change P, if P maximizes ex-ante payoffs for all equilibrium choices ρ', where the equilibrium choice well depends on the distribution P. The consequences of this definition are more exclusive in the sense that it is easy to find a game such that no stable state ρ has a robust distribution P.[13]

6.1 Results

Consider again the general setting, with $\Gamma = \{\mathcal{N}, S, f\}$,
$\mathcal{I} = \{\{\Omega, \mathcal{P}(\Omega), P\}, \{\mathcal{A}\}_{i\in\mathcal{N}}\}$ yielding the expanded game $\mathcal{G}_P = \{\mathcal{N}, \mathcal{R}, \mathcal{F}_P\}$ (making the dependence on P explicit). Define $\Delta_P^{\sim} \subset \Delta(\mathcal{R})$ as the set of rules ρ such that P is robust to manipulation. Define $\Delta_P^{CE} \subset \Delta(\mathcal{R})$ as the set of correlated equilibria given P. Our first result is immediate:

Proposition 5
$\Delta_P^{\sim} \cap \Delta_P^{CE} \neq \emptyset \;\forall\; P \;\in \Delta(\Omega)$.

Proof
Consider a Nash equilibrium $\sigma \in \Delta(S)$ of the original game Γ. Define the rule ρ such that for all $i \in \mathcal{N}$ and $s^i \in S^i$, $\rho^i(r^i) = \sigma^i(s^i)$ for $r^i : r^i(\omega) \equiv s^i$. Clearly, ρ is a correlated equilibrium of Γ given P, hence $\rho \in \Delta_P^{CE}$. No population conditions the choice of strategies on signals, hence $f^i(\rho(\omega)) = f^i(\sigma)\ \forall \omega \in \Omega$ and therefore no population has an incentive to manipulate the generation of the signals. Hence, P is robust given ρ, $\rho \in \Delta_P^{\sim}$. $\square$

We argue that there always is a trivial correlated equilibrium in which agents choose Nash equilibrium strategies ignoring any signals. Since all agents ignore any signals, no agent has an incentive to manipulate the signals.

We cannot give a full characterization of $\Delta_P^{\sim} \cap \Delta_P^{CE}$, the set of correlated equilibria given P that induce P to be robust against manipulation. How-

[13] For example, the chicken game, the battle of the sexes game,...

ever, I suspect that it is a subset of rules that induce a distribution λ on the set of outcomes S that lies in the convex hull of Nash equilibria (see the Chicken Game example below). I leave this characterization to future work.

We can give a full characterization of $\Delta_P^{\sim} \cap F_P$, F_P being an asymptotically stable set of rules given P if we impose a further assumption on the payoffs of the original game Γ. Suppose Γ has the generic property that $f_i(s) \neq f_i(s') \; \forall s, s' \in S, s \neq s'$. Then all asymptotically stable sets are singleton. In this case, I can state that a probability measure P is robust to manipulation given ρ if and only if ρ puts probability one on rules that choose one strict Nash equilibrium.

Proposition 6

Suppose for each population i, the frequency $\rho^i(r^i)$ updates according to (1) and that (1) is convex monotonic. Suppose further that the original game Γ is generic. A set $F_P \subset \Delta(\mathcal{R})$ is asymptotically stable given (1) and a distribution P with full support on Ω, and P is robust given a $\rho \in F_P$, if and only if ρ attaches probability one to a rule that maps all signals to the same strict Nash equilibrium.

Proof

Suppose F_P is asymptotically stable and suppose P is robust given any $\rho \in F_P$. According to Proposition 2, F_P is a SEset, from genericity follows that F_P is singleton, i.e. $\rho = F_P$ puts probability one to a strict correlated equilibrium $r \in \mathcal{R}$. Because no population has an incentive to manipulate P given r, it must be that $f^i(r(\omega)) = f^i(r(\omega')) \; \forall \omega, \omega' \in \Omega, \forall \; i$. Since Γ is generic, it must be $r(\omega) = r(\omega') = s \; \forall \; \omega, \omega' \in \Omega$ and some $s \in S$. Since r is a strict correlated equilibrium, $\mathcal{F}^i(r(\omega)) > \mathcal{F}^i(\tilde{r}^i(\omega), r^{-i}(\omega)) \; \forall \; \tilde{r}^i \neq r^i, \forall \; \omega, \forall \; i \Rightarrow f^i(s) > f^i(\tilde{s}^i, s^{-i}) \; \forall \; \tilde{s}^i \in S^i, \; \forall \; i$. s is a strict Nash equilbrium of Γ. $\square$

Suppose ρ attaches probability one to a rule $r \in \mathcal{R}$ that maps all signals to a strict Nash equilibrium $s \in S$ of Γ, $r(\omega) = s \; \forall \; \omega \in \Omega$. Then $f^i(r(\omega)) = f^i(r(\omega')) \; \forall \; \omega, \omega' \in \Omega, \forall \; i$ and no population has an incentive to manipulate P. Further $\mathcal{F}^i(r) > \mathcal{F}^i(\tilde{r}^i, r^{-i})) \; \forall \tilde{r}^i \in \mathcal{R}^i \; \forall \; i$, ie r is a strict correlated equilbrium. From Proposition 3, r is asymptotically stable. □

Proposition 6 claims that if the game Γ is generic, i.e. if one considers the payoffs as random draws and disregards those payoffs that appear with probability zero, if the agents update their rules boundedly rational and if no population would have an incentive to change the signal generating process if it could, then there is nothing we can learn from the concept of correlated equilibrium. Strict Nash equilibria sufficiently explain behavior under such conditions. The proof makes use of the fact that in generic games no two outcomes provide the same payoff. If a population has the capacity to choose certain signals at will, the population will do so as to maximize ex post payoffs.

6.2 Example: Chicken Game

We elaborate on this subject for the Chicken example, for which we can characterize $\Delta_P^{\sim} \cap \Delta_P^{CE}$. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{A}^1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$ and $\mathcal{A}^2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$. Consider a $P \in \Delta(\Omega)$ with full support. The Chicken game is generic, all asymptotically stable sets are singleton and therefore strict correlated equilibria. We list the expected payoffs and the best replies in Appendix B. For any P, there exist two strict correlated equilibria: (uu, rr) and (dd, ll). If $P(\omega_1) > \max\left\{1 - 3P(\omega_3), \frac{1}{3} - \frac{1}{3}P(\omega_3)\right\}$, also (ud, rl) is a strict correlated equilibrium. If further $P(\omega_1) > \max\left\{1 - \frac{3}{2}P(\omega_3), \frac{2}{3} - \frac{2}{3}P(\omega_3)\right\}$, there exists a fourth strict correlated equilibrium: (du, lr). The equilibria (uu, rr) and (dd, ll) correspond to the two strict Nash equilibria of Γ. In these equilibria, ex ante payoffs $\mathcal{F}$ equal ex post payoffs $f(\omega)$ for any signal ω, no population has an incentive to manipulate the generation of signals. Suppose P is such that (ud, rl) is a strict equilibrium. Then, the ex ante payoffs for population 1 are $6-4P(\omega_1)+P(\omega_3)$. Population 1 has an incentive to increase $P(\omega_3)$ at the expense of $P(\omega_1)$. After the manipulation, either P is outside the region in which (ud, rl) is a strict correlated equilibrium or the incentives to manipulate P are still intact. Note that population 2 also has incentives to manipulate P in the equilibrium (ud, rl). Analogous arguments hold for the equilibrium (du, lr). Wrapping up we get that P is robust given the rules $r \in \{(uu, ll), (dd, rr)\}$ and that these are the only states that are asymptotically stable. Note that there are other correlated equilibria, that are not asymptotically stable, that are generated by a robust P: $\Delta_P^{\sim} \cap \Delta_P^{CE} = \{\rho \in \Delta(\mathcal{R}) \,|\rho^1 = (\frac{1}{3} + \rho^1(rr), \frac{1}{3} - \rho^1(dd), \frac{1}{3} - \rho^1(dd), \rho^1(dd)), \rho^2 = (\frac{1}{3} + \rho^2(rr), \frac{1}{3} - \rho^2(rr), \frac{1}{3} - \rho^2(rr), \rho^2(rr)),\ \rho^1(dd), \rho^2(rr) \in [0, \frac{1}{3}]\} \cup \{(uu, rr), (dd, ll)\}$. For any mixed correlated equilibrium ρ with a robust P, each population gets a payoff of $4\frac{2}{3}$, which is the outcome of the mixed Nash-equilibrium of the original Chicken game. To conclude for the Chicken game: if agents have the capability to influence the signal generating process, and if the distriution P and the distribution of rules ρ is such that agents do neither have an incentive to manipulate the signals nor to change their behavior, the outcome is a Nash outcome.

7 Conclusions

In Aumann (1987), section 3, a player receives a signal and conditions her strategic choice within a normal form game on this signal. She takes into account that other players receive signals that are potentially correlated to hers and calculates conditional beliefs. Aumann (1987) shows that, if players have a common prior on the signal space and if players choose strategies optimally given their beliefs, the equilibrium outcome is a correlated equilibrium. There are correlated equilibrium outcomes that lie outside the convex hull of the Nash outcomes. In this chapter I pursue the question whether agents can achieve a correlated equilibrium without being capable to calculate conditional expectations, indeed even without being able to optimize. For this purpose, given a signal generating process and a game in strategic form, I define an "expanded game" whose strategies are mappings from the set of the signals to the set of the strategies of the original game. For this expanded game I transfer existing and well established results on regular monotonic dynamics, including the replicator dynamic. Applying a result of Samuelson and Zhang (1992), it follows that an outcome which supports a strictly dominated strategy of the original game receives zero weight in the limit. Analogous to results of Weibull (1995) and Ritzberger and Weibull (1995) I show that a Lyapunov stable state of the expanded game represents a correlated equilibrium of the original game and that such a state is asymptotically stable if and only if it represents a strict correlated equilibrium (also Swinkels (1992)). Furthermore, I make use of the setwise concept "strict equilibrium set" introduced by Balkenborg (1994) and provide a result for convex monotonic dynamics that is analogous to Cressman (2003) and Balkenborg and Schlag (2007): a set of restpoints is asymptotically stable if and only if it is a strict correlated equilibrium set. Therefore I can give a positive answer to the initial question: even if agents are extreme boundedly rational a non-Nash outcome can be robust to random perturbations if agents use simple rules that condition their behavior on observed signals. Finally I discuss endogenous signals. If behavior of the agents can be modelled by convex monotonic dynamics and if the game is generic, I show that an asymptotically stable state has a robust distribution of signals if and only if it corresponds to a strict Nash equilibrium of the original game. I suspect that if the (potentially only Lyapunov stable) state is a correlated equilibrium and if the distribution of signals is robust, then the expected payoffs lie in the convex hull of those produced by Nash equilibria. I illustrate this claim for the Chicken game.

This is not the first attempt linking evolutionary concepts to that of correlated equilibria. Cripps (1991) constructs a model in which nature randomly assigns roles to players in bi-matrix games. Analyzing the statics of the model, he shows that ESS in the symmetrized game represent strict correlated equilibria. Lenzo and Sarver (2006) define a model of subpopulations in which an agent of some subpopulation is non-uniformly matched to agents in other subpopulations. I show that any kind of their subpopulation matching may be represented by a particular signalling structure of our model. Kim and Wong (2007) define an evolutionary stable correlated strategy for symmetric 2×2 games.

Appendix

Proposition 1

Let g be convex monotonic. If $F \subset \Delta_{\mathcal{R}}$ is a Lyapunov stable set of rest points, then each $\rho \in F$ is a *c.e.* .

Proof: Since $\rho \in F$ is a restpoint, $g^i(r^i, \rho) = 0 \; \forall \; r^i \in$ supp (ρ^i). Suppose $\exists r^i_l, r^i_k \in \text{supp}(\rho^i)$ such that $\mathcal{F}^i(r^i_l, \rho^{-i}) > \mathcal{F}^i(r^i_k, \rho^{-i})$. Then, by convex monotonicity, $g^i(r^i_k, \rho) < g^i(r^i_l, \rho) \cdot 1$, a contradiction. Therefore $\mathcal{F}^i(r^i_l, \rho^{-i}) = \mathcal{F}^i(r^i_k, \rho^{-i}) \; \forall \; r^i_l, r^i_k \in \text{supp}(\rho^i)$. If ρ is in the interior of F with respect to $\Delta_{\mathcal{R}}$, we are done. Suppose instead that for some i there exists $r^i_k \notin \text{supp}(\rho^i)$ and suppose that $\mathcal{F}^i(r^i_k, \rho^{-i}) > \mathcal{F}^i(\rho)$. Then, again by convex monotonicity, $g^i(r^i_k, \rho) > \sum_{r^i \in \mathcal{R}^i} g^i(r^i, \rho) \cdot \rho^i(r^i) = 0$. Since g is (Lipschitz–)continuous, there exists a neighborhood $\mathcal{U}$ of ρ such that $g^i(r^i_k, \tilde{\rho}) > 0 \; \forall \; \tilde{\rho} \in \mathcal{U} \cap \Delta_{\mathcal{R}}$. Define $\mathcal{U}' = \{\tilde{\rho} \in \mathcal{U} \cap \Delta_{\mathcal{R}} \mid \tilde{\rho}^i(r^i_k) > 0\}$. It holds that $\hat{\rho}^i_{r^i_k}(t, \tilde{\rho})$ is strictly increasing in t for any $\tilde{\rho} \in \mathcal{U}'$. However, Lyapunov stability implies that $\hat{\rho}(t, \tilde{\rho}) \in \mathcal{U}' \; \forall \; t \geq 0$ and $\tilde{\rho} \in \mathcal{U}''$ for some neighborhood $\mathcal{U}''$, which can only be the case if $\dot{\tilde{\rho}}^i(r^i_k) \leq 0$ for some $\tilde{\rho} \in \mathcal{U}'$, because $\rho^i(r^i_k) = 0$. Since $g^i(r^i_k, \tilde{\rho}) > 0 \; \forall \; \tilde{\rho} \in \mathcal{U}'$ this is not true for any subset of $\mathcal{U}'$ and $\mathcal{U}''$ does not exist. Therefore, the existence of some $r^i_k \in \mathcal{R}^i$ for some $i \in \mathcal{N}$ such that

$\mathcal{F}^i(r^i_k, \rho^{-i}) > \mathcal{F}^i(\rho)$ contradicts Lyapunov stability of F and we have established the claim. $\square$

Proposition 3

Let (1) be convex monotonic. If a set F is a SEset, then F is an asymptotically stable set of rest points.

Proof: Suppose F is an SEset and suppose that $F \neq \Delta_{\mathcal{R}}$. Each point in F is a restpoint of (1). Further we have that F is a finite union of faces of $\Delta_{\mathcal{R}}$ and therefore is closed. Consider some ρ_* on the boundary of F with respect to $\Delta_{\mathcal{R}}$. For some population there is a pure rule r^i such that $\mathcal{F}^i(r^i, \rho_*^{-i}) < \mathcal{F}^i(\rho_*)$. Since $g^i(\cdot, \rho) \cdot \rho^i = 0 \; \forall \; \rho \in \Delta_{\mathcal{R}}$ it follows that $g^i(r^i, \rho_*) = 0 \; \forall \; r^i \in \text{supp}(\rho^i_*)$. From convex monotonicity we have that $g^i(r^i, \rho_*) < 0 \; \forall \; r^i \notin \text{supp}(\rho^i_*)$ and from continuity follows that there exists some neighborhood $\mathcal{U} : \mathcal{U} \cap int(\Delta_{\mathcal{R}}) \neq \emptyset$ of ρ_* such that $g^i(r^i, \rho) < 0 \; \forall \; r^i \notin \text{supp}(\rho^i_*), \rho \in \mathcal{U}$. Therefore $\dot{\rho}^i_{r^i}(\rho) < 0 \; \forall r^i \notin \text{supp}(\rho^i_*), \forall \; \rho \in \mathcal{U} \setminus F$ and from $g^i(\cdot, \rho) \cdot \rho^i = 0$ I have for at least one $r^i \in \text{supp}(\rho^i_*)$ that $\dot{\rho}^i_{r^i}(\rho) > 0 \; \forall \; \rho \in \mathcal{U} \setminus F$, which establishes the result. $\square$

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