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Abstract

This paper derives necessary and sufficient conditions for the existence of linear equilibria in the Rochet-Vila model of market making. In contrast to most previous work on the existence of linear equilibria in models of market making, we do not impose independence of the underlying random variables. For distributions that are determined by their moments we show that a linear equilibrium exists if and only if the joint distribution of noise trade and asset payoff is elliptical.

Keywords: Market Microstructure, Market Making, Linear Equilibria

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ON THE EXISTENCE OF LINEAR EQUILIBRIA IN THE ROCHET-VILA MODEL OF MARKET MAKING

By Georg Nöldeke and Thomas Tröger

1. Introduction

Bagnoli, Viswanathan and Holden (2001) have presented a detailed investigation of linear equilibria in models of market making that follow either Kyle (1985) or the extension of the Glosten and Milgrom (1985) model due to Easley and O'Hara (1987). The study of linear equilibria in these models is important as most of the market microstructure literature (See O'Hara (1995) for a survey) focusses on such linear equilibria to understand the effects of differences in market microstructure on key variables such as liquidity, the informational content of prices and the ability of informed traders to profit from their private information. By deriving necessary and sufficient conditions under which linear equilibria exist, Bagnoli, Viswanathan and Holden delineate the range of distributions for which the implications obtained from such studies of linear equilibria remain valid.

This paper extends the analysis in Bagnoli, Viswanathan and Holden (2001). We study the existence of linear equilibria in Rochet and Vila's (1994) model of market making. The Rochet-Vila model differs from the more familiar Kyle (1985) model in that a risk-neutral strategic trader observes not only private information about the asset payoff but also the liquidity traders' orders (noise trade) before submitting his own market order. Market makers observe the net order flow, choose prices and then execute every order.[1] In contrast to the analysis in Bagnoli, Viswanathan and Holden, we do not require the underlying random variables to be independent; indeed, we allow for an arbitrary degree of correlation between asset payoff and noise trade. While most models of market making assume independent random variables, this assumption is rather restrictive and has no compelling justification. The Rochet-Vila model provides a convenient starting point for an investigation of models of market making with dependent random variables, as the dependence does not enter the strategic

[1] As demonstrated by Rochet and Vila this model is equivalent to the Kyle (1989) model of competition in a limit order market when there is one risk neutral informed trader and many risk-neutral market makers.

traders' problem, but only affects the inference problem of the market makers.

For the special case of independent random variables, the conditions we obtain for the existence of linear equilibria in the Rochet-Vila model are essentially equivalent to those obtained by Bagnoli, Viswanathan and Holden (2001, Section 3) for the Kyle model. It then follows from Nöldeke and Tröger (2001) that given independent random variables, a linear equilibrium exists in the Rochet-Vila model for two different numbers of strategic traders if and only if the random variables are normally distributed.

Relaxing the independence assumption has profound implications for the existence of linear equilibria. Our main result shows that for distributions that are determined by their moments linear equilibria in the Rochet-Vila model exist for any number of strategic traders if and only if the random variables are elliptically distributed.[2] The most closely related result in the literature is due to Foster and Viswanathan (1993). They show the existence of linear equilibria in the Kyle model for any number of informed traders when asset payoff and noise trading are semi-independent and elliptically distributed. The important difference to our result lies in our demonstration that the restriction to elliptically distributed random variables is not only sufficient but necessary for the existence of linear equilibrium.

The remainder of this paper is organized as follows. Section 2 describes the extension of the Rochet-Vila model to the case of N strategic traders and introduces the definition of linear equilibrium. Section 3 presents a characterization of linear equilibria from which we obtain a necessary and sufficient condition for the existence of linear equilibria for a given number of strategic traders. Section 4 characterizes those distributions which satisfy this condition for any number of informed traders. Section 5 concludes with a discussion. Proofs are in the Appendix.

2. The Model

The model extends Rochet and Vila (1994) by allowing for multiple strategic traders. There are three types of traders: noise traders, risk neutral market makers, and $N \geq 1$ risk neutral strategic traders. The aggregate quantity traded by noise traders and the payoff of the risky asset are given by exogenous random variables. Noise trading is denoted by $\tilde{u}$. The payoff of the

[2]See Fang, Kotz and Ng (1990) for a detailed study of elliptical distributions. Foster and Viswanathan (1993) present many examples of elliptical distributions and provide references to applications in financial economics. A formal definition is given in Section 4.

risky asset is denoted by $\tilde{v}$. In contrast to most of the existing literature we do not assume that $\tilde{u}$ and $\tilde{v}$ are independent. We do however require the following assumption.

Assumption 1 *The distribution of* $(\tilde{u}, \tilde{v})$ *has finite second moments satisfying* $\sigma_u^2 = Var[\tilde{u}] > 0$, $\sigma_v^2 = Var[\tilde{v}] > 0$, *and* $|\sigma_{uv}| < \sigma_u \sigma_v$, *where* $\sigma_{uv} = Cov[\tilde{u}, \tilde{v}]$.

The assumption that second moments are finite is necessary for a linear equilibrium as defined below to exist.[3] The additional assumptions exclude (trivial) special cases in which the distribution has a one-dimensional support.

All strategic traders observe the realization of $(\tilde{u}, \tilde{v})$ and then simultaneously decide on the market order they submit. A strategy for the strategic trader $n = 1, \cdots, N$ is given by a Lebesgue measurable function $X_n : \mathbb{R}^2 \to \mathbb{R}$, determining his market order as a function of the observed values of the underlying random variables. For a given strategy X_n, let $\tilde{x}_n = X_n(\tilde{u}, \tilde{v})$. A strategy combination $(X_1, \cdots, X_N)$ determines the order flow as $\tilde{y} = \sum_n \tilde{x}_n + \tilde{u}$.

Market makers observe the realization of the order flow, but not any of its components, and engage in a competitive auction to serve the order flow. The outcome of this competition is described by a Lebesgue measurable function $P : \mathbb{R} \to \mathbb{R}$, called the pricing rule. Given $(P, X_1, \cdots, X_n)$ define $\tilde{p} = P(\tilde{y})$ and let $\tilde{\pi}_n = (\tilde{v} - \tilde{p})\tilde{x}_n$ denote the resulting trading profit of insider n. To ensure that the expected profit of an insider is well-defined for all feasible $(P, X_1, \cdots, X_n)$, we restrict the strategy set of an insider to $\mathcal{X} = \{X_n : \mathbb{R}^2 \to \mathbb{R} \mid E[\tilde{x}_n^2] < \infty\}$ and the set of pricing rules to $\mathcal{P} = \{P \mid \forall (X_1, \cdots, X_N) \in \mathcal{X}^N : E[\tilde{p}^2] < \infty\}$.

The equilibrium conditions are that the competition between market makers drives their expected profits to zero conditional on the order flow and that each strategic trader chooses his trading strategy to maximize his expected profits.

Definition 1 $(P, X_1, \cdots, X_N) \in \mathcal{X}^N \times \mathcal{P}$ *is an equilibrium for the model* $(\tilde{u}, \tilde{v}, N)$ *if*

$$E[\tilde{v} - \tilde{p} \mid \tilde{y}] = 0 \tag{1}$$

[3] See Bagnoli, Viswanathan and Holden (2001) for an alternative definition of equilibrium under which they obtain existence of linear equilibria in models of market making for distributions of $(\tilde{u}, \tilde{v})$ without finite second moments.

and, for all n and $X \in \mathcal{X}$,

$$E[\tilde{\pi}_n] \geq E[(\tilde{v} - P(\sum_{m \neq n} \tilde{x}_m + X(\tilde{u}, \tilde{v}) + \tilde{u}))X(\tilde{u}, \tilde{v})]. \tag{2}$$

It will be convenient to refer to $(\tilde{p}, \tilde{x}_1, \cdots, x_N)$ as an equilibrium outcome of the model $(\tilde{u}, \tilde{v}, N)$ if there exists an equilibrium $(P, X_1, \cdots, X_N)$ such that $\tilde{p} = P(\tilde{y})$ and $\tilde{x}_n = X_n(\tilde{u}, \tilde{v})$ for all n.

We are especially interested in those equilibria where the pricing rule is a linear function of the observed order flow.[4]

Definition 2 *An equilibrium $(P, X_1, \cdots, X_N)$ for the model $(\tilde{u}, \tilde{v}, N)$ is linear if there exist $\mu, \lambda \in I\!R$ such that*

$$\forall y : P(y) = \mu + \lambda y.$$

3. Existence of Linear Equilibria for given N

We begin our analysis by obtaining a characterization of linear equilibria. Let $\rho = \sigma_{uv}/\sigma_u\sigma_v$ denote the coefficient of correlation between $\tilde{u}$ and $\tilde{v}$. By Assumption 1 we have $| \rho |< 1$. Let

$$\lambda_N = \frac{\sigma_v}{\sigma_u} k_N, \tag{3}$$

where

$$k_N = \sqrt{\frac{(N-1)^2\rho^2}{4} + N} - \frac{(N-1)\rho}{2}, \tag{4}$$

and let

$$\mu_N = E[\tilde{v}] - \lambda_N E[\tilde{u}]. \tag{5}$$

Proposition 1 *If $(P, X_1, \cdots, X_N)$ is a linear equilibrium for the model $(\tilde{u}, \tilde{v}, N)$ then*

$$\forall y : \ P(y) = \mu_N + \lambda_N y.$$

The resulting equilibrium outcome $(\tilde{p}, \tilde{x}_1, \cdots \tilde{x}_n)$ satisfies

$$\tilde{p} = \tilde{v} - \frac{1}{N+1} (\tilde{v} - \lambda_N \tilde{u} - \mu_N) \tag{6}$$

[4]To ease comparison with the existing literature, we follow the convention of referring to any affine function as a linear function throughout this paper.

and

$$\tilde{x}_n = \frac{1}{\lambda_N(N+1)} (\tilde{v} - \lambda_N \tilde{u} - \mu_N), \quad n = 1, \cdots, N, \tag{7}$$

with resulting order flow

$$\tilde{y} = \frac{1}{\lambda_N(N+1)} (N(\tilde{v} - \mu_N) + \lambda_N \tilde{u}). \tag{8}$$

The proof of Proposition 1 proceeds by showing that given any linear pricing rule $P(y) = \mu + \lambda y$ the strategic traders will choose strategies satisfying the counterpart to (7) for the parameters μ and λ. This results in an order flow that is the counterpart to (8). From the market efficiency condition (1) it is immediate that every equilibrium must satisfy $E[\tilde{p}] = E[\tilde{v}]$ and $E[(\tilde{v} - \tilde{p})\tilde{y}] = 0$. These two equations determine the parameters of the pricing rule as given in the proposition.

Proposition 1 shows that whenever it exists, a linear equilibrium is unique in the sense that every linear equilibrium results in the same equilibrium outcome.[5] Concerning the interpretation of the above characterization of linear equilibria we content ourselves with the observation that if a linear equilibrium exists it is completely characterized by the distributional parameters $E[\tilde{u}], E[\tilde{v}], \sigma_u, \sigma_v, \rho$ and the number of strategic traders N. In particular if a linear equilibrium exists for a given $(\tilde{u}, \tilde{v}, N)$ the pricing rule coincides with the one in a model in which $\tilde{u}$ and $\tilde{v}$ are normally distributed (in which a linear equilibrium exists; see Theorem 1 below).

Proposition 1 can be used to obtain a simple necessary and sufficient condition for the existence of a linear equilibrium for given N. To do so, it is convenient to consider a linear transformation of the underlying random variables. Let $\tilde{u}^* = (\tilde{u} - E[\tilde{u}])/\sigma_u$ and $\tilde{v}^* = (\tilde{v} - E[\tilde{v}])/\sigma_v$ denote the standardization of the random variables $(\tilde{u}, \tilde{v})$. Furthermore, let

$$\tilde{z}^* = \frac{1}{\sqrt{1-\rho^2}} (\tilde{u}^* - \rho \tilde{v}^*).$$

The random variable $\tilde{z}^*$ is the (standardized) component of the noise trade which is orthogonal to the payoff of the asset. A straightforward calculation indeed verifies that $E[\tilde{z}^*] = 0$, $Var[\tilde{z}^*] = 1$ and $Cov[\tilde{v}^*, \tilde{z}^*] = 0$.

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[5] For $N = 1$, Rochet and Vila (1994) show that the equilibrium outcome of their model is unique. Consequently, Proposition 1 describes the unique equilibrium outcome of the Rochet-Vila model whenever a linear equilibrium exists. The question whether uniqueness of the equilibrium outcome in the Rochet-Vila model extends to the case $N > 1$ is beyond the scope of this paper.

Proposition 2 *A linear equilibrium exists in the model* $(\tilde{u}, \tilde{v}, N)$ *if and only if*

$$E[\alpha_N \tilde{z}^* - \tilde{v}^* \mid \tilde{z}^* + \alpha_N \tilde{v}^*] = 0, \tag{9}$$

where

$$\alpha_N = \frac{k_N \sqrt{1-\rho^2}}{1 - k_N \rho}. \tag{10}$$

The result in Proposition 2 has a simple interpretation in terms of the distribution of the underlying random variables if $\tilde{u}$ and $\tilde{v}$ are independent. In that case we have $\rho = 0$, implying $\tilde{z}^* = \tilde{u}^*$ and $k_N = \alpha_N = \sqrt{N}$ so that a linear equilibrium exists if and only if

$$E[\sqrt{N}\tilde{u}^* - \tilde{v}^* \mid \tilde{u}^* + \sqrt{N}v^*] = 0.$$

This is identical to the condition for the existence of linear equilibria in the Kyle model with N informed traders in Proposition 1 of Nöldeke and Tröger (2001). Under a fairly mild technical condition on the characteristic function of $\tilde{v}$ it can be shown (cf. Bagnoli, Viswanathan and Holden, 2001) that this condition is satisfied if and only if $\tilde{v}^*$ has the same distribution as

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \tilde{u}_n^*,$$

where the random variables $\tilde{u}_n^*$ are independent and have the same distribution as $\tilde{u}^*$.

Without assuming independence of $\tilde{u}$ and $\tilde{v}$ condition (9) has no such simple interpretation. We thus proceed to an investigation of the distributional implications of the requirement that a linear equilibrium exists for all N.

4. *Existence of Linear Equilibria for all N*

By Proposition 2, we must determine those distributions of $(\tilde{z}^*, \tilde{v}^*)$ or $(\tilde{u}, \tilde{v})$, respectively, that satisfy (9) for all N. To this end, the following definition is useful (Fang, Kotz and Ng, 1990).

Definition 3 *The distribution of a random variable* $(\tilde{a}, \tilde{b})$ *is rotation invariant (or spherical, or radially symmetric) if for all* $\eta \in I\!R$ *the random variable obtained from a rotation by the angle* η *around the origin,*

$$(\tilde{a}_\eta, \tilde{b}_\eta) = (\cos\eta\tilde{a} + \sin\eta\tilde{b}, -\sin\eta\tilde{a} + \cos\eta\tilde{b}),$$

has the same distribution as $(\tilde{a}, \tilde{b})$.

The distribution of any linear transformation of a rotation invariant random variable is elliptical.

Note that a distribution is rotation invariant if it can be decomposed into two independent distributions: the uniform distribution over directions in the plane, and a distribution over Euclidian distances from the origin. In this decomposition any distribution on $[0, \infty)$ can be used as a distribution over Euclidian distances, so that there is a one-to-one correspondence between distributions on $[0, \infty)$ and the set of rotation invariant distributions. For example, the normal distribution with unitary variance covariance matrix or the uniform distribution on any circle centered at the origin is rotation invariant. Therefore, any normal distribution or uniform distribution on an ellipse is elliptical. See Foster and Viswanathan (1993) for further examples.

Using standard results about characteristic functions it is straightforward to show that condition (9) holds for all N if the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is rotation invariant. As we show in the following proposition the reverse implication holds if the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is determined by its moments, that is all moments of the distribution exist and there is no other distribution with the same moments.

Proposition 3 *Suppose the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is determined by its moments. Then condition (9) holds for all N if and only if the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is rotation invariant.*

The proof of the "only if" part of the proposition uses the assumption that the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is determined by its moments. This implies that the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is determined by the higher order partial derivatives of its characteristic function at the origin. Hence, once we have shown that these derivatives are rotation invariant, we can conclude that the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is rotation invariant. More precisely, we have to show that at the origin $(s, t) = (0, 0)$ the higher order partial derivatives of $\Phi^\eta(s, t)$ are independent of η, where $\Phi^\eta(s, t)$ denotes the characteristic function of $(\tilde{z}^*_\eta, \tilde{v}^*_\eta)$.

In terms of the characteristic function of $(\tilde{z}^*, \tilde{v}^*)$, requiring condition (9) for all N provides a set of points (s, t) and η where the partial derivative $(\partial \Phi^\eta(s, t))/(\partial \eta)$ vanishes. In neighborhoods of the origin, this set is sufficiently rich such that for any rotation angle η, the higher order partial derivatives of $(\partial \Phi^\eta(s, t))/(\partial \eta)$ with respect to s and t vanish at $(s, t) = (0, 0)$.

Therefore, at $(s,t) = (0,0)$ the higher order partial derivatives of $\Phi^\eta(s,t)$ are independent of η, as was to be shown.

Using Proposition 3, it is not difficult to provide necessary and sufficient conditions for the existence of linear equilibria in terms of the distribution of $(\tilde{u}, \tilde{v})$. By construction, we have $(\tilde{u}, \tilde{v}) = T(\tilde{z}^*, \tilde{v}^*)$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an injective linear transformation. It is then straightforward to verify that the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is determined by its moments whenever this holds for the distribution of $(\tilde{u}, \tilde{v})$. Furthermore, it is immediate that the distribution of $(\tilde{u}, \tilde{v})$ is elliptical if and only if the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is rotation invariant. Hence, we obtain without further proof:

Theorem 1 *Suppose the distribution of $(\tilde{u}, \tilde{v})$ is determined by its moments. Then a linear equilibrium in the model $(\tilde{u}, \tilde{v}, N)$ exists for all N if and only if the distribution of $(\tilde{u}, \tilde{v})$ is elliptical.*

Remark: In the special case in which $\tilde{v}^*$ and $\tilde{z}^*$ are independent a stronger version of Proposition 3 can be obtained. The arguments from Nöldeke and Tröger (2001) can be applied to dispense with the requirement that the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is determined by its moments and obtain the conclusion that (9) holds for all N if and only if the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is normal. As the distribution of $(\tilde{u}, \tilde{v})$ is normal if and only if the distribution of $(\tilde{z}^*, \tilde{v}^*)$ is normal, one obtains the result that whenever the error term in the linear regression of $\tilde{u}$ on $\tilde{v}$ is independent of $\tilde{v}$ a linear equilibrium exists in the model $(\tilde{u}, \tilde{v}, N)$ for all N if and only if $(\tilde{u}, \tilde{v})$ are normally distributed.

5. Discussion

We have studied necessary and sufficient condition for the existence of linear equilibria in the Rochet-Vila model of market making. This complements Bagnoli, Viswanathan and Holden (2001), who study linear equilibria in other models of market making. Our results show that the conditions for existence of linear equilibria in the Rochet-Vila model are equivalent to those in the Kyle model in case asset payoff and noise trading are independent. Without the independence assumption we find that for distributions that are determined by their moments linear equilibria in the Rochet-Vila model exist for all numbers of informed traders if and only if the random variables are elliptically distributed.

To prove our characterization of the elliptically distributed class of random variables we rely on the assumption that the distribution of $(\tilde{u}, \tilde{v})$ is determined by its moments. While there are distributions which violate this assumption (the log-normal is a prominent example in the one-dimensional case), we note that in many cases of interest this assumption is satisfied. For instance, if the distribution of $(\tilde{u}, \tilde{v})$ has compact support, as it is assumed by Rochet and Vila (1994) except when they consider the example of normally distributed random variables, it is determined by its moments. More generally, Shohat and Tamarkin (1950) show that the distribution of $(\tilde{u}, \tilde{v})$ is determined by its moments if

$$\sum_{k=1}^{\infty}(E[\tilde{v}^{2k}] + E[\tilde{u}^{2k}])^{-1/2k} = \infty,$$

i.e., the higher order moments must not increase too quickly. Given that many of the familiar one-dimensional distributions are determined by their moments, the result in Peterson (1982), who shows that the distribution of $(\tilde{u}, \tilde{v})$ is determined by its moments if the marginal distributions of $\tilde{u}$ and $\tilde{v}$ are determined by their moments, is of particular interest. For more recent results on the multidimensional problem of moments we refer to Berg (1995).

Our method of proof does not work without the assumption that the distribution is determined by its moments: based on the observation that there are asymmetric one-dimensional distributions for which all moments exist and all odd moments are equal to zero, it is straightforward to construct an example of a bivariate distribution which is not rotation invariant, but for which all moments exist and are rotation invariant. It remains an open question whether a different approach could be used to broaden the class of distributions under which Theorem 1 remains true.

Another interesting extension of our analysis would be to investigate whether Theorem 1 holds true in the Kyle model with multiple informed traders. The difficulty in obtaining such a result for the Kyle model lies in excluding the possibility that linear equilibria exist in which strategic traders use non-linear strategies. The existence of such equilibria (which we have not been able to exclude in general) would invalidate the counterpart to Proposition 1 for the Kyle model. The additional assumption that $E[\tilde{u} \mid \tilde{v}]$ is linear in $\tilde{v}$ eliminates the difficulty, but also eliminates much of the interest in the problem. Nevertheless it is noteworthy that under this assumption the Kyle model will yield a counterpart to Proposition 2, albeit—unless the random variables are uncorrelated— with different values of α_N. This implies that even under the additional assumption mentioned above the conditions for the existence of linear equilibria in the Rochet-Vila model and in the Kyle model for given N are not the same.

APPENDIX

Proof of Proposition 1: Suppose $(P, X_1, \cdots, X_N)$ is a linear equilibrium for the model $(\tilde{u}, \tilde{v}, N)$ with $P(y) = \mu + \lambda y$. Because $X_n(u, v)$ is an equilibrium strategy and thus satisfies (2):

$$X_n(u,v) \in \arg\max_x \left(v - \mu - \lambda(\sum_{m \neq n} X_m(u,v) + u + x) \right) x, \quad (\tilde{u}, \tilde{v}) - \text{a.e..}$$

As $\tilde{v}$ is non-degenerate this condition can only be satisfied if $\lambda > 0$. For $\lambda > 0$ the first order condition characterizes a maximum and yields

$$\forall n : \tilde{v} - \mu - \lambda \left(\tilde{u} + \sum_m \tilde{x}_m + \tilde{x}_n \right) = 0. \tag{11}$$

Hence $\tilde{x}_n$ is independent of n. In particular, $\tilde{x}_m = \tilde{x}_n$ in (11), implying

$$\tilde{x}_n = \frac{\tilde{v} - \mu - \lambda \tilde{u}}{\lambda(N+1)}. \tag{12}$$

Consequently, the order flow is given by

$$\tilde{y} = \frac{1}{\lambda(N+1)} [N(\tilde{v} - \mu) + \lambda \tilde{u}] \tag{13}$$

and thus

$$\tilde{p} = \frac{1}{N+1} [N\tilde{v} + \mu + \lambda \tilde{u}]. \tag{14}$$

The market efficiency condition (1) implies

$$E[\tilde{v} - \tilde{p}] = 0 \tag{15}$$

and

$$E[(\tilde{v} - \tilde{p})\tilde{y}] = 0. \tag{16}$$

Using (14) and (15) we obtain

$$\mu = E[\tilde{v}] - \lambda E[\tilde{u}]. \tag{17}$$

Using (17) to eliminate μ from equations (13) and (14), (16) yields

$$N\sigma_v^2 - \lambda^2 \sigma_u^2 - \lambda(N-1)\sigma_{uv} = 0.$$

Solving the last equation for $\lambda > 0$ then yields the equilibrium value of λ_N as given by (3) and (4). Equation (5) is then immediate from (17) and (6) - (8) follow from (12) - (14).

Proof of Proposition 2: From Proposition 1 every linear equilibrium satisfies (6) and (8). Substituting into (1) we obtain

$$E[\tilde{v} - \lambda_N \tilde{u} - \mu_N \mid N\tilde{v} + \lambda_N \tilde{u}] = 0 \tag{18}$$

as a necessary condition for the existence of a linear equilibrium. Condition (18) is also sufficient for the existence of a linear equilibrium. To see that, note that given the pricing rule $P(y) = \mu_N + \lambda_N y$ the strategies

$$X_n(u, v) = \frac{1}{\lambda_N(N+1)}(v - \mu_N - \lambda_N u)$$

satisfy condition (2), cf. the proof of Proposition 1. By (18) the induced order flow $\tilde{y}$ and price $\tilde{p}$ then satisfy condition (1) and $(P, X_1, \cdots, X_n)$ is a linear equilibrium.

It thus remains to show that condition (18) is equivalent to (9). Using (3) it is straightforward to see that (18) is equivalent to

$$E[k_N \tilde{u}^* - \tilde{v}^* \mid k_N \tilde{u}^* + N\tilde{v}^*] = 0.$$

Substituting $\tilde{u}^* = \sqrt{1-\rho^2}\tilde{z}^* + \rho v^*$ we obtain the equivalent condition

$$E[k_N\sqrt{1-\rho^2}\tilde{z}^* - (1 - k_N\rho)\tilde{v}^* \mid k_N\sqrt{1-\rho^2}\tilde{z}^* + (N + \rho k_N)\tilde{v}^*] = 0. \tag{19}$$

Because k_N as defined by (4) satisfies the relation

$$N - k_N^2 - k_N(N-1)\rho = 0$$

we have $1 - k_N\rho \neq 0$ (as otherwise $\rho = 1$) and

$$\frac{k_N\sqrt{1-\rho^2}}{1 - k_N\rho} = \alpha_N = \frac{N + k_N\rho}{k_N\sqrt{1-\rho^2}}. \tag{20}$$

Using (20) one sees that (19) is equivalent to (9).

Proof of Proposition 3:

The following fact about the sequence defined by (10) is needed below.

$$\alpha_N \text{ is strictly increasing in } N. \tag{21}$$

To prove this, note first that α_N is strictly increasing in k_N. It thus suffices to show that k_N is strictly increasing in N. For $\rho \leq 0$ this is immediate from the definition of k_N. Hence, consider $\rho > 0$.

The function $k : [1, \infty) \to \mathbb{R}$ given by

$$k(x) = \sqrt{\frac{(x-1)^2\rho^2}{4} + x} - \frac{(x-1)\rho}{2}$$

is differentiable with derivative given by

$$k'(x) = \frac{\frac{(x-1)\rho^2}{2} + 1 - \rho\sqrt{\frac{(x-1)^2\rho^2}{4} + x}}{2\sqrt{\frac{(x-1)^2\rho^2}{4} + x}}.$$

From $\rho < 1$ we obtain

$$\left(\frac{(x-1)\rho^2}{2} + 1\right)^2 > \rho^2 \left(\frac{(x-1)^2\rho^2}{4} + x\right)$$

and thus $k'(x) > 0$. Consequently, $k(x)$ strictly increasing. As $k_N = k(N)$ for all $N \in \mathbb{N}$ the desired result (21) follows.

For any infinitely differentiable function f of two real variables we use the shortcuts ($s, t \in \mathbb{R}$, $m, n \in \mathbb{N}_0 = \{0, 1, \ldots\}$)

$$f_{(m)(n)}(s,t) = \frac{\partial^{m+n} f}{\partial s^m \partial t^n}(s,t), \quad f_{(m)(n)} = f_{(m)(n)}(0,0).$$

From Theorem 6.1.1. in Lukacs and Laha (1969, p. 103) condition (9) is satisfied for all N if and only if ($i = \sqrt{-1}$ denotes the imaginary unit)

$$\forall s, N: \; E[(\alpha_N \tilde{z}^* - \tilde{v}^*) e^{is(\tilde{z}^* + \alpha_N \tilde{v}^*)}] = 0.$$

Multiplying by i shows that this is equivalent to

$$\forall s, N: \alpha_N E[i\tilde{z}^* e^{is\tilde{z}^* + i\alpha_N s\tilde{v}^*}] = E[i\tilde{v}^* e^{is\tilde{z}^* + i\alpha_N s\tilde{v}}],$$

or

$$\forall s, N: \alpha_N \Phi_{(1)(0)}(s, \alpha_N s) = \Phi_{(0)(1)}(s, \alpha_N s), \tag{22}$$

where Φ denotes the characteristic function of $(\tilde{z}^*, \tilde{v}^*)$.

Now suppose the distribution of $(\tilde{z}^*, \tilde{v}^*)$, and thus its characteristic function, is rotation invariant; i.e., we have $\Phi(s,t) = \xi(s^2 + t^2)$ for some function ξ. This implies (22).

To obtain the reverse implication, suppose that (22) is satisfied. Rotating the random vector $(\tilde{z}^*, \tilde{v}^*)$ by the angle $\eta \in \mathbb{R}$ yields the random vector $(\tilde{z}^*_\eta, \tilde{v}^*_\eta)$ with the characteristic function

$$\begin{aligned} \Phi^\eta(s,t) &= E[e^{is(\cos\eta\tilde{z}^* + \sin\eta\tilde{v}^*) + it(-\sin\eta\tilde{z}^* + \cos\eta\tilde{v}^*)}] \\ &= \Phi(s\cos\eta - t\sin\eta, s\sin\eta + t\cos\eta). \end{aligned}$$

Because all moments exist for $(\tilde{z}^*, \tilde{v}^*)$, all moments exist for $(\tilde{z}^*_\eta, \tilde{v}^*_\eta)$ and these moments are determined by the higher order partial derivatives of the characteristic functions $\Phi(s,t)$ and $\Phi^\eta(s,t)$ at $(0,0)$. Because $(\tilde{z}^*, \tilde{v}^*)$ is determined by its moments, it is rotation invariant if for all rotation angles η the moments of $(\tilde{z}^*_\eta, \tilde{v}^*_\eta)$ equal those of $(\tilde{z}^*, \tilde{v}^*)$. I.e., it is sufficient to show that

$$\forall \eta \in \mathbb{R},\ m,n \in \mathbb{N}_0 :\ \Phi^\eta_{(m)(n)} = \Phi_{(m)(n)}. \tag{23}$$

For all $\hat{s}, \hat{t} \in \mathbb{R}$, define

$$f(\hat{s},\hat{t}) = -\hat{t}\Phi_{(1)(0)}(\hat{s},\hat{t}) + \hat{s}\Phi_{(0)(1)}(\hat{s},\hat{t}).$$

It is straightforward to verify that

$$\forall s,t,\eta \in \mathbb{R} :\ \frac{\partial}{\partial\eta}\Phi^\eta(s,t) = f(\hat{s}(s,t,\eta), \hat{t}(s,t,\eta)), \tag{24}$$

where

$$\hat{s}(s,t,\eta) = s\cos\eta - t\sin\eta, \quad \hat{t}(s,t,\eta) = s\sin\eta + t\cos\eta.$$

From (22) we get

$$\forall \hat{s} \in \mathbb{R}, N \in \mathbb{N} :\ f(\hat{s}, \alpha_N\hat{s}) = 0.$$

Therefore, for all $k \in \mathbb{N}_0$ we have

$$\forall \hat{s} \in \mathbb{R}, N \in \mathbb{N} :\ \frac{\mathrm{d}^k}{\mathrm{d}\hat{s}^k} f(\hat{s}, \alpha_N\hat{s}) = 0. \tag{25}$$

Using induction over k, we get the following formula for higher order derivatives:

$$\forall k \in \mathbb{N}_0, \hat{s} \in \mathbb{R}, N \in \mathbb{N} :\ \frac{\mathrm{d}^k}{\mathrm{d}\hat{s}^k}\left(f(\hat{s}, \alpha_N\hat{s})\right) = \sum_{i=0}^{k} \alpha^i \binom{k}{i} f_{(k-i)(i)}(\hat{s}, \alpha\hat{s}).$$

For $\hat{s} = 0$, the r.h.s. is a polynomial in α. Because of (25) and $\{\alpha_N \mid N \in \mathbb{N}\}$ is infinite by (21), the polynomial has infinitely many zeros and is therefore identically zero; i.e., we have

$$\forall m, n \in \mathbb{N}_0 : \ f_{(m)(n)} = 0. \tag{26}$$

The chain rule and induction over $m + n$ imply that

$$\frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial t^n} f(\hat{s}(s,t,\eta), \hat{t}(s,t,\eta))$$

is a linear combination of

$$\{f_{(k)(l)}(\hat{s}(s,t,\eta), \hat{t}(s,t,\eta)) \mid k + l \leq m + n\}.$$

This together with $(\hat{s}(0,0,\eta), \hat{t}(0,0,\eta)) = (0,0)$ and (26) implies that

$$\frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial t^n} f(\hat{s}(s,t,\eta), \hat{t}(s,t,\eta))\Big|_{(s,t,\eta)=(0,0,\eta)} = 0. \tag{27}$$

From (24) and (27) we get

$$\forall m, n \in \mathbb{N}_0, \eta \in \mathbb{R} : \ \frac{\partial}{\partial \eta} \Phi^{\eta}_{(m)(n)} = 0,$$

which implies (23).

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