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Convertible Bonds: Risks and Optimal Strategies

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Convertible Bonds: Risks and Optimal Strategies

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Abstract

Within the structural approach for credit risk models we discuss the optimal exercise of the callable and convertible bonds. The Vasiček–model is applied to incorporate interest rate risk into the firm's value process which follows a geometric Brownian motion. Finally, we derive pricing bounds for convertible bonds in an uncertain volatility model, i.e. when the volatility of the firm value process lies between two extreme values.

Keywords: Convertible bond, game option, uncertain volatility, interest rate risk
JEL: G12, G33

1 Introduction

Callable and convertible bonds have attracted substantial research attention due to their exposure to both credit and market risk and the corresponding optimal conversion and call strategies. The bondholder receives coupons plus the return of principal at maturity, given that the issuer (usually the shareholder) does not default on the obligations. Moreover, prior to and at the maturity the bondholder has the right to convert the bond into a given number of stocks. On the other hand, the bond is also callable by the issuer, i.e. the bondholder can be enforced to surrender the bond to the issuer for a previously agreed price. In the context of the structural model the arbitrage free pricing problem was first treated by Brennan and Schwarz (1977) and Ingersoll (1977). Recent articles of Sirbu, Pilovsky and Schreve (2004) and Kallsen and Kühn (2005) treat the optimal behavior of the contract partners more rigorously. The valuation of callable and convertible bond is explicitly related to the game option.

Empirical research indicates that firms that issue convertible bonds often tend to be highly leveraged, the default risk may play a significant role. Moreover, the equity and default risk cannot be treated independently and their interplay must be modeled explicitly. Default risk models can be categorized into two fundamental classes: firm's value models or *structural models*, and *reduced-form* or default-rate models. In the structural model, one constructs a stochastic process of the firm's value which indirectly leads to default, while in the reduced-form model the default process is modeled directly. In the structural models default risk depends mainly on the stochastic evolution of the asset value and default occurs when the random variable describing the firm's value is insufficient for repayment of debt. Instead of asking why the firm defaults, in the reduced-form model formulation, the intensity of the default process is modeled exogenously by using both market-wide as well as firm-specific factors, such as stock prices and default intensities. While both approaches have certain shortcomings, the strength of the structural approach is that it provides economical explanation of the capital structure decision, default triggering, influence of dividend payments and of the behaviors of debtor and creditor.

It describes why a firm defaults and it allows for the description of the strategies of the debtor and creditor. Especially for complex contracts where the strategic behaviors of the debtor and the creditor play an important role, structural models are well suited for the analysis of the relative power of shareholders and creditors. Another reason that we work with the structural approach is because it allows for an integrated model of equity and default risk through common dependence on stochastic variables.

Callable and convertible bonds are American-style contract, meaning that conversion is allowed at any time during the life of the contract, and by existence of a call provision for the issuer this leads to a problem of optimal stopping for both bondholder and issuer. Therefore when we compute the no-arbitrage price of such a contract, we have to take into account the aspect of strategic optimal behaviors which are the study focus of this paper. Based on the results of Kifer (2000) and Kallsen and Kühn (2005) we show that the optimal strategy for the bondholder is to select the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, while the issuer will choose the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This *max-min strategy* of the bondholder leads to the lower value of the convertible bond, whereas the *min-max strategy* of the issuer leads to the upper value of the convertible bond. The assumption that the call value is always larger than the conversion value prior to maturity T and they are the same at maturity T ensures that the lower value equals the upper value such that there exists a unique solution. The no-arbitrage price can be approximated numerically by means of backward induction. In absence of interest rate risk, the recursion procedure is carried out on the Cox-Ross-Rubinstein binomial lattice. To incorporate the influence of the interest rate risk, we use a combination of an analytical approach and a binomial tree approach developed by Menkveld and Vorst (1998) where the interest rate is Gaussian and correlation between the interest rate process and the firm's value process is explicitly modeled. We show that the influence of interest rate risk is small. This can be explained by the fact that the volatility of the interest process is in comparison with that of the firm's value process relatively low, moreover, both parties have the possibility of early exercise.

In practice it is often a difficult problem to calibrate a given model to the available data. Here one major drawback of the structural model is that it specifies a certain firm's value process. As the firm's value, however, is not always observable, e.g. due to incomplete information, determining the volatility of this process is a non-trivial problem. In this paper, we circumvent this problem by applying the uncertain volatility model of Avellaneda, Levy and Parás (1995) and combining it with the results of Kallsen and Kühn (2005) on game option in incomplete market to derive certain pricing bounds for convertible bonds. Hereby we only known that the volatility of the firm's value process lies between two extreme values. The bondholder selects the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, and the expectation is taken with the most pessimistic estimate from the aspect of the bondholder. Thus the optimal strategy of the bondholder and his choice of the pricing measure determine the lower bound of the no-arbitrage price. Whereas the issuer chooses the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This expectation is also the most pessimistic one but from the aspect of the issuer, thus the upper bound of the no-arbitrage price can be derived. Numerically, to make the computation tractable a constant interest rate is assumed. The pricing bounds can be calculated with recursions on a recombining trinomial tree.

The remainder of the paper is structured as follows. Section 2 introduces the model framework: market assumptions, dynamics of the interest rate and firm's value processes, capital structure and the default mechanism are established. The contract feature of the callable and convertible bond is described in Section 3. Section 4 focuses on the description of the optimal strategies and the determination of the no-arbitrage value of the callable and convertible bond. The formulation and solution of the optimization problem are first presented with constant interest rate in Section 5, then the interest rate risk is incorporated in Section 6. In Section 7 uncertain volatilities of the firm value are introduced and pricing bounds are derived. Section 8 concludes the paper.

2 Model Framework

2.1 Model assumptions

We adopt a first passage model and the model assumptions are made mainly according to Briys and de Varenne (1997) and Bielecki and Rutkowski (2004)[1], with some slight modifications. It covers both the firm specific default risk and the market interest rate risk and correlation of them. The financial market is frictionless, which means there are no transactions costs, bankruptcy costs and taxes, and all securities in the market are arbitrarily divisible. Every individual can buy or sell as much of any security as he wishes without affecting the market price. Risk-free assets earn the instantaneous risk-free interest rate. One can borrow and lend at the same interest rate and take short positions in any securities. The Modigliani-Miller theorem is valid, i.e. the firm's value is independent of the capital structure of the firm. In particular, the value of the firm does not change at the time of conversion and is reduced by the amount of the call price paid to the bondholder at the time of the call. Trading takes place continuously. Under these assumptions, the financial market is complete, and according to Harrison and Kreps (1979) there exists a unique probability measure P^* under which the continuously discounted price of any security is a P^*-martingale.

2.2 Dynamic of the firm's value

The Vasiček–model, in its simplest form, a *one-factor* mean-reverting model is applied to incorporate interest rate risk into the process of the firm's value. The conform short rate follows an Ornstein–Uhlenbeck process

$$dr(t) = (a_r - b_r r(t))dt + \sigma_r dW_1^*(t), \tag{1}$$

with constant volatility $\sigma_r > 0$, and the short rate is pulled to the long-run mean $\frac{a_r}{b_r}$ at a speed rate of b_r. $W_1^*(t)$ is a 1-dimensional standard Brownian motion under the martingale measure P^*. Accordingly, the value of a default free zero coupon bond $B(t,T)$ follows the dynamic

$$dB(t,T) = B(t,T)(r_t dt - b(t,T)dW_1^*(t)) \tag{2}$$

where the volatility of the zero coupon bond has the following form

$$b(t,T) = \frac{\sigma_r}{b_r}(1 - e^{-b_r(T-t)}).$$

The firm's value V is assumed to follow a geometric Brownian motion under the martingale measure P^* of the form

$$\frac{dV_t}{V_t} = (r_t - \kappa)dt + \sigma_V(\rho dW_1^*(t) + \sqrt{1-\rho^2}dW_2^*(t)) \tag{3}$$

[1]See, Section 3.4 of their book.

where $W_2^*(t)$ is a 1-dimensional standard Brownian motion, independent of $W_1^*(t)$ and $\rho \in [-1,1]$ is the correlation coefficient between the interest rate and the firm's value. The volatility $\sigma_V > 0$ and the payout rate κ are assumed to be constant. The amount $\kappa V_t dt$ is used to pay coupons and dividends.

Under the martingale measure P^* the no-arbitrage price of a contingent claim is derived as expected discounted payoff, but in the case of stochastic discount factor the calculation can be quite complicated. The calculation can be simplified if the T-forward risk adjusted martingale measure P^T is applied.

Definition 2.1. A *T-forward risk adjusted martingale measure* P^T on $(\Omega, \mathcal{F}_T)$ is equivalent to P^* and the Radon-Nikodým derivative is given by the formula

$$\frac{dP^T}{dP^*} = \frac{\exp\{-\int_0^T r(u)du\}}{\mathbb{E}_{P^*}\left[\exp\{-\int_0^T r(u)du\}\right]} = \frac{\exp\{-\int_0^T r(u)du\}}{B(0,T)},$$

and when restricted to the $\sigma-$ field $\mathcal{F}_t$,

$$\frac{dP^T}{dP^*}|_{\mathcal{F}_t} := \mathbb{E}_{P^*}\left[\frac{\exp\{-\int_0^T r(u)du\}}{B(0,T)}\Big|\mathcal{F}_t\right] = \frac{\exp\{-\int_0^t r(u)du\}B(t,T)}{B(0,T)}.$$

Especially for Gaussian term structure model, when the zero bond price is given by Equation (2), an explicit density function exists. Namely,

$$\frac{dP^T}{dP^*}|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t b^2(u,T)du - \int_0^t b(u,T)dW_1^*(u)\right\}.$$

Furthermore,

$$W_1^T(t) = W_1^*(t) + \int_0^t b(u,T)du \tag{4}$$

follows a standard Brownian motion under the forward measure P^T.

Thus the forward price of the firm's value $F_V(t,T) := V_t/B(t,T)$ satisfies the following dynamics under the T-forward risk adjusted martingale measure P^T [2],

$$\begin{aligned}\frac{dF_V(t,T)}{F_V(t,T)} &= -\kappa dt + (\rho\sigma_V + b(t,T))dW_1^T(t) + \sigma_V\sqrt{1-\rho^2}dW_2^*(t) \\ &= -\kappa dt + \sigma_F(t,T)dW^T(t),\end{aligned} \tag{5}$$

where $W_1^T(t)$ is given by Equation (4) and

$$\sigma_F^2(t,T) = \int_0^t \left(\sigma_V^2 + 2\rho\sigma_V b(u,T) + b^2(u,T)\right)du, \tag{6}$$

and $W^T(t)$ is a 1-dimensional standard Brownian motion that arises from the independent Brownian motions $W_1^T(t)$ and $W_2^*(t)$ [3] by the following equality in law $aW_1^T(t) + bW_2^*(t) \sim \sqrt{a^2+b^2}W^T(t)$, where a, b are constant. Thus the auxiliary process

$$F_V^\kappa(t,T) := F_V(t,T)e^{\kappa t} \tag{7}$$

[2]The dynamic of the forward firm value is derived by application of Itô's Lemma.

[3]The independence of $W_1^T(t)$ and $W_2^*(t)$ is due to the assumption that $W_1^*(t)$ and $W_2^*(t)$ are independent and this property remains after the change of measure acted on $W_1^*(t)$.

is a martingale under P^T and is log-normally distributed. Specifically, we have

$$dF_V^{\kappa}(t,T) = F_V^{\kappa}(t,T) \cdot \sigma_F(t,T) dW^T(t). \tag{8}$$

According to Equation (3) a constant payout rate of κ is assumed, and $\kappa V_t dt$ is the sum of the continuous coupon and dividend payments. Thus the firm's value $F_V(t,T)$ is not a martingale under the T-forward risk adjusted martingale measure P^T , but after compensated with the payout, the auxiliary process $F_V^{\kappa}(t,T)$ is a martingale under P^T.

2.3 Capital structure and default mechanism

The equity price may drop at time of conversion, as the equity-holders may own a smaller portion of the equity after bondholders convert their holdings and become new equity-holders. To capture this effect, we assume that until time of conversion, at time t , the firm's asset consists of m identical stocks with value S_t and of n identical callable and convertible bonds with value CCB_t, thus

$$V_t = m \cdot S_t + n \cdot CCB_t.$$

Especially, at time $t = 0$, the initial firm's value satisfies

$$V_0 = m \cdot S_0 + n \cdot CCB_0. \tag{9}$$

Moreover, we set the principal that the firm must pay back at maturity T to be L for each bond and assume that bondholders are protected by a safety covenant that allows them to trigger early default. The firm defaults as soon as its value hits a prescribed barrier and the default time τ is defined in a standard way by

$$\tau = \inf\{t > 0 : V_t \leq \nu_t\}. \tag{10}$$

3 Contract Feature

In the following we assume that the bond matures at time $T \in \mathbb{R}_+$. The coupons are paid out continuously with a constant rate of c, given that the firm's value is above the level η_t . The contract terminates either at maturity T or, in case of premature default, at the default time τ, which is the first hitting time of the barrier ν_t by the firm's value. Moreover, the contract stops also by conversion or call. The bondholder can stop and convert the bond into equities according to the prescribed conversion ratio γ. The conversion time of the bondholder is denoted as $\tau_b \in [0,\tau]$. The shareholder can stop and buy back the bond at a price given by the maximum of the deterministic call level H_t and the current conversion price. This ensures that the payoff by call is never lower than the conversion payoff. This assumption makes the aspect of game option relevant and interesting for the valuation of callable convertible bonds. The call time of the seller is denoted as $\tau_s \in [0,\tau]$.

3.1 Discounted payoff

First, we introduce the notation $\beta(s,t) = \exp\{-\int_s^t r(u)du\}$ which is the discount factor, where $r(t)$ is the instantaneous risk-free interest rate. The discounted payoff of a callable and convertible bond can be distinguished in four cases.

(i) Let $\tau_b < \tau_s \leq T$, such that the contract begins at time 0 and is stopped and converted by the bondholder. In this case, the discounted payoff $conv(0)$ of the callable and convertible bond at time 0 is composed of the accumulated coupon payments and the payoff through conversion

$$\begin{aligned} conv(0) &= c\int_0^{\tau_b\wedge\tau} \beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds + \frac{\nu_\tau}{n}\cdot\beta(0,\tau)\mathbf{1}_{\{\tau\le\tau_b\}} \\ &\quad +\beta(0,\tau_b)\mathbf{1}_{\{\tau_b<\tau\}}\Big(\frac{\gamma V_{\tau_b}}{m+\gamma n}\Big). \end{aligned} \tag{11}$$

(ii) Let $\tau_s < \tau_b \le T$, such that the contract is bought back by the shareholder before the bondholder converts. In this case, the discounted payoff $call(0)$ of the callable and convertible bond at time 0 is composed of the accumulated coupon payments and the payoff through call,

$$\begin{aligned} call(0) &= c\int_0^{\tau_s\wedge\tau} \beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds + \frac{\nu_\tau}{n}\cdot\beta(0,\tau)\mathbf{1}_{\{\tau\le\tau_s\}} \\ &\quad +\beta(0,\tau_s)\mathbf{1}_{\{\tau_s<\tau\}}\max\Big\{H_{\tau_s},\frac{\gamma V_{\tau_s}}{m+\gamma n}\Big\}. \end{aligned} \tag{12}$$

(iii) If $\tau_s = \tau_b < T$ the discounted payoff of the bond equals the smaller value, i.e. the discounted payoff with conversion.

(iv) For $\tau_b \ge T$ and $\tau_s \ge T$, the discounted payoff of a callable and convertible bond at time 0 is

$$\begin{aligned} term(0) &= c\int_0^{\tau\wedge T} \beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds + \frac{\nu_\tau}{n}\cdot\beta(0,\tau)\mathbf{1}_{\{\tau\le T\}} \\ &\quad +\beta(0,T)\mathbf{1}_{\{T<\tau\}}\max\left\{\frac{\gamma V_T}{m+\gamma n},\min\left\{\frac{V_T}{n},L\right\}\right\}. \end{aligned}$$

Note that $\dfrac{V_T}{n} > \dfrac{\gamma V_T}{m+\gamma n}$ since $n,m\in\mathbb{N}_+$ and $\gamma\in\mathbb{R}_+$. Hence in the case $\dfrac{V_T}{n}\le L$ the bondholder would not convert and

$$\mathbf{1}_{\{V_T\le nL\}}\max\left\{\frac{\gamma V_T}{m+\gamma n},\min\left\{\frac{V_T}{n},L\right\}\right\} = \frac{V_T}{n}.$$

Thus, in the case (iv), we can rewrite the discounted payoff $term(0)$ as

$$\begin{aligned} term(0) &= c\int_0^{\tau\wedge T} \beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds + \frac{\nu_\tau}{n}\cdot\beta(0,\tau)\mathbf{1}_{\{\tau\le T\}} \\ &\quad +\beta(0,T)\mathbf{1}_{\{T<\tau,V_T>nL\}}\max\left\{\frac{\gamma V_T}{m+\gamma n},L\right\} + \beta(0,T)\mathbf{1}_{\{T<\tau,V_T\le nL\}}\frac{V_T}{n}. \end{aligned} \tag{13}$$

Denote the minimum of conversion and call time by $\zeta = \tau_s\wedge\tau_b$. Then, all in all, the discounted payoff of a callable and convertible bond $ccb(0)$ is given as the sum of the payoffs in the former four cases and amounts to

$$
\begin{aligned}
cbb(0) \;=\; & \mathbf{1}_{\{\zeta<\tau\}}\Bigg(c\int_0^{\zeta\wedge T}\beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds + \mathbf{1}_{\{\zeta=\tau_s<\tau_b\le T\}}\beta(0,\zeta)\max\left\{H_\zeta, \frac{\gamma V_\zeta}{m+\gamma n}\right\} \\
& +\mathbf{1}_{\{\zeta=\tau_b<\tau_s<T\}}\beta(0,\zeta)\frac{\gamma V_\zeta}{m+\gamma n} + \mathbf{1}_{\{\zeta=T\}}\beta(0,\zeta)\max\left\{\frac{\gamma V_T}{m+\gamma n}, L\right\}\Bigg) \\
& + \mathbf{1}_{\{\tau\le\zeta\}}\Bigg(c\int_0^{\tau\wedge T}\beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds + \mathbf{1}_{\{\tau\le T\}}\beta(0,\tau)\frac{\nu_\tau}{n} \\
& +\mathbf{1}_{\{T<\tau\}}\beta(0,T)\min\left\{\frac{V_T}{n}, L\right\}\Bigg). \qquad (14)
\end{aligned}
$$

3.2 Decomposition of the payoff

The callable and convertible bond can be decomposed into a straight bond component and an option component. The decomposition enables us to investigate the pure effect caused by the conversion and call rights.

Theorem 3.1. The payoff of a callable and convertible bond can be decomposed into a straight bond $d(0)$ and a defaultable game option component $g(0)$.

$$ccb(0) = d(0) + g(0) \qquad (15)$$

with

$$d(0) := c\int_0^{\tau\wedge T}\beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds + \mathbf{1}_{\{\tau\le T\}}\beta(0,\tau)\frac{\nu_\tau}{n} + \mathbf{1}_{\{T<\tau\}}\beta(0,T)\min\left\{\frac{V_T}{n}, L\right\},$$

and

$$
\begin{aligned}
g(0) \;:=\; & \mathbf{1}_{\{\zeta<\tau\}}\beta(0,\zeta)\Bigg\{\mathbf{1}_{\{\zeta=\tau_b<\tau_s<T\}}\left(\frac{\gamma V_\zeta}{m+\gamma n} - \phi_\zeta\right) \\
& +\mathbf{1}_{\{\zeta=\tau_s<\tau_b\le T\}}\left(\max\left\{H_\zeta, \frac{\gamma V_\zeta}{m+\gamma n}\right\} - \phi_\zeta\right) + \mathbf{1}_{\{\zeta=T\}}\left(\frac{\gamma V_T}{m+\gamma n} - L\right)^+\Bigg\},
\end{aligned}
$$

where

$$\phi_\zeta := c\int_\zeta^{\tau\wedge T}\beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds + \mathbf{1}_{\{\tau\le T\}}\beta(\zeta,\tau)\frac{\nu_\tau}{n} + \mathbf{1}_{\{T<\tau\}}\beta(\zeta,T)\min\left\{\frac{V_T}{n}, L\right\} \qquad (16)$$

is the discounted value (discounted to time ζ) of the sum of the remaining coupon payments and the principal payment of a straight coupon bond given that it has not defaulted till time ζ.

Proof 3.2. We can reformulate $ccb(0)$ in Equation 14 as follows

$$
\begin{aligned}
ccb(0) &= \mathbf{1}_{\{\zeta<\tau\}}\beta(0,\zeta)\Bigg(\mathbf{1}_{\{\zeta=\tau_b<\tau_s<T\}}\frac{\gamma V_\zeta}{m+\gamma n}+\mathbf{1}_{\{\zeta=\tau_s<\tau_b\le T\}}\max\left\{H_\zeta,\frac{\gamma V_\zeta}{m+\gamma n}\right\}\\
&\qquad +\mathbf{1}_{\{\zeta=T\}}\max\left\{\frac{\gamma V_T}{m+\gamma n},L\right\}\Bigg)\\
&\quad+\underbrace{\left(c\int_0^{\tau\wedge T}\beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds+\mathbf{1}_{\{\tau\le T\}}\beta(0,\tau)\frac{\nu_\tau}{n}+\mathbf{1}_{\{T<\tau\}}\beta(0,T)\min\left\{\frac{V_T}{n},L\right\}\right)}_{=d(0)}\\
&\quad-\mathbf{1}_{\{\zeta<\tau\}}\underbrace{\left(c\int_\zeta^{\tau\wedge T}\beta(0,s)\mathbf{1}_{\{V_s>\eta_s\}}ds+\mathbf{1}_{\{\tau\le T\}}\beta(0,\tau)\frac{\nu_\tau}{n}+\mathbf{1}_{\{T<\tau\}}\beta(0,T)\min\left\{\frac{V_T}{n},L\right\}\right)}_{:=\beta(0,\zeta)\phi_\zeta}.
\end{aligned}
$$

Since $\frac{V_T}{n}\ge L$ if $\zeta\le T$, otherwise the bondholder would not make use of his conversion right.
□

4 Optimal Strategies

After the inception of the contract, the bondholder's aim is to maximize the value of the bond by means of optimal exercise of the conversion right. The incentive of the issuer to call a bond is to limit the bondholder's participation in rising stock prices. The embedded option rights owned by both of the bondholder and issuer can be treated with the well developed theories on the game option.

4.1 Game option

In this section we summarize the valuation problem of game options and highlight some important results derived by Kifer (2000). .

Definition 4.1. Let $T\in\mathbb{R}_+$. Consider a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},P)$. A *game option* is a contract between a seller A and a buyer B which enables A to terminate it and B to exercise it at any time $t\in[0,T]$ up to the maturity date T. If B exercises at time t, he obtains from A the payment X_t. If A terminates the contract at time t before it is exercised by B, then he has to pay B the amount Y_t, where X_t and Y_t are two stochastic processes which are adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, and satisfy the following condition

$$X_t\le Y_t,\quad \text{for } t\in[0,T],\qquad \text{and}\qquad X_T=Y_T. \tag{17}$$

Moreover, if the seller A terminates and the buyer B exercises at the same time, A only has to pay the lower value X_t. Loosely speaking, the seller must pay certain penalty if he terminates the contract before the buyer exercises it.

Game options include both American and European options as special cases. Formally, if we set $Y_t=\infty$ for $t\in[0,T)$, then we obtain an American option. A European option is obtained by setting $X_t=0$ for $t\in[0,T)$ and X_T is a nonnegative $\mathcal{F}_T$-measurable random variable.

If the seller A selects a stopping time τ_A as termination time and the buyer B chooses a stopping time τ_B as exercise time, then A promises to pay B at time $\tau_A \wedge \tau_B$ the amount

$$g(\tau_A, \tau_B) := X_{\tau_B} \mathbf{1}_{\{\tau_B \leq \tau_A\}} + Y_{\tau_A} \mathbf{1}_{\{\tau_A < \tau_B\}}, \tag{18}$$

which denotes the payoff of a game option.

The aim of the buyer B is to maximize the payoff $g(\tau_A, \tau_B)$, while the seller A tends to minimize the payoff. The optimal strategy for the buyer is therefore to select the stopping time which maximizes his expected discounted payoff given the minimizing strategy of the seller, while the seller will choose the stopping time that minimizes the expected discounted payoff given the maximizing strategy of the buyer. This *max-min strategy* of the buyer leads to the lower value of the game option, whereas the *min-max strategy* of the seller leads to the upper value of the game option. In a complete market the condition described by Equation (17) ensures that the lower value equals the upper value such that there exists a solution for the pricing problem of a game option.

The existence and uniqueness of the no-arbitrage price in a complete market where the filtration $\{\mathcal{F}_u\}_{0 \leq u \leq T}$ is generated by a standard one-dimensional Brownian motion is proved in Kifer (2000), Theorem 3.1. The no-arbitrage price of a game option equals $G(0)$,

$$\begin{aligned} G(0) &= \sup_{\tau_B \in \mathcal{F}_{0T}} \inf_{\tau_A \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B)] \\ &= \inf_{\tau_A \in \mathcal{F}_{0T}} \sup_{\tau_B \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B)] \end{aligned} \tag{19}$$

where $\mathcal{F}_{0T}$ is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{0 \leq u \leq T}$ with values in $[0, T]$. After the inception of the contract, the value process $G(t)$, $t \in (0, T]$ satisfies

$$\begin{aligned} G(t) &= \text{esssup}_{\tau_B \in \mathcal{F}_{tT}} \text{essinf}_{\tau_A \in \mathcal{F}_{tT}} \mathbb{E}_{P^*}[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B) | \mathcal{F}_t] \\ &= \text{essinf}_{\tau_A \in \mathcal{F}_{tT}} \text{esssup}_{\tau_B \in \mathcal{F}_{tT}} \mathbb{E}_{P^*}[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B) | \mathcal{F}_t]. \end{aligned} \tag{20}$$

Where $\mathcal{F}_{tT}$ is the set of stopping times with values in $[t, T]$. Further, the optimal stopping times for the seller A and buyer B respectively are

$$\begin{aligned} \tau_A^* &= \inf\{t \in [0, T] \mid e^{-rt} Y_t \leq G(t)\} \\ \tau_B^* &= \inf\{t \in [0, T] \mid e^{-rt} X_t \geq G(t)\}. \end{aligned} \tag{21}$$

It is optimal for the seller A to buy back the option as soon as the current exercise value $e^{-rt} Y_t$ is equal to or smaller than the value function $G(t)$, while the optimal strategy for the buyer B is to exercise the option as soon as the current exercise value $e^{-rt} X_t$ is equal to or greater than the value function $G(t)$.

4.2 Optimal stopping and no-arbitrage value of callable and convertible bond

The discounted conversion value of the callable and convertible bond, described with Equation (11), contains expressions about default times. But in the structural approach, the default time is a predictable stopping time, and adapted to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ generated by the firm's value. Thus the discounted conversion value is adapted to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$. And the same is valid for the discounted call value and the discounted terminal payoff, described with Equations (12) and (13) respectively. Moreover, the call value is always larger than the conversion value for $t < T$, and they coincide at maturity T. Hence, the payoffs in the case of conversion and call satisfy the requirements on the payoffs of the game option. Furthermore, the market in our structural approach is assumed to be complete. Therefore the theory on game option developed by Kifer (2000) can be applied to derive the unique no-arbitrage value and the optimal strategies.

Proposition 4.2. Plugging the payoff functions $ccb(0)$ in Equation (19), the unique no-arbitrage price $CCB(0)$ at time $t = 0$ of the callable and convertible bond is given by

$$CCB(0) = \sup_{\tau_b \in \mathcal{F}_{0T}} \inf_{\tau_s \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[ccb(0)] = \inf_{\tau_s \in \mathcal{F}_{0T}} \sup_{\tau_b \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[ccb(0)]. \tag{22}$$

After the inception of the contract, the value process $CCB(t)$ satisfies

$$\begin{aligned} CCB(t) &= \text{esssup}_{\tau_b \in \mathcal{F}_{tT}} \text{essinf}_{\tau_s \in \mathcal{F}_{tT}} \mathbb{E}_{P^*}[ccb(0)|\mathcal{F}_t] \\ &= \text{essinf}_{\tau_s \in \mathcal{F}_{tT}} \text{esssup}_{\tau_b \in \mathcal{F}_{tT}} \mathbb{E}_{P^*}[ccb(0)|\mathcal{F}_t]. \end{aligned} \tag{23}$$

The optimal strategy for the bondholder is to select the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, while the issuer will choose the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This *max-min strategy* of the bondholder leads to the lower value of the convertible bond, whereas the *min-max strategy* of the issuer leads to the upper value of the convertible bond. The assumption that the call value is always larger than the conversion value prior to the maturity and they are the same at maturity T ensures that the lower value equals the upper value such that there exists a unique solution.

Furthermore, the optimal stopping times for the equity holder and bondholder respectively are

$$\begin{aligned} \tau_b^* &= \inf\{t \in [0,T] \mid conv(0) \geq CCB(t)\} \\ \tau_s^* &= \inf\{t \in [0,T] \mid call(0) \leq CCB(t)\}. \end{aligned} \tag{24}$$

It is optimal to convert as soon as the current conversion value is equal to or larger than the value function $CBB(t)$, while the optimal strategy for the issuer is to call the bond as soon as the current call value is equal to or smaller than the value function $CBB(t)$.

Remark 4.3. The no-arbitrage value of the callable and convertible bond and the optimal stopping times described by Equation (22) and (24) incorporate also the case of stochastic interest rate. Kifer (2000) assumes that the interest rate is constant, but this assumption is not necessary, because the game option is essentially a zero-sum Dynkin stopping game and the *min-max* and *max-min* strategies are also valid for the stochastic discount factor. For details, see e.g. Kifer (2000) and Cvitanić and Karatzas (1996).

In section 3.2 it has been shown that the callable and convertible bond can be decomposed into a straight bond and a game option component

$$ccb(0) = d(0) + g(0).$$

Therefore the no-arbitrage price of the callable and convertible bond can also be derived in the following way

$$CCB(0) = \mathbb{E}_{P^*}[d(0)] + \mathbb{E}_{P^*}[g(0)].$$

The no-arbitrage price of the game option component $G(0)$ equals

$$\begin{aligned} G(0) &:= \mathbb{E}_{P^*}[g(0)] \\ &= \sup_{\tau_b \in \mathcal{F}_{0T}} \inf_{\tau_s \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[g(0)] = \inf_{\tau_s \in \mathcal{F}_{0T}} \sup_{\tau_b \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[g(0)]. \end{aligned} \tag{25}$$

5 Deterministic Interest Rates

In general, closed-form solutions of the optimization problems stated in Equations (22) and (25) are not available. One alternative solution is to approximate the continuous time problem with a discrete time one. The no-arbitrage value of the callable and convertible bond can then be derived by a recursion formula. In order to focus on the recursion procedure, we assume in the first step that the interest rate is constant. Theorem 2.1 of Kifer (2000) illustrates the recursion method for the game option and the optimal stopping strategies of both counterparts. The discretization method and its convergence is proved in Proposition 3.2 of the same paper. We will apply and adapt this recursion method to determine the no-arbitrage value and optimal stopping times of the callable and convertible bond.

5.1 Discretization and recursion schema

The time interval $[0,T]$ is discretized into N equidistant time steps $0 = t_0 < t_1 < \ldots < t_N = T$, with $t_i - t_{i-1} = \Delta$. Assume that the bondholder does not receive the coupon for the period in which the bond is converted, while receives the dividends for the converted shares, though. If the bond is called, coupon will be paid. $CCB(t_n)$, the recursion value of the callable and convertible bond at time t_n, can be derived by means of the *max-min* or *min-max* recursion, illustrated in Figure 1 and 2. Note that in complete markets the *max-min* strategy leads to the same value as the *min-max* strategy. Hence it does not matter whether we carry out the recursion according to the strategy of the bondholder or that of the shareholder. In the recursion schema $V_{t_n^+}$ is the firm's value just before payout and ν_{t_n} is the default barrier. The discretized coupon c_{t_n} equals $c\Delta$, and will only be paid out if the firm's value is above certain level, i.e. $V_{t_n^+} > \eta_{t_n}$, therefore c_{t_n} is path-dependent.

For $n = 0, 1, ..., N-1$,

$$CCB(t_n) = \begin{cases} \min\left\{ e^{-rt_n} \max\left\{ H + c_{t_n}, \dfrac{\gamma V_{t_n^+}}{m+\gamma n} \right\}, \right. & \\ \quad \left. \max\left\{ e^{-rt_n} \dfrac{\gamma V_{t_n^+}}{m+\gamma n}, \mathbb{E}_{P^*}[CCB(t_{n+1})|\mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} \right\} \right\} & if \quad V_{t_n^+} > \nu_{t_n} \\ e^{-rt_n} \dfrac{V_{t_n^+}}{n} & if \quad V_{t_n^+} \leq \nu_{t_n} \end{cases} \tag{26}$$

and

$$CCB(T) = \begin{cases} e^{-rT} \max\left\{ \dfrac{\gamma V_{T^+}}{m+\gamma n},\ L + c_{t_N} \right\} & if \quad V_{T^+} > n(L + c_{t_N}) \\ e^{-rT} \dfrac{V_{T^+}}{n} & if \quad V_{T^+} \leq n(L + c_{t_N}) \end{cases} \tag{27}$$

Figure 1: *Min-max* recursion callable and convertible bond, strategy of the issuer

For $n = 0, 1, ..., N-1$,

$$CCB(t_n) = \begin{cases} \max\Big\{ e^{-rt_n} \dfrac{\gamma V_{t_n^+}}{m+\gamma n}, \min\Big\{ e^{-rt_n} \max\Big\{ H + c_{t_n}, \dfrac{\gamma V_{t_n^+}}{m+\gamma n} \Big\}, \\ \qquad \mathbb{E}_{P^*}[CCB(t_{n+1})|\mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} \Big\}\Big\} & if \quad V_{t_n^+} > \nu_{t_n} \\ e^{-rt_n} \dfrac{V_{t_n^+}}{n} & if \quad V_{t_n^+} \leq \nu_{t_n} \end{cases} \tag{28}$$

and

$$CCB(T) = \begin{cases} e^{-rT} \max\Big\{ \dfrac{\gamma V_{T^+}}{m+\gamma n}, \ L + c_{t_N} \Big\} & if \quad V_{T^+} > n(L + c_{t_N}) \\ e^{-rT} \dfrac{V_{T^+}}{n} & if \quad V_{T^+} \leq n(L + c_{t_N}) \end{cases} \tag{29}$$

Figure 2: *Max-min* recursion callable and convertible bond, strategy of the bondholder

Furthermore, for each $i = 0, 1, ..., N-1$, the rational conversion time after time t_i equals

$$\tau_b^*(t_i) \quad = \quad \min\Big\{ t_k \in \{t_i, ..., t_{N-1}\} \ \Big| \ e^{-rt_k} \frac{\gamma V_{t_k^+}}{m+\gamma n} = CCB(t_k) \Big\},$$

the rational call time after time t_i equals

$$\tau_s^*(t_i) \quad = \quad \min\Big\{ t_k \in \{t_i, ..., t_{N-1}\} \ \Big| \ e^{-rt_k} \max\Big\{ H + c_{t_k}, \ \frac{\gamma V_{t_k^+}}{m+\gamma n} \Big\} = CCB(t_k) \Big\}.$$

Therefore at time t_k, it is optimal to convert or call when the current conversion or call value equals the payoff function $CCB(t_k)$.

Remark 5.1. For convenience of notation, the call value H is assumed be constant, but the same recursion formulas also hold in the case of a deterministic and time dependent call level $H(t)$. In that case H has to be replaced by $H(t_n^+)$ in the above formulas.

Analogously, the no-arbitrage value of the pure *game option component* $G(t_n)$ at time t_n can be derived through the recursion shown in Figure 3 with ϕ_{t_n} as discretized value defined by Equation (16).

5.2 Implementation with binomial tree

As the firm's value in our structural model follows a geometric Brownian motion, in absence of interest rate risk, it can be approximated by the Cox-Ross-Rubinstein model. The time interval $[0, T]$ is divided in N subintervals of equal lengths, the distance between two periods is $\Delta = T/N$. The stochastic evolution of the firm's value is then modeled by

$$V(i,j) = V(0) u^j d^{i-j} \hat{\kappa}^i, \quad \text{for all} \quad j = 0, ..., i, \quad i = 1, ..., N, \tag{30}$$

and

$$u = e^{\sigma_V \sqrt{\Delta}}, \quad d = e^{-\sigma_V \sqrt{\Delta}}, \quad \hat{\kappa} = e^{-\kappa \Delta},$$

where $V(i,j)$ denotes the firm's value at time t_i after j up movements, and less the amount to be paid out. And according to Equation (30), the firm's value just before the payment equals $\dfrac{V(i,j)}{\hat{\kappa}}$, and the total amount to be paid out at time t_i is $V(i,j)\dfrac{\hat{\kappa}}{1-\hat{\kappa}}$. We see that u, d

For $n = 0, 1, ..., N-1$,

$$G(t_n) = \begin{cases} \min\left\{e^{-rt_n}\left(\max\left\{H + c_{t_n}, \dfrac{\gamma V_{t_n^+}}{m+\gamma n}\right\} - \phi_{t_n}\right),\right. & \\ \quad \left.\max\left\{e^{-rt_n}\left(\dfrac{\gamma V_{t_n^+}}{m+\gamma n} - \phi_{t_n}\right), \mathbb{E}_{P^*}\left[G(t_{n+1})|\mathcal{F}_{t_n}\right]\right\}\right\} & if \;\; V_{t_n^+} > \nu_{t_n} \\ 0 & if \;\; V_{t_n^+} \leq \nu_{t_n} \end{cases}$$

or

$$G(t_n) = \begin{cases} \max\left\{e^{-rt_n}\left(\dfrac{\gamma V_{t_n^+}}{m+\gamma n} - \phi_{t_n}\right),\right. & \\ \quad \min\left\{e^{-rt_n}\left(\max\left\{H + c_{t_n}, \dfrac{\gamma V_{t_n^+}}{m+\gamma n}\right\} - \phi_{t_n}\right),\right. & \\ \quad\quad \left.\left.\mathbb{E}_{P^*}\left[G(t_{n+1})|\mathcal{F}_{t_n}\right]\right\}\right\} & if \;\; V_{t_n^+} > \nu_{t_n} \\ 0 & if \;\; V_{t_n^+} \leq \nu_{t_n} \end{cases}$$

and

$$G(T) = \begin{cases} e^{-rT}\max\left\{\dfrac{\gamma V_{T^+}}{m+\gamma n} - L - c_N, 0\right\} & if \;\; V_{T^+} > n(L + c_N) \\ 0 & if \;\; V_{T^+} \leq n(L + c_N) \end{cases}$$

Figure 3: *Max-min* and *min-max* recursion game option component

and $\hat{\kappa}$ are time and state independent. The equivalent martingale measure P^* exists if the periodical discount factor $d < 1 + \hat{r} = e^{r\Delta} < u$. The transition probability is given by

$$p^* := \frac{1 + \hat{r} - d}{u - d}.$$

Concretely, the recursion procedure of the *min-max* strategy [4] of the issuer of a callable and convertible bond, described by Equations (26) and (27), can be implemented within the Cox-Ross-Rubinstein model with Algorithm I (Figure 4). The no-arbitrage price of the callable and convertible bond is then given by $CCB(0,0)$.

The first loop in Algorithm I (Figure 4) determines the optimal strategy and thus the optimal terminal value $CCB(N,j)$. While the second loop determines the value of $CCB(i,j)$ according to the *min-max* strategy at node (i,j) of the tree. The value of each $CCB(i,j)$ is stored in a data matrix, and the event of conversion, call or continuation of the contract is recorded for each node (i,j). Then given the development, i.e. the path of the firm's value $V(i,j)$, the bondholder and issuer can determine their optimal stopping times by moving forward alongside the tree. At the time the contract is terminated, i.e. converted, called or default at the node (i,j), $CCB(i,j)$ is then the discounted payoff of the callable and convertible bond for this realization of the firm's value.

5.3 Influences of model parameters illustrated with a numerical example

The no-arbitrage value of the callable and convertible bond is affected by the randomness of the firm's value, and the randomness of the termination time. It is a complex contract and

[4] The algorithm of *max-min* strategy and recursion of the best strategy of the game option component can be written in the similar way, therefore we omit these cases.

$$
\begin{aligned}
&\textit{for}\ \ j = 0, 1, \ldots, N, \\
&\qquad \textit{if}\ \ \frac{V(N,j)}{\hat{\kappa}} > nL + nc_{N,j}, \\
&\qquad\quad \textit{then}\ \ CCB(N,j) = \max\Big\{\frac{\gamma}{m+\gamma n}\cdot\frac{V(N,j)}{\hat{\kappa}},\ L + c_{N,j}\Big\} \\
&\qquad \textit{else},\ \ CCB(N,j) = \frac{V(N,j)}{n\cdot\hat{\kappa}} \\
&\textit{for}\ \ i = N-1, \ldots, 0, \\
&\quad \textit{for}\ \ j = i, \ldots, 0, \\
&\qquad \textit{if}\ V(i,j) > K,\ \ \textit{then} \\
&\qquad CCB(i,j) = \min\Bigg\{\max\Big[H + c_{i,j},\ \frac{\gamma}{m+\gamma n}\cdot\frac{V(i,j)}{\hat{\kappa}}\Big], \\
&\qquad\quad \max\Big[\frac{\gamma}{m+\gamma n}\cdot\frac{V(i,j)}{\hat{\kappa}},\ \frac{1}{1+\hat{r}}\Big(p^*\cdot CCB(i+1,j) \\
&\qquad\qquad +(1-p^*)\cdot CCB(i+1,j+1)\Big) + c_{i,j}\Big]\Bigg\}, \\
&\qquad \textit{else},\ \ CCB(i,j) = \frac{V(i,j)}{n\cdot\hat{\kappa}}
\end{aligned}
$$

Figure 4: Algorithm I: *Min-max* recursion American-style callable and convertible bond

influenced by a number of parameters: e.g. the value of coupon and principal, default barrier, volatility of the firm's value, conversion ratio, call level, maturity, etc. The firm's value in total follows a diffusion process, while the bond and equity value are results of a strategic game, which are not simple diffusion processes. Change of one parameter causes simultaneous changes of the value of bond and equity. For example, intuitively, the increment of the conversion ratio causes the rise of conversion value, thus the rise of the bond price, but at the same time the reduction of the equity value, and consequently the decline of the conversion value. The direction and quantity of the total effect cannot be determined without numerical evaluation. Moreover, to design a meaningful callable and convertible bond, the parameters should in accordance with each other. The situation such that the bond will be converted or called immediately after the start of the contract, should not happen. In the following, we will illustrate the influences of the model parameters and their interactions with a close study of a numerical example.

Example 5.2. As an explicit numerical example we choose the following parameters: $T = 8,\ \sigma_V = 0.2,\ r = 0.06,\ V(0) = 1000,\ K = 400,\ \omega = 1300,\ L = 100,\ \gamma = 1.5,\ m = 10,\ n = 8,\ H = 120$.

The results in Table 1 are derived for different payout ratios κ and coupons c[5]. They illustrate first that the value of the game option component decreases when coupon payment rises. The reason is that the value of the remaining coupon and principal payment defined by Equation (16) can be thought as the strike of the game option, which is an increasing function of coupon rate c, and the value of the game option component decreases in strike. The large price difference of game option component $G(0)$ in the case $\kappa = 0,\ c = 0$, to the case $\kappa = 0.04,\ c = 2$ is due to

[5]The coupons are to be paid if the firm's value is above $\eta_t = \omega \cdot e^{-r(T-t)}e^{-\kappa t}$, The default barrier is $\nu_t = K \cdot e^{-r(T-t)}e^{-\kappa t}$

		$\sigma_V = 0.2$				$\sigma_V = 0.4$			
κ	c	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$
0	0	59.40	16.92	76.32	38.94	48.01	26.85	74.86	40.11
0.04	2	65.15	8.65	73.80	40.96	52.38	20.41	72.79	41.77
0.04	3	69.83	7.82	77.65	37.88	56.39	18.72	75.12	39.91
0.04	4	74.50	6.99	81.50	34.80	60.40	17.03	77.44	38.06

Table 1: Influence of the volatility of the firm's value and coupons on the no-arbitrage price of the callable and convertible bond (384 steps)

the increment of payout ratio and coupon rate. Both factors together result in a large drop of the value of $G(0)$. The second effect shown by Table 1 is that the more volatile the firm's value, the larger the default probability, hence the smaller the value of straight bond $SB(0)$. But on the other side the game option component $G(0)$ becomes more valuable. In our example, the value of the callable and convertible bond $CCB(0)$ which is the sum of the both components decreases in volatility[6].

	$\sigma_V = 0.2$			$\sigma_V = 0.4$		
Δ	$SB(0)$	$CCB(0)$	$G(0)$	$SB(0)$	$CCB(0)$	$G(0)$
1	69.37	77.00	7.64	54.30	73.83	19.54
1/12	69.82	77.65	7.82	56.14	75.12	18.96
1/48	69.83	77.64	7.81	56.39	75.12	18.72
1/100	69.83	77.64	7.81	56.45	75.11	18.66
1/250	69.83	77.64	7.81	56.51	75.12	18.61

Table 2: Stability of the recursion

Remark 5.3. The stability of the recursion is demonstrated with Table 2. The recursions are carried out alongside trees with different steps for $\sigma_V = 0.2$ and $\sigma_V = 0.4$. We can see that the numerical results stabilized at $\Delta = 1/48$. Further refinements ($\Delta = 1/100$ and $\Delta = 1/250$) of the tree do not change the numerical results considerably while much more time are needed for the calculation. Therefore, for the further calculations in this example Δ is always set to be 1/48, which approximately corresponds to a weekly valuation. By $\Delta = 1/48$ and a maturity of $T = 8$ it corresponds to a tree with 384 steps.

Table 3 has the same structure as Table 1 and shows the influence of the conversion ratio γ on $G(0)$ and $CCB(0)$. The volatility of the firm's value is kept to be constant, i.e. $\sigma_V = 0.2$. The change of conversion ratio γ does not affect the price of the straight coupon bond and it only changes the value of $G(0)$. The increase of γ from 1.5 to 2.0 makes the game option component more valuable, thus in total the callable and convertible bond more valuable[7]. The case by $\kappa = 0.04$, $c = 2$ and $\gamma = 2$ is not a good contract design. As with $CCB(0) = 78.27$, and $S(0) = 37.38$, the initial price of the bond is almost equal to the initial conversion value, which means that the conversion may take place very quickly after the inception of the contract, because a slight increase of the firm's value will make conversion the optimal choice of the

[6]In Example 5.2, the value of the callable and convertible bond increases in volatility, but one cannot argue it generally, as it depends also on other factors e.g. default barrier and maturity.

[7]Again we cannot take it as a general result, as it depends also on the parameters m and n.

		$\gamma = 1.5$				$\gamma = 2$			
κ	c	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$
0	0	59.40	16.92	76.32	38.94	59.40	22.96	82.35	34.12
0.04	2	65.15	8.65	73.80	40.96	65.15	13.12	78.27	37.38
0.04	3	69.83	7.82	77.65	37.88	69.83	11.71	81.54	34.77
0.04	4	74.50	6.99	81.50	34.80	74.50	10.39	84.90	32.08

Table 3: Influence of the conversion ratio on the no-arbitrage price of the callable and convertible bond (384 steps)

bondholder. Usually it is not the intention of the issuer to issue a bond which will be converted or called immediately after the inception of the contract.

		$T = 8$				$T = 6$			
κ	c	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$
0	0	59.40	16.92	76.32	38.94	67.21	11.91	79.12	36.71
0.04	2	65.15	8.65	73.80	40.96	71.75	6.22	77.97	37.62
0.04	3	69.83	7.82	77.65	37.88	75.57	5.75	81.33	34.94
0.04	4	74.50	6.99	81.50	34.80	74.50	5.27	84.67	32.26

Table 4: Influence of the maturity on the no-arbitrage price of the callable and convertible bond (384 steps)

Table 4 is also structured in the same way as Tables 3 and 1. It demonstrates the influence of the maturity T on $G(0)$ and $CCB(0)$. The volatility of the firm's value and conversion ratio are $\sigma_V = 0.2$ and $\gamma = 1.5$. Comparing the case $T = 8$ with $T = 6$, we observe that the straight bond is more valuable with shorter maturity, because the default probability is lower and by positive interest rate the principal is more valuable if it is paid earlier. The game option component $G(0)$ is less valuable in the case of shorter maturity. It is due to two effects: first, shorter maturity means less conversion chances for the bondholder, and secondly, an increase of the value of the straight bond reduces the value of the equity thus the conversion value. The reduction of $G(0)$ may in turn increase the value of equity, here the final result is that reduction in maturity increases the value of the callable and convertible bond $CCB(0)$.

The value of the game option component can be restricted when the call level is reduced. This effect is confirmed by the results in Table 5. The reduction of the call level is achieved by making the call level to be time dependent

$$H(t) = e^{-\omega(T-t)H}, \qquad \omega \geq 0. \tag{31}$$

The value of $H(t)$ increases in time and reaches H at maturity T. By $\omega = 0$, the call level reaches its maximum and is a constant H. The impact of the call level on the no-arbitrage price of game option component is stronger in the case of higher coupon rate c and lower volatility of the firm's value σ_V.

κ	c	$\omega = 0, \sigma_V = 0.2$	$\omega = 0.04, \sigma_V = 0.2$	$\omega = 0, \sigma_V = 0.4$	$\omega = 0.04, \sigma_V = 0.4$
0	0	16.92	14.84	26.85	24.40
0.04	2	8.65	7.67	20.41	19.00
0.04	4	6.99	3.64	17.03	13.58

Table 5: Influence of the call level on the no-arbitrage price of the game option component (384 steps)

6 Stochastic Interest Rate

6.1 Recursion schema

In this section we solve the optimization problems stated in Equations (22) and (25) by allowing stochastic interest rate. Similar as in Section 5, the continuous time problem is approximated with a discrete time one and the no-arbitrage value is derived by a recursive formula. We discretize the forward price of the firm's value process modeled in Section 2.2. Accordingly, the call level and coupons are adjusted to the forward value. The recursion is carried out on the T-forward adjusted values, see Figure 5, where $F_V(t_n^+, T)$ is the forward price of the firm's value just before payout and $CCB_F(t_n)$ is the T-forward value of the callable and convertible bond at time t_n. At the terminal date T, $F_V(T,T) = V_T$ thus $CCB_F(T) = CCB(T)$. ν_{t_n} is the default barrier. The coupon c_{t_n} will only be paid out if the firm's value is above certain level, i.e. $V_{t_n^+} > \eta_{t_n}$. The no-arbitrage price of the callable and convertible bond equals $B(0,T)CCB_F(0)$.

For $n = 0, 1, ..., N-1$,

$$CCB_F(t_n) = \begin{cases} \min\left\{ \max\left\{ \dfrac{H + c_{t_n}}{B(t_n,T)}, \dfrac{\gamma F_V(t_n^+,T)}{m+\gamma n} \right\}, \right. \\ \quad \max\left\{ \dfrac{\gamma F_V(t_n^+,T)}{m+\gamma n}, \right. \\ \quad\quad \left.\left. \mathbb{E}_{P^T}[CCB_F(t_{n+1})|\mathcal{F}_{t_n}] + \dfrac{c_{t_n}}{B(t_n,T)} \right\} \right\} & if \quad F_V(t_n^+,T) > \nu_{t_n} \\ \dfrac{F_V(t_n^+,T)}{n} & if \quad F_V(t_n^+,T) \le \nu_{t_n} \end{cases} \tag{32}$$

and

$$CCB(T) = \begin{cases} \max\left\{ \dfrac{\gamma V_{T^+}}{m+\gamma n}, \ L + c_{t_N} \right\} & if \quad V_{T^+} > n(L + c_{t_N}) \\ \dfrac{V_{T^+}}{n} & if \quad V_{T^+} \le n(L + c_{t_N}) \end{cases} \tag{33}$$

Figure 5: *Min-max* recursion callable and convertible bond, T-forward value

6.2 Some conditional expectations

The recursion formula, Equation (32) contains both $F_V(t_n,T)$ and $B(t_n,T)$ as variables. In order to circumvent a two-dimensional tree, we solve $CCB_F(t_n,T)$ as conditional expectation given $F_V(t_n,T)$. To achieve the analytical closed-form solution, we first explore the relationship between $F_V(t,T)$ and $B(t,T)$.

According to the assumptions on the firm's value made in Section 2.2, under P^T the auxiliary forward price of the firm's value $F_V^\kappa(t,T)$ and the T-forward price of the default free zero coupon bond $F_B(t,s,T) := \dfrac{B(t,s)}{B(t,T)}$, $t \le s < T$ are both martingales, and satisfy

$$\begin{aligned} dF_V^\kappa(t,T) &= F_V^\kappa(t,T) \cdot \sigma_F(t,T) dW_t^T. \\ dF_B(t,s,T) &= F_B(t,s,T) \cdot \sigma_B(t,s,T) dZ_t^T \end{aligned}$$

with

$$\begin{aligned} \sigma_F^2(t,T) &= \int_0^t \sigma_V^2 + 2\rho\sigma_V b(u,T) + b^2(u,T) du \\ \sigma_B^2(t,s,T) &= \int_0^t (b(u,s) - b(u,T))^2 du \end{aligned}$$

and

$$b(t,s) = \frac{\sigma_r}{b}(1 - e^{-b(s-t)}).$$

W_t^T and Z_t^T are two correlated standard Brownian motion with constant coefficient of correlation equals ρ.

Hence $F_V^\kappa(t,T)$ and $F_B(t,t,T) = \dfrac{B(t,t)}{B(t,T)} = \dfrac{1}{B(t,T)}$ are bivariate normally distributed and have the following variances, expectations and covariances[8]

$$\begin{aligned} \sigma_1^2 &:= \mathbb{V}_{P^T}[\ln F_V^\kappa(t,T)] &&= \int_0^t (\sigma_V^2 + 2\rho\sigma_V b(s,T) + b^2(s,T)) ds \\ \sigma_2^2 &:= \mathbb{V}_{P^T}[\ln F_B(t,t,T)] &&= \frac{1}{2b^3}(1 - e^{-2bt}) b(t,T)^2 \\ \mu_1 &:= \mathbb{E}_{P^T}[\ln F_V^\kappa(t,T)] &&= \ln F_V^\kappa(0,T) - \frac{1}{2}\sigma_1^2 \\ \mu_2 &:= \mathbb{E}_{P^T}[\ln B(t,T)] &&= \mathbb{E}[-\ln F_B(t,t,T)] = \ln \frac{B(0,T)}{B(0,t)} + \frac{1}{2}\sigma_2^2 \end{aligned}$$

and

$$\begin{aligned} \gamma &:= \mathrm{Cov}_{P^T}(\ln F_V^\kappa(t,T), \ln B(t,T)) \\ &= -\mathrm{Cov}_{P^T}(\ln F_V^\kappa(t,T), \ln F_B(t,T)) \\ &= \int_0^t \Big(\rho\sigma_V(b(u,T) - b(u,t)) + (b(u,t)b(u,T) - b(u,t)^2)\Big) du. \end{aligned}$$

Given these relationships the expectation and variance of $\ln B(t,T)$ conditional on the forward price of the firm's value can be derived with the following formulas

$$\mu_3 := \mathbb{E}\Big[\ln B(t,T) \mid \ln F_V^\kappa(t,T) = \bar{w}\Big] = \mu_2 + \frac{\gamma}{\sigma_1^2}(\ln \bar{w} - \mu_1), \tag{34}$$

$$\sigma_3^2 := \mathbb{V}\Big[\ln B(t,T) \mid \ln F_V^\kappa(t,T) = \bar{w}\Big] = \sigma_2^2 - \frac{\gamma^2}{\sigma_1^2}. \tag{35}$$

Therefore, conditional on $\ln F_V^\kappa(t,T) = \bar{w}$ the random variable $\ln(B(t,T))$ equals

$$\ln B(t,T)\Big(\ln F_V^\kappa(t,T) = \bar{w}\Big) = \mu_3 + \sigma_3 x$$

[8]For details see Menkveld and Vorst (2000).

where x is a standard normal random variable. Thus the following conditional expectation can be derived after some elementary integration

$$\mathbb{E}\Big[\frac{1}{B(t,T)}\ \Big|\ \ln F_V^\kappa(t,T)=\bar{w}\Big] = \exp\left(-\mu_3+\frac{1}{2}\sigma_3^2\right) \tag{36}$$

$$\mathbb{E}\left[\Big(\frac{p}{B(t,T)}-q\Big)^+\ \Big|\ \ln F_V^\kappa(t,T)=\bar{w}\right] = \int_{-\infty}^{h}\Big(\frac{p}{e^{\mu_3+\sigma_3\cdot x}}-q\Big)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$

$$= p\cdot e^{-\mu_3+\frac{\sigma_3^2}{2}}N(h+\sigma_3)-q\cdot N(h) \tag{37}$$

with $h=(\ln(p/q)-\mu_3))/\sigma_3$ for some $p,q\in\mathbb{R}_+$. Here, $N(\cdot)$ denotes the cumulative distribution function of a standard normal distribution.

6.3 Implementation with binomial tree

For the implementation of the recursion schema displayed in Figure 5 we apply the method developed by Menkveld and Vorst (1998) which is a combination of an analytical approach and a one-dimensional binomial tree approach. A simple recombining binomial tree for the forward price $F_V(t,T) := V_t/B(t,T)$ of the firm's value can be constructed with the trick that the interval $[0,T]$ is not divided into periods of equal length, but into periods of equal volatility. Recursion is then carried out alongside the T-forward risk adjusted tree. The interval $[0,T]$ is divided into periods $0=t_0<t_1<...<t_N=T$ of equal volatility

$$\sigma_F^N := \frac{1}{N}\int_0^T(\sigma_V^2+2\rho\sigma_V b(s,T)+b^2(s,T))ds.$$

The stochastic evolution of the forward price of the firm's value is then modeled by

$$F_V(n,j) = F(0)u^jd^{n-j}\hat{\kappa}_n, \quad \forall j=0,...,n, \quad n=1,...,N$$

with $F(0)=V(0)/B(0,T)$ and

$$u=e^{\sigma_F^N}, \quad d=e^{-\sigma_F^N}, \quad \hat{\kappa_n}=e^{-\kappa\Delta_n}, \quad \Delta_n=t_n-t_{n-1},$$

where $F_V(n,j)$ denotes the forward price of the firm's value after payout, at time t_n after j up-movements. $F(0)$ is the initial forward price of the firm's value. The expressions show that u and d are time and state independent. $\hat{\kappa}_n$ is time dependent as the time steps are no longer of equal length. The (time dependent) coupon payment is given by $c(n)=c\Delta_n$. The forward martingale measure P^T exists because $d<1<u$ and the transition probability is given by

$$p^T := \frac{1-d}{u-d}.$$

Thus the conditional expectation in the recursion schema can be calculated as

$$EV(n,j) := p^T\cdot CCB_F(n+1,j)+(1-p^T)\cdot CCB_F(n+1,j+1)$$

The forward price of the firm's value at time t_n after j up movements and just before payout is

$$F_V(n+,j) := \frac{F_V(n,j)}{1-\hat{\kappa}_n}.$$

At each node (n,j) we calculate the expected value of the *min-max* strategy under the measure P^T conditional on the available information $F_V(n,j)$. The calculation is tedious but can be solved analytically. We make first some simplifications of the notations which are only used for the calculation of $CCB_F(n,j)$. $H(n,j)$ and $c(n,j)$ are written as H and c, and

$$CV := \frac{\gamma F_V(n+,j)}{m+\gamma n} \qquad EV := EV(n,j)$$

which are conversion and simple recursion value. According to the recursion formula Equation (32),

$$\begin{aligned} CCB_F(n,j) &= \min\left\{\max\left\{\frac{H+c}{B(t_n,T)}, CV\right\}, \max\left\{CV,\ EV+\frac{c}{B(t_n,T)}\right\}\right\} \\ &= \min\left\{\left[\frac{H+c}{B(t_n,T)}-CV\right]^+ + CV,\ \left[EV+\frac{c}{B(t_n,T)}-CV\right]^+ + CV\right\} \\ &= CV+\left[EV+\frac{c}{B(t_n,T)}-CV\right]^+ \\ &\quad -\left[\frac{H}{B(t_n,T)}-EV\right]^+ \mathbf{1}_{\{\frac{H+c}{B(t_n,T)}>CV\}}\mathbf{1}_{\{EV+\frac{c}{B(t_n,T)}>CV\}}. \end{aligned} \tag{38}$$

Equation (38) can be further calculated in two cases.

(i) $CV \le EV$

$$CCB_F(n,j) = EV + \frac{c}{B(t_n,T)} - \left[\frac{H}{B(t_n,T)} - EV\right]^+ \tag{39}$$

because in this case the second term of Equation (38) is certainly positive and $\frac{H}{B(t_n,T)} > CV$ includes also the case $\frac{H+c}{B(t_n,T)} > CV$.

(ii) $CV > EV$

$$\begin{aligned} CCB_F(n,j) &= CV + \left[\frac{c}{B(t_n,T)} - (CV-EV)\right]^+ \\ &\quad - \left[\frac{H}{B(t_n,T)} - EV\right]^+ \mathbf{1}_{\{B(t_n,T)>MIN\}} \end{aligned} \tag{40}$$

where

$$MIN := \min\left[\frac{H}{EV}, \frac{H+c}{CV}, \frac{c}{CV-EV}\right].$$

According to the conditional expectations given in Equations (36) and (37), the analytical solution of Equations (39) and (40) can be derived as conditional expectations given $F_V^\kappa(n,j) = F_V(n,j)e^{\kappa t_n} = \bar{w}$.

(i) $CV \le EV$

$$CCB_F(n,j) = EV + c\cdot\exp\left[-\mu_3+\frac{\sigma_3^2}{2}\right] - H\cdot\exp\left[-\mu_3+\frac{\sigma_3^2}{2}\right]N(h_1+\sigma_3) + EV\cdot N(h_1)$$

where

$$h_1 := \frac{\ln\frac{H}{EV}-\mu_3}{\sigma_3}.$$

(ii) $CV > EV$

$$\begin{aligned} CCB_F(n,j) &= CV + c\cdot\exp\Big[-\mu_3+\frac{\sigma_3^2}{2}\Big]\cdot N(h_2+\sigma_3) - (CV-EV)N(h_2) \\ &\quad +H\cdot\exp\Big[-\mu_3+\frac{\sigma_3^2}{2}\Big]\cdot N(h_3+\sigma_3) - EVN(h_3) \end{aligned}$$

where

$$\begin{aligned} h_2 &:= \frac{\ln\frac{c}{CV-EV}-\mu_3}{\sigma_3} \\ h_3 &:= \frac{\ln MIN-\mu_3}{\sigma_3}. \end{aligned}$$

And μ_3 and σ_3 have been defined in Equations (34) and (35).

In the following numerical example we compute the no-arbitrage price of a callable and convertible bond with stochastic interest rates.

Example 6.1. The initial term structure is flat, choose $T=8$, $\sigma_V=0.2$, $K=400$, $\omega=1300$, $\sigma_r=0.02$, $b=0.1$, $V(0)=1000$, $L=100$, $K=400$, $m=10$, $n=8$, $H=120$, $\gamma=1.5$, $r_0=0.06$.[9] The recursions are carried out alongside a tree with 384 steps.

The no-arbitrage prices of a straight bond, a callable and convertible bond and the game option component in American-style with and without stochastic interest rates are presented in Table 6. "Non" stands for no interest rate risk, "-0.5" and "0.5" give the correlation coefficient of the interest rate and firm's value. The values are derived for different payout and coupon combinations.

		G(0)			CCB(0)			$SB(0)$		
κ	c	Non	-0.5	0.5	Non	-0.5	0.5	Non	-0.5	0.5
0	0	16.92	15.80	19.07	76.32	76.00	76.49	59.40	60.21	57.41
0.04	2	8.65	7.42	9.97	73.80	73.97	73.40	65.15	66.56	62.35
0.04	4	6.99	6.03	8.88	81.05	82.09	80.29	74.50	76.06	71.41

Table 6: No-arbitrage prices of the non-convertible bond, callable and convertible bond and game option component in American-style with stochastic interest rate (384 steps)

Increasing correlation between the interest rate and the firm's value causes increasing volatility of the forward price of the firm's value. The default probability rises with increasing volatility, which results in a reduction of the value of the straight bond $SB(0)$. But on the other side, the value of the game option component $G(0)$ increases in volatility. Therefore in general the total effect is uncertain, in our concrete example the total value declines with increasing correlation. Moreover, the influence of the interest rate risk is relatively small which is recognized by the value of the convertible bond, the results listed in the columns under $CCB(0)$.

[9] The default barrier is $\eta_t = KB(t,T)e^{-\kappa t}$, and the coupons are to be paid if the firm's value is above $\eta_t=\omega B(t,T)e^{-\kappa t}$.

7 Uncertain Volatility of Firm Value

In practice it is often a difficult problem to calibrate a model to the available data. Here one major drawback of the structural model approach is that it specifies a certain firm's value process. As the firm's value, however, is not always observable, e.g. due to incomplete information, determining the volatility of this process is a non-trivial problem. Moreover, the interest rate risk and the uncertainty about the correlation of the interest rate and firm value process are other contributors to the uncertainty of the volatility.

To relax the assumption of constant volatility of the firm's value, one can specify volatility as a particular function of the firm's value, or model volatility itself with a stochastic process. However, specification of a reasonable model for the volatility dynamics and precise estimation of the parameters would be a difficult task. We circumvent these problems by assuming that the volatility of the firm's value process lies between two extreme values. The volatility is no longer assumed to be constant or a function of underlying and time. It is instead assumed to lie between two extreme values $\sigma_{\min}$ and $\sigma_{\max}$, which can be viewed as a confidence interval for the future volatilities. This assumption is less stringent compared to the approaches where the volatility is modeled as a function of the underlying or as a stochastic process. It needs also less parameter inputs.

We treat the callable and convertible bond with uncertain volatility by applying the model of Avellaneda et al. (1995) and Lyons (1995) and combining it with the results of Kallsen and Kühn (2005) on game option in incomplete market such that certain pricing bounds can be derived. The bondholder selects the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, and the expectation is taken with the most pessimist estimate from the aspect of the bondholder. The optimal strategy of the bondholder and his choice of the pricing measure determine the lower bound for the no-arbitrage price. Whereas the issuer chooses the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder and the expectation is also the most pessimist one but from the aspect of the issuer, thus the upper bound of the no-arbitrage price can be derived. The volatility is selected dynamically from the two values $\sigma_{\min}$ and $\sigma_{\max}$ in a way that always the one with the worse effect, thus the most pessimist pricing measure is chosen.

7.1 Uncertain volatility model

The *uncertain volatility* model is first proposed independently by Avellaneda et al. (1995) and Lyons (1995). It is an extension of the Black-Scholes framework to deal with the biased estimate of the historical volatility or the smile effect of the implied volatility[10]. Avellaneda et al. (1995) study the case of derivatives written on a single underlying asset. The volatility of the asset is not assumed to be a constant or a function of the underlying or rather stochastic. Instead, it is only assumed to lie between two extreme values $\sigma_{\min}$ and $\sigma_{\max}$, which can be viewed as a confidence interval for the future volatilities. This assumption is less stringent compared to other approaches and it needs also less parameter inputs. The derivation of a no-arbitrage pricing bound is based on a super-hedging strategy which is a worst case estimation. At each (t, x) the volatility is selected dynamically from the two values $\sigma_{\min}$ and $\sigma_{\max}$ in a way that always the one with the worse effect on the value of the derivative from aspect of seller or buyer is chosen.

For a given martingale measure Q, suppose the stock price evolves according to the following

[10]The volatility implied from the traded options, plotted as a function of the strike price, often exhibits a specific *U-shape*, which is referred to as the *smile effect*.

dynamic

$$dS_t = S_t(rdt + \sigma_t dW_t^*),$$

where, for simplification the interest rate r is assumed to be constant. The super-hedge, i.e. the worst case scenario leads to the solution of a non-linear PDE, which is called Black-Scholes-Barenblatt equation

$$\frac{\partial f}{\partial t} + r\left(S\frac{\partial f}{\partial S} - f\right) + \frac{1}{2}\Sigma^2\left[\frac{\partial^2 f}{\partial S^2}\right]S^2\frac{\partial^2 f}{\partial S^2} = 0, \tag{41}$$

with terminal value $f(S,T) = F(S)$, and $\Sigma^2[x]$ stands for a volatility parameter which depends on x, the convexity of function f. For example, the super-hedge price for the seller of a call option can be obtained by setting

$$\Sigma^2[x] = \begin{cases} \sigma^2_{\max} & \text{if} \quad x \geq 0 \\ \sigma^2_{\min} & \text{else.} \end{cases}$$

The authors provide also a simple algorithm for solving the equation by a trinomial tree and prove the convergence of this discrete scheme. In case of vanilla European options, the pricing bounds can be derived simply with the Black-Scholes equations using the extreme values of the volatility parameter, thus the nonlinear solution is reduced to the linear Black-Scholes solution.

Lyons (1995) treats the case of derivatives written on multiple assets. The volatility is assumed to lie in some convex region depending on the prices of the underlying and time. Same as Avellaneda et al. (1995), the volatility matrix is chosen such that the worst effect on the derivative is achieved. However, vanilla European options written on multi-assets, in general, cannot be derived simply by using the extreme values of the volatility parameter. Moreover, it is only possible under particular conditions to reduce the nonlinear solution to the linear Black-Scholes solution.

7.2 No-arbitrage pricing bounds

The relax of the assumption of deterministic volatility and the adoption of the uncertain volatility introduce market incompleteness. There would be a set of possible equivalent martingale measures which are compatible with the no arbitrage requirement. The holder and issuer of a callable and convertible bond must not only decide their optimal stopping strategies but also the proper pricing measure.

This problem has been considered by Kallsen and Kühn (2005) in context of game option in incomplete market. Theorem 2.2 of their paper tells us that: suppose that only a *buy-and-hold* strategy is allowed in the game option, while the underlying risky asset and the savings account can be traded dynamically, the set of initial no-arbitrage prices is determined by super hedging and lies in the interval $[G_{low}(0), G_{up}(0)]$ with

$$G_{low}(0) = \sup_{\tau_B \in \mathcal{F}_{0T}} \inf_{\tau_A \in \mathcal{F}_{0T}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B)] \tag{42}$$

$$G_{up}(0) = \inf_{\tau_A \in \mathcal{F}_{0T}} \sup_{\tau_B \in \mathcal{F}_{0T}} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B)] \tag{43}$$

where $\mathcal{Q}$ is the family of equivalent martingale measures, $\mathcal{F}_{0T}$ is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{0 \leq u \leq T}$ with values in $[0,T]$, and $g(\tau_A, \tau_B)$ is defined in Section 4.1 by Equation (18). The bondholder selects the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, and the expectation is taken with the most pessimistic estimate from the aspect of the bondholder. The optimal strategy of the bondholder and his choice of the pricing measure determine the lower bound of the no-arbitrage price. Whereas the issuer chooses the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This expectation is also the most pessimistic one but from the aspect of the issuer, thus the upper bound of the no-arbitrage price can be derived.

Suppose that the callable and convertible bond is not traded dynamically, applying the results from the theory of game option which are given in Equations (42) and (43), the set of initial no-arbitrage prices can be determined. It is given by the interval $[CCB_{low}(0), CCB_{up}(0)]$ with

$$CCB_{low}(0) = \sup_{\tau_B \in \mathcal{F}_{0T}} \inf_{\tau_A \in \mathcal{F}_{0T}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[ccb(0)] \tag{44}$$

and

$$CCB_{up}(0) = \inf_{\tau_A \in \mathcal{F}_{0T}} \sup_{\tau_B \in \mathcal{F}_{0T}} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[ccb(0)] \tag{45}$$

where $\mathcal{Q}$ is the family of equivalent martingale measures, $\mathcal{F}_{0T}$ is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{0 \le u \le T}$ with values in $[0, T]$, $cbb(0)$ is defined in Section 3.1 by Equation (14).

7.3 Discretization and recursion schema

The upper and lower bound $CCB_{up}(0)$ and $CCB_{low}(0)$ can be approximated with the recursions demonstrated in Figures 6 and 7.

For $n = 0, 1, ..., N-1$,

$$CCB_{up}(t_n) = \begin{cases} \min\Big\{ e^{-rt_n} \max\Big\{ H + c_{t_n}, \dfrac{\gamma V_{t_n^+}}{m + \gamma n} \Big\}, \max\Big\{ e^{-rt_n} \dfrac{\gamma V_{t_n^+}}{m + \gamma n}, \\ \qquad \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[CCB_{up}(t_{n+1}) | \mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} \Big\} \Big\} & if \quad V_{t_n^+} > \nu_{t_n} \\ e^{-rt_n} \dfrac{V_{t_n^+}}{n} & if \quad V_{t_n^+} \le \nu_{t_n} \end{cases}$$

and

$$CCB(T) = \begin{cases} e^{-rT} \max\Big\{ \dfrac{\gamma V_{T^+}}{m + \gamma n}, \ L + c_{t_N} \Big\} & if \quad V_{T^+} > n(L + c_{t_N}) \\ e^{-rT} \dfrac{V_{T^+}}{n} & if \quad V_{T^+} \le n(L + c_{t_N}) \end{cases}$$

Figure 6: Recursion: upper bound for callable and convertible bond

7.4 Implementation with trinomial tree

To make the computation tractable, we make some simplifications on the firm's value process and the default mechanism defined in Section 2. The interest rate r, the payout rate κ and the default barrier K are assumed to be constant. The volatility of the firm's value lies between two extreme values $\sigma_{\min}$ and $\sigma_{\max}$ which are two constant. The firm's value process can thus be described with the following diffusion process

$$dV_t = V_t((r - \kappa)dt + \sigma_t dW_t)$$

For $n = 0, 1, ..., N-1$,

$$CCB_{low}(t_n) = \begin{cases} \max\left\{e^{-rt_n}\dfrac{\gamma V_{t_n^+}}{m+\gamma n}, \min\left\{e^{-rt_n}\max\left\{H + c_{t_n}, \dfrac{\gamma V_{t_n^+}}{m+\gamma n}\right\}, \right.\right. & \\ \left.\left. \qquad \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[CCB_{low}(t_{n+1})|\mathcal{F}_{t_n}] + e^{-rt_n}c_{t_n}\right\}\right\} & if \quad V_{t_n^+} > \nu_{t_n} \\ e^{-rt_n}\dfrac{V_{t_n^+}}{n} & if \quad V_{t_n^+} \leq \nu_{t_n} \end{cases}$$

and

$$CCB(T) = \begin{cases} e^{-rT}\max\left\{\dfrac{\gamma V_{T^+}}{m+\gamma n},\ L + c_{t_N}\right\} & if \quad V_{T^+} > n(L + c_{t_N}) \\ e^{-rT}\dfrac{V_{T^+}}{n} & if \quad V_{T^+} \leq n(L + c_{t_N}) \end{cases}$$

Figure 7: Recursion: lower bound for callable and convertible bond

and

$$\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}.$$

Avellaneda et al. (1995) provide also a simple algorithm for solving the Black-Scholes-Barenblatt equation by a trinomial tree. According to this discretization, the time interval $[0, T]$ is divided in N subintervals of equal lengths. The distance between two periods is $\Delta = T/N$. After each period Δ, the firm's value will go up, in the middle way, or down, and then has the corresponding value

$$V_{t_{n+1}} = u \cdot V_{t_n}, \quad V_{t_{n+1}} = m \cdot V_{t_n}, \quad V_{t_{n+1}} = d \cdot V_{t_n},$$

where

$$u = e^{\sigma_{\max}\sqrt{\Delta}+(r-\kappa)\Delta}, \quad m = e^{(r-\kappa)\Delta}, \quad d = e^{-\sigma_{\max}\sqrt{\Delta}+(r-\kappa)\Delta}.$$

The so constructed tree is recombining because $m^2 = u \cdot d$. The stochastic evolution of the firm's value is then modeled by

$$V(n,j) = V(0) \cdot e^{j\cdot\sigma_{\max}\sqrt{\Delta}+n\cdot(r-\kappa)\Delta}, \quad \forall j = 0, ..., 2n, \quad n = 1, ..., N,$$

where $V(n,j)$ denotes the firm's value at time $t_n := n\Delta$ in state j. At time t_{n+1} there are three possible nodes conditional on (n,j) : in case of an up-movement we have $(n+1, j+1)$, in case of a down-movement $(n+1, j-1)$ and in case of the middle way $(n+1, j)$. Thus higher j indicates a higher firm's value at time t_n. $V(0)$ is the initial firm's value. The transition probability for the up- and down-movement is, respectively, given by

$$\begin{aligned} p_u(p) &:= p \cdot \left(1 - \frac{\sigma_{\max}\sqrt{\Delta}}{2}\right) \\ p_d(p) &= p \cdot \left(1 + \frac{\sigma_{\max}\sqrt{\Delta}}{2}\right) \\ p_m(p) &= 1 - 2p \end{aligned}$$

where the parameter p varies in the range $\sigma^2_{\min}/(2\sigma^2_{\max}) \leq p \leq 1/2$.[11] This condition ensures that the uncertain volatility σ takes values such that $\sigma_{min} \leq \sigma \leq \sigma_{max}$. The trinomial tree

[11]The transition probabilities depend on p because otherwise we would have a deterministic volatility model.

has one degree of freedom at each node, thus the choice of risk-adjusted probabilities is not unique. This freedom is used to model heteroskedasticity. For $p = 1/2$, highest probabilities are assigned to the extreme nodes u and d which yields the largest volatility. While for $p = \sigma_{\min}^2/(2\sigma_{\max}^2)$ highest probability is assigned to center node m, thus the lowest volatility is achieved. Therefore, by fixing u, d and m and allowing the risk-adjusted probabilities to vary over a one-dimensional set, a range of variances within the volatility band $[\sigma_{\min}, \sigma_{\max}]$ can be modeled.

Define

$$EV^+(t_n) := \sup_{Q\in\mathcal{Q}} \mathbb{E}_Q[CCB_{up}(t_{n+1})|\mathcal{F}_{t_n}]$$

and

$$EV^-(t_n) := \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[CCB_{up}(t_{n+1})|\mathcal{F}_{t_n}],$$

at each node (n, j)

$$EV^+(n,j) = \begin{cases} CCB_{up}(n+1,j) + \frac{1}{2}Z^+(n+1,j) & if\ Z^+(n+1,j) > 0 \\ CCB_{up}(n+1,j) + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}Z^+(n+1,j) & if\ Z^+(n+1,j) \le 0 \end{cases}$$

and

$$EV^-(n,j) = \begin{cases} CCB_{low}(n+1,j) + \frac{1}{2}Z^-(n+1,j) & if\ Z^-(n+1,j) < 0 \\ CCB_{low}(n+1,j) + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}Z^-(n+1,j) & if\ Z^-(n+1,j) \ge 0 \end{cases}$$

where $Z^+(n+1,j)$ and $Z^-(n+1,j)$ are the approximations of the second-derivative and are defined as

$$\begin{aligned} Z^+(n+1,j) &:= (1 - \frac{\sigma_{\max}\sqrt{\Delta}}{2})CCB_{up}(n+1,j+1) + (1 + \frac{\sigma_{\max}\sqrt{\Delta}}{2})CCB_{up}(n+1,j-1) \\ &\quad - 2CCB_{up}(n+1,j) \end{aligned}$$

$$\begin{aligned} Z^-(n+1,j) &:= (1 - \frac{\sigma_{\max}\sqrt{\Delta}}{2})CCB_{low}(n+1,j+1) + (1 + \frac{\sigma_{\max}\sqrt{\Delta}}{2})CCB_{low}(n+1,j-1) \\ &\quad - 2CCB_{low}(n+1,j). \end{aligned}$$

7.5 Illustration with a numerical example

Example 7.1. Let $T = 8$, $\sigma_{\min} = 0.2$, $\sigma_{\max} = 0.4$, $V = 1000$, $L = 100$, $K = 300$, $m = 10$, $n = 8$, $H = 120$, $\gamma = 1.5$, $r = 0.06$. In Table 7 the call level is kept constant with H while in Table 8 the call level is time dependent with

$$H(t) = e^{-w(T-t)}H, \qquad w = 0.04.$$

The pricing bounds of the callable and convertible bonds with uncertain volatility which lies in the interval $[0.2, 0.4]$ are summarized in Tables 7 and 8. These price bounds are compared with the results if they are calculated with the extreme values of the volatility. Since we chose a relatively wide range of volatilities, $\sigma_{\min} = 0.2$ and $\sigma_{\max} = 0.4$, the price differential of the lower and upper bound is relatively large. Moreover, the lower (upper) bounds are smaller (larger) than the results calculated with extreme volatilities.
The reduction of the call level is achieved in Table 8 by making it time dependent. Comparing the results in Tables 7 and 8, we see that both lower and upper bound are lower in Table 8.

κ	c	$\sigma_V \in [0.2, 0.4]$		$\sigma_V = 0.2$	$\sigma_V = 0.4$
0	0	73.68	78.67	76.33	74.90
0.04	2	69.20	75.70	73.81	71.60
0.04	3	71.23	79.22	77.66	73.35
0.04	4	73.20	82.94	81.50	75.08

Table 7: Pricing bounds callable and convertible bond with uncertain volatility and constant call level H (384 steps)

κ	c	$\sigma_V \in [0.2, 0.4]$		$\sigma_V = 0.2$	$\sigma_V = 0.4$
0	0	71.97	75.06	74.25	72.37
0.04	2	69.04	73.56	72.83	70.57
0.04	3	70.26	76.24	75.51	71.49
0.04	4	71.20	78.94	78.16	72.41

Table 8: Pricing bounds callable and convertible bond with uncertain volatility and time dependent call level $H(t)$ (384 steps)

It is intuitive as the callable and convertible bond is less valuable by a lower call level. The reduction of the call level has larger impact on the upper bound. For example, for $\kappa = 0.04$ and $c = 4$, the lower bound goes from 73.20 to 71.20 while the upper bound drops from 82.94 to 78.94.

8 Conclusion

Modeling of the callable and convertible bond as a defaultable game option within structural approach has been studied by Sirbu et al. (2004) and further developed in a companion paper of Sirbu and Schreve (2006). In their models the volatility of the firm's value and the interest rate are constant. The bond earns continuously a stream of coupon at a fixed rate. The dynamic of the firm's value does not follow a geometric Brownian motion, but a more general one-dimensional diffusion due to the fixed rate of coupon payment. Default occurs if the firm's value falls to zero which means both equity and bond have zero recovery. The no-arbitrage price of the bond is characterized as the result of a two-person zero-sum game. Viscosity solution concept is used to determine the no-arbitrage price and optimal stopping strategies. Our model differs from theirs mainly by allowing non-zero recovery rate of the bond and default occurs if the firm's value hit a low but positive boundary. The dynamic of the firm's value follows a geometric Brownian motion which means that the underlying process, the evolution of the firm's value, does not depend on the solution of the game option. Therefore the results of Kifer (2000) can be applied to the valuation of the bond. Simple recursion with a binomial tree can be used to derive the value of the bond and the optimal strategies. Moreover, stochastic interest rate and uncertain volatility can be incorporated into our model.

Our idealized model illustrates how the optimal strategies work and what are the important underlying factors. For practical use other features have to be taken into account. For example, a firm issues usually several different kinds of debt with different priorities. Convertible bonds are usually junior debt. The mutual dependence of the different debts and stocks must also be modeled. We derived pricing bounds for convertible and callable bonds under the assumption that the volatility of the firm value process lies between two extreme values. The pricing bounds can be improved if a narrower confidence interval of the volatility of the firm value is available. Otherwise we need more knowledge of the risk preferences of the bondholder and issuer. Randomization of the strategies and partial exercises could be the subject of the future study.

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