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# On an Alternative Approach to Pricing General Barrier Options

by

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# On an Alternative Approach to Pricing General Barrier Options

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**Abstract.** In this paper, an alternative approach to pricing barrier options is presented that relies on the use of the first hitting time density to the barrier. The lateral Chapman-Kolmogorov relation is used as a major tool in order to determine option prices. It turns out that this approach allows for pricing barrier options with more general payoffs and with general continuous Markovian stochastic processes as underlying (at least numerically). As an illustrative example, a simple down-and-in call option is considered and its well-known closed form pricing formula is obtained.

**Keywords:** Barrier options, first passage time density, first hitting time density, lateral Chapman-Kolmogorov relation

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# 1 Introduction

Barrier options have become one of the most popular financial instruments in the area of options[1]. They were traded sporadically in 1967 on the US-American over-the-counter (OTC) market and in 1991 also with the S&P 500-Index as underlying on the Chicago Board Options Exchange (CBOE). Nowadays, barrier options are mainly established on the OTC market. Although traditionally these options fall under the category of exotic options, today they are no longer considered „exotic“. Barrier options belong to the class of path-dependent options, i.e. not only the value of the underlying (stock, stock index, interest rate, commodity, exchange rate, currency) at maturity is important, but also the path that the underlying price has taken up to this moment in time. The special feature of barrier options is that if the price of the underlying hits the barrier specified in the option contract, then something happens to the option. Therefore the barrier is a critical value for the underlying price process. In the event that the price of the underlying hits or breaks the barrier, either a plain-vanilla option begins to exist[2] (for knock-in options) or a plain-vanilla option ceases to exist (for knock-out options). In the case that no plain-vanilla option exists at maturity, there is the possibility to include so-called rebate payments in the option contract. These lump-sum cash payments help to dampen the loss due to a suddenly knocked out knock-out option. Therefore, barrier options are more flexible than standard plain-vanilla options, because one can explicitly take into account the preferences of the investor when choosing the value of the barrier and the value of the rebate.

Although barrier options have been traded since 1967, the first valuation formula for a down-and-out call was presented in 1973 by Merton (1973). Several years later, Rubinstein and Reiner (1991) published the fundamental article presenting analytical formulas for all standard types of single-barrier options. The complete formulas can also be found in the survey article by Rich (1994) and in almost all standard books on option pricing[3]. In this original approach, the discounted expected value of the option's payoff is computed by integrating with respect to the state variable, i.e. the underly-

[1]According to Thomas (1996), barrier options account for e.g. 10% of all currency options trades (by volume) between banks and their clients.

[2]A plain-vanilla option is either a standard put or a standard call option.

[3]See Haug (1998), Hull (2003) and Nelken (1996).

ing asset price process. An alternative approach is mentioned in El Karoui and Jeanblanc (1999) who thought of using the first hitting time density and computing the discounted expected value of the option's payoff by integrating with respect to time. This approach is more flexible since it allows to compute (in some cases only numerically) the prices for general barrier options, e.g. one can think of barrier options with more exotic payoffs. Another advantage of this approach is that it remains valid for all underlying stochastic processes as long as they are continuous and Markovian. In this paper, we develop this approach further and provide the still missing proof for the valuation formula. In section 2, we present the lateral Chapman-Kolmogorov relation which constitutes an important tool for the solution of the problem. Then, in section 3, the general framework is described. In section 4, the first passage time valuation approach is presented and the analytical solution for a down-and-in call follows in section 5. Section 6 concludes the paper.

# 2 The lateral Chapman-Kolmogorov relation

In this section, we develop the idea of the lateral Chapman-Kolmogorov relation introduced by Carr (2002). We will closely follow the derivation in his paper. First, we present the idea in full generality, i.e. we only assume that the stochastic process $X$ that we consider is Markovian. Then we restrict ourselves to the example of a Brownian Motion with drift since this will be of interest in the following sections. Let $q(x,t;y,T)$ denote the transition probability density function, where $x$ and $t$ are the backward variables and $y$ and $T$ are the forward variables. Furthermore, let $f(x,t;h,u)$ denote the probability density function for the first passage time to $h$, $x$ and $t$ are again the backward variables and $h$ and $u$ are the forward variables.
Suppose that $x < h < y$. Then we define the lateral Chapman-Kolmogorov relation to be:

$$q(x,t;y,T) = \int_t^T f(x,t;h,u)q(h,u;y,T)du. \tag{1}$$

From a probabilistic point of view, this result is obvious.

Now suppose that $X$ is a Brownian motion with drift $\mu$ and diffusion parameter $\sigma$ started at $x$. The transition probability density function is given by:

$$q(x,t;y,T) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{1}{2}\frac{(y-x-\mu(T-t))^2}{\sigma^2(T-t)}}. \tag{2}$$

The first passage time density function is given by:

$$f(x,t;h,u) = \frac{h-x}{\sqrt{2\pi\sigma^2(u-t)^3}} e^{-\frac{1}{2}\frac{(h-x+\mu(u-t))^2}{\sigma^2(u-t)}}. \tag{3}$$

We want to verify that the lateral Chapman-Kolmogorov relation holds for a Brownian motion with drift. Without loss of generality, we choose $t = 0$. Let:

$$\begin{aligned} I &\equiv \int_0^T f(x,0;h,u)q(h,u;y,T)du \\ &= \int_0^T \frac{h-x}{\sqrt{2\pi\sigma^2u^3}} e^{-\frac{1}{2}\frac{(h-x+\mu u)^2}{\sigma^2 u}} \frac{1}{\sqrt{2\pi\sigma^2(T-u)}} e^{-\frac{1}{2}\frac{(y-h-\mu(T-u))^2}{\sigma^2(T-u)}} du. \end{aligned} \tag{4}$$

Consider the following change of variable from $u$ to $\tau = \frac{T}{u} - 1$. If $\tau = \frac{T}{u} - 1$, then $\frac{T}{u} = 1 + \tau$, so $u = \frac{T}{1+\tau}$. Hence $T - u = T - \frac{T}{1+\tau} = T\left(1 - \frac{1}{1+\tau}\right) = \frac{T\tau}{1+\tau}$ and $du = -\frac{T}{(1+\tau)^2}d\tau$.

Consequently:

$$\begin{aligned} I &= \frac{h-x}{2\pi\sigma^2} \int_0^\infty \frac{e^{-\frac{1}{2}\frac{\left(h-x-\mu\left(\frac{T}{1+\tau}\right)\right)^2(1+\tau)}{\sigma^2 T}}}{\left(\frac{T}{1+\tau}\right)^{\frac{3}{2}}} \frac{e^{-\frac{1}{2}\frac{\left(y-h-\mu\left(\frac{T\tau}{1+\tau}\right)\right)^2(1+\tau)}{\sigma^2 T\tau}}}{\left(\frac{T\tau}{1+\tau}\right)^{\frac{1}{2}}} \left(\frac{T}{(1+\tau)^2}\right) d\tau \\ &= \frac{h-x}{2\pi\sigma^2} \frac{e^{-\frac{1}{2}\frac{(h-x)^2}{\sigma^2 T}}}{T^{\frac{3}{2}}} \frac{e^{-\frac{1}{2}\frac{(y-h)^2}{\sigma^2 T}}}{T^{\frac{1}{2}}} e^{\frac{2\mu T(h-x+y-h)}{2\sigma^2 T}} e^{-\frac{\mu^2 T^2}{2\sigma^2 T}} T \int_0^\infty \frac{1}{\sqrt{\tau}} e^{-\frac{1}{2}\frac{(h-x)^2\tau}{\sigma^2 T}} e^{-\frac{1}{2}\frac{(y-h)^2}{\sigma^2 T\tau}} d\tau \\ &= \frac{h-x}{\sqrt{2\pi}\sigma^2 T} e^{-A-\frac{k^2}{2}} e^{\frac{2\mu T(h-x+y-h)-\mu^2T^2}{2\sigma^2 T}} \int_0^\infty e^{-A\tau} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{k^2}{2\tau}} d\tau, \end{aligned} \tag{5}$$

where $A \equiv \frac{(h-x)^2}{2\sigma^2 T}$ and $k \equiv \frac{y-h}{\sigma\sqrt{T}}$. The integral $\int_0^\infty e^{-A\tau}\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{k^2}{2\tau}}d\tau$ is recognized as the Laplace transform (since $A > 0$) of the probability density function $q(0,0;k,\tau)$ for a standard Brownian motion without drift. To evaluate this Laplace transform in complete generality, we drop the requirement that $y > h$, or equivalently that $k > 0$. The probability density function $q(0,0;k,\tau)$ satisfies Kolmogorov's forward equation,

$$\frac{\partial}{\partial t} q(0,0;k,\tau) = \frac{1}{2}\frac{\partial^2}{\partial k^2} q(0,0;k,\tau), \quad k \in \Re, \tau > 0, \tag{6}$$

where $\tau$ is now the forward time and $k$ is the forward spatial variable. This PDE must be solved subject to the initial condition

$$q(0,0;k,0) = \delta(k), \quad k \in \Re, \tag{7}$$

and the boundary conditions

$$\lim_{k \to \pm\infty} q(0,0;k,\tau) = 0, \quad \tau > 0. \tag{8}$$

Multiplying the PDE by $e^{-A\tau}$ and integrating with respect to $\tau$ from 0 to $\infty$ implies that the Laplace transform

$$\mathcal{L}(k,A) \equiv \int_0^\infty e^{-A\tau} q(0,0;k,\tau) d\tau \tag{9}$$

solves the ordinary differential equation (ODE)

$$\frac{1}{2}\frac{\partial^2}{\partial k^2}\mathcal{L}(k,A) - A\mathcal{L}(k,A) = \delta(k) \tag{10}$$

subject to the boundary conditions

$$\lim_{k \to \pm\infty} \mathcal{L}(k,A) = 0. \tag{11}$$

The pair of linearly independent solutions to the second order linear homogeneous ODE

$$\frac{1}{2}\frac{\partial^2}{\partial k^2} h(k,A) - Ah(k,A) = 0 \tag{12}$$

are exponentials

$$h(k,A) = c_1 e^{\sqrt{2A}k} + c_2 e^{-\sqrt{2A}k},$$

where $c_1$ and $c_2$ are constants to be determined. We set $\mathcal{L}(k, A) = h(k, A)$ on $k > 0$ and $k < 0$:

$$\mathcal{L}(k, A) = \begin{cases} c_{11}e^{\sqrt{2A}k} + c_{12}e^{-\sqrt{2A}k} & \text{, if } k < 0 \\ \\ c_{21}e^{\sqrt{2A}k} + c_{22}e^{-\sqrt{2A}k} & \text{, if } k > 0. \end{cases} \tag{13}$$

Imposing the boundary conditions implies that $c_{12} = c_{21} = 0$:

$$\mathcal{L}(k, A) = \begin{cases} c_{11}e^{\sqrt{2A}k} & \text{, if } k < 0 \\ \\ c_{22}e^{-\sqrt{2A}k} & \text{, if } k > 0. \end{cases} \tag{14}$$

Continuity at $k = 0$ implies $c_{11} = c_{22} = c$. To satisfy the ODE, we want to choose $c$ so that the jump in the first derivative at $k = 0$ is $-2$. The final solution is

$$\mathcal{L}(k, A) = \begin{cases} \frac{e^{\sqrt{2A}k}}{\sqrt{2A}} & \text{, if } k < 0 \\ \\ \frac{e^{-\sqrt{2A}k}}{\sqrt{2A}} & \text{, if } k > 0 \end{cases} \tag{15}$$

or more simply

$$\mathcal{L}(k, A) = \frac{e^{-\sqrt{2A}|k|}}{\sqrt{2A}}. \tag{16}$$

Multiplying by $\frac{h-x}{\sqrt{2\pi\sigma^2 T}}e^{-A-\frac{k^2}{2}}e^{\frac{2\mu T(h-x+y-h)-\mu^2T^2}{2\sigma^2 T}}$:

$$I = \frac{h-x}{\sqrt{2\pi\sigma^2 T}}e^{-A-\frac{k^2}{2}}e^{\frac{2\mu T(h-x+y-h)-\mu^2T^2}{2\sigma^2 T}}\frac{e^{-\sqrt{2A}|k|}}{\sqrt{2A}}. \tag{17}$$

Recalling that $A \equiv \frac{(h-x)^2}{2\sigma^2 T}$ and $k \equiv \frac{y-h}{\sigma\sqrt{T}}$ implies:

$$I = \frac{1}{\sqrt{2\pi\sigma^2 T}}e^{-\frac{(h-x)^2}{2\sigma^2 T}-\frac{(y-h)^2}{2\sigma^2 T}}e^{\frac{2\mu T(h-x+y-h)-\mu^2T^2}{2\sigma^2 T}}e^{-\frac{2(h-x)|y-h|}{2\sigma^2 T}}. \tag{18}$$

Now impose the requirement that $y > h$:

$$\begin{aligned} I &= \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{h^2-2hx+x^2+y^2-2hy+h^2-2\mu Ty+2\mu Tx+\mu^2T^2+2hy-2h^2-2xy+2xh}{2\sigma^2 T}} \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{x^2+y^2-2\mu T(y-x)+\mu^2T^2}{2\sigma^2 T}} \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(y-x-\mu T)^2}{2\sigma^2 T}}. \end{aligned} \tag{19}$$

Hence $I = q(x, 0; y, T)$ as was to be shown.

# 3 General Framework

Although most of the derivations are valid for general continuous Markov processes, we assume for simplicity that the underlying risk-neutral stock price process $S_t$ follows a geometric Brownian motion,

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t, t \in [0, T], S_0 = S > 0, \tag{20}$$

where $r \geq 0$ and $q \geq 0$ are assumed to be the constant interest rate and the constant dividend rate, respectively. $\sigma$ denotes the constant volatility rate and $W_t$ is a standard Brownian motion.

# 4 First passage time valuation approach to barrier options

As an example, we will concentrate on the valuation of a down-and-in call, $DIC$, written on $S_t$ with strike $K$, expiration date $T$ and constant barrier $B$.

With down-and-in options, the price of the underlying is initially above the barrier, i.e. $S > B$. If the price hits or breaks the barrier before maturity, the investor gets, depending on the specification, a plain-vanilla call or a plain-vanilla put, respectively. If the price stays above the barrier during the entire time to maturity, then at maturity the investor gets a rebate which can also be zero and which is pre-specified in the option contract.

Pricing formulas for all other barrier option types can be derived similarly or can be obtained by using the well-known in-out-parity:

up-and-in option=plain-vanilla option - up-and-out option
down-and-in option=plain-vanilla option - down-and-out option.

The proof follows immediately from the payoff profiles of the different options and is therefore omitted.

Following El Karoui and Jeanblanc (1999), we denote by $\tau$ the first passage time (= first hitting time) of the underlying stock price process $S_t$ to the barrier $B$. We assume $S > B > 0$, $K > B > 0$. If this process never hits the barrier, $\tau = \infty$. The probability density function $f(t)$ for the first passage time $\tau$ of a Brownian motion with drift to a constant barrier can be found e.g. in Cox and Miller (1965), Ingersoll (1987), Karatzas and Shreve (1991) or Borodin and Salminen (2002). In our case:

$$P(\tau \in dt) = f(t)dt = \frac{\ln(S/B)}{\sqrt{\sigma^2 t^3}} n\left[\frac{\ln(S/B) + (r - q - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right] dt, \qquad (21)$$

where $n[y] = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$ is the standard normal density function. Notice that when $r - q > \frac{\sigma^2}{2}$, then the density function is defective, i.e. integrating $\tau$ over 0 to $\infty$ leads to a value strictly less than 1. Hence $r - q \leq \frac{\sigma^2}{2}$.

The Black and Scholes (1973) time $t$ value of a plain-vanilla European call option is given by:

$$C(S,t) = Se^{-q(T-t)}N[d_1] - Ke^{-r(T-t)}N[d_2], \qquad (22)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

and where $N[\bullet]$ is the cumulative standard normal distribution function.

Let $DIC_0$ denote the initial value of a down-and-in call. If the first passage time occurs before time $T$, i.e. $\tau < T$, then a down-and-in call becomes a plain-vanilla call at that time:

$$\tau = t \in [0, T] \Rightarrow DIC_t = C(B, t). \tag{23}$$

If the underlying never hits the barrier until maturity, the down-and-in call expires worthless:

$$\tau > T \Rightarrow DIC_T = 0. \tag{24}$$

Hence the time 0 value of a European down-and-in call is given by:

$$\begin{aligned} DIC_0 &= \int_0^T e^{-rt} DIC_t P(\tau \in dt) \\ &= \int_0^T e^{-rt} C(B, t) f(t) dt. \end{aligned} \tag{25}$$

Of course, it is possible to evaluate this last expression numerically since both the value of a European call and the first passage time density are given in closed form above. However, it is more instructive to solve the integral analytically. The result is of course the same as if the original approach of Rubinstein and Reiner (1991) were used:

$$DIC_0 = Se^{-qT}(B/S)^{2\lambda} N[y] - Ke^{-rT}(B/S)^{2\lambda-2} N[y - \sigma\sqrt{T}], \tag{26}$$

where $y = \frac{\ln\left(\frac{B^2}{SK}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$ and $\lambda = \frac{r-q+\sigma^2/2}{\sigma^2}$.

Notice that the alternative approach allows for the (numerical) valuation of more general barrier options as long as the underlying price process is continuous and Markovian. For example, we might use a different volatility in the first passage time density and in the formula for the call option, i.e. we could model a change in volatility occurring at the moment when the barrier is hit.

Alternatively, a smile consistent deterministic volatility model might be used in order to model the underlying (since observed call option prices are a direct input in the valuation formula). If the first hitting time to a flat barrier for CEV process were known we could use this approach in order to price barrier options on the CEV process. Furthermore, instead of a European call option, we might use any type of option payoff and thus obtain a (numerical) pricing formula for this exotic barrier option. And what about using an analytical approximation for an American call option instead of the European call? However, these extensions to more exotic barrier options will be considered in a subsequent paper.

# 5 Analytical solution for a DIC

In the preceding section, we claimed that the initial value of a European down-and-in call is given by:

$$DIC_0 = \int_0^T e^{-rs} C(B,s) f(s) ds. \tag{27}$$

Substituting the formulas for $C(B,s)$ and $f(s)$ yields:

$$\begin{aligned} DIC_0 &= \int_0^T \left\{ e^{-rs} \left( Be^{-q(T-s)} N \left[ \frac{\ln(B/K) + (r - q + \frac{1}{2}\sigma^2)(T-s)}{\sigma\sqrt{T-s}} \right] \right.\right. \\ &\quad \left. - Ke^{-r(T-s)} N \left[ \frac{\ln(B/K) + (r - q - \frac{1}{2}\sigma^2)(T-s)}{\sigma\sqrt{T-s}} \right] \right) \\ &\quad \left. \times \frac{\ln(S/B)}{\sqrt{\sigma^2 s^3}} n \left[ \frac{\ln(S/B) + (r - q - \frac{1}{2}\sigma^2)s}{\sigma\sqrt{s}} \right] \right\} ds. \end{aligned}$$

To ease notation, let $x = \ln(S/B)$, $k = \ln(K/B)$ and $\mu = r - q - \frac{1}{2}\sigma^2$ and rewrite:

$$\begin{aligned} DIC_0 &= \int_0^T \left\{ e^{-rs} \left( Be^{-q(T-s)} N \left[ \frac{-k + (\mu + \sigma^2)(T-s)}{\sigma\sqrt{T-s}} \right] \right.\right. \\ &\quad \left.\left. - Ke^{-r(T-s)} N \left[ \frac{-k + \mu(T-s)}{\sigma\sqrt{T-s}} \right] \right) \frac{x}{\sqrt{\sigma^2 s^3}} n \left[ \frac{x + \mu s}{\sigma\sqrt{s}} \right] \right\} ds \\ &= \int_0^T e^{-rs} c(\sigma) \frac{x}{\sqrt{\sigma^2 s^3}} n \left[ \frac{x + \mu s}{\sigma\sqrt{s}} \right] ds \end{aligned} \tag{28}$$

using

$$c(\sigma) = Be^{-q(T-s)}N\left[\frac{-k+(\mu+\sigma^2)(T-s)}{\sigma\sqrt{T-s}}\right] - Ke^{-r(T-s)}N\left[\frac{-k+\mu(T-s)}{\sigma\sqrt{T-s}}\right]. \tag{29}$$

Now take the first derivative of $c(\sigma)$ with respect to $\sigma$, i.e. compute the vega of a European call:

$$\begin{aligned} c'(\sigma) &= Be^{-q(T-s)}N\left[\frac{-k+(\mu+\sigma^2)(T-s)}{\sigma\sqrt{T-s}}\right] - Ke^{-r(T-s)}N\left[\frac{-k+\mu(T-s)}{\sigma\sqrt{T-s}}\right] \\ &\quad \times\frac{\partial}{\partial\sigma}\left(\frac{-k+\mu(T-s)}{\sigma\sqrt{T-s}}\right) + Be^{-q(T-s)}n\left[\frac{-k+(\mu+\sigma^2)(T-s)}{\sigma\sqrt{T-s}}\right]\sqrt{T-s} \\ &= Ke^{-r(T-s)}n\left[\frac{-k+\mu(T-s)}{\sigma\sqrt{T-s}}\right]\sqrt{T-s}, \end{aligned} \tag{30}$$

based on the fact that

$$Be^{-q(T-s)}n\left[\frac{-k+(\mu+\sigma^2)(T-s)}{\sigma\sqrt{T-s}}\right] = Ke^{-r(T-s)}n\left[\frac{-k+\mu(T-s)}{\sigma\sqrt{T-s}}\right]. \tag{31}$$

See appendix 2 for the derivation of this equality.

By the fundamental theorem of calculus:

$$\begin{aligned} c(\sigma) &= c(0) + \int_0^\sigma c'(v)dv \\ &= \max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] \\ &\quad + \int_0^\sigma Ke^{-r(T-s)}n\left[\frac{-k+(r-q-\frac{1}{2}v^2)(T-s)}{v\sqrt{T-s}}\right]\sqrt{T-s}dv. \end{aligned} \tag{32}$$

Now, let us rewrite the entire integral expression using the above decomposition:

$$
\begin{aligned}
DIC_0 &= \int_0^T e^{-rs}\Big(\max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] \\
&+ \int_0^\sigma Ke^{-r(T-s)} n\left[\frac{-k+(r-q-\frac{1}{2}v^2)(T-s)}{v\sqrt{T-s}}\right]\sqrt{T-s}dv\Big)\frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \\
&= \int_0^T e^{-rs}\max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right]\frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \\
&+ \int_0^T e^{-rs}\int_0^\sigma Ke^{-r(T-s)} n\left[\frac{-k+(r-q-\frac{1}{2}v^2)(T-s)}{v\sqrt{T-s}}\right]\sqrt{T-s}dv \\
&\times \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \qquad (33) \\
&= I_1 + I_2.
\end{aligned}
$$

### 5.1 Computation of the integral $I_1$

Remember that we assumed $K > B$ for the option to be priced. In order to compute the first of the two integrals,

$$
\begin{aligned}
I_1 &= \int_0^T e^{-rs}\max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right]\frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \quad (34) \\
&= \int_0^T e^{-rT}\max\left[0, \hat{B}(s) - K\right]\frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds,
\end{aligned}
$$

where $\hat{B}(s) = Be^{(r-q)(T-s)}$ denotes the forward value of the barrier at time $s$, let us first have a closer look at the maximum operator $\max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right]$. This term is positive if $Be^{-q(T-s)} - Ke^{-r(T-s)} > 0$ or equivalently $\hat{B}(s) > K$, otherwise it is equal to zero. Under the assumption that $r - q > 0$, it follows:

$$\begin{aligned}
&Be^{-q(T-s)} - Ke^{-r(T-s)} > 0 \Leftrightarrow Be^{-q(T-s)} > Ke^{-r(T-s)} \\
&\Leftrightarrow \frac{B}{K}e^{-q(T-s)} > e^{-r(T-s)} \Leftrightarrow \frac{B}{K} > e^{-(r-q)(T-s)} \\
&\Leftrightarrow \ln\left(\frac{B}{K}\right) > -(r-q)(T-s) \Leftrightarrow -k + (r-q)T > (r-q)s \\
&\Leftrightarrow T - \frac{k}{(r-q)} > s \Leftrightarrow T^* > s.
\end{aligned}$$

Thus, $\max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] > 0$ for all $s < T^*$. Denote $z = k - (r-q)T$. Notice, that $T^* < 0$ if $z > 0$ and $(r-q) > 0$, i.e. $T^* \notin [0,T]$. As a consequence $I_1 = 0$ for this case. For further purposes, this case will be called ,,case 1".

If $z \leq 0$ and $r - q > 0$, then along the same lines $\max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] > 0$ for all $s < T^*$. However, in this case $0 \leq T^* \leq T$, i.e. $T^* \in [0,T]$. This case which we call ,,case 2" for further reference, requires the computation of the integral $I_1$, but with $T^*$ as upper integration limit.

For $r - q < 0$, it follows that:

$$\begin{aligned}
&Be^{-q(T-s)} - Ke^{-r(T-s)} > 0 \Leftrightarrow Be^{-q(T-s)} > Ke^{-r(T-s)} \\
&\Leftrightarrow \frac{B}{K}e^{-q(T-s)} > e^{-r(T-s)} \Leftrightarrow \frac{B}{K} > e^{-(r-q)(T-s)} \\
&\Leftrightarrow \ln\left(\frac{B}{K}\right) > -(r-q)(T-s) \Leftrightarrow -k + (r-q)T > (r-q)s \\
&\Leftrightarrow T - \frac{k}{(r-q)} < s \Leftrightarrow T^* < s.
\end{aligned}$$

So, $\max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] > 0$ for all $T^* > s$. It can also be shown that for this parameter constellation $T^* > T > 0$ always holds, i.e. $T^* \notin [0,T]$, and hence $I_1 = 0$ always. This case will be referred to as ,,case 3".

If $r - q = 0$, it follows that

$$Be^{-q(T-s)} - Ke^{-r(T-s)} > 0 \Leftrightarrow \qquad B > K,$$

which is always violated under the current parameter constellation $K > B$, hence $I_1 = 0$ always. This is ,,case 4“.

To summarize, for later computations, we have to distinguish the four cases: $T^* < 0$, $T \geq T^* \geq 0$, $T^* > T > 0$ and $r = q$.

But first, we compute the closed form solution for the integral $I_1$ for the case in which $I_1 > 0$:

$$I_1 = \int_0^T e^{-rs} \max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x + \mu s}{\sigma\sqrt{s}}\right] ds. \quad (35)$$

Using the above notation, the difference inside the maximum is strictly positive on the interval $[0, T^*]$ for $T^* \in [0, T]$. Otherwise, this term equals zero. Thus, we only have to consider:

$$\hat{I}_1 = \int_0^{T^*} e^{-rs}(Be^{-q(T-s)} - Ke^{-r(T-s)}) \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x + \mu s}{\sigma\sqrt{s}}\right] ds. \quad (36)$$

From the computations shown in appendix 3, we finally obtain that this integral is equal to:

$$\begin{aligned}
\hat{I}_1 &= \int_0^{T^*} e^{-rs}(Be^{-q(T-s)} - Ke^{-r(T-s)}) \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x + \mu s}{\sigma\sqrt{s}}\right] ds \\
&= Se^{-qT} N[e_1] - Ke^{-rT} N[e_2] \\
&\quad + Be^{-qT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N[e_3] - Ke^{-rT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N[e_4], \quad (37)
\end{aligned}$$

where

$$e_1 = \frac{\ln(B/S) - (r - q + \frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}, \quad e_2 = e_1 + \sigma\sqrt{T^*}$$

$$e_3 = \frac{\ln(B/S) + (r - q + \frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}, \quad e_4 = e_3 - \sigma\sqrt{T^*}.$$

Hence, we have:

$$
\begin{aligned}
I_1 &= \int_0^T e^{-rs} \max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \\
&= \begin{cases}
\hat{I}_1 = Se^{-qT} N[e_1] - Ke^{-rT} N[e_2] & \\
+Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N[e_3] & \\
-Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N[e_4] & \text{, if } T \geq T^* \geq 0 \\
 & \\
0 & \text{, if } T^* < 0 \text{ or } T^* > T > 0 \text{ or } r = q,
\end{cases}
\end{aligned} \quad (38)
$$

where

$$
e_1 = \frac{\ln(B/S) - (r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}, \quad e_2 = e_1 + \sigma\sqrt{T^*}
$$

$$
e_3 = \frac{\ln(B/S) + (r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}, \quad e_4 = e_3 - \sigma\sqrt{T^*}.
$$

## 5.2 Computation of the integral $I_2$

Now we proceed to the computation of the second integral,

$$
I_2 = \int_0^T e^{-rs} \int_0^\sigma Ke^{-r(T-s)} n\left[\frac{-k + (r-q-\frac{1}{2}v^2)(T-s)}{v\sqrt{T-s}}\right] \sqrt{T-s}dv \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds. \quad (39)
$$

In order to evaluate the inner integral, consider the following change of variable and let

$$
\begin{aligned}
& t = \frac{v^2}{\sigma^2}(T-s) + s \\
& \Rightarrow \sigma\sqrt{t-s} = v\sqrt{T-s} \Rightarrow v = \frac{\sigma}{\sqrt{T-s}}\sqrt{t-s} \Rightarrow dv = \frac{\sigma}{\sqrt{T-s}2\sqrt{t-s}} dt.
\end{aligned}
$$

Consequently:

$$
I_2 = Ke^{-rT} \int_0^T \int_s^T \frac{\sigma}{2\sqrt{t-s}} n\left[\frac{-k+\mu(t-s)}{\sigma\sqrt{t-s}}\right] dt \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds. \quad (40)
$$

Apply Fubini's theorem and use the fact that $n(-y) = n(y)$ to obtain:

$$\begin{aligned} I_2 &= \int_0^T \frac{\sigma^2 K e^{-rT}}{2} \int_0^t \frac{1}{\sigma\sqrt{t-s}} n\left[\frac{k-(r-q)T+(r-q)s-(r-q)t+(r-q)t+\frac{1}{2}\sigma^2(t-s)}{\sigma\sqrt{t-s}}\right] \\ &\quad \times \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds dt. \end{aligned} \tag{41}$$

This simplifies to

$$\begin{aligned} I_2 &= \frac{\sigma^2 K e^{-rT}}{2} \int_0^T \int_0^t \frac{1}{\sigma\sqrt{t-s}} n\left[\frac{k-(r-q)(T-t)-(r-q-\frac{1}{2}\sigma^2)(t-s)}{\sigma\sqrt{t-s}}\right] \\ &\quad \times \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds dt \\ &= \frac{\sigma^2 K e^{-rT}}{2} \int_0^T I_3 dt \end{aligned} \tag{42}$$

such that we can consider $I_3$ separately.

With a closer look, $I_3$ is very similar to the integral $I$ that we considered in section 2. Thus, the solution can be obtained in the same way:

$$I_3 = \int_0^t \frac{x}{\sqrt{2\pi\sigma^2 s^3}} e^{-\frac{(x+\mu s)^2}{2\sigma^2 s}} \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} e^{-\frac{(k-(r-q)(T-t)-\mu(t-s))^2}{2\sigma^2(t-s)}} ds. \tag{43}$$

With $k-(r-q)T = z$, $(r-q) = b$ and using the analog change of variables as in section 2, i.e. the change of variable from $s$ to $\tau = \frac{t}{s} - 1$. If $\tau = \frac{t}{s} - 1$, then $\frac{t}{s} = 1+\tau$, so $s = \frac{t}{1+\tau}$. Consequently, $t - s = t - \frac{t}{1+\tau} = t\left(1 - \frac{1}{1+\tau}\right) = \frac{t\tau}{1+\tau}$ and $ds = -\frac{t}{(1+\tau)^2} d\tau$. Therefore:

$$\begin{aligned} I_3 &= \frac{x}{2\pi\sigma^2} \int_0^\infty \frac{e^{-\frac{1}{2}\frac{\left(x+\mu\left(\frac{t}{1+\tau}\right)\right)^2(1+\tau)}{\sigma^2 t}}}{\left(\frac{t}{1+\tau}\right)^{\frac{3}{2}}} \frac{e^{-\frac{1}{2}\frac{\left(z+bt-\mu\left(\frac{t\tau}{1+\tau}\right)\right)^2(1+\tau)}{\sigma^2 t\tau}}}{\left(\frac{t\tau}{1+\tau}\right)^{\frac{1}{2}}} \left(\frac{t}{(1+\tau)^2}\right) d\tau \\ &= \frac{xt}{2\pi\sigma^2} \frac{e^{-\frac{1}{2}\frac{x^2}{\sigma^2 t}}}{t^{\frac{3}{2}}} \frac{e^{-\frac{1}{2}\frac{(z+bt)^2}{\sigma^2 t}}}{t^{\frac{1}{2}}} e^{\frac{2\mu t(z+bt-x)-\mu^2 t^2}{2\sigma^2 t}} \int_0^\infty \frac{1}{\sqrt{\tau}} e^{-\frac{1}{2}\frac{x^2\tau}{\sigma^2 t}} e^{-\frac{1}{2}\frac{(z+bt)^2}{\sigma^2 t\tau}} d\tau \\ &= \frac{x}{\sqrt{2\pi\sigma^2 t}} e^{-\alpha-\frac{\kappa^2}{2}} e^{\frac{2\mu t(z+bt-x)-\mu^2 t^2}{2\sigma^2 t}} \int_0^\infty e^{-\alpha\tau} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{\kappa^2}{2\tau}} d\tau, \end{aligned} \tag{44}$$

where $\alpha \equiv \frac{x^2}{2\sigma^2 t}$ and $\kappa \equiv \frac{z+bt}{\sigma\sqrt{t}}$. The integral $\displaystyle\int_0^\infty e^{-\alpha\tau}\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{\kappa^2}{2\tau}}d\tau$ is recognized as the Laplace transform (since $\alpha > 0$) of the probability density function $q(0,0;B,\tau)$ for a standard Brownian motion without drift.

Using the solution for the Laplace transform obtained in section 2, we have that:

$$
\begin{aligned}
I_3 &= \frac{x}{\sqrt{2\pi\sigma^2 t}}e^{-\alpha-\frac{\kappa^2}{2}}e^{\frac{2\mu t(z+bt-x)-\mu^2t^2}{2\sigma^2 t}}\frac{e^{-\sqrt{2\alpha}|\kappa|}}{\sqrt{2\alpha}} \\
&= \frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{2t\mu x-2t\mu(z+bt)+(z+bt)^2+x^2+\mu^2t^2}{2t\sigma^2}}e^{-\frac{x}{\sigma\sqrt{t}}\left|\frac{z+bt}{\sigma\sqrt{t}}\right|}. \qquad (45)
\end{aligned}
$$

In order to get rid of the absolute value, we distinguish the following cases for different values of $z$ and $b$.

#### 5.2.1 Case 1: $T^* < 0$

With $z > 0$ and $b > 0$, implying $-\frac{z}{b} = T^* < 0$:

$$
\begin{aligned}
I_3 &= \frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{(z+x+(b-\mu)t)^2}{2t\sigma^2}}e^{-\frac{2\mu x}{\sigma^2}} \qquad (46) \\
&= e^{-\frac{2\mu x}{\sigma^2}}\frac{1}{\sqrt{\sigma^2 t}}n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right].
\end{aligned}
$$

#### 5.2.2 Case 2: $T \geq T^* \geq 0$

With $z \leq 0$ and $b > 0$, implying $T \geq -\frac{z}{b} = T^* \geq 0$, the integral $I_3$ is more tedious to compute. We have to distinguish the following two cases for $z \leq 0$:

$$
\left|\frac{(z+bt)}{\sigma}\right| = \begin{cases} \frac{(-z-bt)}{\sigma} & \text{, if } z+bt \leq 0 \\ \frac{(z+bt)}{\sigma} & \text{, if } z+bt > 0. \end{cases}
$$

Then:

$$\begin{aligned}
I_3 &= e^{-\frac{2\mu x}{\sigma^2}} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] \\
&\quad - \left(e^{\frac{-2\mu x}{\sigma^2}} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] - \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{-z+x-(b-\mu)t}{\sigma\sqrt{t}}\right]\right) 1_{\{t\leq T^*\}}.
\end{aligned} \tag{47}$$

### 5.2.3 Case 3: $T^* > T > 0$

With $z > 0$ and $b < 0$, implying $-\frac{z}{b} = T^* > T > 0$, the integral $I_3$ is also tedious to compute. We have to distinguish the following two cases for $z > 0$:

$$\left|\frac{(z+bt)}{\sigma}\right| = \begin{cases} \frac{(-z-bt)}{\sigma} & \text{, if } z+bt < 0 \\ \frac{(z+bt)}{\sigma} & \text{, if } z+bt \geq 0. \end{cases}$$

Actually, the case $z + bt < 0$ needs not to be considered any further since $z + bt < 0 \Leftrightarrow t > T^* > T$. We have however:

$$\begin{aligned}
I_3 &= e^{-\frac{2\mu x}{\sigma^2}} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] \\
&\quad - \left(e^{\frac{-2\mu x}{\sigma^2}} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] - \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{-z+x-(b-\mu)t}{\sigma\sqrt{t}}\right]\right) 1_{\{t>T^*\}}.
\end{aligned} \tag{48}$$

### 5.2.4 Case 4: $r = q$

Finally for the case $b = 0$, i.e. $r = q$:

$$\begin{aligned}
I_3 &= \frac{e^{-k}}{\sqrt{\sigma^2 t}} n\left[\frac{k+x-\frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right] \qquad (49)\\
&= \frac{B}{K}\frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{\ln\left(\frac{K}{B}\right)+\ln\left(\frac{S}{B}\right)-\frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right].
\end{aligned}$$

## 5.3 Computation of the option price

Now that we have computed the inner integrals for all possible cases, we can proceed with the computation of the option price.
From the preceding sections, we know that the price of a down-and-in call is the sum of two integrals taking on different values depending on whether $T^*$ is positive or negative and on whether $r = q$ or not. Of course, regardless of the case we are considering the same valuation formula should be reached.

Again we start with the integral representation from section 4,

$$\begin{aligned} DIC_0 &= \int_0^T e^{-rs} \max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \\ &+ \int_0^T e^{-rs} \int_0^\sigma Ke^{-r(T-s)} n\left[\frac{-k+\mu(T-s)}{v\sqrt{T-s}}\right] \sqrt{T-s}dv \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds. \end{aligned} \quad (50)$$

### 5.3.1 Case $T^* < 0$

For $z > 0$, we know from section 5.1 that for this case:

$$I_1 = \int_0^T e^{-rs} \max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds = 0. \quad (51)$$

From section 5.2, especially subsection 5.2.1, we know that for this case:

$$\begin{aligned} I_2 &= \int_0^T e^{-rs} \int_0^\sigma Ke^{-r(T-s)} n\left[\frac{-k+\mu(T-s)}{v\sqrt{T-s}}\right] \sqrt{T-s}dv \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \\ &= \frac{\sigma^2 Ke^{-rT}}{2} \int_0^T I_3 dt \\ &= \frac{\sigma^2 Ke^{-rT}}{2} \int_0^T e^{-\frac{2\mu x}{\sigma^2}} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] dt \\ &= \frac{\sigma^2 Ke^{-rT}}{2} e^{-\frac{2\mu x}{\sigma^2}} \int_0^T \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] dt. \end{aligned} \quad (52)$$

It can be shown (see appendix 4) that:

$$\begin{aligned} I_2 &= Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\ &\quad -Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]. \quad (53) \end{aligned}$$

Hence the value at $t=0$ of a down-and-in call is:

$$\begin{aligned} DIC_0 &= I_1 + I_2 \\ &= Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\ &\quad -Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]. \quad (54) \end{aligned}$$

As expected, this is identical to the formula of Rubinstein and Reiner (1991).

### 5.3.2 Case $T \geq T^* \geq 0$

We know from section 5.1 that for this case:

$$\begin{aligned} I_1 &= \int_0^T e^{-rs}\max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right]\frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \\ &= Se^{-qT}N[e_1] - Ke^{-rT}N[e_2] \\ &\quad + Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N[e_3] - Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N[e_4], \quad (55) \end{aligned}$$

where

$$e_1 = \frac{\ln(B/S)-(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}, \quad e_2 = e_1 + \sigma\sqrt{T^*}$$

$$e_3 = \frac{\ln(B/S)+(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}, \quad e_4 = e_3 - \sigma\sqrt{T^*}.$$

From section 5.2, especially subsection 5.2.2, we know that for this case:

$$
\begin{aligned}
I_2 &= \int_0^T e^{-rs}\int_0^\sigma Ke^{-r(T-s)}n\left[\frac{-k+\mu(T-s)}{v\sqrt{T-s}}\right]\sqrt{T-s}dv\frac{x}{\sqrt{\sigma^2 s^3}}n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right]ds \\
&= \frac{\sigma^2 Ke^{-rT}}{2}\int_0^T I_3 dt \\
&= \frac{\sigma^2 Ke^{-rT}}{2}\int_0^T e^{-\frac{2\mu x}{\sigma^2}}\frac{1}{\sqrt{\sigma^2 t}}n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] \\
&\quad -\left(e^{\frac{-2\mu x}{\sigma^2}}\frac{1}{\sqrt{\sigma^2 t}}n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right]-\frac{1}{\sqrt{\sigma^2 t}}n\left[\frac{-z+x-(b-\mu)t}{\sigma\sqrt{t}}\right]\right)1_{\{t\leq T^*\}}dt \\
&= \frac{\sigma^2 Ke^{-rT}}{2}e^{\frac{-2\mu x}{\sigma^2}}\int_0^T \frac{1}{\sqrt{\sigma^2 t}}n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right]dt \\
&\quad -\frac{\sigma^2 Ke^{-rT}}{2}e^{\frac{-2\mu x}{\sigma^2}}\int_0^{T^*} \frac{1}{\sqrt{\sigma^2 t}}n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right]dt \\
&\quad -\frac{\sigma^2 Ke^{-rT}}{2}\int_0^{T^*} \frac{1}{\sqrt{\sigma^2 t}}n\left[\frac{-z+x-(b-\mu)t}{\sigma\sqrt{t}}\right]dt. \qquad (56)
\end{aligned}
$$

It can be shown (see appendix 5) that this expression is equal to:

$$
\begin{aligned}
I_2 &= Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
&\quad -Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
&\quad -Se^{-qT}N\left[\frac{\ln\left(\frac{B}{S}\right)-(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right]+Ke^{-rT}N\left[\frac{\ln\left(\frac{B}{S}\right)-(r-q-\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right] \\
&\quad -Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B}{S}\right)+(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right] \\
&\quad +Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B}{S}\right)+(r-q-\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right]. \qquad (57)
\end{aligned}
$$

Hence the value at $t = 0$ of a down-and-in call is:

$$\begin{aligned}
DIC_0 &= I_1 + I_2 \\
&= Se^{-qT} N\left[e_1\right] - Ke^{-rT} N\left[e_2\right] \\
&\quad + Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[e_3\right] - Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[e_4\right] \\
&\quad + Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
&\quad - Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
&\quad - Se^{-qT} N\left[e_1\right] + Ke^{-rT} N\left[e_2\right] \\
&\quad - Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[e_3\right] + Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[e_4\right], \qquad (58)
\end{aligned}$$

where

$$e_1 = \frac{\ln(B/S)-(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}, \quad e_2 = e_1 + \sigma\sqrt{T^*}$$

$$e_3 = \frac{\ln(B/S)+(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}, \quad e_4 = e_3 - \sigma\sqrt{T^*}.$$

This simplifies because of cancellation of some terms to:

$$\begin{aligned}
DIC_0 &= Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
&\quad - Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]. \qquad (59)
\end{aligned}$$

As expected, this is identical to the formula of Rubinstein and Reiner (1991).

### 5.3.3 Case $T^* > T > 0$

We know from section 5.1 that for this case:

$$I_1 = \int_0^T e^{-rs} \max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds = 0. \quad (60)$$

From section 5.2, especially subsection 5.2.3, we know that for this case:

$$\begin{aligned} I_2 &= \int_0^T e^{-rs} \int_0^\sigma Ke^{-r(T-s)} n\left[\frac{-k+\mu(T-s)}{v\sqrt{T-s}}\right] \sqrt{T-s} dv \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right] ds \\ &= \frac{\sigma^2 Ke^{-rT}}{2} \int_0^T I_3 dt \\ &= \frac{\sigma^2 Ke^{-rT}}{2} \int_0^T e^{-\frac{2\mu x}{\sigma^2}} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] \\ &\quad - \left(e^{\frac{-2\mu x}{\sigma^2}} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] - \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{-z+x-(b-\mu)t}{\sigma\sqrt{t}}\right]\right) 1_{\{t>T^*\}} dt \\ &= \frac{\sigma^2 Ke^{-rT}}{2} e^{\frac{-2\mu x}{\sigma^2}} \int_0^T \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right] dt, \end{aligned} \quad (61)$$

since $T^* > T$.

As in subsection 5.3.1, it can be shown (see appendix 4) that this expression is equal to:

$$\begin{aligned} I_2 &= Be^{-qT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\ &\quad -Ke^{-rT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]. \end{aligned} \quad (62)$$

Hence the value at $t = 0$ of a down-and-in call is:

$$\begin{aligned} DIC_0 &= I_1 + I_2 \\ &= Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\ &\quad -Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]. \quad (63) \end{aligned}$$

As expected, this is identical to the formula of Rubinstein and Reiner (1991).

### 5.3.4 Case $r = q$

For $r = q$, we know from section 5.1 that for this case:

$$I_1 = \int_0^T e^{-rs} \max\left[0, Be^{-q(T-s)} - Ke^{-r(T-s)}\right] \frac{x}{\sqrt{\sigma^2 s^3}} n\left[\frac{x + \mu s}{\sigma\sqrt{s}}\right] ds = 0. \quad (64)$$

From section 5.2, especially subsection 5.2.4, we know that for this case:

$$\begin{aligned} I_2 &= \frac{\sigma^2 Ke^{-rT}}{2}\int_0^T I_3 dt \\ &= \frac{\sigma^2 Ke^{-rT}}{2}\int_0^T e^{-k}\frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{k + x - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right] dt \\ &= \frac{\sigma^2 Ke^{-rT}}{2}\left(\frac{B}{K}\right)\int_0^T \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{\ln\left(\frac{K}{B}\right) + \ln\left(\frac{S}{B}\right) - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right] dt. \quad (65) \end{aligned}$$

Applying formula (3) of appendix 1, the following result is obtained:

$$I_2 = Be^{-qT} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right] - Ke^{-rT}\left(\frac{S}{B}\right) N\left[\frac{\ln\left(\frac{B^2}{SK}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right]. \quad (66)$$

Hence:

$$DIC_0 = Be^{-qT} N \left[ \frac{\ln\left(\frac{B^2}{SK}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right] - Ke^{-rT} \left( \frac{S}{B} \right) N \left[ \frac{\ln\left(\frac{B^2}{SK}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right]. \quad (67)$$

This is exactly the Rubinstein and Reiner (1991) formula for $r = q$.

# 6 Conclusion

In this paper, an alternative approach to the valuation of barrier options is presented that relies on the first passage time to the barrier. This extends the work of El Karoui and Jeanblanc (1999), where only an integral formula is given. The lateral Chapman-Kolmogorov relation is used as an important tool in order to prove that the integral formula indeed yields the original formula given in Rubinstein and Reiner (1991). The advantage of this approach is that it allows for the valuation of barrier options with general payoffs and that it is valid for any continuous Markovian underlying stochastic process if the first passage time density is known. The disadvantage is that it may involve more difficult calculations than the original approach by Rubinstein and Reiner (1991).

# 7 References

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# 8 Appendices

## 8.1 Appendix 1: Some useful formulas

**Formula (1)** (see e.g. Berger (1996)):
For general $\alpha$, $\beta$, $\gamma$ and for $n[y] = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$:

$$e^{-\gamma s} n\left[\frac{\alpha}{\sqrt{s}} + \beta\sqrt{s}\right] = e^{-\alpha\beta+\alpha\sqrt{\beta^2+2\gamma}} n\left[\frac{\alpha}{\sqrt{s}} + \sqrt{\beta+2\gamma}\sqrt{s}\right]. \tag{68}$$

**Formula (2)** (see e.g. Berger (1996)):
For general $\alpha$, $\beta$, $T$ and for $n[y] = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$ and where $N[\bullet]$ is the cumulative standard normal distribution function:

$$\int_0^T \frac{1}{\sqrt{s^3}} n\left[\frac{\alpha+\beta s}{\sqrt{s}}\right] ds = \frac{1}{|\alpha|}\left\{N\left[\frac{-|\alpha| - \mathrm{sgn}[\alpha]\beta\, T}{\sqrt{T}}\right] + e^{-2\alpha\beta} N\left[\frac{-|\alpha| + \mathrm{sgn}[\alpha]\beta\, T}{\sqrt{T}}\right]\right\}. \tag{69}$$

**Formula (3)** (see e.g. Berger (1996)):
For general $\alpha$, $\beta$, $T$ and for $n[y] = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$ and where $N[\bullet]$ is the cumulative standard normal distribution function:

$$\int_0^T \frac{1}{\sqrt{s}} n\left[\frac{\alpha+\beta s}{\sqrt{s}}\right] ds = \frac{\mathrm{sgn}[\alpha]}{\beta}\left\{N\left[\frac{|\alpha| + \mathrm{sgn}[\alpha]\beta\, T}{\sqrt{T}}\right] + e^{-2\alpha\beta} N\left[\frac{-|\alpha| + \mathrm{sgn}[\alpha]\beta\, T}{\sqrt{T}}\right] - 1\right\}. \tag{70}$$

**Formula (4)** (see e.g. Berger (1996)):
For general $\alpha$, $\beta$ and for $n[y] = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$:

$$n\left[\frac{\alpha}{\sqrt{s}} - \beta\sqrt{s}\right] = e^{2\alpha\beta} n\left[\frac{\alpha}{\sqrt{s}} + \beta\sqrt{s}\right]. \tag{71}$$

## 8.2 Appendix 2: Trick for the vega of a European call

Remember that $k = \ln(K/B)$. We have to show that

$$\begin{aligned}
Be^{-q(T-s)} n\left[\frac{-k + (\mu+\sigma^2)(T-s)}{\sigma\sqrt{T-s}}\right] &= Be^{-q(T-s)} n[d_1] \\
&= Ke^{-r(T-s)} n\left[\frac{-k + \mu(T-s)}{\sigma\sqrt{T-s}}\right] \\
&= Ke^{-r(T-s)} n[d_2] \\
&= Ke^{-r(T-s)} n\left[\frac{-k + (\mu+\sigma^2)(T-s)}{\sigma\sqrt{T-s}} - \sigma\sqrt{T-s}\right] \\
&= Ke^{-r(T-s)} n[d_1 - \sigma\sqrt{T-s}].
\end{aligned} \tag{72}$$

Consider now the following expression:

$$\begin{aligned}
& \ln\left(\frac{Bn[d_1]}{e^{-r(T-s)}Kn[d_1-\sigma\sqrt{T-s}]}\right) \\
= \; & \ln(B)+\ln\left(n[d_1]\right)+(r-q)(T-s)-\ln(K)-\ln(n[d_1-\sigma\sqrt{T-s}]) \\
= \; & \ln(B)-\frac{1}{2}d_1^2+(r-q)(T-s)-\ln(K)+\frac{1}{2}(d_1-\sigma\sqrt{T-s})^2 \\
= \; & \ln(B)-\frac{1}{2}d_1^2+(r-q)(T-s)-\ln(K)+\frac{1}{2}d_1^2-d_1\sigma\sqrt{T-s}+\frac{1}{2}\sigma^2(T-s) \\
= \; & \ln(B)-\ln(K)+(r-q+\frac{1}{2}\sigma^2)(T-s)-d_1\sigma\sqrt{T-s}=0. \qquad (73)
\end{aligned}$$

## 8.3 Appendix 3: Detailed computation of $\hat{I}_1$

$$\begin{aligned}
\hat{I}_1 &= \int_0^{T^*} e^{-rs}(Be^{-q(T-s)}-Ke^{-r(T-s)})\frac{x}{\sqrt{\sigma^2 s^3}}n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right]ds \\
&= Be^{-qT}\int_0^{T^*} e^{-(r-q)s}\frac{x}{\sqrt{\sigma^2 s^3}}n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right]ds - Ke^{-rT}\int_0^{T^*}\frac{x}{\sqrt{\sigma^2 s^3}}n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right]ds. \qquad (74)
\end{aligned}$$

Making use of formula (1) of appendix 1, we have:

$$\begin{aligned}
\hat{I}_1 &= Be^{-qT}\frac{x}{\sigma}\int_0^{T^*} e^{-\frac{x\mu}{\sigma^2}+\frac{x}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2}+2(r-q)}}\frac{1}{\sqrt{s^3}}n\left[\frac{x+\sqrt{\mu^2+2(r-q)}s}{\sigma\sqrt{s}}\right]ds \\
&\quad -Ke^{-rT}\int_0^{T^*}\frac{x}{\sqrt{\sigma^2 s^3}}n\left[\frac{x+\mu s}{\sigma\sqrt{s}}\right]ds. \qquad (75)
\end{aligned}$$

Making use of the formula (2) of appendix 1, we have:

$$\begin{aligned}
\hat{I}_1 &= Be^{-qT}\frac{x}{\sigma}e^{-\frac{x\mu}{\sigma^2}+\frac{x}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2}+2(r-q)}}\frac{1}{\sqrt{s^3}}\frac{\sigma}{x}\left(N\left[\frac{-x-\sqrt{\mu^2+2(r-q)\sigma^2}T^*}{\sigma\sqrt{T^*}}\right]\right.\\
&\quad\left.+\left(\frac{S}{B}\right)^{\frac{-2(r-q)}{\sigma^2}-1}N\left[\frac{-x+\sqrt{\mu^2+2(r-q)\sigma^2}T^*}{\sigma\sqrt{T^*}}\right]\right)\\
&\quad -Ke^{-rT}\left(N\left[\frac{-x-(r-q-\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right]+\left(\frac{S}{B}\right)^{\frac{-2(r-q)}{\sigma^2}+1}N\left[\frac{-x+(r-q-\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right]\right).
\end{aligned} \tag{76}$$

After simplification and the use of $x = \ln(S/B)$, $\mu = (r-q-\frac{1}{2}\sigma^2)$ and $\sqrt{\mu^2+2(r-q)\sigma^2} = (r-q+\frac{1}{2}\sigma^2)$, we finally obtain the result:

$$\begin{aligned}
\hat{I}_1 &= Se^{-q(T-s)}N\left[\frac{\ln\left(\frac{B}{S}\right)-(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right]-Ke^{-r(T-s)}N\left[\frac{\ln\left(\frac{B}{S}\right)-(r-q-\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right]\\
&\quad +Be^{-q(T-s)}\left(\frac{B}{S}\right)^{\frac{-2(r-q)}{\sigma^2}}N\left[\frac{\ln\left(\frac{B}{S}\right)+(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right]\\
&\quad -Ke^{-r(T-s)}\left(\frac{B}{S}\right)^{\frac{-2(r-q)}{\sigma^2}+1}N\left[\frac{\ln\left(\frac{B}{S}\right)+(r-q-\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right].
\end{aligned} \tag{77}$$

## 8.4 Appendix 4: Detailed computation of $I_2$ for cases 1 and 3

We start with the equation:

$$\begin{aligned}
I_2 &= \frac{\sigma^2Ke^{-rT}}{2}e^{-\frac{2\mu x}{\sigma^2}}\int_0^T\frac{1}{\sqrt{\sigma^2 t}}n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right]dt\\
&= \frac{\sigma Ke^{-rT}}{2}e^{-\frac{2\mu x}{\sigma^2}}\int_0^T\frac{1}{\sqrt{t}}n\left[\frac{z+x+(b-\mu)t}{\sigma\sqrt{t}}\right]dt.
\end{aligned} \tag{78}$$

Using formula (3) of appendix 1, we have

$$\begin{aligned} I_2 &= \frac{\sigma}{2} K e^{-rT} e^{-\frac{2\mu x}{\sigma^2}} \left( \frac{2\mathrm{sgn}\left[(x+z)\frac{1}{\sigma}\right]}{\sigma} \left\{ N\left[\frac{|x+z|}{\sigma\sqrt{T}} + \mathrm{sgn}\left[(x+z)\frac{1}{\sigma}\right]\frac{1}{2}\sigma^2 T\right] \right.\right. \\ &\left.\left. + e^{-(x+z)} N\left[\frac{-|x+z|}{\sigma\sqrt{T}} + \mathrm{sgn}\left[(x+z)\frac{1}{\sigma}\right]\frac{1}{2}\sigma^2 T\right] - 1 \right\}\right), \end{aligned} \tag{79}$$

and using the fact that sgn $\left[(x+z)\frac{1}{\sigma}\right]$ is positive for $z > 0$, we get:

$$I_2 = K e^{-rT} e^{-\frac{2\mu x}{\sigma^2}} \left\{ N\left[\frac{x+z}{\sigma\sqrt{T}} + \frac{1}{2}\sigma^2 T\right] + e^{-(x+z)} N\left[\frac{-(x+z)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma^2 T\right] - 1 \right\}. \tag{80}$$

Applying $N[-y] = 1 - N[y] \Leftrightarrow N[-y] - 1 = -N[y] \Leftrightarrow -N[-y] + 1 = N[y]$,
we obtain:

$$I_2 = K e^{-rT} e^{-\frac{2\mu x}{\sigma^2}} \left\{ -N\left[\frac{-(x+z)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma^2 T\right] + e^{-(x+z)} N\left[\frac{-(x+z)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma^2 T\right] \right\}. \tag{81}$$

Re-establishing the original notation:

$$\begin{aligned} I_2 &= K e^{-rT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} \left( \left(\frac{B^2}{SK}\right) e^{-(r-q)T} N\left[\frac{-\ln\left(\frac{SK}{B^2}\right) + (r-q)T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma^2 T\right] \right. \\ &\left. - N\left[\frac{-\ln\left(\frac{SK}{B^2}\right) + (r-q)T}{\sigma\sqrt{T}} - \frac{1}{2}\sigma^2 T\right]\right). \end{aligned} \tag{82}$$

Then:

$$\begin{aligned} I_2 &= B e^{-qT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\ &\quad - K e^{-rT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\ &= DIC_0. \end{aligned} \tag{83}$$

## 8.5 Appendix 5: Detailed computation of $I_2$ for case 2

We start with:

$$
\begin{aligned}
I_2 \;=\; & \frac{\sigma^2 K e^{-rT}}{2} e^{\frac{-2\mu x}{\sigma^2}} \int_0^T \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z + x + (b-\mu)t}{\sigma\sqrt{t}}\right] dt \\
& - \frac{\sigma^2 K e^{-rT}}{2} e^{\frac{-2\mu x}{\sigma^2}} \int_0^{T^*} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{z + x + (b-\mu)t}{\sigma\sqrt{t}}\right] dt \\
& + \frac{\sigma^2 K e^{-rT}}{2} \int_0^{T^*} \frac{1}{\sqrt{\sigma^2 t}} n\left[\frac{-z + x - (b-\mu)t}{\sigma\sqrt{t}}\right] dt. \qquad (84)
\end{aligned}
$$

Re-establishing the original notation and using formula (3) of appendix 1, we have:

$$
\begin{aligned}
I_2 \;=\; & K e^{-rT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} \left( -N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q)T - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right] \right. \\
& \left. - \frac{B^2}{SK} e^{(r-q)T} N\left[\frac{\ln\left(\frac{SK}{B^2}\right) - (r-q)T - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right] + 1 \right) \\
- \; & K e^{-rT} \left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} \left( -N\left[\frac{\ln\left(\frac{B^2}{SK}\right) + (r-q)T - \frac{1}{2}\sigma^2 T^*}{\sigma\sqrt{T^*}}\right] \right. \\
& \left. - \frac{B^2}{SK} e^{(r-q)T} N\left[\frac{\ln\left(\frac{SK}{B^2}\right) - (r-q)T - \frac{1}{2}\sigma^2 T^*}{\sigma\sqrt{T^*}}\right] + 1 \right) \\
+ \; & K e^{-rT} \left( -N\left[\frac{\ln\left(\frac{S}{K}\right) + (r-q)T - \frac{1}{2}\sigma^2 T^*}{\sigma\sqrt{T^*}}\right] \right. \\
& \left. - \frac{S}{K} e^{(r-q)T} N\left[\frac{\ln\left(\frac{K}{S}\right) - (r-q)T - \frac{1}{2}\sigma^2 T^*}{\sigma\sqrt{T^*}}\right] + 1 \right). \qquad (85)
\end{aligned}
$$

Then we get:

$$
\begin{aligned}
I_2 \;=\;& Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
& -Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
& +Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q)T+\frac{1}{2}\sigma^2T^*}{\sigma\sqrt{T^*}}\right] \\
& -Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q)T-\frac{1}{2}\sigma^2T^*}{\sigma\sqrt{T^*}}\right] \\
& +Se^{-qT}N\left[\frac{\ln\left(\frac{K}{S}\right)-(r-q)T-\frac{1}{2}\sigma^2T^*}{\sigma\sqrt{T^*}}\right]-Ke^{-rT}N\left[\frac{\ln\left(\frac{K}{S}\right)-(r-q)T+\frac{1}{2}\sigma^2T^*}{\sigma\sqrt{T^*}}\right].
\end{aligned}
\tag{86}
$$

Using the definition of $T^* = \frac{\ln\left(\frac{B}{K}\right)+(r-q)T}{r-q}$, we finally obtain:

$$
\begin{aligned}
I_2 \;=\;& Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
& -Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B^2}{SK}\right)+(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \\
& +Be^{-qT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}} N\left[\frac{\ln\left(\frac{B}{S}\right)+(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right] \\
& -Ke^{-rT}\left(\frac{S}{B}\right)^{-\frac{2(r-q)}{\sigma^2}+1} N\left[\frac{\ln\left(\frac{B}{S}\right)+(r-q-\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right] \\
& +Se^{-qT}N\left[\frac{\ln\left(\frac{B}{S}\right)-(r-q+\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right]-Ke^{-rT}N\left[\frac{\ln\left(\frac{B}{S}\right)-(r-q-\frac{1}{2}\sigma^2)T^*}{\sigma\sqrt{T^*}}\right].
\end{aligned}
\tag{87}
$$