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A TREE IMPLEMENTATION OF A CREDIT SPREAD MODEL FOR CREDIT DERIVATIVES

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ABSTRACT. In this paper we present a tree model for defaultable bond prices which can be used for the pricing of credit derivatives. The model is based upon the two-factor Hull-White (1994) model for default-free interest rates, where one of the factors is taken to be the credit spread of the defaultable bond prices. As opposed to the tree model of Jarrow and Turnbull (1992), the dynamics of default-free interest rates and credit spreads in this model can have any desired degree of correlation, and the model can be fitted to any given term structures of default-free and defaultable bond prices, and to the term structures of the respective volatilities. Furthermore the model can accommodate several alternative models of default recovery, including the fractional recovery model of Duffie and Singleton (1994) and recovery in terms of equivalent default-free bonds (see e.g. Lando (1998)). Although based on a Gaussian setup, the approach can easily be extended to non-Gaussian processes that avoid negative interest-rates or credit spreads.

1. INTRODUCTION

In this paper we present a tree model for defaultable bond prices which can be used for the pricing of credit derivatives. The model is based upon the two-factor Hull-White (1994) model for default-free interest rates, where one of the factors is taken to be the credit spread of the defaultable bond prices. As opposed to the tree model of Jarrow and Turnbull (1992), the dynamics of default-free interest rates and credit spreads in this model can have any desired degree of correlation, and the model can be fitted to any given term structures of default-free and defaultable bond prices, and to the term structures of the respective volatilities. Furthermore the model can accommodate several alternative models of default recovery, including the fractional recovery model of Duffie and Singleton (1994) and recovery in terms of equivalent default-free bonds (see e.g. Lando (1998)). Although based on a Gaussian setup, the approach can easily be extended to non-Gaussian processes that avoid negative interest-rates or credit spreads.

The model contributes to the existing literature in two respects: First, it provides an implementation framework for most of the existing intensity-based credit risk models, and second, it enables a quantitative comparison of the properties of these models and the relative importance

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of input parameters like recovery rates, volatility of credit spreads and the correlation between credit spreads and interest rates.

The paper is structured as follows:

In the first section we discuss the credit risk model that we are going to use. The time of default will be a (discretisation of a) totally inaccessible stopping time with a stochastic intensity λ, and the recovery rate of a defaulted bond will be determined by using one of two models: the fractional recovery model (where a bond loses a fraction of ist pre-default value in default, see e.g. Duffie and Singleton), or the equivalent recovery model (where a defaulted bond recovers a certain number of equivalent default-free bonds). It is also show, how the model can be extended to incorporate stochastic recovery rates or recovery in terms of a fraction of the par value of the defaulted bond. We discuss the relative merits of all recovery models, the extension to stochastic recovery, and their influence on the results later on. Taking one node of the tree as example it is show how these continuous-time models are incorporated into a discrete-time tree setup and the branching scheme for default risk is demonstrated.

The next section demonstrates how to incorporate stochastic credit spreads into the model assuming independence of credit spreads and risk-free interest rates. Because of the independence assumption, a separate tree for the credit risk can be built and fitted to the term structure of credit spreads. The tree building and fitting procedure are demonstrated for the case of a mean-reverting Gaussian diffusion process for the default intensity of the form

$$d\lambda = (\overline{k}(t) - \overline{a}\lambda)dt + \overline{\sigma}(t)d\overline{W}$$

where $\overline{k}(t)$ and $\overline{\sigma}(t)$ are used to fit the tree. It is also shown how to combine this tree with a tree for the default-free interest rates, where the default-free short rate follows a similar process of the form $dr = (k(t) - ar)dt + \sigma(t)dW$.

In the following step correlation between credit spreads and default-free interest rates is introduced by introducing correlation between dW and $d\overline{W}$. The tree-building and fitting procedure now has to be done sequentially: First, the tree for the default-free interest rates is built and fitted to the default-free term structures, then the tree for the credit spreads is built, then both trees are combined and correlation between spreads and interest-rates is introduced, and finally, the combined tree is fitted to the term structure of defaultable bond prices. For this fitting procedure a new set of defaultable state prices has to be introduced. The section is concluded by examples that demonstrate the use of this model for the pricing of credit derivatives. The credit derivatives are credit default swaps, callable credit default swaps, credit spread options and asset swaptions. Finally, the extension of the model to the valuation of first-to-default basket credit derivatives is discussed, and it is shown how to modify the model to ensure positive interest rates and credit spreads.

In the last section the model is used to analyse numerically the input parameters to the intensity-based credit risk models that have been proposed in the literature. The effect on implied default probabilities (and default swap prices) of correlation between credit spreads and default-free interest rates is analysed and it is compared to the effect of misspecification in the expected recovery rate. Then we look at the effects of the different recovery models and how they influence the prices of default swaps and the implied default probabilities. The paper is concluded with a summary of the main results.

2. THE CREDIT RISK MODEL

2.1. **Model Setup and Notation.** The model is set up in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{(t\geq 0)}, P)$ where P is a pre-specified martingale measure. We assume the filtration $(\mathcal{F}_t)_{(t\geq 0)}$ satisfies the usual conditions[1] and the initial filtration $\mathcal{F}_0$ is trivial. We also assume a finite time horizon $\bar{T}$ with $\mathcal{F} = \mathcal{F}_{\bar{T}}$, all definitions and statements are understood to be only valid until this time horizon $\bar{T}$. The notation used is:

- $B(t,T)$: default free zero coupon bond price,
- $r(t)$: default free short rate,
- $\beta_{t,T}$: discount factor over $[t,T]$,
- $\overline{B}(t,T)$: defaultable zero coupon bond price,
- $P(t,T)$: survival probability for $[t,T]$.

2.2. **The Time of Default.** Although we are going to use two different models to model the *recovery* of defaulted bonds, the model for the *time* of the default(s) is the same for both:

We assume that the times of default τ_i are generated by a Cox process. Intuitively, a Cox Process is defined as a Poisson process with stochastic intensity λ (see Lando (1998), p.101). Formally the definition is:

Definition 1. *N is called a Cox process, if there is a nonnegative adapted stochastic process $\lambda(t)$ (called the* intensity *of the Cox process) with $\int_0^t \lambda(s)ds < \infty \quad \forall\ t > 0$, and conditional on the realization $\{\lambda(t)\}_{\{t>0\}}$ of the intensity, $N(t)$ is a time-inhomogeneous Poisson process with intensity $\lambda(t)$.*

This definition follows Lando (1998) and differs from the usual definition of a Cox process where the intensity process $\lambda(t)$ is *fully* revealed immediately after time 0 (i.e. the intensity is $\mathcal{F}_0$-measurable see e.g. Brémaud (1981)). Mathematically, it is not necessary to reveal all information about the future development of the intensity, and from the point of view of realism and for the valuation of derivatives this modelling approach would even introduce pricing errors[2].

Assumption 1. *(i) The default counting process*

$$N(t) := \max\{i|\tau_i \leq t\} = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i \leq t\}} \tag{1}$$

is a Cox process with intensity process $\lambda(t)$.

[1] See Jacod and Shiryaev (1988).

[2] Consider e.g. an American Put option on a defaultable bond in a world with constant zero risk-free interest rates. If all information about $\lambda(t)$ is revealed at $t = 0$ this would enable the investor to condition his optimal exercise policy on the future development of λ which is not realistic.

(ii) In the equivalent recovery model the time of default is the time of the first jump of N. To simplify notation the time of the first *default will be referred to with $\tau := \tau_1$.*

(iii) In the fractional recovery model the times of default are the times of the jumps of N.

Remark 1. By standard properties of inhomogenous Poissson processes, given the realisation of λ, the probability of having exactly n jumps is

$$\mathbf{P}\left[\, N(T) - N(t) = n \mid \{\lambda(s)\}_{\{T \geq s \geq t\}} \,\right] \;=\; \frac{1}{n!}\left(\int_t^T \lambda(s)ds\right)^n \exp\{-\int_t^T \lambda(s)ds\}. \tag{2}$$

The probability of having n jumps (without knowledge of the realisation of λ) is found by conditioning on the realisation of λ *within* an outer expectation operator:

$$\begin{aligned} \mathbf{P}_t\left[\, N(T) - N(t) = n \,\right] &= \mathbf{E}_t\left[\; \mathbf{P}\left[\, N(T) - N(t) = n \mid \{\lambda(s)\}_{\{T \geq s \geq t\}} \,\right] \;\right] \\ &= \mathbf{E}_t\left[\; \frac{1}{n!}\left(\int_t^T \lambda(s)ds\right)^n \exp\{-\int_t^T \lambda(s)ds\} \;\right], \end{aligned} \tag{3}$$

Define the process $P(t,T)$

$$P(t,T) = \mathbf{E}_t\left[\, e^{-\int_t^T \lambda(s)ds} \,\right]. \tag{4}$$

For $\tau > t$ (before default), $P(t,T)$ can be interpreted as the *survival probability* from time t until time T. In general,

$$\mathbf{1}_{\{\tau > t\}} P(t,T) = \mathbf{E}_t\left[\, \mathbf{1}_{\{\tau > T\}} \,\right]. \tag{5}$$

Given $\tau > t$ the density of the time of the first default as seen from t is for $T > t$

$$p(t,T) = \mathbf{E}_t\left[\, \lambda(T) \exp\{-\int_t^T \lambda(s)ds\} \,\right], \tag{6}$$

and $p(t,T) = 0$ for $T \leq t$. [3]

The law of iterated expectations as it was used above is extremely useful in Cox process based default models, it was first used in a credit risk context by Lando (1998).

The specification of the default trigger process as a Cox process precludes a dependence of the default intensity on previous defaults[4] and also ensures totally inaccessible stopping times τ_i as times of default. Apart from this it allows rich dynamics of the intensity process, specifically, we can reach stochastic credit spreads. If only the time of the *first* jump of N is of interest, the Cox-process specification is completely without loss of generality within the totally inaccessible stopping times.

In the following sections we will consider time t as 'today', and assume that no default has happened so far $\tau > t$. (The statements for $\tau < t$ are trivial.)

[3] If a default has already happened, $p(t,T) = \epsilon_\tau$ the density of the first default reduces to the Dirac measure at τ.

[4] It is therefore not possible to specify an intensity that jumps at defaults.

2.3. **The Fractional Recovery Model.** The version of the fractional recovery model used here is an extension of the Duffie-Singleton (1994) model to multiple defaults. More details to the model can be found in Schönbucher (1996; 1998). The new feature of this model is that a default does not lead to a liquidation but a reorganisation of the issuer: defaulted bonds lose a fraction q of their face value and continue to trade. This feature enables us to consider European-type payoffs in our derivatives without necessarily needing to specify a payoff of the derivative at default (although we will consider this case, too). The next assumption summarises the fractional recovery model:

Assumption 2. *There is an increasing sequence of stopping times $\{\tau_i\}_{i\in\mathbb{N}}$ that define the times of default. These times are given in definition 1 and assumption 1 as the times of the jumps of the Cox process N.*
At each default τ_i the defaultable bond's face value is reduced by a factor q_i, where q_i may be a random variable itself. A defaultable zero coupon bond's final payoff is the product

$$Q(T) := \prod_{\tau_i \le T} (1 - q_i) \tag{7}$$

of the face value reductions after all defaults until the maturity T of the defaultable bond. The loss quotas q_i can be random variables drawn from a distribution $K(dq)$ at time τ_i, but for the first calculations we will assume $q_i = q$ to be constant.

It is now easily seen[5] that in this setup the price of a defaultable zero coupon bond is given by

$$\overline{B}(t,T) = Q(t)\, \mathbf{E}_t \left[\, e^{-\int_t^T \overline{r}(s)ds} \, \right]. \tag{8}$$

The process $\overline{r}$ is called the defaultable short rate $\overline{r}$ and it is defined by

$$\overline{r} = r + \lambda q. \tag{9}$$

Here r is the default-free short rate, λ the hazard rate of the defaults and q is the loss quota in default. If q is stochastic then q has to replaced by its (local) expectation $q_t^e = \int qK_t(dq)$ in equation (9).

It is convenient to decompose the defaultable bond price $\overline{B}$ as follows:

$$\overline{B}(t,T) = Q(t) B(t,T) \widetilde{P}(t,T). \tag{10}$$

Here $Q(t)$ represents the face-value reduction due to previous defaults (before time t). Frequently we will be able to set $t = 0$ and thus $Q(t) = 1$, but for the analysis at intermediate times it is important to be clear about the notation[6]. The defaultable bond price $\overline{B}(t,T)$ is thus the product of $Q(t)$, the influence of previous defaults, and the product of the default-free bond price $B(t,T)$ and the third factor $\widetilde{P}(t,T)$ which is uniquely defined by equation (10), or equivalently:

$$\widetilde{P}(t,T) = \frac{1}{Q(t)} \frac{\overline{B}(t,T)}{B(t,T)}. \tag{11}$$

[5]Using the iterated expectations, see also Duffie and Singleton (1994) and Schönbucher (1996; 1998) for a more general proof.

[6]For example, at the expiry date of an option we would like to separate previous defaults and credit spreads in the price of the underlying.

Remark 2. $\widetilde{P}(t,T)$ is related to the *survival probability* $P(t,T)$ of the defaultable bond: If r and λ are independent and there is a total loss ($q=1$) at default then $\widetilde{P}(t,T)$ is the probability (under the martingale measure) that there is no default in $[t,T]$.
If r and λ are not independent and $q=1$, then $\widetilde{P}(t,T)$ is the survival probability under the T-forward measure P^T

$$\widetilde{P}(t,T) = \mathbf{E}_t^{P^T}\left[\, e^{-\int_t^T q\lambda(s)ds} \,\right]. \tag{12}$$

(Under independence $\mathbf{E}_t^T\left[\, e^{-\int_t^T q\lambda(s)ds} \,\right]$ and $\mathbf{E}_t\left[\, e^{-\int_t^T q\lambda(s)ds} \,\right]$ coincide.)
If r and λ are not independent and there is positive recovery ($q<1$), then $\widetilde{P}(t,T)$ is the *expected final payoff* under the T-forward measure, but the implied survival probability cannot be recovered without more knowledge about the distribution of λ, q and r.

2.4. **The Equivalent Recovery Model.** The equivalent recovery model has been proposed by several authors, amongst them Jarrow and Turnbull (1995) Lando (1998) and Madan and Unal (1998). Here the recovery of defaulted debt is treated as follows:

Assumption 3. *At the time of default τ, one defaultable bond $\overline{B}(\tau,T)$ with maturity T has a payoff of c equivalent (i.e. with the same maturity and face value) default free bonds $B(\tau,T)$, where c may be random, too.*

Under the equivalent recovery model (with constant c and given no default so far $\tau>t$) the price of a defaultable bond can be decomposed into c default-free bonds and $(1-c)$ defaultable bonds with zero recovery

$$\begin{aligned}\overline{B}(t,T) &= \mathbf{E}_t\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} + c\beta_{t,\tau}B(\tau,T)\mathbf{1}_{\{\tau\le T\}} \,\right]\\ &= \mathbf{E}_t\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \,\right] + c\,\mathbf{E}_t\left[\, \beta_{t,T} \,\right] - c\,\mathbf{E}_t\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \,\right]\\ &= (1-c)\overline{B}_0(t,T) + cB(t,T), \end{aligned} \tag{13}$$

where $\overline{B}_0(t,T)$ is the price of a defaultable bond under zero recovery:

$$\mathbf{E}_t\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \,\right] = \mathbf{1}_{\{\tau>t\}}\,\mathbf{E}_t\left[\, e^{-\int_t^T r(s)+\lambda(s)ds} \,\right]$$

It should be pointed out that the equivalent recovery model is not able to fit all term structures of credit spreads with a given fixed common recovery rate c. Assume $\tau>t$ and the term structure of credit spreads is at a constant credit spread h for all maturities T. Then

$$\frac{\overline{B}(t,T)}{B(t,T)} = e^{-h(T-t)}$$

and for large enough $T-t$ (such that $T-t > -(\ln c)/h$),

$$\widetilde{P}(t,T) = \frac{1}{1-c}\left(\frac{\overline{B}(t,T)}{B(t,T)} - c\right) = \frac{1}{1-c}\left(e^{-h(T-t)} - c\right) < 0,$$

the survival probability (see below) that can be implied from the zero-recovery bond $\overline{B}_0(t,T)$ would become negative, which is obviously not sensible. In the equivalent recovery model there is a lower bound on the ratio of defaultable bond prices to default-free bond prices and this bound is the recovery rate c. Therefore the zero coupon yield spread must satisfy

$$\overline{y}(t,T) - y(t,T) < -\frac{\ln c}{T-t},$$

which may not be satisfied by market prices for longer times to maturity $T-t$ and high credit spreads. E.g. for a recovery rate of $c = 50\%$ and a time to maturity of $T-t = 10$ years the maximal (continuously compounded) credit spread is $h = 6.93\%$.

Despite these different properties of the two modelling approaches, with a suitable choice of (time dependent or stochastic) parameters, both models can be transformed into each other: The value of the security in default is only expressed in different numeraires, once in terms of defaultable bonds and once in terms of default-free bonds. Both approaches are therefore equivalent and one should use the specification that is best suited for the issue at hand.

2.5. **Implied Survival Probabilities.** In the equivalent recovery model it is easy to recover *implied survival probabilities* from a given term structure of defaultable bond prices and a given value for c. From equation (13) we have

$$\widetilde{P}(t,T) := \frac{\overline{B}_0(t,T)}{B(t,T)} = \frac{1}{1-c}\left(\frac{\overline{B}(t,T)}{B(t,T)} - c\right). \tag{14}$$

$\widetilde{P}(t,T)$ is the probability of survival from t to T under the T-forward measure (and also under the spot martingale measure for independence of credit spreads and interest rates).

This survival probability and the prices of defaultable zero coupon bonds $\overline{B}_0(t,T)$ under zero recovery are very useful to value survival contingent payoffs. For many pricing applications knowledge of $\overline{B}_0(t,T)$ is already sufficient. It is a great advantage of the equivalent recovery model that it allows to derive the value of a survival contingent payoff just from the defaultable and default-free term structures and an assumption about recovery rates c.

In the fractional recovery model it is not possible to derive the value of a zero-recovery defaultable bond just from knowledge of the recovery rate q, the defaultable bond price and the default-free bond prices unless the recovery rate is zero. Here a full specification of the dynamics of r and λ is needed.

Given independence of interest rates and the default intensity, the implied survival probability under the spot martingale measure is the ratio of the zero coupon bond prices:

$$P(t,T) = \frac{\overline{B}_0(t,T)}{B(t,T)}$$

Typically the survival probability $P(t,T)$ will change over time because of two effects: First, if there was no default in $[t, t+\Delta t]$ this reduces the possible default times, information has arrived via the (non)-occurrence of the default. Secondly, additional default-relevant information could have arrived in the meantime.

For the analysis of the local default probability in some future time interval it is instructive to consider the *conditional* probability of survival. The probability of survival in $[T_1, T_2]$, given that there was no default until T_1 and given the information at time t is:

$$P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)} = \frac{\overline{B}_0(t,T_2)}{B(t,T_2)}\frac{B(t,T_1)}{\overline{B}_0(t,T_1)}.$$

This is a simple consequence of Bayes' rule. The probability of survival until T is the probability of survival until $s < T$ times the conditional probability of survival from s until T:

$$P(t,T) = P(t,s)P(t,s,T).$$

There is a close connection between forward rates and conditional survival / default probabilities.

Definition 2. *The* default-free simply compounded forward rate *over the period* $[T_1, T_2]$ *as seen from* t *is:*

$$F(t,T_1,T_2) = \frac{B(t,T_1)/B(t,T_2) - 1}{T_2 - T_1}$$

The zero-recovery defaultable simply compounded forward rate *over the period* $[T_1, T_2]$ *as seen from* t *is:*

$$\overline{F}(t,T_1,T_2) = \frac{\overline{B}_0(t,T_1)/\overline{B}_0(t,T_2) - 1}{T_2 - T_1}$$

Proposition 1. *Under independence, the* conditional probability of default *over* $[T_1, T_2]$ *is given by:*

$$\frac{P^{\text{def}}(t,T_1,T_2)}{T_2 - T_1} = \frac{\overline{F}(t,T_1,T_2) - F(t,T_1,T_2)}{1 + (T_2 - T_1)\overline{F}(t,T_1,T_2)}.$$

The marginal probability of default *at time* T *is the spread of the continuously compounded defaultable forward rate over the default-free forward rate:*

$$\lim_{\Delta t \searrow 0} \frac{P^{\text{def}}(t,T,T+\Delta t)}{\Delta t} = \overline{f}(t,T) - f(t,T).$$

Proof. (dropping the t-index)

$$\begin{aligned}
P^{\text{def}}(T_1,T_2) = 1 - P(T_1,T_2) &= 1 - \frac{\overline{B}_0(T_2)B(T_1)}{B(T_2)\overline{B}_0(T_1)} \\
&= \frac{B(T_2)\overline{B}_0(T_1) - \overline{B}_0(T_2)B(T_1)}{B(T_2)\overline{B}_0(T_1)} \\
&= \frac{B(T_2)[\overline{B}_0(T_1) - \overline{B}_0(T_2)] - \overline{B}_0(T_2)[B(T_1) - B(T_2)]}{B(T_2)\overline{B}_0(T_1)} \\
&= \frac{\overline{B}_0(T_2)}{\overline{B}_0(T_1)}\frac{\overline{B}_0(T_1) - \overline{B}_0(T_2)}{\overline{B}_0(T_2)} - \frac{\overline{B}_0(T_2)}{\overline{B}_0(T_1)}\frac{B(T_1) - B(T_2)}{B(T_2)}
\end{aligned}$$

therefore

$$\frac{P^{\text{def}}(T_1,T_2)}{T_2 - T_1} = \frac{\overline{B}_0(T_2)}{\overline{B}_0(T_1)}\Big(\overline{F}(T_1,T_2) - F(T_1,T_2)\Big),$$

and from definition 2 follows that

$$\frac{\overline{B}_0(T_1)}{\overline{B}_0(T_2)} = 1 + (T_2 - T_1)\overline{F}(T_1, T_2).$$

The result for the marginal default probability follows directly from taking the limit. □

The default probability over the interval $[T_1, T_2]$ equals *the length of the interval* times *the spread of the simply compounded forward rates over the interval* times *discounting with the defaultable forward rates*.

For small time intervals, the probability of default in $[T, T + \Delta t]$ is approximately *proportional* to the length of the interval with proportionality factor $(\overline{f}(t, T) - f(t, T))$.

These results highlight two points. First, there is an intimate connection between default probabilities and credit spreads. A full term structure of credit spreads contains a wealth of information about the market's perception of the likelihood of default at each point in time. The equivalent recovery model has the advantage of making this information more easily accessible than the fractional recovery model. Unfortunately, to reach this information in a practical application, an assumption about the expected recovery rate c is needed, and independence of defaults and default-free term structure of interest rates must be assumed. There is a large degree of uncertainty about recovery rates with variation between 20% and 80%.

The second observation is the reason why processes like Poisson or Cox processes are so well suited for credit-spread based default modelling. These processes have intensities, and the probability of jump of a point process with an intensity is approximately proportional to the length of the time interval considered (for small intervals). The proportionality factor is the intensity at that point. This property is exactly equivalent to the second equation in proposition 1, and it also gives a link to models of defaultable forward credit spreads as for example in Schönbucher (1998). But proposition 1 is also valid for default models that are not based on an intensity model.

2.6. **Comparison of Recovery Mechanisms.** In real-world applications the recovery rate of a defaulted bond is expressed as the fraction of its *par value* that is paid out to the creditor. A model that uses this approach can be found e.g. in Duffie (1998). Although it seems more natural there are some complications as this recovery mechanism only makes sense for coupon bonds, and not for zero-coupon bonds. To fit this model to observed bond prices we would like to strip observed coupon bonds into coupon strips and principal. These two components now have different recoveries in default, only the principal of the bond has a positive recovery while the coupons recover nothing. Thus we have to model recovery in two different ways which makes this modelling approach more complicated.

In figure 1 the effects of the different recovery models on zero coupon bonds of different maturities are shown. Here default-free interest rates are $r = 7\%$, credit spreads are $h = 4\%$ and the recovery rate is 50%. The recovery models are equivalent recovery, fractional recovery and recovery of par.

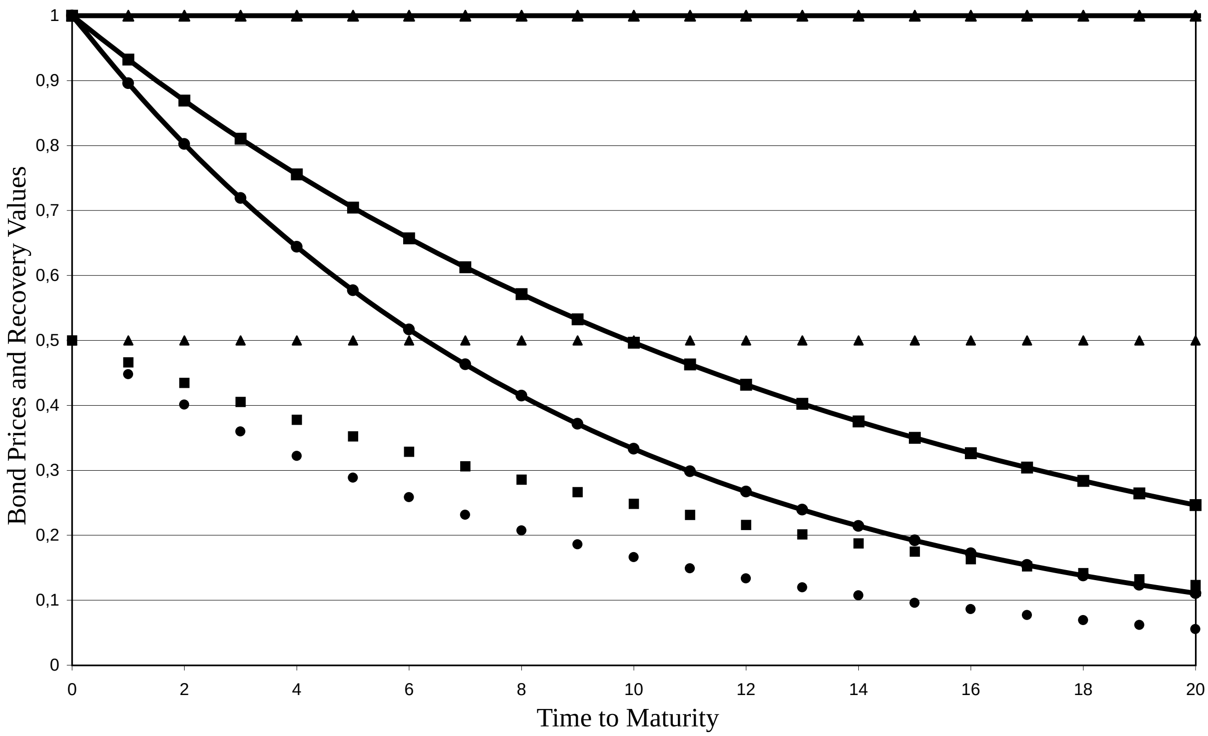


FIGURE 1. Equivalent, fractional and par recovery for different maturities. Parameter values: default-free interest rates $r = 7\%$, credit spread $h = 4\%$, recovery 50%. The continuous line with circles are the defaultable bond prices, continuous line with squares are the default-free bond prices and continuous line with triangles are par values for different maturities. The recovery values for 50% equivalent recovery are given by the squares, for 50% fractional recovery by circles and for 50% recovery of par by triangles.

In all recovery models, the recovery of a full term structure of defaulted zero coupon bonds can be represented as recovery rate times a certain reference price curve. In equivalent recovery, the payoff to the defaultable bonds is 50% times the equivalent default-free bond price (the default-free bond prices are shown as continuous line with squares and the corresponding recovery values are the dotted line with squares). The reference prices curve for the fractional recovery model are the defaultable bond prices (shown with circles) and the reference price for the par recovery model are marked with triangles.

The differences between the models increase with time to maturity, the further the defaultable bond price is from par, the larger the differences in the recovery values. For times to maturity of 6.5 years and more the recovery of par model is inconsistent with the defaultable bond prices as the recovery value exceeds the pre-default price of the defaultable bonds; for times to maturity larger than 17 years the same problem occurs with the equivalent recovery model. This problem was already discussed in section 2.4.

3. IMPLEMENTATION: FIRST STEPS

3.1. **Inputs to the Model.** As mentioned in the introduction the aim of this paper is to provide a tree implementation algorithm that can be fitted to both defaultable and default-free term structures of bond prices and volatilities. We therefore need as inputs to the model (for all $T \geq 0$):

- $B(0,T)$: the initial default-free term structure of zero coupon bond prices. The construction of such zero-coupon curves from market prices is now standard in interest-rate literature.
- $a, k(T)$ and $\sigma(T)$: the parameters of the dynamics of the default-free short rate. Here we use the extended Vasicek (1977) model

$$dr(t) = (k(t) - ar)dt + \sigma(t)dW(t). \tag{15}$$

  The level of mean reversion $k(t)$ will be used to fit the tree to the initial term structure of bond prices and is therefore already implicitly defined. The spot volatility function $\sigma(t)$ can be used to fit an initial term structure of volatilities.
- $\overline{B}(0,T)$: the initial term structure of defaultable bond prices.
- $\overline{a}, \overline{k}(T)$ and $\overline{\sigma}(T)$: the parameters of the dynamics of the default intensity λ. We also use the extended Vasicek (1977) model for the intensity

$$d\lambda(t) = (\overline{k}(t) - \overline{a}\lambda)dt + \overline{\sigma}(t)d\overline{W}(t). \tag{16}$$

  We make provisions for the fitting of the volatility $\overline{\sigma}(T)$ of the default intensity to an initial term structure of volatilities for the defaultable bonds although in typical applications there will not be sufficient data to support this fitting. In this case one can set the volatility to a constant: $\overline{\sigma}(T) = \overline{\sigma} =$ const.
- ρ: The correlation between the Brownian motions W and $\overline{W}$: $dW d\overline{W} = \rho dt$. The value of this parameter will also introduce correlation between the motion of the credit spreads and the default-free interest rates.
- c or q: A choice of recovery model (equivalent recovery or fractional recovery) and the respective recovery rate (c) for equivalent recovery or loss quota (q) for fractional recovery. If recovery is *stochastic*, one must also specify the distribution function of the recovery rate and (derived from that) the expected recovery rate c^e or loss quota q^e.
- Finally, some numerical parameters like the time step size Δt and the number of time steps have to be chosen.

3.2. **Pre-Processing.**

3.2.1. *Equivalent Recovery to Zero Recovery Conversion.* If the equivalent recovery model is used, a first pre-processing step is required to derive an initial term structure $\overline{B}_0(0,T)$ of defaultable bonds with zero recovery (see equation (13)):

$$\overline{B}_0(0,T) = \frac{1}{1-c}(\overline{B}(0,T) - cB(0,T)). \tag{17}$$

If the recovery rate c is stochastic, the expected recovery rate c^e must be used in equation (17). These zero recovery defaultable bond prices can now also be viewed as defaultable bond prices under zero fractional recovery, i.e. a loss quota of $q = 1$. It is therefore sufficient to demonstrate the implementation for the fractional recovery model, the modifications for the equivalent recovery model are given where they are necessary.

3.2.2. *Bond Volatility Fitting.* The specification of a time-dependent interest-rate volatility $\sigma(t)$ translates into time-dependent bond price volatilities via

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt - \sigma(t)\frac{1}{a}(1 - e^{-a(T-t)})dW(t) \tag{18}$$

and forward rate volatilities via

$$df(t,T) = \frac{\sigma(t)^2}{a}e^{-a(T-t)}(1 - e^{-a(T-t)})dt + \sigma(t)e^{-a(T-t)}dW(t) \tag{19}$$

where the drift of the forward rates follows from the Heath-Jarrow-Morton (1992) drift restriction. The parameters a and $\sigma(t)$ can now be used to find a fit to a given volatility structure of the bond prices or forward rates. As $\sigma(t)$ enters the model as a multiplicative factor we can thus capture time dependence in the general interest-rate and bond price volatility, but the shape of the forward volatilities of different maturities T at the same time t remains of the exponential form.

3.2.3. *Closed-Form Solutions.* To specify the payoffs of the derivative securities in the tree we need the prices of the corresponding underlying security at the nodes of the tree. Often the underlying security are simple coupon bonds with defaultable or default-free payoffs at fixed dates far in the future. To avoid building a ten-year tree for an option that expires in one year, just because the underlying bond has a maturity of ten years, it is useful to have closed-form solutions for these simple payoffs.

The price of a default-free bond for short rate $r(t)$ and the dynamics (15) is given by

$$B(t,T) = e^{\mathcal{A}(t,T) - \mathcal{B}(t,T)r(t)} \tag{20}$$

where

$$\mathcal{B}(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}) \tag{21}$$

$$\mathcal{A}(t,T) = \frac{1}{2}\int_t^T \sigma^2(s)\mathcal{B}(t,s)^2 ds - \int_t^T \mathcal{B}(t,s)k(s)ds. \tag{22}$$

The price of a defaultable bond for: short rate $r(t)$, default intensity $\lambda(t)$, survival until t and dynamics (15) and (16) is

$$\overline{B}(t,T) = B(t,T)e^{\overline{\mathcal{A}}(t,T) - \overline{\mathcal{B}}(t,T)\lambda(t)} \tag{23}$$

where

$$\overline{\mathcal{B}}(t,T) = \frac{1}{\overline{a}}(1 - e^{-\overline{a}(T-t)}) \tag{24}$$

$$\overline{\mathcal{A}}(t,T) = \frac{1}{2}\int_t^T \sigma^2(s)\overline{\mathcal{B}}(t,s)^2 ds - \int_t^T \overline{\mathcal{B}}(t,s)\tilde{k}(s)ds \tag{25}$$

$$\tilde{k}(t) = \overline{k}(t) + \rho\overline{\sigma}(t)\sigma(t)\mathcal{B}(s,T). \tag{26}$$

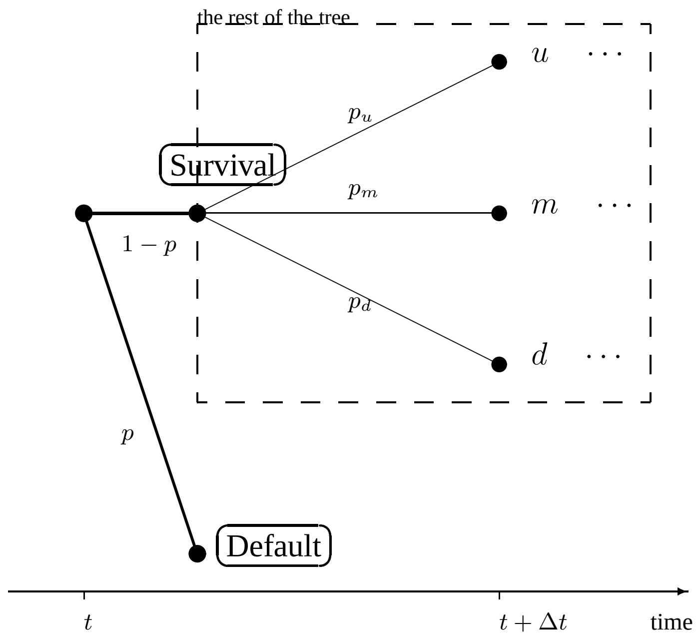


FIGURE 2. The branching to default at a typical node in the tree. Over one time step from t to $t+\Delta t$ there is first a branch to default or survival, and only in the survival node the tree is continued.

3.3. **The Default Branching.** In the following sections we are going to construct a tree model for the development of the short term interest rate and the default intensity, and this tree has to be joined with a model-consistent default and recovery mechanism. At each node in the tree we will know the current defaultable and default-free bond price structures and thus the current default intensity λ. By equation (4) the survival probability from t to $t+\Delta t$ is given by

$$1-p = \mathbf{E}\left[e^{-\int_t^{t+\Delta t} \lambda(s)ds} \;\middle|\; \mathcal{F}_t \right].$$

The default intensity is constant over $[t, t+\Delta t[$, thus

$$1-p = e^{-\lambda(t)\Delta t} \tag{27}$$

is (by equation (4)) the survival probability over the next time interval $[t, t+\Delta t[$, and p is the corresponding default probability. If the time step Δt is not too large, we can assume without much loss of accuracy that the default happens at the left end $\tau = t$ of the time interval (*if* it happens in the time interval). If more precision is required one can use the expected time of default, given that there is a default in $[t, t+\Delta t[$. This is

$$\tau^e = \mathbf{E}\left[\, \tau \mid \tau \in [t, t+\Delta t[\, \right] = t + \frac{1}{\lambda} - \Delta t \frac{e^{-\lambda\Delta t}}{1-e^{-\lambda\Delta t}}.$$

To incorporate the default an additional branching point has to be added to the tree in the way indicated in figure 3.3. Thus, at each node of the tree, it is *first* decided, whether a default has happened (branch down to default) or not (branch across), and then, *given survival*, the 'normal' tree continues with the evolution of interest-rates and default intensities. The default state is a 'leaf' of the tree and apart from the calculation of the payoffs in default the tree ends there[7].

Although the tree ends in default it is still possible to value default-free securities in this framework (or payoff components that are unaffected by defaults): When the backwards induction reaches the survival node S with a (local) value of V for the default-free security, the payoff in the default node D must be set to V, too. Thus the default branching will be effectively ignored. Alternatively, by adding two lines of code to the program one can ensure that the default branching is ignored altogether.

The probability of reaching node u *and* surviving over the next time interval is now $p \cdot p_u$, the probability of reaching m and surviving is $p \cdot p_m$ and for node d this is $p \cdot p_d$. Consider now a survival contingent security with payoffs x_u, x_m and x_d in nodes u, m and d, and zero at default. Without the possibility of default this security would have the price[8] $x' = x_u p_u + x_m p_m + x_d p_d$. The price with default is on the other hand $x = (1-p)x'$, the possibility of default introduces an additional discounting with the survival probability $(1-p)$ in each node. This fact can also be proven in the continuous-time setup.

3.4. **Recovery Modelling in the Default Branch.** As the default-free interest rates are known in the survival branch they are also known for the default branch. Therefore specifying the equivalent recovery mechanism is straightforward in this setup.[9]

For fractional recovery the mechanism is slightly more complex because in the continuous-time model there can be multiple defaults. There are two alternative ways of approximating this model in discrete-time: Either, the number of defaults is restricted to one default per time interval $[t, t+\Delta t[$, or multiple defaults are allowed even within the interval $[t, t+\Delta t[$.

Let V_n be the value of a defaultable security at $t = n\Delta t$, and V_n^* its value if it survived until $t = n\Delta t$. If only one default is allowed, the following recursion holds for V_n (ignoring the discounting by default-free interest rates)

$$V_n = e^{-\lambda_n \Delta t} V_{n+1}^* + (1 - e^{-\lambda_n \Delta t})(1-q) V_{n+1}^* = (1 - q(1 - e^{-\lambda_n \Delta t})) V_{n+1}^*. \tag{28}$$

If the full multiple default model is used over the interval $[t, t+\Delta t[$ the value is given by

$$V_n = e^{-q\lambda_n \Delta t} V_{n+1}^*. \tag{29}$$

The dynamics of equation (28) converge to (29) as $\Delta t \to 0$, and for reasonably small time step sizes the difference is negligible. If the time-step size is large (e.g. larger than 1/12), the approach in equation (29) is more appropriate.

Stochastic recovery rates can be incorporated into the pricing algorithm by a direct specification of the distribution of the recovery rate in default. This distribution has to be evaluated at all

[7] The branching method and the termination of the tree at default are different from the tree implementation in Jarrow and Turnbull (1995). The procedure chosen here avoids an unnecessary expansion of the tree.

[8] Assuming zero default-free interest rates.

[9] The equivalent recovery model only has to be implemented for the pricing runs through the tree. For the tree setup and fitting in the equivalent recovery model we will only use a term structure of zero-recovery defaultable bond prices.

branches to default. If the payoff in default is a function $f(q)$ of the loss quota, the value that has to be used in the algorithm is the average payoff

$$V^e = \int f(q)K(dq), \tag{30}$$

where $K(dq)$ is the distribution function of the loss quota *given a default has happened.* The implementation of stochastic recovery with the equivalent recovery model is similar. For the pricing of defaultable bonds with stochastic recovery it is sufficient to use the *expected* recovery rate.

4. Implementation: The Independence Case

4.1. **Pricing Relationships.** In this section we assume that the dynamics of the default-free interest rates is independent from the credit spread and default processes. This enables us to decouple defaults and discounting in most pricing problems:

Defaultable zero coupon bond prices (see equation (10))

$$\overline{B}(t,T) = \mathbf{E}_t\left[\,\beta_{t,T}Q(T)\,\right] = \mathbf{E}_t\left[\,\beta_{t,T}\,\right]\mathbf{E}\left[\,Q(T)\,\right] = B(t,T)\widetilde{P}(t,T) \tag{31}$$

where

$$\widetilde{P}(t,T) = \mathbf{E}_t\left[\,e^{-\int_t^T q\lambda(s)ds}\,\right]. \tag{32}$$

Zero-recovery defaultable zero coupon bond prices decouple to

$$\overline{B}_0(t,T) = \mathbf{E}_t\left[\,\beta_{t,T}\mathbf{1}_{\{\tau>T\}}\,\right] = B(t,T)P(t,T)$$

Payoffs *at* default can be decoupled: Receiving X at t if $\tau = t$ (a default happens at t), has the value

$$\mathbf{E}\left[\,\beta_{0,\tau}X\mathbf{1}_{\{\tau<T\}}\,\right] = \int_0^T B(0,t)Xp(0,t)dt,$$

where $p(t,T)$ is the density of the default time as seen from time t.

In general, the payoffs can be decoupled if [credit spreads and defaults] and interest rates appear as a *sum of products* in the payoff function:

$$f(r,t)g(\lambda,\tau,t)$$

where f and g can be functionals that depend on the whole path of r or λ. Payoffs *at* default also fall into this category because – like in the preceding paragraph – they can be rewritten as integral over the time horizon weighted with the density of the time of default.

There are also cases where the independence will not help to decouple the payoffs. A simple example is a call option on a defaultable bond. The payoff is $(\overline{B}(T_1,T) - K)^+$, and the defaultable bond price depends on both interest rates and credit spreads. The nonlinearity of the function $(\,\cdot\,)^+$ does not allow to separate the payoff function into two factors.

The simplifications also carry through to the discrete-time tree model. Although the pricing of some credit derivatives will not necessarily decouple, the prices of the defaultable bonds decouple and therefore the *fitting* of the interest-rate and the credit spread trees can be done separately. The implementation goes in the following steps:

1. If the pricing of the credit derivative does not decouple, build the tree for the default-free interest rate.
2. Fit the interest-rate tree to the initial default-free bond prices

$$B(0,T)$$

3. Build the tree for the short credit spread $q\lambda$.
4. Fit the credit spread tree to the initial term structure of credit spreads, i.e. to

$$\widetilde{P}(0,T) \qquad \forall\, T > 0.$$

5. Add the branches to default.
6. If the payoff function of the credit derivative decouples, price it directly using only the tree for the credit spreads. Use the default-free bond prices $B(0,T)$ for discounting.
7. Otherwise combine both trees and price using the combined tree.

4.2. **Building the Tree: The Hull-White Algorithm.** The tree building and the tree-fitting algorithm is based upon the Hull-White (1994a; 1994b; 1996) algorithm for default-free interest rate modelling. As these algorithms are already well-known we restrict ourselves to a concise summary, which is already extended to incorporate time dependency in the volatility parameter.

All direct references to interest-rates were avoided and the algorithm is presented for a process x (which can be thought of as the short rate process) and fitted to a term structure $C(0,T)$ (which can be seen as bond prices). This was done to point out the general nature of the algorithm which we will use alternatively regarding x either as short-term interest rate $x = r$, or as default intensity $x = \lambda$, or as short term credit spread in the fractional recovery model $x = \lambda q$. Furthermore, a common modification of this algorithm is to define the short rate as a *function* of the process x

$$r = f(x), \tag{33}$$

so that now the direct interpretation of x as interest-rate is lost, too. This trick can be used to ensure positive interest rates (if $f(x) > 0\ \forall x$, e.g. $f(x) = e^x$).

The Hull-White algorithm is an algorithm for the discrete-time implementation of diffusion models of the form:

$$dx = [k(t) - ax]dt + \sigma dW. \tag{34}$$

The aim is to find a discrete-time version of the model that has the following properties: It has a recombining trinomial tree structure, it converges to the continuous-time model (34), and it replicates a given initial term structure of expectations of the bond-price type:

$$C(0,T) = \mathbf{E}\left[\, e^{-\int_0^T x(t)dt} \,\right]. \tag{35}$$

This is achieved in two steps: First, a discrete-time tree is built for the modified process x^* with the dynamics

$$dx^* = -ax^*dt + \sigma dW. \tag{36}$$

Because $x^*(t) + \int_0^t k(s)ds = x(t)$ we can reach a tree for x in a second step by shifting the tree for x^* by a time-dependent offset $\alpha(t)$. A suitable choice of $\alpha(t)$ will enable us to fit it to the initial term structure of interest rates.

Step 1: Building the Tree. First, a time step size Δt has to be chosen. This determines the size of the step in x

$$\Delta x = \widehat{\sigma}\sqrt{3}\sqrt{\Delta t}, \tag{37}$$

where $\widehat{\sigma} = \max_t \sigma(t)$ is the largest σ that we will encounter.

To describe the nodes of the tree we will use the following notation: Node (n, j) denotes the node at time $t = n\Delta t$ and $x = j\Delta r$. The time index n ranges from zero through the positive integers, while the ‘space’ index j can take both positive and negative values[10]. The discretised (grid) version of $x(t)$ will be denoted with x^n where the time-index n indicates that this is the discretisation of the process. The value of x^n at node j will be denoted with x_j^n, and similar notation applies to x^*.

To achieve consistency with the continuous-time dynamics (36) we require at all nodes (n, j) that the first two moments of the discrete and the continuous process coincide (possibly up to terms of order Δt^2 and larger[11]), and that the branching probabilities add up to one:

$$\mathbf{E}\left[\, x^{*n+1} - x^{*n} \,\right] = p_u\Delta x_u + p_m\Delta x_m + p_d\Delta x_d = -ax_j^{*n}\Delta t \tag{38}$$

$$\mathbf{E}\left[\, (x^{*n+1} - x^{*n})^2 \,\right] = p_u\Delta x_u^2 + p_m\Delta x_m^2 + p_d\Delta x_d^2 = \sigma^2\Delta t + a^2(x_j^{*n})^2\Delta t^2 \tag{39}$$

$$p_u + p_m + p_d = 1, \tag{40}$$

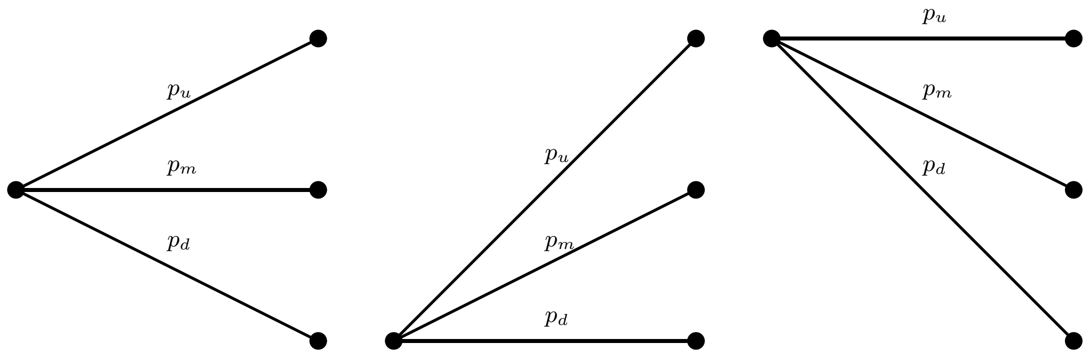
where $\Delta x_u, \Delta x_m$ and Δx_d are the changes in x^{*n} depending on whether the next move in the trinomial tree takes x^{*n} to the upper, the middle or the lower branch. Given the structure of the tree these three equations uniquely determine the branching probabilities at each node.

There are three possible trinomial branches in the tree (see figure 4.2): The typical case is the up-across-down branch (a) with $\Delta x_u = +\Delta x$, $\Delta x_m = 0$ and $\Delta x_d = -\Delta x$. This branch is used at nodes in the interior of the tree.

The dynamics (36) of x^* incorporate a mean reversion to zero, where the strength of the mean reversion is proportional to the value of x^*. Therefore for large $x \geq j_{\max}\Delta x$, the mean $-ax_j^{*n}\Delta t$ will be smaller than the lower branch $-\Delta x$ and equation (38) cannot be satisfied without having negative probabilities. The opposite will happen at a very low branch, such that there are lower and upper limits $j_{\min}$ and $j_{\max}$ at which we have to use the branching methods (b) and (c) respectively. Thus, for each time level n, we will use the following branching methods:

[10] For two- or three-dimensional variables the time-index n is written as superscript, and the space-indices (j for interest-rates and i for spreads or intensities) are written as subscripts. If the variable depends on time alone, the index n is written as subscript.

[11] The convergence to the continuous-time process will still be ensured if terms of order Δt^2 are ignored.



(a) standard branching (b) ‘up’ branching (c) ‘down’ branching

FIGURE 3. The three branching types of the Hull-White trinomial tree. (a) is the standard branching method at inner nodes of the tree, (b) is used at the lower edge of the tree, and (c) is used at the upper edge of the tree.

(c) at the top node $j_{\max}$
(a) at intermediate nodes
(b) at the bottom node $j_{\min}$

For constant σ we can choose the boundaries of the tree as

$$j_{\max} \geq \frac{0.184}{a\Delta t} \quad \text{and} \quad j_{\min} = -j_{\max}. \tag{41}$$

The branching probabilities are given by the solution of equations (38) to (40) which are for constant σ

at node (a)

$$p_u = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2}$$
$$p_m = \frac{2}{3} - a^2 j^2 \Delta t^2$$
$$p_d = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2}$$

at node (b)

$$p_u = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2}$$
$$p_m = -\frac{1}{3} - a^2 j^2 \Delta t^2 - 2aj\Delta t$$
$$p_d = \frac{7}{6} + \frac{a^2 j^2 \Delta t^2 + 3aj\Delta t}{2}$$

at node (c)

$$\begin{aligned} p_u &= \frac{7}{6} + \frac{a^2 j^2 \Delta t^2 - 3aj\Delta t}{2} \\ p_m &= -\frac{1}{3} - a^2 j^2 \Delta t^2 + 2aj\Delta t \\ p_d &= \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2}. \end{aligned}$$

Here it was used that $\Delta x^2 = 3\sigma^2 \Delta t$ which is only true for constant σ. If $\sigma(t)$ is time-dependent, equations (38) to (40) have to be solved with σ as parameter:

at node (a)

$$\begin{aligned} p_u &= \frac{1}{2}\left[\sigma^2 \frac{\Delta t}{\Delta x^2} + a^2 j^2 \Delta t^2 - aj\Delta t\right] \\ p_m &= 1 - p_u - p_d \\ p_d &= \frac{1}{2}\left[\sigma^2 \frac{\Delta t}{\Delta x^2} + a^2 j^2 \Delta t^2 + aj\Delta t\right] \end{aligned}$$

at node (b)

$$\begin{aligned} p_u &= \frac{1}{2}\left[\sigma^2 \frac{\Delta t}{\Delta x^2} + a^2 j^2 \Delta t^2 + aj\Delta t\right] \\ p_m &= -\sigma^2 \frac{\Delta t}{\Delta x^2} - a^2 j^2 \Delta t^2 - 2aj\Delta t \\ p_d &= 1 - p_u - p_d \end{aligned}$$

at node (c)

$$\begin{aligned} p_u &= 1 - p_m - p_d \\ p_m &= -\sigma^2 \frac{\Delta t}{\Delta x^2} - a^2 j^2 \Delta t^2 + 2aj\Delta t \\ p_d &= \frac{1}{2}\left[\sigma^2 \frac{\Delta t}{\Delta x^2} + a^2 j^2 \Delta t^2 - aj\Delta t\right]. \end{aligned}$$

Furthermore, for time-dependent σ it has to be decided at each time step where the limits of the tree are, i.e. at which level $\pm j_{\max}$ the branching of type (b) and (c) becomes necessary. If $\sigma(t)$ is strongly decreasing it can happen that branching of type (b) and (c) will not only be necessary on the outermost level of the tree ($\pm j_{\max}$), but also one or more levels further to the middle of the tree (on levels $\pm(j_{\max} - 1), \pm(j_{\max} - 2), \ldots$). This can be decided for each node by checking whether a branching of type (a) would lead to negative transition probabilities in one of the nodes.

A small example trinomial tree is shown in figure 4.2 on the left. At the time level $t + 2\Delta t$ the special branching is shown for the top and bottom nodes of the tree. In a typical application this branching back would happen at a later time level. If this tree is to be used for default risk

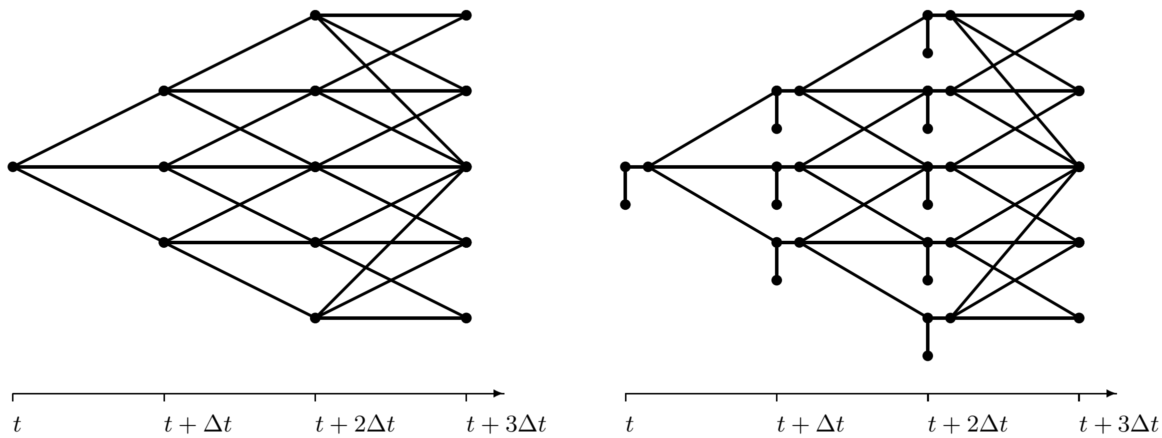


FIGURE 4. The Hull-White trinomial tree with and without the additional branches to default. At time level $t + 2\Delta t$ there is special branching at the top and bottom nodes.

modelling, it has to be extended for branches to default as explained in the previous section, resulting in the tree on the right in figure 4.2.

It is not necessary to save all transition probabilities in one large array. There are some properties that reduce memory requirements:
For every n, the transition probabilities are symmetric around $j = 0$, i.e.

$$p_{u}{}_{j}^{n} = p_{d}{}_{-j}^{n}, \qquad p_{m}{}_{j}^{n} = p_{m}{}_{-j}^{n}.$$

For constant σ the transition probabilities do not depend on n:

$$p_{u}{}_{j}^{n} = p_{u}{}_{j}, \qquad p_{d}{}_{j}^{n} = p_{d}{}_{j}, \qquad p_{m}{}_{j}^{n} = p_{m}{}_{j}.$$

The transition probabilities for time-dependent $\sigma(t)$ are easily calculated from the transition probabilities for constant σ because they only have to be adjusted for a time-dependent difference. For example, let the 'up' probability for time-dependent $\sigma(t)$ be $\tilde{p}_{u}{}_{j}^{n}$ and the 'up' probability for constant σ be $p_{u}{}_{j}$. Then

$$\tilde{p}_{u}{}_{j}^{n} = p_{u}{}_{j} - \frac{1}{6} + (\sigma_n)^2 \frac{1}{2} \frac{\Delta t}{\Delta x^2}.$$

Similar formulae apply to the 'middle' and 'down' probabilities. Thus, only four one-dimensional arrays are needed: The constant-σ transition probabilities (p_u, p_m, p_d three arrays in j), and the volatilities σ_n (one array in n). This will require much less memory than one $(j \times n)$ array, and the loss in computing time will be small. Furthermore, the adjustments do not depend on the branching type used.

Step 2: Fitting the Tree. Now that the tree for x^* has been constructed it must be converted into a tree for x via

$$x(t) = \alpha(t) + x^*(t),$$

such that the initial term structure is recovered:

$$C(0,T) = \mathbf{E}\left[e^{-\int_0^T x(s)ds} \right].$$

The tree for x then has the same transition probabilities and links between the nodes as the tree for x^*, but the values of x_j^n at time level $t = n\Delta t$ have been shifted from x_j^{*n} by $\alpha_n := \alpha(n\Delta t)$:

$$x_j^n = x_j^{*n} + \alpha_n \qquad \forall j_{\min} \le j \le j_{\max} \ \ \forall n \ge 0.$$

Note that for a given time level n, all nodes are shifted by the *same* amount α_n. To shorten notation, we denote with $C_n := C(0, n\Delta t)$ the price of the zero coupon bond maturing at $n\Delta t$.

In the continuous-time model a closed-form solution exists for this problem in terms of the forward rates $f(0,T) := -\frac{\partial}{\partial T} \ln C(0,T)$

$$\alpha(t) = f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2,$$

but in the discrete tree model the solution for the continuous-time model will not exactly reproduce the initial term structure. Furthermore, if a function of x is used as short rate to ensure positive interest rates as suggested in equation (33), a closed-form solution for $\alpha(t)$ may not exist.

Define π_j^n to be the *state price* of node (n,j), i.e.

(42) $$\pi_j^n := \mathbf{E}\left[\mathbf{1}_{\{x^n = x_j^n\}} \prod_{m=0}^{n-1} e^{-x^m \Delta t} \right].$$

π_j^n equals the probability that the discretised process x^m on the tree hits the node (n,j), discounted with the intermediate values of x^m.
If x is a short term interest rate r, then π_j^n is the value of a payoff of 1 at node (n,j), and zero otherwise.
If x is a default intensity λ, then π_j^n is the probability of reaching node (n,j) without having defaulted before.
If x is a short credit spread λq in the fractional recovery model, then π_j^n is the expected payoff of a claim of 1 that is only paid out iff the node (n,j) is reached.

The tree is now fitted in a procedure which is known as *forward induction.* Starting from the initial node $n = 0$, it is show how to fit the next time-level $n \to n+1$ to the given price C_{n+1}.

Initialisation $n = 0$*:*
For $n = 0$ the state price and the offset α_0 follow immediately

(43) $$\pi_0^0 = 1 \qquad\qquad \alpha_0 = -\frac{1}{\Delta t} \ln C_1.$$

Iteration: $n \to n+1$*:*
Assume the tree has been fitted up to level n, i.e. we know α_m and π_j^m for all $m \le n$ and all j. Then the new state prices for level $n+1$ are:

(44) $$\pi_j^{n+1} = \sum_{k \in \text{Pre}(n+1,j)} p_{kj}^n \pi_k^n e^{-r_k^n \Delta t}.$$

The sum is over all nodes (n,k) which are predecessors of $(n+1,j)$, and p_{kj}^n is the transition probability of going from (n,k) to $(n+1,j)$. Equation (44) can be derived by rewriting the definition of π_j^n as a sum over all possible paths that x can take to (n,j), weighted with the probability of that path and discounting with x along this path.

Equation (44) can also be implemented by writing a loop over the nodes (n,j) at time level n. Each of these nodes (n,j) contributes to the state prices of its three successor nodes $e^{-x\Delta t}\pi_j^n$ times the respective branching probability p_u, p_m or p_d. This loop may be more efficient as it will not be necessary to keep track of predecessor nodes.

The new α_{n+1} is given by:

$$C_{n+2} = \sum_j \pi_j^{n+1} e^{-x_j^{n+1}\Delta t} = \sum_j \pi_j^{n+1} e^{-x_j^{*n+1}\Delta t} e^{-\alpha_{n+1}\Delta t}. \tag{45}$$

thus

$$\alpha_{n+1} = \frac{\ln\Big(\sum_j \pi_j^{n+1} e^{-x_j^{*n+1}\Delta t}\Big) - \ln C_{n+2}}{\Delta t}. \tag{46}$$

If the short rate / intensity / spread is a function of the parameter that is modelled (see e.g. (33)), then it will be necessary to fit the tree by numerically finding a solution α_{n+1} to equation (45) (or its equivalent).

4.3. **The Tree for Credit Risk.** Now all the tools are in place for the tree model of default risk. We will describe here the implementation of the tree for the fractional recovery model. The adaptation to the equivalent recovery model only requires one additional pre-processing step to reach zero-recovery defaultable bond prices. The implementation steps are:

1. Build a tree with nodes (n,j) for the default-free short rate r, where
$$dr = (k(t) - ar)dt + \sigma dW.$$
2. Fit this tree to the default-free bond prices $B(0,T)$.
3. Build a tree with nodes (n,i) for the default intensity λ or the the short credit spread λq.
$$d\lambda = (\overline{k}(t) - \overline{a}\lambda)dt + \overline{\sigma}d\overline{W}.$$
4. Fit this tree to
$$\widetilde{P}(0,T) = \mathbf{E}\left[\, e^{-\int_0^T q\lambda(s)ds} \,\right] = \frac{\overline{B}(0,T)}{B(0,T)}.$$
5. Incorporate default branches into the credit spread tree.
6. Combine the two trees.
7. Price derivatives.

Remarks: The independence of dW and $d\overline{W}$ (i.e. default-free interest rates and the default intensity) allows us to express $\mathbf{E}\left[\, e^{-\int_0^T q\lambda(s)ds} \,\right]$ in terms of observed market prices and to fit the λ-tree separately.
It makes no difference, if λ or $q\lambda$ are modelled, as one is only a linear multiple of the other and the dynamics of both are Gaussian.

		r-move			Marginal
		down	middle	up	
	up	$p'_u p_d$	$p'_u p_m$	$p'_u p_u$	p'_u
λ-move	middle	$p'_m p_d$	$p'_m p_m$	$p'_m p_u$	p'_m
	down	$p'_d p_d$	$p'_d p_m$	$p'_d p_u$	p'_d
	Marginal	p_u	p_m	p_d	1

Default probability: p

TABLE 1. Combined branching probabilities (independence). The table gives the branching probabilities in the combined tree for the indicated combined movements of r and λ. These must be multiplied with $(1-p)$ to reach the full probabilities of the indicated moves *and* survival over the next time interval. The original probabilities are: r: up p_u; middle p_m; down p_d. λ: up p'_u; middle p'_m; down p'_d. Default p.

Both trees should have the same time step size Δt, but they can have different space steps Δr and $\Delta\lambda$ and different numbers of nodes $j_{\max} - j_{\min}$ and $i_{\max} - i_{\min}$.

To incorporate default branches to the credit spread tree (step 5), the additional branch to default has to be added to each node (n, i) as described in section 3.3. If the short credit spread is $q\lambda_i^n$ in this node, the survival probability is $1 - p = e^{-\lambda_i^n \Delta t}$, and the default probability is $p = 1 - e^{-\lambda_i^n \Delta t}$. The branching probabilities must also be updated with the survival probability.

The key step in the full implementation is the combination of the two trees (step 6). The combined tree is a tree in *three dimensions:* two space dimensions (r and $q\lambda$) and the time dimension. Nodes (n, i, j) carry therefore three indices: n for the time $t = n\Delta t$, i for the credit spread $q\lambda = \alpha_n^\lambda + i\Delta\lambda$, and j for the default-free short rate $r = \alpha_n^r + j\Delta r$.

At time-level n, the tree has $(i_{\max} - i_{\min}) \times (j_{\max} - j_{\min})$ survival nodes and the same number of 'default' nodes. From node (n, i, j) there are 10 different branches: Both rates r and intensities λ have three possible branches which gives nine possible combinations, and there is a tenth branch to default.

As shown in table 1, the branching probabilities simply multiply: If in node (n, j) of the interest rate tree the probability for an 'up' move in r was p_u, and in node (n, i) of the tree for $q\lambda$ the probability for a 'down' move in $q\lambda$ was p'_d and the survival probability was $(1-p)$, then in the combined tree the probability of a move from node (n, i, j) to $(n+1, i+1, j-1)$ (i.e. 'up' in r *and* 'down' in $q\lambda$) is $p_u p'_d$. The default probability p remains unchanged, therefore the probability of this move *and survival* is $(1-p)p_u p'_d$. The probabilities p'_u, p'_m, p'_d for the λ-movements in table 1 are the original branching probabilities from the tree for λ, before it was extended for defaults.

		r-move			Marginal
		down	middle	up	
λ-move	up	$p'_u p_d - \epsilon$	$p'_u p_m - 4\epsilon$	$p'_u p_u + 5\epsilon$	p'_u
	middle	$p'_m p_d - 4\epsilon$	$p'_m p_m + 8\epsilon$	$p'_m p_u - 4\epsilon$	p'_m
	down	$p'_d p_d + 5\epsilon$	$p'_d p_m - 4\epsilon$	$p'_d p_u - \epsilon$	p'_d
	Marginal	p_u	p_m	p_d	

TABLE 2. Combined branching probabilities (positive correlation). The table gives the probabilities of the indicated combined movements of r and λ in the combined tree for a given positive correlation $\rho = 36\epsilon$. To reach the probabilities for the movements with survival over the next time interval multiply them with $(1-p)$. The original probabilities are: r: up p_u; middle p_m; down p_d. λ: up p'_u; middle p'_m; down p'_d. Default p.

The combined tree is now fully described: It inherits and combines the branching possibilities and the branching probabilities from the two original trees, and it is fully fitted to both the default-free term structure of bond prices and the defaultable term structure of bond prices.

5. IMPLEMENTATION: CORRELATION

If there is correlation $\rho \neq 0$ between dW and $d\overline{W}$ in the dynamics of interest rates and default intensities, the defaultable bond prices do not decouple any more as easily as in equation (31), which makes fitting the tree to a. Therefore the strategy of the preceding section has to be modified, the fitting of the defaultable term structure must be postponed. The new strategy is:

1. Build a tree for the default-free short rate r, and Fit this tree to the default-free bond prices $B(0,T)$.
2. Build a tree for the short credit spread λq. Do *not* fit the tree yet.
3. Combine the two trees and incorporate the correlation.
4. Incorporate default branches into the tree.
5. Fit the combined tree to the defaultable bond prices $\overline{B}(0,T)$, while preserving the fit to the default-free bond prices.
6. Price derivatives.

The algorithm was modified in points 3 and 5.

5.1. **Combining the trees.**

The problem of introducing correlation into a two-dimensional tree model has been treated in a similar context by Hull and White (1994b). For positive correlation $\rho > 0$ they propose to modify the transition probabilities of table 1 as shown in tables 2 and 3.

		r-move			Marginal
		down	middle	up	
λ-move	up	$p'_u p_d + 5\epsilon$	$p'_u p_m - 4\epsilon$	$p'_u p_u - \epsilon$	p'_u
	middle	$p'_m p_d - 4\epsilon$	$p'_m p_m + 8\epsilon$	$p'_m p_u - 4\epsilon$	p'_m
	down	$p'_d p_d - \epsilon$	$p'_d p_m - 4\epsilon$	$p'_d p_u + 5\epsilon$	p'_d
	Marginal	p_u	p_m	p_d	

TABLE 3. Combined branching probabilities (negative correlation). The table gives the probabilities of the indicated combined movements of r and λ in the combined tree for a given negative correlation $\rho = -36\epsilon$. To reach the probabilities for the movements with survival over the next time interval multiply them with $(1-p)$. The original probabilities are: r: up p_u; middle p_m; down p_d. λ: up p'_u; middle p'_m; down p'_d. Default p.

First, an auxiliary parameter ϵ is defined

$$\epsilon = \begin{cases} \frac{1}{36}\rho & \text{for } \rho > 0 \\ -\frac{1}{36}\rho & \text{for } \rho < 0. \end{cases}$$

Tables 2 and 2 give the probabilities of the indicated combined movements of r and λ in the combined tree for a given positive (table 2) or negative (table 3) correlation $\rho = \pm 36\epsilon$. Default and survival are ignored in these tables, to reach the probabilities for the movements *and survival over the next time interval*, the probabilities must be multiplied with $(1-p)$. The original probabilities are: r: up p_u; middle p_m; down p_d. λ: up p'_u; middle p'_m; down p'_d. Default p.

The adjustment for correlation in tables 2 and 3 only work if ϵ is not too large. Thus there is a maximum value for the correlation that can be implemented for a given time step size Δt. As the refinement is increased ($\Delta t \to 0$) this restriction becomes weaker and the maximum correlation approaches one.

5.2. **Fitting to the defaultable bond prices.** As in section 4 the idea behind the fitting algorithm is to shift the tree by a deterministic amount $\overline{\alpha}_n$. If the shift only takes place in the λ-dimension, the development of the default-free interest rate r remains unaffected and the fit to the default-free term structure is preserved.

We define the fitting algorithm recursively over the time-step n. Inputs are: a combined tree (n, i, j) (indices: n time, i intensity, j interest rate) which is fitted to a term structure of default-free bond prices $B(0,T)$ by a shift α_n in the r-dimension. Define the *defaultable state price* π^n_{ij} to be the state price of node (n, i, j), i.e. the value of a defaultable claim on \$ 1 at node (n, i, j). The tree is built for the default intensity λ directly (and not for the short credit spread λq).

Initialisation $n = 0$*:*
Set

$$\pi_{00}^{0} = 1 \qquad \overline{\alpha}_0 = -\frac{1}{q\Delta t}\ln(\overline{B}(1) - B(1)). \tag{47}$$

Iteration: $n \to n+1$*:*
The tree has been fitted up to level n, i.e. we know $\overline{\alpha}_m$ and π_{ij}^{m} for all $m \leq n$ and all i and j. The new state prices for level $n+1$ are:

$$\pi_{ij}^{n+1} = \sum_{(k,l)\in \text{Pre}(n+1,i,j)} p_{(kl)(ij)}^{n}\pi_{kl}^{n} e^{-(q\lambda_k^n + r_l^n)\Delta t}. \tag{48}$$

Again we sum over all state prices of the predecessors of the node (n, i, j). The predecessors' state prices are weighted with the transition probabilities $p_{(kl)(ij)}^{n}$, the discounting with the risk-free interest rate $e^{-r\Delta t}$ and the discounting with the fractional recovery factor $e^{-q\lambda\Delta t}$.

The fractional recovery factor $e^{-q\lambda\Delta t}$ reflects the expectation of a defaultable payoff Δt in the future if the fractional recovery model is used as a continuous-time model (with defaults at any time in $[t, t+\Delta t[$). If one assumes that defaults happen only at the beginning of the interval, then the factor $1 - q(1 - e^{-\lambda\Delta t})$ has to be used. This factor gives the expectation of 1 in survival and $(1-q)$ in default. For normal parameter values both approaches yield almost the same results.

Again it will be simpler to implement equation (48) using a loop over the nodes on time level n and adding up the contributions to the successor nodes at level $n+1$.

The new α_{n+1} is given by:

$$\overline{B}(n+2) = \sum_{ij} \pi_{ij}^{n+1} e^{-(r_j^{n+1} + q\lambda_i^{n+1})\Delta t} = \sum_{ij} \pi_{ij}^{n+1} e^{-(r_j^{n+1} + q\lambda_i^{*n+1})\Delta t} e^{-\overline{\alpha}_{n+1}\Delta t}. \tag{49}$$

thus

$$\overline{\alpha}_{n+1} = \frac{1}{\Delta t}\ln\Big(\frac{\sum_{ij}\pi_{ij}^{n+1} e^{-(r_j^{n+1} + q\lambda_i^{*n+1})\Delta t}}{\overline{B}(n+2)}\Big). \tag{50}$$

Again, if a function of the short intensity is modelled (see (33)) a numerical solution of (49) becomes necessary.

6. Using the Tree

Once the tree is constructed and fitted to the initial bond prices it can be used to price other derivative securities. A derivative security is characterised by its payoff in default, in survival and by American / Bermudan early exercise features:

- f_{ij}^{n} The payoff of the derivative if a default happens in node (n, i, j).
- F_{ij}^{n} The payoff of the derivative if node (n, i, j) is reached.
- G_{ij}^{n} The early exercise payoff in node (n, i, j).

Fees and other payments are specified as negative payoffs. Very frequently these payoffs will be in terms of underlying securities whose prices cannot be derived directly from the values of the state variables r and λ in the node. In this case the underlying securities must be priced in the tree first, and only then they can be substituted as payoffs to the derivative. This can be done in the same backwards induction as the pricing of the derivative, one only has to keep track of the prices of both securities.

Sometimes it may be inefficient to value these payoffs in the full tree model: One might end up with building a ten-year tree for an option that expires in one year, just because the underlying bond has a maturity of ten years. Here the computational effort can be reduced by increasing the time step size from year one onwards. Furthermore, if the prices of the underlying security are the values of defaultable or default-free fixed payoffs (e.g. the underlying is a coupon bond) and if the model uses the original specification (15) and (16), we can use the closed-form solutions given in equations (20) and (23) to reach the prices of these bonds directly.

Having specified all payoffs the price V_{ij}^n of the credit derivative is derived by standard backwards induction:

Initialisation: $n = N$
At the final level of the tree set its value to the final payoff

$$V_{ij}^N := F_{ij}^N.$$

Iteration: $n + 1 \to n$
For every node (n, i, j) the value of the credit derivative at the survival node of the default branch is given by

$$V''^n_{ij} = \sum_{k,l \in \text{Succ}(n,i,j)} p_{kl}^n e^{-r_j^n \Delta t} V_{kl}^{n+1}, \tag{51}$$

where Succ(n, i, j) gives the successor nodes of (n, i, j) (except the default node) and p_{kl}^n is the transition probability from node (n, i, j) to node $(n + 1, k, l)$. If there is no early exercise, the value at node (n, i, j) is then

$$V'^n_{ij} = e^{-\lambda_i^n \Delta t} V''^n_{ij} + (1 - e^{-\lambda_i^n \Delta t}) f_{ij}^n + F_{ij}^n. \tag{52}$$

With early exercise the value is

$$V_{ij}^n = \max(V'^n_{ij}\,,\, G_{ij}^n), \tag{53}$$

where we assumed that the early exercise right is with us (i.e. the person that receives any positive payoffs) and that we can exercise *before* we receive or pay F_{ij}^n. (For early exercise rights of the counterparty we would have to use a minimum-function.)

To exemplify the usage of the tree we will show which specifications have to be used for some popular credit derivatives. We will call counterparty **A** the protection buyer, and counterparty **B** the protection seller, and we will take the point of view of counterparty **A**.

6.1. **Default Digital Swap.** In a default digital swap, counterparty **B** pays \$ 1 to counterparty **A** if a default happens and at the time of default. Counterparty **A** pays a periodic fee of s per annum for this protection.

This is one of the most basic credit derivatives one can imagine. It would not be necessary to price this security in a tree as closed-form solutions can be derived within this model setup easily (see e.g. Schönbucher (submission fall 1999)).

The payoffs in the tree model are

- The payoff of the derivative if a default happens in node (n, i, j): $f_{ij}^n = 1$.
- The payoff of the derivative if node (n, i, j) is reached: $F_{ij}^n = -s$ if $n\Delta t$ is a fee payment date[12] , $F_{ij}^n = 0$ otherwise.
- The early exercise payoff in node (n, i, j): $G_{ij}^n = -\infty$: early exercise does not apply.

6.2. **Default Swap.** In the default swap, counterparty **B** pays [par]-[recovery of a reference bond $\overline{B}^*$] to counterparty **A** if a default happens, payment is at the time of default. Again Counterparty **A** pays a periodic fee of s per annum for this protection.

Next to the total return swap, the default swap is one of the most common credit derivatives. Often its pricing can be reduced to the pricing of a default digital swap, but we will use the tree model. Because the payoff is conditioned on a defaultable reference bond[13] $\overline{B}^*$, we need the value of this reference bond in every node of the tree, which can be done for fixed-coupon bonds using the closed-form solutions in equations (20) and (23). The payoffs are then

- The payoff of the derivative if a default happens in node (n, i, j): $f_{ij}^n = 1 - (1 - q)\overline{B}^{*n}_{ij}$ for fractional recovery $f_{ij}^n = 1 - c\overline{B}^{*n}_{ij}$ for equivalent recovery.
- The payoff of the derivative if node (n, i, j) is reached: $F_{ij}^n = -s$ if $n\Delta t$ is a fee payment date, $F_{ij}^n = 0$ otherwise.
- The early exercise payoff in node (n, i, j): $G_{ij}^n = -\infty$: early exercise does not apply.

6.3. **Callable Default Swap.** A callable default swap is a default swap where counterparty **A** has the right to cancel the default swap at pre-determined dates. Usually this is combined with an increasing fee schedule which will make the security a *callable step-up default swap*. The motivation is often that for regulatory capital reasons counterparty **A** needs a default swap whose maturity matches that of the underlying reference asset, although economically she only wants protection for a shorter period. With a sufficiently steep step-up schedule counterparty **B** can be almost certain that counterparty **A** will exercise early but the regulatory requirements are satisfied.

This very simple variation on the classical default swap is already impossible to price in closed-form with pencil and paper, and can be priced with Monte-Carlo methods only at a prohibitive cost in computation time.

[12] Here some care has to be taken when payment dates do not fall on the time-grid.
[13] Note that the reference bond is not a zero-coupon bond.

To model the callability we must specify the early exercise payoff in the nodes where a cancellation is possible:
$G^n_{ij} = 0$: early exercise if the value of the default swap is negative.

Because of the increasing fee structure early exercise will become optimal after some time.

6.4. **Credit Spread Options.** In a credit spread put option, counterparty **A** has the right to sell the defaultable reference bond $\overline{B}^*$ at a given time T in the future at a given strike credit spread k over a default-free reference bond B^* to counterparty **B**.

This credit derivative has two underlying securities: $\overline{B}^*$ and B^*. To explicitly calculate the payoffs at $T = N\Delta t$ we must calculate for each possible interest rate $r = j\Delta r$ at time T the corresponding default-free reference bond price B^* and the strike price of the option: the price K^N_j that is equivalent to the price of the defaultable reference bond $\overline{B}^*$ at a credit spread of k over B^*.

If the option is knocked out at default, we specify zero payoffs $f^n_{ij} = 0$ at the default nodes, and the option payoff at the final nodes:

$$F^N_{ij} = \max(K^N_j - \overline{B}^{*N}_{ij}, 0).$$

If the option survives defaults, we have to add the payoff that the option will have in default. If a default has happened we can be sure that the option will be exercised. Counterparty **A** will get K^N_j for sure in T and has to deliver a defaultable (and defaulted) bond $\overline{B}^*$ for that. Call K^n_{ij} the node-(n, i, j)-value of receiving K^N_j for sure at time T (this has to be valued recursively, too). The payoff in default is then
$f^n_{ij} = K^n_{ij} - (1 - q)\overline{B}^{*n}_{ij}$ for fractional recovery, and
$f^n_{ij} = K^n_{ij} - c\overline{B}^{*n}_{ij}$ for equivalent recovery.

6.5. **Asset Swaptions.** An *asset swap package* is a combination of a (defaultable) fixed coupon bond $\overline{B}^*$ with coupon x, and a fixed-for-floating interest-rate swap where the fixed side pays x and the floating side pays $R + s$ LIBOR R plus a spread s (the *asset swap spread*). The spread s is chosen such that the whole package is valued at 1 (par). This instrument allows the investor to change the cash-flow of the defaultable fixed coupon bond into a floating coupon plus the asset swap spread. This only works as long as there is no default because the swap is a plain interest-rate swap which remains in place even if a default happens on the underlying bond $\overline{B}^*$.

An *asset swaption* is an option on an asset swap package. It gives counterparty **A** the right to enter an asset swap package at time T at spread $\hat{s}$ (call option), or the right to put the asset swap package to counterparty **B** at time T for $\hat{s}$ over LIBOR (put option).

To price the asset swaption we first have to find the fair asset swap spread at the nodes in our tree. Basic calculations yield that

$$s = \frac{1}{A}(B^* - \overline{B}^*) \tag{54}$$

where B^* is the value of a default-free bond with the same coupon x, principal and maturity as the defaultable bond $\overline{B}^*$; and A is the value of an annuity, $A = \sum_i B(0, T_i)$ is the (default-free) value of receiving 1 at every coupon date T_i.

The value of the right to enter an asset swap package at $\hat{s}$ if the fair asset swap rate is s is then

$$F_{ij}^N = A \max(\hat{s} - s, 0) = \max(A\hat{s} - B^* + \overline{B}^*, 0) = \max(\overline{B}^* - (B^* - A\hat{s}), 0), \tag{55}$$

where all quantities are evaluated at the node (N, i, j). The call asset swaption is thus equivalent to an option to exchange the defaultable bond for an equivalent default-free bond whose coupon is reduced by the asset swap spread. All the quantities in the payoff function (defaultable coupon bond price, default-free coupon bond price, value of default-free annuity) are given in closed-form in the model.

Next we need to consider the payoff in default. A call asset swaption will be worthless if the underlying asset has defaulted, but the put asset swaption will be exercised for sure. Thus we know at the time of default that counterparty **A** will receive at maturity $\overline{B}^* - (B^* - A\hat{s})$, we have to deliver a defaultable (and defaulted) bond and receive a default-free bond with adjusted coupon. This payoff can be reached by investing in the respective bonds[14], thus its value at time $t = n\Delta t$ is
$f_{ij}^n = (1-q)\overline{B}^{*n}_{ij} - (B^{*n}_{ij} - A_{ij}^n \hat{s})$ for fractional recovery, and
$f_{ij}^n = c\overline{B}^{*n}_{ij} - (B^{*n}_{ij} - A_{ij}^n \hat{s})$ for equivalent recovery.

6.6. **First-to-Default Baskets.** The tree model can be extended to a model for several defaultable issuers by sequentially building a credit spread tree for each issuer, and then combining the trees similarly to the procedure demonstrated in section 5. This brute-force approach would lead to a very high-dimensional tree and an exponential increase in computation time and memory requirement.

To reduce this computation requirements we can use the fact that

$$\lambda(t) = \sum_{m=1}^{M} \lambda_m(t) \tag{56}$$

is the intensity of the *first*-jump process N if the individual jumps are driven by Cox processes $N_m(t)$ with intensities $\lambda_m(t)$. In equation (56) the individual intensities λ_m can be correlated, but given the intensities λ_m, the jump processes N_m must be independent inhomogeneous Poisson processes with intensities λ_m.

Thus the problem of the pricing of a first-to-default swap can be reduced to the problem of a default swap with a modified intensity process. Unfortunately, if the default intensities are not independent, the market prices are not given in the form that we need to apply the fitting algorithm of section 5. This is an area of further research. For independent default intensities the combined model (56) can be fitted to

$$\overline{B}(0,T) = B(0,T) \prod_{m=1}^{M} \frac{\overline{B}_m(0,T)}{B(0,T)}, \tag{57}$$

[14]When valuing these bonds we have to ignore any coupon payments before the maturity of the option.

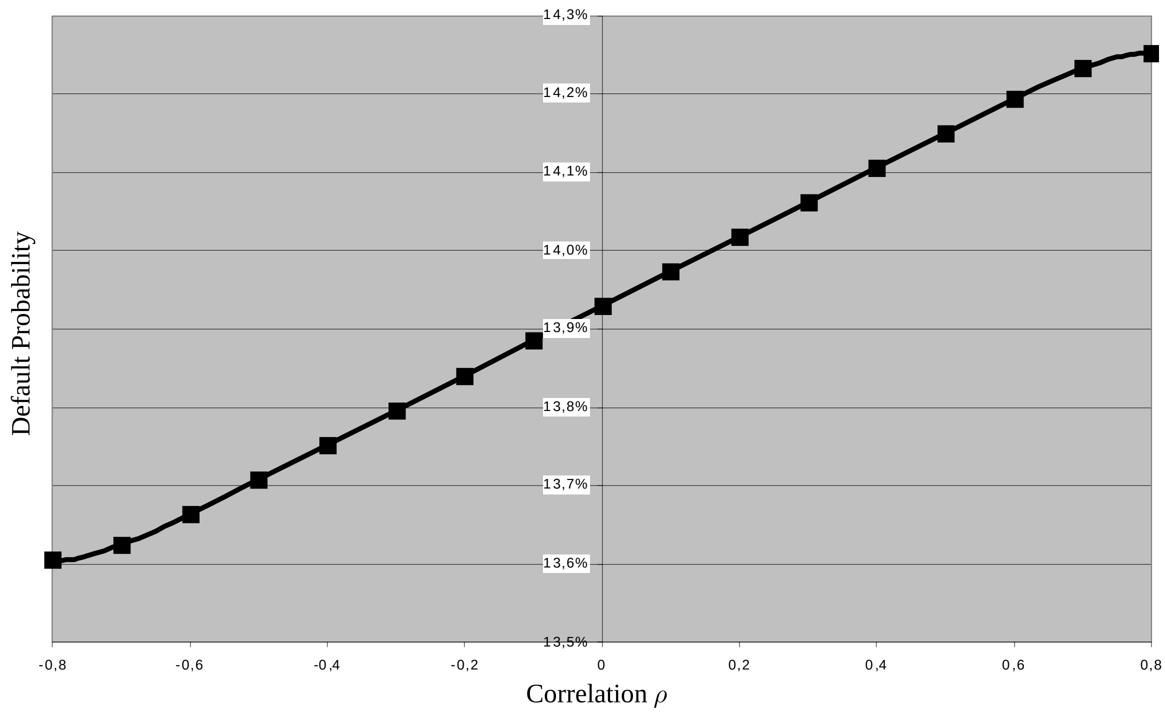


FIGURE 5. Implied default probabilities as a function of the correlation ρ between r and λ.

using the methods of section 4.

7. NUMERICAL RESULTS

7.1. **Numerical Analysis of Parameter Sensitivities.** Unless otherwise stated, the calculations were performed with the following inputs: Default-free continuously compounded zero bond yield curve flat at 6%; short rate volatility $\sigma = 0.02$, short rate mean reversion $a = 0.15$; defaultable continuously compounded zero coupon bond yields flat at 9%; intensity volatility $\overline{\sigma} = 0.01$, intensity mean reversion $\overline{a} = 0.10$; correlation $\rho = 0$; zero recovery; time horizon: $T = 5$ years; 21 time-steps.

Figure 5 shows the 5-year default probability that is implied by the model as a function of the correlation ρ between the dynamics of the intensity and the default-free interest-rates. It can be seen, that the implied default probability increases with increasing correlation. There is an intuitive explanation of the direction of the effect:

If interest rates and credit spreads are positively correlated ($\rho > 0$) this means that defaults are slightly more likely in states of nature when interest rates are high. Because of the higher interest rates these states are discounted more strongly when they enter the price of the defaultable bond, and conversely states with low interest rates enter with less discounting and simultaneously fewer defaults. To reach a *given* price for a defaultable bond, the absolute default

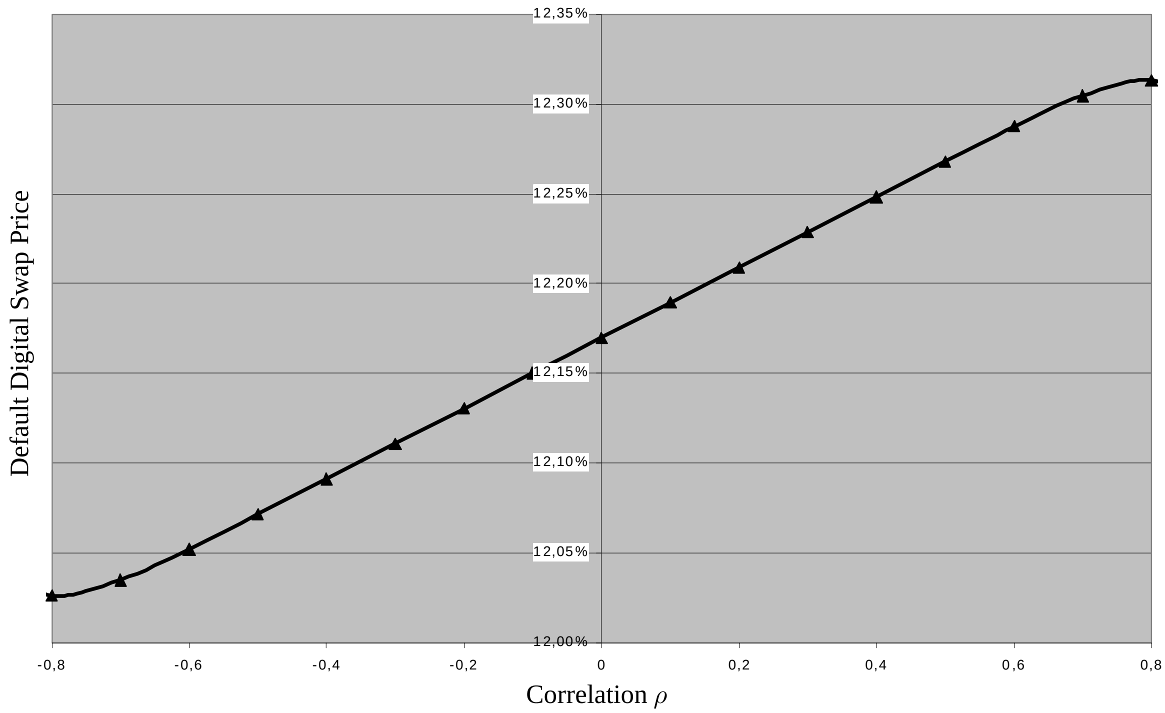


FIGURE 6. Default digital swap prices as a function of the correlation ρ between r and λ.

likelihood must therefore be higher. This implies a lower survival probability which is also the result of the numerical simulation. The argument runs conversely for negative correlation $\rho < 0$.

Figure 6 shows the corresponding default digital swap prices. Here the influence of the correlation parameter is much smaller, because default digital swap prices contain the *discounted* expected payoffs, and the discounting counteracts the effect of the correlation on the implied default probabilities.

To get a feeling for the order of magnitude of the error that is committed when a wrong correlation is specified, we show in figures 7 and 8 the effect of the specification of the (equivalent or fractional) recovery rate on the prices of a default digital swap and a default swap. A higher expected recovery rate means a higher likelihood of default for given defaultable bond prices. This in turn increases the value of the default digital swap.

In figure 8 we show the prices for a default swap with the different recovery rates in both the fractional and the equivalent recovery model. An increase in the expected recovery rate leads to an increase in the implied default probability, but it also leads to a lower payoff of the default swap in default. These two effects cancel out to a large extent which makes the default swap more robust to errors in the expected recovery rate than the default digital swap. Interestingly, for the fractional recovery model the increase in default probability dominates and leads to an increasing function, while for the equivalent recovery model the both effects exactly cancel.

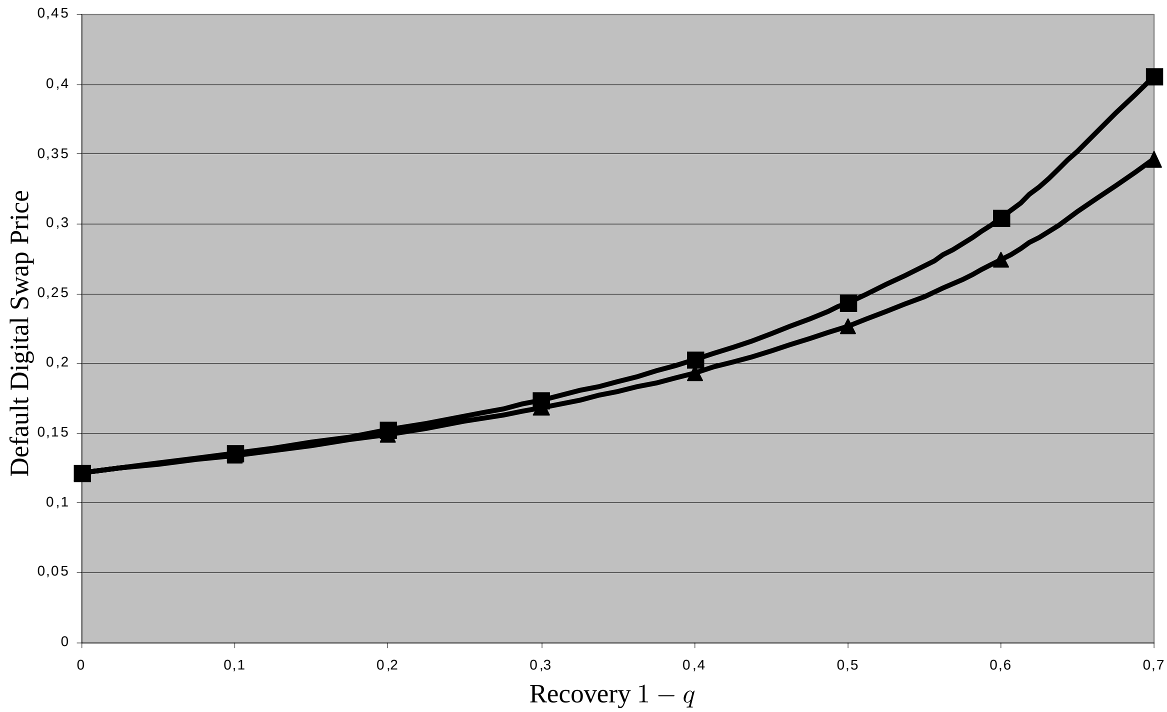


FIGURE 7. Five year default digital swap price (up-front) as a function of the fractional recovery rate $1 - q$ or the equivalent recovery rate c of the defaultable bond. (Prices for equivalent recovery are shown with squares, fractional recovery with triangles.)

The recovery rate is one of the most uncertain input parameters in the model and it can be seen that its influence is much larger than the influence of the correlation. It will therefore be more important to improve the estimate of the recovery rate than the correlation.

8. CONCLUSION

This article offers several conclusions. First, a viable approach was presented to build and fit a combined tree model for defaultable and default-free bonds. We discussed the mathematical theory, on which the model is based, and showed how to apply the model to real-world pricing problems. In the last section the model was used to explore some of the subtler aspects of recovery modelling.

Secondly, the implementation method which was presented in this paper is not restricted to the Hull-White model for interest-rates alone. Along similar lines almost any tree model for default-free interest rates can be adapted to a tree model for the default intensity and thus to a combined tree model for defaultable and default-free bond prices. The only modifications are the addition of default branches to the intensity tree and the combination of both trees.

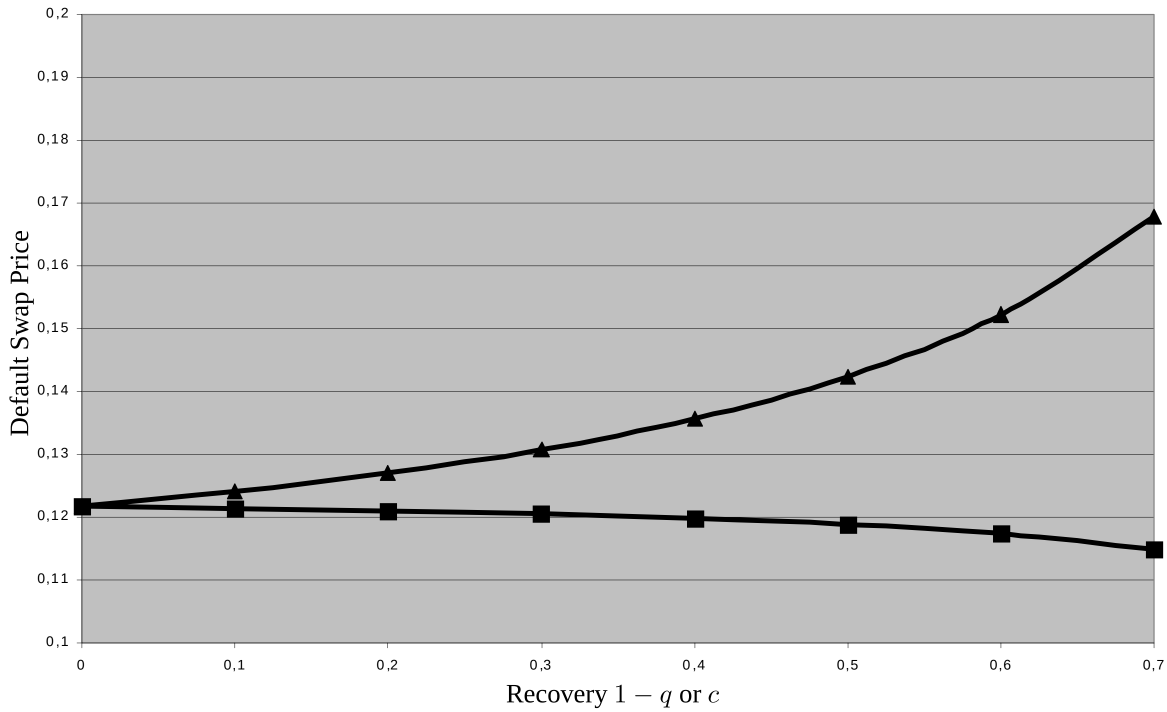


FIGURE 8. Five-year default swap price (up-front) as a function of the fractional recovery rate $1-q$ or the equivalent recovery rate c of the defaultable bond. (Prices for equivalent recovery are shown with squares, fractional recovery with triangles.)

The Hull-White model can also be extended in many directions, notably to ensure positive interest-rates and intensities, or to reach more realistic dynamics for the factors. Many extensions of this kind have been proposed in the literature for default-free interest-rate models and their adaptation to the defaultable case is usually straightforward. Nevertheless these extensions have been designed for problems arising in the default-free interest-rate world (like the fitting to cap and swaption prices) which need not be of first importance in the world of defaultable bonds. Here it may be more important to address the problems of recovery modelling, rating transitions and the dynamics of the credit spreads in a crisis.

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