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by

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Dynamic E¢ciency and Pareto Optimality in a Stochastic OLG Model with Production and Social Security

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Abstract

We analyze the interaction between risk sharing and capital accumulation in a stochastic OLG model with production. We give a complete characterization of interim Pareto optimality. Our characterization also subsumes equilibria with a PAYG social security system. In a competitive equilibrium interim Pareto optimality is equivalent to intergenerational exchange e¢ciency, which in turn implies dynamic e¢ciency. Furthermore, dynamic e¢ciency does not rule out a Pareto-improving role for a social security system. Social security can provide insurance against macroeconomic risk, namely aggregate productivity risk in the second period of life (old age) through dynamic risk sharing.

We brie‡y relate our results to models without uncertainty where the notions of exchange e¢ciency, dynamic e¢ciency and interim Pareto optimality are all equivalent in a competitive equilibrium.

JEL classi…cation: D61, H55

Keywords: Stochastic OLG Model, Dynamic E¢ciency, Interim Pareto Optimality, Social Security, Risk Sharing

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1 Introduction

Social security old age insurance has been a policy issue for decades in all countries with pay-as-you-go public pension systems. A key policy question is whether it is better to finance social security on a pay-as-you-go (PAYG) or fully funded basis. Until recently, academics and policymakers have mostly used deterministic models to analyze this fundamental question. Recent proposals to privatize social security in the US, in addition to less radical proposals such as the social security administration's plan to invest a portion of the Social Security Trust Fund in equities, have highlighted that the deficient treatment of risk and uncertainty might seriously flaw policy analysis. The reason for this flaw is that many proposed policy reforms involve subtle changes in who bears various risks. It has only been in the last few years that theoretical and numerical models have incorporated uncertainty in order to analyze these issues. This has lead to a renewed interest in overlapping generations models.

Most of the theoretical literature is based on the Diamond (1965) OLG model with capital. Under certainty, the question of whether to finance social security on a fully funded basis or whether there is a Pareto-improving role for a PAYG system then translates to the question of dynamic efficiency of competitive equilibria (Bose (1974), Breyer (1989), Bose and Ray (1993)).

Under uncertainty, the analysis of social security becomes conceptually more complex. The reason for this complexity is that, apart from the issue of efficient capital accumulation, risk sharing issues have to be considered. We analyze the interaction between risk sharing and capital accumulation in a stochastic OLG model with production. This allows us to derive implications about the Pareto optimality of competitive equilibria with a redistributive transfer scheme like a PAYG social security system. We characterize conditions under which the dynamic risk sharing opportunities of a PAYG social security system may lead to a Pareto-improvement of a pure market allocation.

In the development of the OLG literature three themes concerning efficiency can be identified: efficient intergenerational exchange, efficient production (overaccumulation of capital) and Pareto optimality. The first theme of efficient intergenerational exchange was already mentioned in the seminal paper by Samuelson (1958). As is well known, in pure exchange OLG economies the first welfare theorem may fail to hold, i.e. competitive equilibria may fail to be Pareto optimal. The contributions to this theme are concerned with the reasons for this failure. A characterization of efficient exchange was given by Balasko and Shell (1980) and Okuno and Zilcha (1980) in a pure exchange OLG model under certainty.

The second theme was introduced in OLG models in the celebrated contribution by Diamond (1965) where the question of overaccumulation of capital (dynamic efficiency) is examined. The first complete characterization of dynamic efficiency was given in the context of an infinite horizon production model by Cass (1972). Tirole (1985) has analyzed the relationship between dynamic efficiency and the existence of bubbles as well as the Pareto optimality of bubbly equilibria in an OLG model with production under certainty. An extension of the dynamic efficiency issue under certainty to a setting with uncertainty has been given by Zilcha (1990) and Dechert and Yamamoto (1992). They derived complete characterizations of dynamic efficiency in stochastic OLG models. However, they do not deal with risk sharing issues and hence Pareto optimality.

A third theme that has received considerable interest is that of Pareto optimality of equilibria in OLG models. As can be seen from the discussion above, this question is closely related to the other two themes. First, in the pure exchange case under certainty, the question of exchange efficiency (theme 1) and that of Pareto optimality (theme 3) are equivalent. In fact, the characterizations given in Balasko and Shell (1980) and Okuno and Zilcha (1980) are stated in terms of Pareto optimality. Our terminology, exchange efficiency, is introduced to highlight that Pareto optimality in a general setting (with production) consists of three distinct issues: efficient exchange, efficient production and impossibility of improving by joint changes in distribution and production of commodities. A fourth issue, the efficient allocation of risk, enters once uncertainty is introduced. The risk sharing issues that have to be considered in a stochastic setting are of course closely related to exchange efficiency, since the notion of exchange efficiency involves the consumers' preferences which reflect their attitudes towards risk. Second, under certainty it can be shown that efficient production (theme 2) already implies Pareto optimality (theme 3) in a competitive equilibrium (see Bose and Ray (1993) for a discussion). Under uncertainty, however, the relationship becomes more complex because risk sharing issues have to be considered.

At this point we have to be more precise about the notion of Pareto optimality adopted under uncertainty. The definition most often used in the literature is that of interim Pareto optimality[1] which means that agents born in different states are considered distinct agents. Agents' welfare is thus evaluated by conditioning their utility on the date of their birth. We also make use of this concept of optimality.

The use of interim Pareto optimality excludes Pareto-improvements through risk sharing that arise from the market incompleteness implied by the missing insurance possibilities against the state in which an individual is born. If it is assumed that markets are complete once an individual is born (sequentially complete markets), then the remaining risk sharing possibilities (if any) follow from the dynamic structure of the economy. It may be possible to introduce a kind of intertemporal insurance which works as an intergenerational transfer under certainty with the exception that transfers in the second period of life may be different in distinct states of the world and thus also incorporate an insurance aspect.[2]

[1] The terms interim and conditional Pareto optimality are not used consistently in the literature. We follow Demange and Laroque (1999) and use interim Pareto optimality for allocations which are optimal among all feasible allocations, not only stationary ones.

[2] An alternative notion of optimality would be ex ante Pareto optimality. Here agents' utility is evaluated before they are born, i.e. in expected terms. A good discussion of alternative concepts of Pareto optimality can be found in Dutta and Polemarchakis (1990) and also in Chattopadhyay and Gottardi (1999).

A first result concerning interim Pareto optimality was obtained in an influential paper by Abel, Mankiw, Summers and Zeckhauser (1989). They derive a strong sufficient condition for interim Pareto optimality of a market equilibrium. A characterization of exchange efficiency in a pure exchange model under uncertainty was derived by Chattopadhyay and Gottardi (1999), extending the result by Balasko and Shell (1980). As under certainty, exchange efficiency is of course equivalent to interim Pareto optimality in a pure exchange setting under uncertainty.

Our first main result gives a complete characterization of interim Pareto optimality in a stochastic OLG model with production. It turns out that the concept of interim Pareto optimality is equivalent to exchange efficiency in a competitive equilibrium. Futhermore, exchange efficiency implies dynamic efficiency in a competitive equilibrium. In particular this means that extending a pure exchange model to production does not improve the possibilities of risk sharing, although the redistributional possibilities improve in comparison to a pure exchange model. This implies that under exchange efficiency, there exists no pure redistributive transfer system like a PAYG social security system that is Pareto improving. Our second main result shows that, contrary to the case of certainty, the conditions for dynamic efficiency and exchange efficiency do not coincide under uncertainty. Our analysis shows that under the interim Pareto optimality concept the important efficiency benchmark is exchange efficiency. This means that the possibility of overaccumulation of capital is not necessarily related to the risk sharing part of the efficiency problem. As argued above, under certainty, dynamic efficiency implies exchange efficiency, hence the concepts of exchange efficiency, dynamic efficiency and interim Pareto optimality are all equivalent under certainty. Under uncertainty however, dynamic efficiency is weaker than interim Pareto optimality. This means that dynamic efficiency does not rule out dynamic risk sharing possibilities which could be implemented by a PAYG social security system. A PAYG system can thus be interpreted as an insurance against aggregate productivity risk in the second period of life (old age).

The characterization does not only answer the question whether a market equilibrium without social security is suboptimal. It also applies to equilibria with social security. This can be seen by noticing that a redistributive policy like social security can replicate monetary equilibria in an OLG model where money is a pure store of value. The fact that the efficiency characterization applies to monetary equilibria as well has been used by Balasko and Shell (1981) in a pure exchange OLG model under certainty and Bose and Ray (1993) in an OLG model with production under certainty. Under uncertainty, Manuelli (1990) examined optimality of monetary equilibria in a pure exchange setup. Aiyagari and Peled (1991) and Demange and Laroque (1999) carry out a similar analysis in an economy with a linear storage technology, where the latter analysis explicitly considers social security equilibria. The contributions under uncertainty, however, restrict attention to stationary allocations, whereas our analysis (with neoclassical production technology and uncertainty) does not require this.

The paper is organized as follows. In section 2 the model is developed. In section 3, we introduce the notions of exchange efficiency and dynamic efficiency in our setting. Section 4 features our first main result, the characterization of interim Pareto optimality in a stochastic OLG model with production. Section 5 presents our second main result, a generic example in which dynamic efficiency and interim Pareto optimality do not coincide. Section 6 gives a sufficient condition for interim Pareto optimality and relates the condition to the existence of land. Section 7 presents a second welfare theorem for our economy. Section 8 concludes.

2 The Model

We consider a stochastic version of the Diamond model (Diamond (1965)). Uncertainty enters the model via shocks to the production technology. Time is discrete, starts at 0 and extends infinitely into the future. There is production and a consumption/saving decision at every point of time. The production technology at time is described by a function $F : \mathbb{R}^2_+ \times A_t \to \mathbb{R}_+$ where $F(K_t; L_t; \theta_t)$ is the output produced at time t given the capital stock is K_t; labor input is L_t and the current stochastic shock is θ_t: The perishable good produced by the technology is the only good in the economy and is used for production and consumption. There is one representative consumer born per period of time and state of the economy who inelastically supplies one unit of labor in his youth and lives for two periods and trades the consumption good on sequentially complete markets. Due to the production shocks, he has an uncertain second period of life. There is no population growth. These assumptions are only made for the simplicity of exposition. It can easily, at the cost of some additional notation, be dispensed with. Further, again for simplicity, the depriciation rate rate is assumed to be 1. More specifically, the production function satisfies:

- $F(K_t; L_t; \theta_t)$ is homogenous of degree 1 in $K_t; L_t$, strictly increasing, strictly concave and twice continuously differentiable in $K_t; L_t$: Further $F(0; L_t; \theta_t) = F(K_t; 0; \theta_t) = 0$: It also satisfies the Inada conditions $\lim_{K_t \to 0} F_K(K_t; L_t; \theta_t) = \infty$ and $\lim_{K_t \to \infty} F_K(K_t; L_t; \theta_t) = 0$: As usual, define $f(k_t; \theta_t) = F\left(\frac{K_t}{L_t}; 1; \theta_t\right)$; the per capita production function. It inherits from F the following properties:

  $f' = \frac{\partial f}{\partial k} > 0$; $f'' < 0$; $f'(0; \theta_t) = \infty$; $f'(\infty; \theta_t) = 0$:

For each period in time t; the set of production shocks, A_t; is assumed to have finite cardinality with A_0 being single valued. Consider all sequences of the form $(\theta_0; \theta_1; \theta_2; \ldots)$ where $\theta_i \in A_i$: These sequences form an uncountable set, denoted Ω: Let $\mathcal{A}$ denote the $\sigma-$algebra generated by the product topology on Ω; if each A_i is endowed with the discrete topology. Let $(\Omega; \mathcal{A}; \nu)$ be a measure space with probability measure ν which is assumed to satisfy:

If $\{\theta_0\} \times \{\theta_1\} \times \dots \times \{\theta_t\} \times \{\theta_{t+1}\} \times A_{t+2} \times \dots \in A$ is given, then the conditional probability $q_{t+1}(\theta_{t+1}|\sigma_t) = \frac{\mu(\{\theta_0\} \times \{\theta_1\} \times \dots \times \{\theta_t\} \times \{\theta_{t+1}\} \times A_{t+2} \times \dots)}{\mu(\{\theta_0\} \times \{\theta_1\} \times \dots \times \{\theta_t\} \times A_{t+1} \times \dots)}$ with $\sigma_t = (\theta_0, \theta_1, \theta_2, \dots, \theta_t)$ is well defined and strictly positive for every $\theta_{t+1} \in A_{t+1}$.

Given that the production shocks are the only source of uncertainty in the economy, it is possible to describe the uncertainty by a date-event tree, where $\sigma_0 = \{\theta_0\}$ is the root, $\sigma_t = (\theta_0, \theta_1, \theta_2, \dots, \theta_t)$ is a node at time t. The set of nodes at time t is therefore $A_0 \times A_1 \times A_2 \times \dots \times A_t$ and denoted by Σ_t. The date-event tree Γ is therefore equal to $\Sigma = \cup_{t \geq 0} \Sigma_t$. The generic element of Γ will be denoted by σ. Further we can define the functions $g^t : \Sigma_{t+1} \to \Sigma_t$ by $g^t(\theta_0, \theta_1, \theta_2, \dots, \theta_t, \theta_{t+1}) = (\theta_0, \theta_1, \theta_2, \dots, \theta_t)$, i.e. g^t assigns to each node in Σ_{t+1} its predecessor in Σ_t. The unique predecessor of a node σ will also be denoted by σ_{-1}. σ^+ denotes the set of nodes which are successors of node σ, i.e. the set of all nodes for which σ is the predecessor. Since the sets A_t are finite, the number of successors of a node is always finite. A path is a sequence of nodes $\{\sigma_t\}$ such that $\sigma_t = g^t(\sigma_{t+1})$ and a generic path will be denoted by σ^∞. $t(\sigma)$ denotes the period of time at which event $\sigma \in \Sigma_t$ occurs. For more details on the event-date tree see Chattopadhyay and Gottardi (1999).

There is one commodity available at each node in the tree and one consumer is born, who lives for two periods. So agents are here distinguished according to date and state of nature in which they are born. Therefore agents can be identified with the node at which they are born, so that in the rest of the paper the agent born in node σ will be called agent σ.

The consumption set of agent σ is $R_+^{1+S(\sigma)}$, $S(\sigma)$ is the cardinality of A_{t+1} if the agent is born in period t. His preferences will be described by a utility function $u_\sigma\left(x(\sigma,\sigma), (x(\sigma',\sigma))_{\sigma' \in \sigma^+}\right)$ where $x(\sigma) = \left(x(\sigma,\sigma), (x(\sigma',\sigma))_{\sigma' \in \sigma^+}\right)$ denotes the consumption vector of agent σ, $x(\sigma,\sigma)$ is his consumption in his youth, $(x(\sigma',\sigma))_{\sigma' \in \sigma^+}$ is his consumption in the different states of nature in his old age. In his youth, each agent inelastically supplies one unit of labor. In old age, agents receive interest payments from capital. Let $w(\sigma)$ denote the wage paid in node σ and $R(\sigma)$ the capital income. Throughout the paper assume that $w(\sigma) > 0$ and $R(\sigma) > 0$. The preferences of the agents are assumed to satisfy the following assumptions:

1. The agent born in period -1 has preferences which are strictly monotone in the single consumption good in period 0.

2. The preferences of other agents are described by a twice continuously differentiable (in the interior of its domain), strictly increasing and strictly differentiably quasiconcave $u_\sigma : R_+^{1+S(\sigma)} \to R_+$.

The firm problem is to decide at each node σ how much capital to invest, i.e. after the shock realization in period t. This capital is then used at the successor nodes of σ to produce output. Given the probabilities and prices, the firm tries to maximize expected profits. Let prices $\pi_t(\sigma_t)$ for all $\sigma_t \in \Sigma_t$ for all $t \geq 0$ be given. Let $k(\sigma_t)$ denote the investment undertaken by the firm in state σ_t. The firm's problem is then

$$\max_{k(\sigma)\geq 0} \sum_{\theta_{t+1}\in A_{t+1}} q_{t+1}(\theta_{t+1}|\sigma_t)\cdot\psi_{t+1}(\theta_{t+1};\sigma_t)\cdot f(k(\sigma_t);\theta_{t+1}) - \psi_t(\sigma_t)\cdot k(\sigma_t) \tag{1}$$

The prices can, given the investment decisions, be defined as follows: set $\psi_{-1}\equiv 1$ and define recursively $\psi_t(\sigma_t)\cdot\frac{1}{f'(k(\sigma_t);\theta_{t+1})}=\psi_{t+1}(\sigma')$ for $\sigma'\in\sigma_t^+$.

>From the prices used in the firm problem we derive the contingent claim prices for the sequentially complete markets by setting

$$p(\sigma_t)=\psi_t(\sigma_t)\cdot\prod_{i=0} q_i(\theta_i|(\theta_{i-1};\theta_{i-2};\ldots;\theta_0)) \tag{2}$$

for $\sigma_t=(\theta_0;\ldots;\theta_t)$. With this contingent prices, by setting $w(\sigma)=p(\sigma)\cdot\left[f\left(k\left(g^{t-1}(\sigma)\right);\theta_t\right)-f'\left(k\left(g^{t-1}(\sigma)\right);\theta_t\right)\cdot k\left(g^{t-1}(\sigma)\right)\right]$ and $R(\sigma')=p(\sigma')\cdot[f'(k(\sigma);\theta_{t+1})\cdot k(\sigma)]$ the consumer problem for consumers born in $t\geq 0$ can be written as

$$\max_{\left(x(\sigma;\sigma);(x(\sigma';\sigma))_{\sigma'\in\sigma^+}\right)\in\mathbb{R}_+^{1+S(\sigma)}} u_\sigma\left(x(\sigma;\sigma);(x(\sigma';\sigma))_{\sigma'\in\sigma^+}\right) \tag{3}$$

$$\text{s.t.}\quad p(\sigma)\cdot x(\sigma;\sigma)+\sum_{\sigma'\in\sigma^+}p(\sigma')\cdot x(\sigma';\sigma)\leq w(\sigma)+\sum_{\sigma'\in\sigma^+}R(\sigma')$$

Next, we define feasible allocations in this economy, the notion of interim Pareto optimality and a competitive equilibrium.

Definition 1 A feasible allocation (given initial capital $\bar{k}$) is a tuple $(x;k)=\left(\left(x(\theta_0;-1);\left(\left(x(\sigma;\sigma);(x(\sigma';\sigma))_{\sigma'\in\sigma^+}\right)_{\sigma\in\Sigma}\right)\right);\left((k(\sigma))_{\sigma\in\Sigma};\bar{k}\right)\right)$ such that

1. $x(\sigma_0;-1)+x(\sigma_0;\sigma_0)+k(\sigma_0)=f\left(\bar{k};\theta_0\right)$
2. For $\sigma\in\Sigma$: $x(\sigma';\sigma)+x(\sigma';\sigma')+k(\sigma')=f(k(\sigma);\theta)\quad\forall\sigma'=(\theta;\sigma)\in\sigma^+$

For notational convenience allocations will be simply denoted by $(x;k)$ in the rest of paper, unless confusion arises.

The concept of Pareto optimality for the economy adopted in this paper is now introduced (Muench (1977), Peled (1982)).

Definition 2 A feasible allocation $(x;k)$ is called interim Pareto-optimal if there exists no other feasible allocation $\left(\hat{x};\hat{k}\right)$ such that $\hat{x}(\theta_0;-1)\geq x(\theta_0;-1)$ and $u_\sigma\left(\hat{x}(\sigma;\sigma);(\hat{x}(\sigma';\sigma))_{\sigma'\in\sigma^+}\right)\geq u_\sigma\left(x(\sigma;\sigma);(x(\sigma';\sigma))_{\sigma'\in\sigma^+}\right)$ for all $\sigma\in\Sigma$; with at least one strict inequality.

Now, we introduce the concept of a competitive equilibrium for the economy. As a price system p we define a list of contingent prices $(p(\sigma))_{\sigma \in \Sigma}$:

Definition 3 $(x^*; k^*; p^*)$ is a competitive equilibrium if

1. $(x^*; k^*)$ is a feasible allocation.
2. given the price system p^* and k^*; household σ solves (3).
3. given the price system p^*; $k^*(\sigma)$ solves (1) $\forall \sigma \in \Sigma$:

Remark 4 Given that the shocks each period are finite, the tree describing the uncertainty has a countable number of nodes. Existence of a competitive equilibrium in this economy can be proved by first showing that after a suitable transformation of the single consumer problem with a given youth wage, the competitive prices in such a "static" two period problem exist. The argument can then be extended by induction (see for example Zilcha (1990)).

A convenient and standard assumption we will make is to assume that $\sup_{\theta \in \cup_{t \geq 0} A_t} \{k \mid f(k;\theta) = k\}$ is finite, so that our economy is bounded, and therefore, all allocations will be bounded above.

3 Dynamic Efficiency and Exchange Efficiency

In the literature on efficiency in OLG models, most models deal only with a characterization of efficiency of exchange economies, i.e. economies without production (Balasko and Shell (1980), Okuno and Zilcha (1980), Geanakoplos and Polemarchakis (1991)). There are, on the other hand, a number of results that deal with characterization of efficiency in infinite production problems, but without considering specific preferences for consumers and thus without explicitly dealing with Pareto optimality (Cass (1972), Benveniste and Gale (1975)). Both results have been extended to uncertainty (Chattopadhyay and Gottardi (1999) extend the former; Zilcha (1990) and Dechert and Yamamoto (1992) the latter). In the rest of the paper, we will call the latter form of efficiency dynamic efficiency and refer to the former as exchange efficiency. Note that under uncertainty, efficient exchange incorporates an efficient allocation of risk, since exchange efficiency is related to consumers' preferences, which in turn reflects their attitudes towards risk. The purpose of this paper is to examine how the two concepts are related to interim Pareto optimality.

To start, we give the definition of dynamic efficiency and note that a sequence of investment decisions is called dynamically inefficient if it is not dynamically efficient.

Definition 5 A sequence of investment decisions $\left(k(\sigma)_{\sigma \in \Sigma}\right)$ is dynamically efficient (given initial capital $\bar{k}$) if there exists no other sequence of investment decisions $\left(\hat{k}(\sigma)\right)_{\sigma\in\Sigma}$ such that

$$f\left(\hat{k}(\sigma);\theta\right) - \hat{k}(\sigma') \geq f(k(\sigma);\theta) - k(\sigma') \qquad \forall\sigma' \in \sigma^{+}\,\forall\sigma \in \Sigma$$

$$f\left(\hat{k};\theta_1\right) - \hat{k}(\sigma) \geq f\left(\hat{k};\theta_1\right) - k(\sigma) \qquad \forall\sigma \in \Sigma_1$$

with at least one strict inequality.

Remark 6 In the definition for uncertainty given e.g. in Dechert and Yamamato (1992) the inequalities are only required to hold with probability one with respect to the probability measure μ defined over the set of paths Ω. In the context of interim optimality, however, it seems natural to change the definition in the way mentioned above. Consider e.g. the case in which there is a shock each period according to a continuously distributed random variable. Then the probability of a certain shock occuring at a certain time is zero, i.e. that the probability of a certain "node" σ occuring is zero. Nevertheless there is a continuum of consumers, namely those born at successor "nodes" of σ who have this single shock (respectively node σ) in their history. According to the probability one definition of dynamic efficiency, the production in their life could be zero, so the resulting allocation would generally not be interim Pareto optimal. In this sense the definition in Dechert and Yamamoto (1992) takes the ex ante period zero standpoint.

Dynamic efficiency thus rules out overaccumulation of capital in the sense that a decrease in savings at one or more nodes would allow for a permanently higher consumption level. For dynamic efficiency necessary and sufficient conditions can be derived with the following assumptions on the production function (see Mitra (1979)):

There are positive constants $m_1; m_2; m_3; m_4$ such that for all $k > 0$ and $\theta \in \bigcup_{t\geq 0} A_t$ the following holds:

$$m_1 \leq \frac{kf'(k;\theta)}{f(k;\theta)} \leq m_2 \quad \text{and} \quad m_3 \leq \frac{-kf''(k;\theta)}{f'(k;\theta)} \leq m_4 \qquad (4)$$

These elasticity conditions will be used later in the proof of our characterization of interim Pareto optimality.

Before we state a characterization of dynamic efficiency some additional notation is required. Given $\hat{\sigma} \in \Gamma$; we define a subtree (of Γ) with root $\hat{\sigma}$, denoted by $\Gamma_{\hat{\sigma}}$; as a collection of nodes such that $\Gamma_{\hat{\sigma}} \subset \Gamma$ and $\Gamma_{\hat{\sigma}}$ is itself a tree with $\hat{\sigma}$ as its root. Given a path σ^{∞}; σ^{∞}_t denotes the $t-$th coordinate of the path. Given an arbitrary subtree $\Gamma_{\hat{\sigma}}$; we define a path in the subtree $\Gamma_{\hat{\sigma}}$ as a path with the property that for $t \geq t(\hat{\sigma})$ all the nodes are elements of the subtree and denote it by $\sigma^{\infty}(\Gamma_{\hat{\sigma}})$; i.e. $\sigma^{\infty}(\Gamma_{\hat{\sigma}}) \subset \left\{\sigma^{\infty}_1; \ldots; \sigma^{\infty}_{t(\hat{\sigma})-1}\right\} \cup \Gamma_{\hat{\sigma}}$. For any $\hat{\sigma} \in \Gamma$; we denote by $\Gamma(\hat{\sigma};\Gamma)$ the subtree that has $\hat{\sigma}$ as its root and includes all successor nodes of $\hat{\sigma}$.

Under this assumption and the standard assumptions made above, Zilcha (1990) derives the following characterization of dynamic efficiency under uncertainty. The fact that we have slightly changed the definition of dynamic efficiency does obviously not alter the result.

Theorem 7 Under the above assumptions on the production function an interior feasible allocation $(x^*;k^*)$ for which k^* is bounded below, i.e. there exists $\varepsilon > 0$ such that $k^*(\sigma) \geq \varepsilon$ for all $\sigma \in \Gamma$; is dynamically inefficient if and only if there exists a node $\sigma_\tau \in \Gamma$ and some $C > 0$ such that

$$\sum_{t=\tau}^{\infty} \frac{1}{\pi(\sigma_t)} \leq C$$

along every path $\sigma^\infty \in \left\{\sigma_1^\infty;\ldots;\sigma_{\tau-1}^\infty\right\} \cup \Gamma(\sigma_\tau;\Gamma)$; where $\pi_t(\sigma_t)$ are the non-contingent prices defined above.[3]

Let us now illustrate the concept of (intergenerational) exchange efficiency. We can reduce our economy with production to a pure exchange economy by fixing the production and investment/saving decisions of a given competitive equilibrium by taking the now fixed (competitive equilibrium) output as aggregate endowment at each node. If it is not possible to achieve an interim Pareto-improvement by pure transfers of commodities, the allocation is called exchange efficient. More formally, we give the following definition of exchange efficiency in a setting with capital.

Definition 8 An allocation $(x^*;k^*)$ is exchange efficient (given initial capital $\hat{k}$) if there is no other feasible allocation $(\hat{x};k^*)$ which improves upon $(x^*;k^*)$ in the sense of Definition 2.

Note that the fixed level of initial capital stock is not so important here since capital remains fixed anyway. Necessary and sufficient conditions for exchange efficiency are usually given by imposing restrictions on the curvature of indifference surfaces or the production technology. A sufficient condition for exchange efficiency can be derived if the following assumption about the curvature of the utility functions is fulfilled.

Definition 9 An interior competitive equilibrium $(x^*;k^*;p^*)$ satisfies the non-vanishing Gaussian curvature condition if there exists a $\underline{\rho} > 0$ such that for all feasible allocations $(\hat{x};k^*)$

$$u_\sigma(\hat{x}(\sigma)) \geq u_\sigma(x(\sigma)) \Rightarrow \sum_{\sigma' \in \sigma^+} \delta_2(\sigma';\sigma) \geq -\delta_1(\sigma) + \underline{\rho}\frac{(\delta_1(\sigma))^2}{p(\sigma)} \quad \forall \sigma \in \Sigma$$

[3] Note that due to the the strict monotonicity of preferences and the properties of the technology equilibrium prices will be strictly positive.

where $\delta_1(\sigma) = p(\sigma)\cdot[\hat{x}(\sigma;\sigma) - x^*(\sigma;\sigma)]$ and $\delta_2(\sigma';\sigma) = p(\sigma')\cdot[\hat{x}(\sigma';\sigma) - x^*(\sigma';\sigma)]$ for $\sigma' \in \sigma^+$. $\underline{\rho}$ is called the lower curvature coefficient.

Note that in the case without uncertainty, the non-vanishing Gaussian curvature condition is imposing a lower bound on the curvature of the indifference curve by approximating it from below by a quadratic polynominal.

Remark 10 The assumption of non-vanishing Gaussian curvature is not very restrictive. This can be seen in many applications. It will be fulfilled if the preferences are identical across nodes, satisfy the other assumptions made in this paper and the competitive equilibrium allocation $(x^*;k^*)$ is uniformly bounded away from 0 and alternative allocations $(\hat{x};k^*)$ are restricted to a neighborhood of $(x^*;k^*)$: i.e. $\|x^*(\sigma) - \hat{x}(\sigma)\| < \epsilon$ for some ϵ sufficiently small, where $\|.\|$ denotes the euclidian norm. Then our assumptions about preferences imply non-vanishing Gaussian curvature. Note that restricting the allocation $(\hat{x};k^*)$ to neighborhoods of $(x^*;k^*)$ is without loss of generality when considering whether a Pareto-improvement exists, given the strict quasiconcavity of preferences and the convexity of technology. The same applies for the bounded Gaussian curvature defined next.

Definition 11 An interior competitive equilibrium $(x^*;k^*;p^*)$ satisfies the bounded Gaussian curvature condition if there exists a $\bar{\rho} > 0$ such that for all feasible allocations $(\hat{x};k^*)$

$$p(\sigma)\cdot[\hat{x}(\sigma) - x^*(\sigma)] < 0 \quad \text{and} \quad \sum_{\sigma' \in \sigma^+} \delta_2(\sigma';\sigma) \geq -\delta_1(\sigma) + \bar{\rho}\frac{(\delta_1(\sigma))^2}{p(\sigma)}$$

$$\text{imply} \quad u_\sigma(\hat{x}(\sigma)) \geq u_\sigma(x(\sigma))$$

where $\delta_1(\sigma) = p(\sigma)\cdot[\hat{x}(\sigma;\sigma) - x^*(\sigma;\sigma)]$ and $\delta_2(\sigma';\sigma) = p(\sigma')\cdot[\hat{x}(\sigma';\sigma) - x^*(\sigma';\sigma)]$ for $\sigma' \in \sigma^+$. $\bar{\rho}$ is called the upper curvature coefficient.

In order to characterize exchange efficiency we need some more definitions. Given a subtree $\Gamma_{\hat{\sigma}}$, a weight function is a function $\lambda_{\Gamma_{\hat{\sigma}}} : \Gamma_{\hat{\sigma}} \to [0;1]$ such that $\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\hat{\sigma}}} \lambda_{\Gamma_{\hat{\sigma}}}(\sigma') = 1$ for all $\sigma \in \Gamma_{\hat{\sigma}}$. Given a pair $(\Gamma_{\hat{\sigma}};\lambda_{\Gamma_{\hat{\sigma}}})$ the induced weight function, denoted $\hat{\lambda}_{\Gamma_{\hat{\sigma}}} : \Gamma_{\hat{\sigma}} \times \lambda_{\Gamma_{\hat{\sigma}}} \to [0;1]$, is defined as $\hat{\lambda}_{\Gamma_{\hat{\sigma}}}(\hat{\sigma}) = 1$; $\hat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma) = \lambda_{\Gamma_{\hat{\sigma}}}(\sigma)\cdot\hat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_{-1})$ for $\sigma_{-1} \in \Gamma_{\hat{\sigma}}$. Now we can restate a simplified form of a result due to Chattopadhyay and Gottardi (1999).[4]

[4] The result in Chattopadhyay and Gottardi (1999) is proved under somewhat weaker assumptions. Under their assumptions, the necessary and sufficent conditions do not coincide (see Theorem 1 and 2 in Chattopadhyay and Gottardi (1999) for the more general statement). We chose our stronger assumptions in order to simplify the exposition.

Theorem 12 Let $(x^*; k^*; p^*)$ be an interior competitive equilibrium which is bounded below, satisfying the non-vanishing Gaussian curvature condition and the bounded Gaussian curvature condition. Then a necessary and sufficent condition for the allocation not to be (interim) exchange efficient is that there exists a subtree $\Gamma_{\hat{\sigma}}$, a weight function $\lambda_{\Gamma_{\hat{\sigma}}}$ and a finite number A such that for every path $\sigma^1(\Gamma_{\hat{\sigma}})$ in the subtree

$$\sum_{t=t(\hat{\sigma})} \frac{\hat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^1)}{p(\sigma_t^1)} \le A$$

4 Characterizing Interim Pareto Optimality

In this section, we present our first main result. It gives a characterization of interim Pareto optimality in a stochastic OLG model with production. Compared to pure exchange, a setting with production allows for the analysis of many real world problems as they often involve capital accumulation. In particular, it allows us to analyze the interplay between capital accumulation and a redistributive transfer scheme like PAYG social security and its risk sharing opportunities. Other applications include the role of government debt or the role of monetary policy as insurance. A priori, it is not clear how the rich additional redistributional possibilities (compared to a pure exchange setting) of an economy with production influence the characterization of interim Pareto optimality.

Theorem 13 Let $(x^*; k^*; p^*)$ be an interior competitive equilibrium which is bounded below. Assume that both the non-vanishing and bounded Gaussian curvature assumptions hold as well as the elasticity assumption on the production function holds. Then a necessary and sufficent condition for the competive equilibrium allocation not be interim Pareto-optimal is that here exists a subtree $\Gamma_{\hat{\sigma}}$, a weight function $\lambda_{\Gamma_{\hat{\sigma}}}$ and a finite positive number A such that for every path $\sigma^1(\Gamma_{\hat{\sigma}})$ in the subtree

$$\sum_{t=t(\hat{\sigma})} \frac{\hat{\lambda}_{\Gamma_{\hat{\sigma}}}(\sigma_t^1)}{p(\sigma_t^1)} \le A. \quad (5)$$

Therefore exchange efficiency and interim Pareto optimality are equivalent.

Proof. See the Appendix. ■

The main idea of the proof is to show that a competitive equilibrium is exchange efficient if and only if there is no possibility of Pareto improvement by joint feasible deviations (lower consumption and investment of the young) of the initial competitive consumption-investmemt plan.

Remark 14 Our result implies as a special case the equivalence of Pareto optimality, exchange e¢ciency and dynamic e¢ciency in an econmy without uncertainty under the curvature assumptions on preferences and technology made above (Bose (1975), Bose and Ray (1993)). To see this, note that under certainty the condition in theorem 13 reduces to $\sum_{t=1} \frac{1}{p_t} = 1$; because the degenerate date-event tree is the only subtree of itself. But this condition is equivalent to both exchange and dynamic e¢ciency (see Balasko and Shell (1980) and Cass (1972)).

Remark 15 It should be noted that the e¢ciency characterization also carries over to monetary competive equilibria as can easily be seen from the proof. This was also used to characterize the e¢ciency of monetary equilibria by Balasko and Shell (1981) and Bose and Ray (1993) under certainty.

Note that the characterization in the theorem was derived for an economy without a social security system. The previous remark, however, shows that the theorem also applies to a model where money is a pure store of value. An alternative interpretation of money can be given in terms of a redistributive scheme like PAYG social security (or government debt). The reason for this is that monetary and …scal policy are equivalent in this class of models. This was already noted by Balasko and Shell (1981). This equivalence carries over to models with uncertainty under the interim Pareto optimality criterion.

Thus our result indicates the scope for a Pareto-improving role for PAYG social security under uncertainty. If the condition for exchange e¢ciency is violated, a well designed social security system may improve the allocation of risk relative to the pure market outcome (or to some initially given social security system). In the special case of certainty, the corresponding characterization of interim Pareto optimality also indicates a Pareto-improving role for social security. There, however, the role for social security is restricted to simple Ponzi schemes which roll over debt in…nitely into the future. The condition for Pareto optimality then exactly rules out this kind of schemes. Under uncertainty, social security may additionally serve as a means of intergenerational risk sharing. The contributions and bene…ts of the social security system may have to be conditioned on the di¤erent realizations of uncertainty to achieve this goal. That there are risk sharing opportunities left in the competitive equilibrium does not derive from some sort of market incompleteness, but from the dynamic structure of the economy. The crucial point is that these risk sharing opportunities do not derive from the fact that markets are incomplete, but from the dynamic structure of the economy. In particular, we are not considering risk sharing against the state in which an individual is born. This kind of risk sharing is ruled out by the choice of our criterion of interim Pareto optimality combined with our assumption that markets are complete once an individual is born (sequentially complete markets). Thus risk sharing in our model works through a mechanism that closely resembles a Ponzi scheme, but is more sophisticated. Instead of rolling over debt, we can interpret our scheme as one that collects contributions and then rolls over an insurance contract in exchange for the contributions. Therefore we use the term dynamic risk sharing.

Three things are remarkable about this insurance scheme in the Diamond model under uncertainty. First, the condition under which such an insurance contract is feasible is the same as in a model without production (as in Chattopadhyay and Gottardi (1999)). In particular this means that extending a pure exchange model to production does not improve the possibilities of risk sharing. This is surprising because, in comparison to a pure exchange model, the redistributional possibilities considerably improve by introducing joint (consumption and investment) deviations from a competitive equilibrium as a source of Pareto-improvements.

Second, this kind of social security scheme provides an insurance that cannot be replicated in a capital market. In fact, social security may provide insurance against macroeconomic risk which is often considered to be uninsurable. To be more precise, it provides insurance against the aggregate productivity risk in the second period of life (old age).

Third, contrary to the pure exchange case, we can relate this theoretical possibility of insurance against aggregate risk to an empirically testable efficiency criterion, namely dynamic efficiency. This is important because real world economies are usually considered to be dynamically efficient (Abel, Summers, Mankiw and Zeckhauser (1989)). Thus simple Ponzi schemes are not a feasible source of Pareto-improvements. The question is then whether the described dynamic risk sharing possibilities, which we interpreted above as sophisticated Ponzi schemes, are relevant in real world economies or whether they are ruled out by the fact that simple Ponzi schemes are infeasible. We answer this question in the next section.

5 Dynamic Efficiency versus Interim Pareto Optimality

In this section we provide our second main result. We demonstrate that under uncertainty dynamic efficiency is not sufficient for interim Pareto optimality in a competitive equilibrium, but it is a strictly weaker efficiency benchmark.

Theorem 16 In a stochastic OLG model with production, dynamic efficiency does not rule out (interim) Pareto-improvements in a competitive equilibrium. Thus, even under dynamic efficiency, there may be a Pareto-improving role for a government by introducing a well designed social security system.

Proof. In the following example we construct a generic competitive equilibrium allocation that is dynamically efficient without being exchange efficient respectively interim Pareto optimal. Consider an economy with two possible shocks each period of time. Let $(\alpha_i)_{i=0}^{\infty}$ be a sequence of real numbers $0 < \alpha_i < 1$ with the property $\prod_{i=0}^{\infty} \alpha_i > 0$; which is equivalent to $\sum_{i=0}^{\infty}(1 - \alpha_i) < \infty$:

Suppose there is a sequence of shocks $(\theta_t)_{t=0}^{\infty}$ with $\theta_t \in A_t$ for all t such that $\alpha_i = q_i(\theta_i|\theta_{i-1},\ldots,\theta_0)$. In other words, there exists a path in the tree which has strictly positive probability. Suppose further that along the path $\sigma^{\infty} = (\theta_0,\theta_1,\ldots)$ we have $\psi(\sigma_i) = 3^i$ and hence $\sum_{i=0}^{\infty} \frac{1}{\psi(\sigma_i)} < \infty$. For every node σ in the tree there exists a path $\tilde{\sigma}^{\infty}$ with $\sigma \in \tilde{\sigma}^{\infty}$ such that $\sum_{i=0}^{\infty} \frac{1}{\psi(\tilde{\sigma}_i)} = \infty$.

Clearly the economy described above is not dynamically inefficient although the series $\sum_{i=0}^{\infty} \frac{1}{\psi(\sigma_i)}$ converges (and is therefore in this case uniformly bounded) on a set of strictly positive measure, since there is no node in the tree at which a dissaving is possible without a later decrease in consumption.

However, there is a interim Pareto-improving pure redistribution possible. We clearly have $\prod_{i=0} \alpha_i \cdot \psi(\sigma_t) \geq \left(\prod_{i=0} \alpha_i\right) \cdot (1-\varepsilon)^{t-T} \cdot \psi(\sigma_t)$ for every $\varepsilon > 0$ and some T when $t > T$. Since for an ε sufficiently small we have $\sum_{t=T+1}^{\infty} \frac{\frac{1}{(1-\varepsilon)^{t-T}}}{\psi(\sigma_t)} = \sum_{t=T+1}^{\infty} \frac{\frac{1}{(1-\varepsilon)^{t-T}}}{3^t} < \infty$ the claim follows immediately by choosing the degenerate subtree consisting of the path σ^{∞} and by applying Theorem 16 .

To make the example more concrete, suppose that preferences of the consumers along the convergent path σ^{∞} consumers have preferences of the form $u_{\sigma_t}(x(\sigma)) = \sqrt{x(\sigma;\sigma)} + b(\sigma_t)\cdot\alpha_t\sqrt{x(\sigma_t;\sigma'_{1t})} + b(\sigma_t)\cdot(1-\alpha_t)\sqrt{x(\sigma_t;\sigma'_{2t})}$, where $b(\sigma_t)$ is a positive real number. Let the technology be given by

$$f(k;\theta) = a(\theta)\cdot k^{\beta}$$

where β will be chosen to satisfy a certain condition and $0 < \beta < 1$.

This technology clearly satisfies the elasticity conditions (4). Suppose the individual born in σ_t faces given interest rates of 3 and $\frac{1}{3}$ in the two possible events in his second period of life. The individual's problem is then

$$\max \quad \sqrt{x(\sigma;\sigma)} + b(\sigma_t)\cdot\alpha_t\sqrt{x(\sigma_t;\sigma'_{1t})} + b(\sigma_t)\cdot(1-\alpha_t)\sqrt{x(\sigma_t;\sigma'_{2t})} \tag{6}$$

$$\begin{aligned} \text{s.t.} \quad x(\sigma;\sigma) + s(\sigma_t) &= w(\sigma_t) \\ x(\sigma_t;\sigma'_{1t}) &= \frac{1}{3}\cdot s(\sigma_t) \\ x(\sigma_t;\sigma'_{2t}) &= 3\cdot s(\sigma_t) \end{aligned}$$

A computation shows that $s(\sigma_t) = \frac{c(\sigma_t)}{1+c(\sigma_t)}\cdot w(\sigma_t)$, where $c(\sigma_t) = (b(\sigma_t))^2 \cdot \left(\alpha_t\sqrt{\frac{1}{3}} + (1-\alpha_t)\sqrt{3}\right)^2$. We want $c(\sigma_t)$ to be independent of the node σ_t and equal an arbitrarily chosen positive real number c. Given the savings decision $s(\sigma_t)$, in order for 3 and $\frac{1}{3}$ to be equilibrium interest rates we must have

$$f'(s(\sigma_t);\theta_1) = a(\theta_1)\cdot\beta\cdot(s(\sigma_t))^{\beta-1} = \frac{1}{3} \tag{7}$$

$$f'(s(\sigma_t);\theta_2) = a(\theta_2)\cdot\beta\cdot(s(\sigma_t))^{\beta-1} = 3$$

We know $w(\sigma_t) = (1-\beta)\cdot a(\theta)\cdot(k(\sigma_{t-1}))^{\beta}$ where $\sigma_t = (\theta;\sigma_{t-1})$. Thus

$$s(\sigma_t) = \frac{c(\sigma_t)}{1+c(\sigma_t)}\cdot(1-\beta)\cdot a(\theta)\cdot(k(\sigma_{t-1}))^{\beta}$$

The function $f(x) = \frac{c}{1+c}\cdot(1-\beta)\cdot a(\theta)\cdot x^{\beta}$ has for fixed $a(\theta)$ a nonzero fixed point, which we call k^*. We can now solve, for an arbitrary given k^*, for the corresponding a, which is given by $a = \frac{1+c}{c}\cdot\frac{1}{1-\beta}\cdot(k^*)^{1-\beta}$. Plugging k^* for $s(\sigma_t)$ and $\frac{1+c}{c}\cdot\frac{1}{1-\beta}\cdot(k^*)^{1-\beta}$ for $a(\theta_1)$ into (7), k^* cancels out and the resulting condition for β is

$$\frac{1+c}{c}\cdot\frac{\beta}{1-\beta} = \frac{1}{3}$$

or equivalently

$$\beta = \frac{\frac{1}{3}\cdot\frac{c}{1+c}}{1+\frac{1}{3}\cdot\frac{c}{1+c}}.$$

If we choose $a(\theta_2) = 9\cdot a(\theta_1)$, the second equation above is also satisfied.

Up to now we have constructed a k^*, so that if along the path σ^1 the same shock always occurs, a capital stock of k^* is maintained. It is clear from our construction that once we deviate from this path, agents have a higher wage income. Hence if the c (which determines the savings behavior together with the wage) remains fixed, households save more. This implies that if the same shocks occur each period, higher capital stock means higher wages and therefore higher savings etc. Thus, as long c remains fixed, the capital stock off the path will never fall below k^*. Furthermore, by $a(\theta_2)\cdot k^{\beta} = k \iff k = a(\theta_2)^{\frac{1}{1-\beta}}$, an upper bound on the maximal possible capital stock is given. Since $a(\theta_1)$ and therefore $a(\theta_2)$ depend on the choice of k^* as described above, k^* can be chosen sufficiently large to ensure $a(\theta_2)\cdot\beta\cdot a(\theta_2)^{\frac{\beta}{1-\beta}}$, the lowest possible interest rate when the "high" shock $a(\theta_2)$ occurs, to be strictly larger than 1. Given this lower bound on interest rates, it is possible to determine bounds on $b(\sigma_t)$ if we fix the probabilities of shocks off the path equal to $\left(\frac{1}{2};\frac{1}{2}\right)$. On the path we have $\alpha_t \to 1$, and therefore $b(\sigma)$ will converge along the path to a constant.

Overall, the examples display all the features described in the first few paragraphs of this section. The Gaussian curvature assumptions are satisfied since the values of the $b(\sigma_t)$ are in a bounded intervall. This is so because in the expression for $c(\sigma_t)$, under the assumptions made, the values of the second bracket is in a intervall, so that to keep $c(\sigma_t)$ constant the $b(\sigma_t)$ can also be chosen from an intervall. Since interest rates are bounded above and below and consumption along σ^1 is constant, the overall equilibrium consumption levels are a subset of a compact set. All of this implies the curvature assumptions. Further, the consumption along the path σ^1 is uniformly bounded away from zero[5]. ■

The basic idea, why under uncertainty dynamic efficiency is weaker than exchange efficiency and also Pareto optimality becomes clear if one realizes that under uncertainty dynamic efficiency is a very coarse efficiency benchmark. The reason for this can be seen by comparing dynamic efficiency and exchange efficiency under certainty to the case of uncertainty. Under certainty both dynamic efficiency and exchange efficiency reduce to the same criterion and therefore rule out Ponzi schemes. Therefore the efficiency characterizations coincide. Under uncertainty, dynamic efficiency essentially rules out a reduction in savings at one node and a (weak) increase in aggregate consumption in all successor nodes (not just along one path succeeding the node where savings were reduced). The crucial point is that an attempt to lower savings at one node will affect capital accumulation in all succesor nodes, even those which lie on efficient paths when these paths are viewed in isolation. This means that the instrument of savings adjustments cannot be finely tuned in the sense that only one (inefficient) path along a subtree can be adjusted. But this is exactly what exchange efficiency requires. Exchange efficiency rules out that there is even a single path in the tree along which a Ponzi scheme can be played. This highlights the difference between efficient capital accumulation and risk sharing. Efficient risk sharing rules out even sophisticated Ponzi schemes, i.e. schemes which roll over debt along isolated paths in a subtree without affecting other paths in the subtree.

Summing up, we showed that sequentially complete markets do not rule out risk sharing even in the presumably empirically relevant case of dynamic efficiency. This result can be interpreted as justification for a well designed PAYG social security system as insurance against macroeconomic risks during old age.

6 Sufficient condition for Interim Pareto Optimality

As for the case under certainty (see Balasko and Shell (1980), Okuno and Zilcha (1980)), it is possible to derive a stronger sufficient condition for Pareto optimality. This stronger criterion has the advantage that it does not require curvature assumptions on preferences and technology. Under certainty the criterion requires $\liminf_{t\to\infty} \|p_t\| = 0$; if the resources of the economy are bounded. What the criterion essentially achieves is making the economy "quasi-finite", by putting a low weight, in form of a low value, on the future (or at least parts of the future). This fact allows one to use again the simple revealed preference proof

[5] For the sake of a simpler presentation we assumed in the theorems that allocations are uniformly bounded away from zero. In fact, we only need the consumption to be uniformly bounded away from zero along the subtree at which the condition on the sum of weighted prices holds (see Chattopadhyay/Gottardi (1999), Theorem 2).

of the first welfare theorem from standard Arrow-Debreu-economies. So with this assumption, the reason for the failure of the first welfare theorem in OLG models, namely that prices do not necessarily form a linear functional on the commodity space, which means that the value of a certain commodity bundle is not necessarily finite,[6] does not have severe consequences in this case. In particular, economies with land (see Rhee (1991), Homburg (1991)), or more generally, economies with non-negligible consumers (see Geanakoplos and Polemarchakis (1991)) display equilibrium prices satisfying this property. We state now the corresponding assumption under uncertainty and prove the first welfare theorem. The assumption is similar to one given by Demange (1998).[7]

$$\liminf_{t\to\infty}\sum_{\sigma\in\Sigma_t} p(\sigma)=0 \tag{8}$$

Under the condition (8) and under standard assumptions on preferences and technology, i.e. under the assumptions made in the paper with exception of the curvature assumptions, we may state:

Proposition 17 *Suppose preferences and production technology satisfy standard assumptions. If (8) holds at a competitive equilibrium, the equilibrium allocation is interim Pareto-efficient.*

Proof. Suppose not. Then there exists an alternative feasible allocation $(\hat{x};\hat{k})$ which interim Pareto improves in comparison with the competitive allocation $(x^*;k^*)$. Since the consumers maximized their utility given the budget constraint in the competitive equilibrium, we must have

$$p(\sigma)\cdot\hat{x}(\sigma;\sigma)+\sum_{\sigma'\in\sigma^+}p(\sigma')\cdot\hat{x}(\sigma';\sigma)\geq p(\sigma)\cdot x^*(\sigma;\sigma)+\sum_{\sigma'\in\sigma^+}p(\sigma')\cdot x^*(\sigma';\sigma) \tag{9}$$

with strict inequality for at least one $\sigma\in\Sigma$. Let $\tilde{\sigma}\in\Sigma$ denote such a node. Further, since firms maximize expected profits in the competitive equilibrium

$$\sum_{\sigma'\in\sigma^+}p(\sigma')\cdot f(k^*(\sigma);\theta)-p(\sigma)\cdot k^*(\sigma)\geq\sum_{\sigma'\in\sigma^+}p(\sigma')\cdot f\left(\hat{k}(\sigma);\theta\right)-p(\sigma)\cdot\hat{k}(\sigma) \tag{10}$$

Let $\varepsilon=p(\tilde{\sigma})\cdot\hat{x}(\tilde{\sigma};\tilde{\sigma})+\sum_{\sigma'\in\tilde{\sigma}^+}p(\sigma')\cdot\hat{x}(\sigma';\tilde{\sigma})-p(\tilde{\sigma})\cdot x^*(\tilde{\sigma};\tilde{\sigma})+\sum_{\sigma'\in\tilde{\sigma}^+}p(\sigma')\cdot x^*(\sigma';\tilde{\sigma})>0.$

[6] As consequence of this OLG economies can exhibit a "lack of market clearing at infinity" (see for example Geanakoplos and Polemarchakis (1991))

[7] A related result in a model with exogenous stochastic development of wages and interest rates can be found in Richter (1993).

Let $\Gamma_{\tilde{\sigma}}$ denote the subtree that contains all successor nodes of $\tilde{\sigma}$. Now fix some $T > t(\tilde{\sigma})$. Summing form 0 to T over all nodes in the tree Γ gives

$$\sum_{\sigma\in\Gamma\cap(\cup_{t\le T}\Sigma_t)} p(\sigma)\cdot\hat{x}(\sigma;\sigma)+\sum_{\sigma'\in\sigma^+} p(\sigma')\cdot\hat{x}(\sigma';\sigma) \tag{11}$$

$$-\sum_{\sigma\in\Gamma\cap(\cup_{t\le T-1}\Sigma_t)}\sum_{\sigma'\in\sigma^+} p(\sigma')\cdot f\left(\hat{k}(\sigma);\theta\right)+p(\sigma)\cdot\hat{k}(\sigma) >$$

$$\sum_{\sigma\in\Gamma\cap(\cup_{t\le T}\Sigma_t)} p(\sigma)\cdot x^*(\sigma;\sigma)+\sum_{\sigma'\in\sigma^+} p(\sigma')\cdot x^*(\sigma';\sigma)-$$

$$\sum_{\sigma\in\Gamma\cap(\cup_{t\le T-1}\Sigma_t)}\sum_{\sigma'\in\sigma^+} p(\sigma')\cdot f(k^*(\sigma);\theta)+p(\sigma)\cdot k^*(\sigma)$$

where the difference between the left-hand and right hand expression is is at least $\varepsilon > 0$. Forming the same sum with the exception of not including the successor nodes of Σ_{T}, by feasibility of both allocations, taking differences we get

$$\sum_{\sigma\in\Sigma_{T+1}} p(\sigma)\cdot\hat{x}(\sigma;\sigma_{-1}) > \sum_{\sigma\in\Sigma_{T+1}} p(\sigma)\cdot x^*(\sigma;\sigma_{-1}) \tag{12}$$

with the difference again being at least ε. But by assumption (8) and the boundness of the economy this is a contradiction. ■

The efficiency of an economy with land is now a corollary of this result. We define land as an object which pays a fixed amount of the single commodity at each time in each state, which we denote by D. Note that this more restrictive than necessary and done for notational simplicity. Under certainty (see Rhee (1991), Homburg (1991)), it is only necessary that the income share of land does not vanish asymptotically. Given the boundness of our economy it would therefore suffice to assume that the return from land is across all nodes bounded away from zero.In equilibrium, the value of land must be finite, i.e. $R\cdot\sum_{\sigma\in\Sigma} p(\sigma) < \infty$. This clearly requires (8). Thus we can state:

Corollary 18 A competitive equilibrium in an economy with land is always interim Pareto optimal.

Remark 19 It can also be shown that the sufficient condition for interim Pareto optimality stated in terms of a condition on the ratio of (net) dividends and the value of the market portfolio in Abel, Mankiw, Summers and Zeckhauser (1989) implies (8).

7 A Second Welfare Theorem

In exchange OLG models, it is possible to prove a version of the second welfare theorem, namely that Pareto optimal allocations can be supported as a competitive equilibrium. In our setup this is generally not possible, since the definition of competitive equilibrium with production requires that the young generation earns income only by selling their endowment (one unit of labor) and that the old generation owns the capital. To support any interim Pareto optimal allocation, we must therefore allow for transfers.

Definition 20 A competitive equilibrium with transfers (given ω) is a tuple $(x^*; k^*; p^*; T^*)$ such that

1. $(x^*; k^*)$ is feasible given ω
2. given the price system p^*; (1) is solved by $k^*(\sigma) \quad \forall \sigma \in \Sigma$:
3. households solve

$$\max_{\left(x(\sigma;\sigma);(x(\sigma';\sigma))_{\sigma'\in\sigma^+}\right)\in\mathbb{R}_+^{1+S(\sigma)}} u_\sigma\left(x(\sigma;\sigma);(x(\sigma';\sigma))_{\sigma'\in\sigma^+}\right) \tag{13}$$

$$\text{s.t.} \quad p(\sigma)\cdot x(\sigma;\sigma) + \sum_{\sigma'\in\sigma^+} p(\sigma')\cdot x(\sigma';\sigma) \le w(\sigma) + \sum_{\sigma'\in\sigma^+} R(\sigma') + T^*(\sigma;\sigma) + \sum_{\sigma'\in\sigma^+} T^*(\sigma;\sigma')$$

4. $T^*(\sigma;\sigma) + T^*(\sigma_{-1};\sigma) = 0 \quad \forall \sigma \in \Sigma$

To prove the second welfare theorem, we want to characterize optimal allocations by first-order conditions and for this we need some boundary conditions on the utility functions. A precise statement is given in the following remark.

Remark 21 We say that utility functions satisfy boundary conditions if for all $\sigma \in \Sigma$ the corresponding utility functions $u_\sigma(\cdot)$ satisfies $\lim_{x(\sigma;\sigma)\to 0} \left(\frac{\partial u_\sigma\left(x(\sigma;\sigma);(x(\sigma';\sigma))_{\sigma'\in\sigma^+}\right)}{\partial x(\sigma;\sigma)} \right) \to \infty$ for all $(x(\sigma';\sigma))_{\sigma'\in\sigma^+} \in \mathbb{R}_{++}^{S(\sigma)}$ and $\lim_{x(\sigma^*;\sigma)\to 0} \frac{\partial u_\sigma\left(x(\sigma;\sigma);(x(\sigma';\sigma))_{\sigma'\in\sigma^+}\right)}{\partial x(\sigma^*;\sigma)} \to \infty$ for $\sigma^* \in \sigma^+$ for all $x(\sigma;\sigma)$; $(x(\sigma';\sigma))_{\sigma'\in\sigma^+\setminus\{\sigma^*\}} \in \mathbb{R}_{++}^{S(\sigma)}$:

With this slightly modified concept of equilibrium a second welfare theorem can be proved.

Proposition 22 Under the boundary conditions on utility functions, any interior interim Pareto optimal allocation can be supported as a competitive equilibrium with transfers.

Proof. Let $(x^*; k^*)$ be the given interior interim Pareto optimal allocation. If we define an Arrow-Debreu price system by using (2) and construct $\psi(\sigma)$ recursively using k^* from the given allocation, optimality conditions for the firm imply that given these prices k^* is profit maximizing. Give the consumer σ the final income $p(\sigma) \cdot x^*(\sigma; \sigma) + \sum_{\sigma' \in \sigma^+} p(\sigma') \cdot x^*(\sigma'; \sigma)$ by constructing an appropriate transfer scheme. First-order conditions for interim Pareto optimality imply that consumer σ chooses indeed $x^*(\sigma; \sigma); (x^*(\sigma'; \sigma))_{\sigma' \in \sigma^+}$: Since the allocation $(x^*; k^*)$ is by definition also feasible, the desired result follows. ■

8 Conclusions

We have given a complete characterization of interim Pareto optimality in a stochastic OLG model with production and social security. We have shown that the risk sharing possibilities in a model with production do not improve compared to a pure exchange model, although the redistributional possibilities improve. Our first main result gives a characterization of interim Pareto optimality. In a competitive equilibrium, interim Pareto optimality is equivalent to intergenerational exchange efficiency, which in turn implies dynamic efficiency. Our characterization subsumes also equilibria with PAYG social security system. If the condition for exchange efficiency is violated, there is a Pareto-improving role for a PAYG social security system. Our second main result shows that dynamic efficiency does not rule out a Pareto-improving role for a social security system to act as insurance against aggregate productivity risk in the second period of life (old age) through dynamic risk sharing. Furthermore, this kind of risk sharing cannot be replicated on the capital market as it insures against aggregate (macroeconomic) risk. We have also provided a stronger sufficient criterion for interim Pareto optimality and showed that economies with land fulfill this criterion. Moreover, we have proved a second welfare theorem for our economy.

It is important to note that the fact that there may be risk sharing possibilities in a competitive equilibrium of our economy are not derived from market incompleteness but from the dynamic structure of the economy. We have interpreted this kind of risk sharing as sophisticated Ponzi scheme as it involves collecting contributions from the young but rolls over an insurance contract rather than debt from generation to generation. The importance of our results stems from the fact that dynamic efficiency is an empirically testable criterion which implies that Ponzi are schemes infeasible. Therefore, if it holds, a wide range of Pareto-improving policies are ruled out. Our results, however, show that this testable criterion is not sufficient to guarantee even the weak notion of interim Pareto optimality of a pure market allocation.

Our results can be extended in several directions. First, it would be interesting to derive conditions for dynamic efficiency and Pareto optimality of competitive allocations that can easily be verified empirically as, for example, the sufficient condition for dynamic efficiency and interim Pareto optimality in Abel, Summers, Mankiw and Zeckhauser (1989). Second, one can hope to find simpler efficiency characterizations for stationary economies. This is at least true for pure exchange models and is likely to carry over to a model with capital (Demange and Laroque (2000)).

Appendix: Proof of Theorem 13

Proof. Clearly, if the above conditions holds, then the allocation is not exchange efficient and therefore not interim Pareto optimal. For the converse recall that the characterizing condition for dynamic efficiency was $\sup_{\sigma^\infty \in \Gamma_\sigma} \sum_{t=\tau}^{\infty} \frac{1}{\psi(\sigma_t)}$; $\sigma_t = (\theta_0; \ldots; \theta_t)$ not being bounded in every subtree Γ_σ: Since $p(\sigma_t) = \psi(\sigma_t) \cdot \prod_{i=0}^{t} q_i(\theta_i | \sigma_{i-1}) \le \psi(\sigma_t)$ exchange efficiency implies dynamic efficiency. To see this consider every subtree and then choose the induced weight functions on it to be equal to the probability of reaching the node. The condition for dynamic efficiency follows from the previous remark and the fact that each node is reached with strictly positive probability. To prove the result of the theorem it therefore remains to be shown that joint deviations from the equilibrium allocation cannot be Pareto-improving. The proof is completed by the following three steps:

Step 1: We will show that it is never optimal to increase savings at any point of time at any node in the tree. Suppose there is an increase in savings at node $\sigma = (\theta_0; \ldots; \theta_t)$(at time t) and node σ is among the first nodes (in terms of time) at which there is an increase in saving. Consider the individual born at this node with utility function $u_\sigma(x(\sigma))$: Then there must be a decrease in the youth consumption of this individual which is defined by the amount of increase in saving. So let $\hat{x}(\sigma;\sigma)$ denote the youth consumption in node σ after the increase in saving and let $x^*(\sigma;\sigma)$, as defined above, denote the equilibrium consumption when young. Since the new allocation is supposed to be Pareto-improving, we must have by the non-vanishing Gaussian curvature condition

$$\sum_{\sigma' \in \sigma^+} \delta_2(\sigma'; \sigma) \ge -\delta_1(\sigma) + \rho \frac{(\delta_1(\sigma))^2}{p(\sigma)} \tag{14}$$

Rewriting this using the relation between $\psi_{t+1}; \psi_t$ and p we get

$$\sum_{\theta \in A_{t+1}} q_{t+1}(\theta | \sigma_t) \cdot \frac{\psi_t(\sigma)}{f'(k^*(\sigma); \theta)} \cdot [\hat{x}(\sigma'; \sigma) - x^*(\sigma'; \sigma)] \ge \tag{15}$$

$$-\psi_t(\sigma) \cdot (\hat{x}(\sigma;\sigma) - x^*(\sigma;\sigma)) + \rho \frac{(\psi_t(\sigma) \cdot (\hat{x}(\sigma;\sigma) - x^*(\sigma;\sigma)))^2}{\psi_t(\sigma)}$$

Noting that $\hat{k}(\sigma) - k^*(\sigma) = -(\hat{x}(\sigma;\sigma) - x^*(\sigma;\sigma)) > 0$; replacing ψ_t with ψ_{t+1}/f'; neglecting the last term on the right hand side and averaging with weights gives

$$\sum_{\theta\in A_{t+1}} q_{t+1}(\theta|\sigma_t)\cdot\psi_{t+1}(\sigma')\cdot[\hat{x}(\sigma';\sigma) - x^*(\sigma';\sigma)] > \tag{16}$$

$$\sum_{\theta\in A_{t+1}} q_{t+1}(\theta|\sigma_t)\cdot\psi_{t+1}(\sigma)\cdot f'(k^*(\sigma);\theta)\cdot\left[\hat{k}(\sigma) - k^*(\sigma)\right]$$

i.e. that due to the strict concavity of the production function, the value of the necessary increase in tomrrow's consumption is strictly larger than the increase in tomorrow's production induced by the increase in saving.

This result shows that a Pareto-improving new allocation can never begin with an increase in saving.

Step 2: Suppose now $\sigma = (\theta_0;\ldots;\theta_t)$(at time t) is among the first nodes in time at which there is a decrease in saving. Consider the individual born at this node with utility function $u_\sigma(x(\sigma))$: Let us, as in the preceding paragraph, denote the new allocation by $\hat{a}$ So suppose in this case $\hat{k}(\sigma) - k^*(\sigma) = -(\hat{x}(\sigma;\sigma) - x^*(\sigma;\sigma)) < 0$: The argument is now similar to the one before. Again, by non-vanishing Gaussian curvature

$$\sum_{\sigma'\in\sigma^+}\delta_2(\sigma';\sigma) \geq -\delta_1(\sigma) + \underline{\rho}\frac{(\delta_1(\sigma))^2}{p(\sigma)} \tag{17}$$

Note that $\sum_{\sigma'\in\sigma^+}\delta_2(\sigma';\sigma) \leq 0$ and $-\delta_1(\sigma) < 0$ in this case, so that

$$-\sum_{\sigma'\in\sigma^+}\delta_2(\sigma';\sigma) < \delta_1(\sigma) \tag{18}$$

Writing out the expressions and substituting as in the argument before

$$\sum_{\theta\in A_{t+1}} q_{t+1}(\theta|\theta_t)\cdot\psi_{t+1}(\sigma')\cdot[x^*(\sigma';\sigma) - \hat{x}(\sigma';\sigma)] < \tag{19}$$

$$\sum_{\theta\in A_{t+1}} q_{t+1}(\theta|\theta_t)\cdot\psi_{t+1}(\sigma)\cdot f'(k^*(\sigma);\theta)\cdot\left[k^*(\sigma) - \hat{k}(\sigma)\right]$$

i.e. the value of reduction in output at the sucessor nodes of σ is larger than the maximal possible reduction in value of consumption that leaves the individual indifferent.

These two facts together imply that if there is an interim Pareto-improving allocation it must be of the following form. At the first node where it differs from the initial allocation there is either a pure redistributive transfer (which necessarily involves a positive transfer for the old generation) or a decrease in saving which gives the amount less saved as consumption to the old generation

, or there is a combination of both. This fact along with the quasi-concavity of utility functions and strict convexity of the technology imply that there will never be an increase in saving at any node in the tree.

Step 3: Assume a Pareto improvement is possible. Consider the following identity, which follows from the resource constraint $x(\sigma';\sigma) + x(\sigma';\sigma') + k(\sigma') = f(k(\sigma);\theta) \quad \forall \sigma' = (\theta;\sigma) \in \sigma^+$

$$\Delta x(\sigma';\sigma) + \Delta x(\sigma';\sigma') + \Delta k(\sigma') = \Delta f(k(\sigma);\theta) \tag{20}$$

where $\Delta x(\sigma';\sigma) = (\hat{x}(\sigma';\sigma) - x^*(\sigma';\sigma))$ etc. if $(x^*;k^*)$ is the initial competitive equilibrium allocation and $\left(\hat{x};\hat{k}\right)$ is the new interim Pareto-improving allocation. Equivalently

$$-\Delta x(\sigma';\sigma') - \Delta k(\sigma') = \Delta x(\sigma';\sigma) - \Delta f(k(\sigma);\theta) \tag{21}$$

If we define $\Delta\varepsilon(\sigma') = -\Delta k(\sigma')$ as the dissaving at node σ' when changing to the new allocation, which is by the argument made above always nonnegative, and $\Delta a(\sigma';\sigma') = -\Delta x(\sigma';\sigma')$ as the decrease in youth consumption when changing to the new allocation, we have $\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma') > 0$[8]

$$\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma') = \Delta x(\sigma';\sigma) - \Delta f(k(\sigma);\theta) \tag{22}$$

Consider all nodes $\sigma' \in \sigma^+$ for which $\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma') > 0$. Since the improving allocation $\left(\hat{x};\hat{k}\right)$ must at some node $\hat{\sigma}$ be different from the initial one, by the arguments made above we must have $\Delta a(\hat{\sigma};\hat{\sigma}) + \Delta\varepsilon(\hat{\sigma}) > 0$. Consider now the successor nodes of $\hat{\sigma}$ for which $\Delta a\left(\sigma'';\sigma''\right) + \Delta\varepsilon\left(\sigma''\right) > 0;\ \sigma'' \in \hat{\sigma}^+$. It is easy to see that if we continue this way we inductively define a subtree, called $\Gamma_{\hat{\sigma}}$. Multiplying with contingent prices $p(\sigma')$ and summing over $\sigma' \in \sigma^+ \cap \Gamma_{\hat{\sigma}}$ gives

$$\sum_{\sigma' \in \sigma^+ \cap \Gamma_{\hat{\sigma}}} p(\sigma') \cdot [\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma')] = \sum_{\sigma' \in \sigma^+ \cap \Gamma_{\hat{\sigma}}} p(\sigma') \cdot [\Delta x(\sigma';\sigma) - \Delta f(k(\sigma);\theta)] \tag{23}$$

Next, we state a technical lemma which follows from the elasticity assumption (4) and gives a quadratic term lower bound for changes in the value of production due to changes in investment behavior similar to the non-vanishing Gaussian curvature assumption for the utility functions. We omit a proof of this lemma. It follows from standard arguments (see Zilcha (1990)).

Lemma 23 Given assumption (4) on the production function and the boundedness of the economy, the following holds for all $0 < \delta < k(\sigma) <$

[8] It is well possible that there is an increase in youth consumption at some later node, however the sum $\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma')$ cannot become zero or negative, since such increases in youth consumption just cancel out with the decrease in saving.

$\sup_{\theta \in \cup_{t \ge 0} A_t} \{k \mid f(k;\theta) = k\}$ for all $\sigma \in \Sigma$ and a price system p consistent with $k(\sigma)$ as defined in (2)

$$\sum_{\sigma' \in \sigma^+} p(\sigma') \cdot [f(k(\sigma);\theta_{t+1}) - f(k(\sigma) - \delta;\theta_{t+1})] \ge p(\sigma) \cdot \delta + c \cdot \frac{(p(\sigma) \cdot \delta)^2}{p(\sigma)}$$

Using now the non-vanishing Gaussian curvature condition for preferences and the lemma above for production functions, we get[9]

$$\sum_{\sigma' \in \sigma^+ \cap \Gamma_\sigma} p(\sigma') \cdot [\Delta x(\sigma';\sigma) - \Delta f(k(\sigma);\theta)] \ge \tag{24}$$

$$-\delta_1(\sigma) + \underline{\rho} \frac{(\delta_1(\sigma))^2}{p(\sigma)} + p(\sigma) \cdot \Delta\varepsilon(\sigma) + c \cdot \frac{(p(\sigma) \cdot \Delta\varepsilon(\sigma))^2}{p(\sigma)}$$

where $\delta_1(\sigma)$ is defined as above.

Replacing $\Delta x(\sigma';\sigma) - \Delta f(k(\sigma);\theta)$ by $\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma')$ gives

$$\sum_{\sigma' \in \sigma^+ \cap \Gamma_\sigma} p(\sigma') \cdot [\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma')] \ge -\delta_1(\sigma) + \underline{\rho} \frac{(\delta_1(\sigma))^2}{p(\sigma)} + p(\sigma) \cdot \Delta\varepsilon(\sigma) + c \cdot \frac{(p(\sigma) \cdot \Delta\varepsilon(\sigma))^2}{p(\sigma)} \tag{25}$$

Like Chattopadhyay and Gottardi (1999) we define a function $\lambda_{\Gamma_\sigma} : \Gamma_\sigma \to [0;1]$ by

$$\lambda_{\Gamma_\sigma}(\sigma') = \frac{p(\sigma') \cdot [\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma')]}{\sum_{\sigma' \in \sigma^+ \cap \Gamma_\sigma} p(\sigma') \cdot [\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma')]} \tag{26}$$

Note that given the way Γ_σ is constructed, λ_{Γ_σ} is well defined, strictly positive and satisfies $\sum_{\sigma' \in \sigma^+ \cap \Gamma_\sigma} \lambda_{\Gamma_\sigma}(\sigma') = 1$.

Now we consider an arbitrary path $\sigma^1(\Gamma_\sigma)$ in the subtree. Define $\gamma = \min\{c; \underline{\rho}\}$. Equation (25) can now be written as

$$\frac{1}{\lambda_{\Gamma_\sigma}(\sigma')} \cdot p(\sigma') \cdot [\Delta a(\sigma';\sigma') + \Delta\varepsilon(\sigma')] \ge \tag{27}$$

$$p(\sigma) \cdot \Delta a(\sigma;\sigma) + \gamma \frac{(p(\sigma) \cdot \Delta a(\sigma;\sigma))^2}{p(\sigma)} + p(\sigma) \cdot \Delta\varepsilon(\sigma) + \gamma \frac{(p(\sigma) \cdot \Delta\varepsilon(\sigma))^2}{p(\sigma)}$$

[9] Note that the inequality in (24) holds if we sum the left-hand side over σ^+ and therefore also holds by summing over the nonnegative terms, what is done in (24).

Inverting both sides of this equation we obtain

$$\frac{\lambda_{-k}(\sigma')}{p(\sigma')\cdot[\Delta a(\sigma';\sigma')+\Delta\varepsilon(\sigma')]} \;. \tag{28}$$

$$\frac{1}{p(\sigma)\cdot\Delta a(\sigma;\sigma)+\gamma\frac{(p(\sigma)\cdot\Delta a(\sigma;\sigma))^2}{p(\sigma)}+p(\sigma)\cdot\Delta\varepsilon(\sigma)+\gamma\frac{(p(\sigma)\cdot\Delta\varepsilon(\sigma))^2}{p(\sigma)}}$$

for all $\sigma \in \Sigma_{-k}$:
This is equivalent to

$$\frac{\lambda_{-k}(\sigma')}{p(\sigma')\cdot[\Delta a(\sigma';\sigma')+\Delta\varepsilon(\sigma')]} \;. \tag{29}$$

$$\frac{1}{p(\sigma)\cdot[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]}\cdot\left(1-\frac{\frac{\gamma}{p(\sigma)}\cdot\left[(p(\sigma)\cdot\Delta a(\sigma;\sigma))^2+(p(\sigma)\cdot\Delta\varepsilon(\sigma))^2\right]}{p(\sigma)\cdot\Delta a(\sigma;\sigma)+\gamma\frac{(p(\sigma)\cdot\Delta a(\sigma;\sigma))^2}{p(\sigma)}+p(\sigma)\cdot\Delta\varepsilon(\sigma)+\gamma\frac{(p(\sigma)\cdot\Delta\varepsilon(\sigma))^2}{p(\sigma)}}\right)$$

Further algebraic manipulations on the right-hand side give

$$\frac{\lambda_{-k}(\sigma')}{p(\sigma')\cdot[\Delta a(\sigma';\sigma')+\Delta\varepsilon(\sigma')]} \;. \tag{30}$$

$$\frac{1}{p(\sigma)\cdot[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]}-\frac{1}{\frac{(p(\sigma)\cdot[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)])^2}{\frac{\gamma}{p(\sigma)}\cdot\left[(p(\sigma)\cdot\Delta a(\sigma;\sigma))^2+(p(\sigma)\cdot\Delta\varepsilon(\sigma))^2\right]}+p(\sigma)\cdot[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]}$$

so that we finally obtain

$$\frac{\lambda_{-k}(\sigma')}{p(\sigma')\cdot[\Delta a(\sigma';\sigma')+\Delta\varepsilon(\sigma')]} \;. \tag{31}$$

$$\frac{1}{p(\sigma)\cdot[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]}-\frac{\gamma}{p(\sigma)}\frac{1}{\frac{[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]^2}{(\Delta a(\sigma;\sigma))^2+(\Delta\varepsilon(\sigma))^2}+\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)}$$

We want to show next that the expression $\frac{[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]^2}{(\Delta a(\sigma;\sigma))^2+(\Delta\varepsilon(\sigma))^2}+\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)$; which by the assumptions made on the subtree Σ_{-k} strictly positive, is also bounded above. By the resource constraint and the assumption that the economy is bounded, $\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)$ is clearly bounded above. Rewrite $\frac{[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]^2}{(\Delta a(\sigma;\sigma))^2+(\Delta\varepsilon(\sigma))^2}$ as

$$\frac{(\Delta a(\sigma;\sigma))^2}{(\Delta a(\sigma;\sigma))^2+(\Delta\varepsilon(\sigma))^2}+\frac{2\Delta a(\sigma;\sigma)\,\Delta\varepsilon(\sigma)}{(\Delta a(\sigma;\sigma))^2+(\Delta\varepsilon(\sigma))^2}+\frac{(\Delta\varepsilon(\sigma))^2}{(\Delta a(\sigma;\sigma))^2+(\Delta\varepsilon(\sigma))^2} \tag{32}$$

The first and the third term are bounded above by 1 and so is the middle term because

$$\frac{2\Delta a(\sigma;\sigma)\,\Delta\varepsilon(\sigma)}{(\Delta a(\sigma;\sigma))^2+(\Delta\varepsilon(\sigma))^2}=\frac{2}{\frac{\Delta a(\sigma;\sigma)}{\Delta\varepsilon(\sigma)}+\frac{\Delta\varepsilon(\sigma)}{\Delta a(\sigma;\sigma)}} \tag{33}$$

and the function $x+\frac{1}{x}$ is bounded below on the positive real line.

So there is a constant $K>0$ such that $\frac{[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]^2}{(\Delta a(\sigma;\sigma))^2+(\Delta\varepsilon(\sigma))^2}+\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)\le K$. Inserting this into (31) gives

$$\frac{\lambda_{\Gamma_{\tilde\sigma}}(\sigma')}{p(\sigma')\cdot[\Delta a(\sigma';\sigma')+\Delta\varepsilon(\sigma')]}\le\frac{1}{p(\sigma)\cdot[\Delta a(\sigma;\sigma)+\Delta\varepsilon(\sigma)]}-\frac{\gamma}{p(\sigma)}\frac{1}{K} \tag{34}$$

If we now iterate this inequality along $\sigma^\infty(\Gamma_{\tilde\sigma})$ starting with $\tilde\sigma$, we obtain

$$\frac{\lambda_{\Gamma_{\tilde\sigma}}(\sigma_T^\infty)\cdots\lambda_{\Gamma_{\tilde\sigma}}\left(\sigma_{t(\tilde\sigma)}^\infty\right)}{p(\sigma_T^\infty)\cdot[\Delta a(\sigma_T^\infty;\sigma_T^\infty)+\Delta\varepsilon(\sigma_T^\infty)]}+\frac{\gamma}{K}\sum_{t=t(\tilde\sigma)}^{T-1}\frac{\lambda_{\Gamma_{\tilde\sigma}}(\sigma_t^\infty)\cdots\lambda_{\Gamma_{\tilde\sigma}}\left(\sigma_{t(\tilde\sigma)}^\infty\right)}{p(\sigma_t^\infty)}$$
$$\le\frac{1}{p(\tilde\sigma)\cdot[\Delta a(\tilde\sigma;\tilde\sigma)+\Delta\varepsilon(\tilde\sigma)]} \tag{35}$$

so that $\lim_{T\to\infty}\sum_{t=t(\tilde\sigma)}^{T-1}\frac{\lambda_{\Gamma_{\tilde\sigma}}(\sigma_t^\infty)\cdots\lambda_{\Gamma_{\tilde\sigma}}\left(\sigma_{t(\tilde\sigma)}^\infty\right)}{p(\sigma_t^\infty)}$ as being increasing in T must converge to a positive real number, call it A. So in the case that an interim Pareto improvement were possible, a subtree $\Gamma_{\tilde\sigma}$, a weight function $\lambda_{\Gamma_{\tilde\sigma}}$ and a finite number A would exist such that (5) would hold. This completes the proof. ∎

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