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by

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Moment characterization of higher-order risk preferences

Sebastian Ebert*

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Abstract Prudence and temperance play key roles in aversion to negative skewness and kurtosis, respectively. This paper puts a new perspective on these relationships and presents a characterization of higher-order risk preferences in terms of statistical moments. An implication is, for example, that prudence implies preference for distributions with higher skewness as defined by *all odd moments.* Moreover, we show that this preference is robust towards variation in kurtosis as defined by *all even moments.* We thus speak of the *kurtosis robustness feature* of prudence. Further, we show that all higher-order risk preferences of odd order imply skewness preference, but for different distributions than prudence. Similar results are presented for temperance and higher-order risk preferences of even order that can be related to kurtosis aversion and have a skewness robustness feature.

Keywords Decision making under risk · Higher-order risk preferences · Kurtosis aversion · Moments · Prudence · Skewness preference · Temperance
JEL D81

1 Introduction

It is well known that risk aversion only partially describes individuals' risk preferences. Numerous behavioral traits stem from higher-order risk preferences such as prudence or temperance. The most prominent one is that prudence is necessary and sufficient for the precautionary savings motive[1] as shown by Kimball (1990). Eeckhoudt and Gollier (2005) analyze the impact of prudence on prevention, i.e. the action un-

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[1] That means the awareness of uncertainty in future payoffs will raise an individual's optimal saving today. The relationship between precautionary savings and the third derivative of the utility function was recognized already by Leland (1968) and Sandmo (1970) and has been of major economic interest ever since.

dertaken to reduce the probability of an adverse effect occuring. These rather general concepts are easily applied to specific problems in various areas of economics and finance. For example, prudence has been shown to be an important factor in preventive care decisions within a medical decision making context (see Courbagé and Rey, 2006). Esö and White (2004) show that there can be precautionary bidding in auctions when the value of the object is uncertain and when bidders are prudent. Likewise, White (2008) analyzes prudence in bargaining. Treich (2010) shows that prudence can decrease rent-seeking efforts in a symmetric contest model. Fagart and Sinclair-Desagné (2007) investigate prudence in a principal-agent model with applications to monitoring and optimal auditing. Temperance generally implies aversion towards mutually aggravating risks (see Kimball, 1992 and 1993). Eeckhoudt and Schlesinger (2008) show that temperance is necessary and sufficient for an increase in downside risk of future labor income always to increase the level of precautionary savings. Within a standard macroeconomic consumption and labor model, Eeckhoudt and Schlesinger (2008) analyze the impact of prudence and temperance on policy decisions such as changes in the interest rate. Other examples for the significant impact of risk preferences of higher order than risk aversion are insurance demand, e.g. Fei and Schlesinger (2008) or life-cycle investment behavior, e.g. Gomes and Michaelides (2005). By necessity this is not a complete list of applications.

These predictions are derived from models based on the expected utility (EU) framework. Under EU, assuming differentiability of a utility function u, risk aversion, prudence and temperance are equivalent to $u'' < 0,\ u''' > 0$ and $u^{(4)} < 0$, respectively. More generally, Ekern (1980) defines a decision maker as being nth-degree risk-averse if and only if $sgn(u^{(n)}) = (-1)^{n+1}$. It is important to note that these specifications are exhibited by "all the commonly used utility functions" (see Brockett and Golden, 1987). They also serve as necessary conditions for numerous stronger preference specifications employed in the economics literature. Prudence, for example, is also widely assumed because it is necessary (but not sufficient) for decreasing absolute risk aversion.

In this paper, we focus on the other reason why higher-order risk preferences are important. This reason is independent of the EU paradigm. Prudence and temperance are linked to preference for high skewness and small kurtosis of distributions, respectively. Menezes et al. (1980) show that an individual dislikes increases in downside risk if and only if she is prudent. A downside risk increase is a mean-variance preserving density transformation shifting variation from the right to the left of the distribution. This is in analogy to the mean-preserving spread of Rothschild and Stiglitz (1970) disliked by a risk-averse individual. The former motivates the definition of prudence as downside risk aversion. Menezes et al. also directly show that prudence, unlike risk aversion, relates to measures of skewness, in particular to the third central moment and semi-target variance. For more on prudence and skewness, see Chiu (2005 and forthcoming). Likewise, Menezes and Wang (2005) show that an individual dislikes increases in outer risks if and only if she is temperate. Edginess (5th-degree risk aversion) has been considered by Lajeri-Chaherli (2004).

However, despite this progress, the relationship between higher-order risk preferences and the statistical moments of a distribution has not been investigated exhaustively. For example, the above mentioned authors presented results on moments up to order n for nth-degree risk aversion only. Statistical moments are interesting because they are among the standard summary statistics which are well understood and applied by a wide audience in various fields of economics and finance. In particular, moments are measures of skewness and kurtosis. Further, numerous risk and performance measures are also based on moments. On the other hand, preference implications based on a *finite* number of moments are generally flawed; see e.g. Brockett and Kahane (1992) and Brockett and Garven (1996). Thus the goal of this paper is to investigate the relationships of higher-order risk preferences to *all* moments.

The approach undertaken makes use of the proper risk apportionment model of Eeckhoudt and Schlesinger (2006). They give another definition of nth-degree risk aversion as a preference over (seemingly) simple lotteries and show equivalence to Ekern's definition. The lottery preferences can be interpreted as the desire to "disaggregate the harms" of unavoidable risks and losses, i.e. to apportion them properly accross different states of nature. These lotteries allow for studying risk attitudes outside the EU framework. Furthermore, one can exploit the simplicity of defining risk preferences via proper risk apportionment for both theoretical and empirical purposes.[2] The remarkable equivalence between the lottery preferences and nth-degree risk aversion motivates the intensive study of their statistical properties.

In this paper, we compute all moments of the proper risk apportionment lotteries of all orders. Thus we actually present a characterization of the lotteries and, implicitly, of higher-order risk preferences. This is because the sequence of moments uniquely determines the distribution of a bounded random variable. This is known as the solution to the Hausdorff moment problem in the probability literature; see Hausdorff (1921).[3] This characterization provides a better understanding of the relationships between higher-order risk preferences, skewness preference and kurtosis aversion. In particular, not only prudence and and temperance, but all higher-order risk preferences of odd and even order, are shown to relate to skewness seeking and kurtosis aversion, respectively. As the measures of skewness and kurtosis used are moments, these results should be accessible to a wide audience. Our results, which are independent of EU, build up on the recent work of Roger (forthcoming), who made an important contribution in achieving this characterization. He computed all moments of the proper risk apportionment lotteries for the special case where the risks that have to be apportioned are *symmetric*. However, we will show that the *asymmetry* of these risks is just the origin of the proper risk apportionment model's statistical generality. We also generalize the early work of Ekern (1980), who considered differences in moments up

[2] For example, the lotteries are used by Gollier (2010) to investigate ecological discounting, by Maier and Rüger (2009) to investigate reference-dependent risk preferences of higher orders and by Jindapon (2009) to define probability premia of higher order. Deck and Schlesinger (forthcoming) and Ebert and Wiesen (2009) employ the lotteries in a laboratory experiment and find strong evidence for third- and fourth-order risk preferences (rather than indifference), which further highlights their importance. Furthermore, the concept of proper risk apportionment can be generalized to the bivariate case, as shown in Eeckhoudt et al. (2007) or Tsetlin and Winkler (2009), which are largely applied in health economics.

[3] The assumption of boundedness is unproblematic from an economic point of view as there is not an infinite amount of money. Thus the assumption is standard in the literature on decision making under risk. A stronger assumption often made is that distributions are defined on a compactum which implies boundedness.

to order n for nth-degree risk aversion.

The paper proceeds as follows. In section 2, we review the proper risk apportionment model of Eeckhoudt and Schlesinger and discuss our notions of skewness and kurtosis and how they relate to *all* odd and even moments, respectively. In particular, we illustrate how skewness and kurtosis manifest in discrete (lottery) distributions. For binary risks we prove that all odd moments (except for the mean) must be of the same sign. This also provides intuition why they are generalized skewness comparable in the sense of Chiu (forthcoming).

In section 3, we explicitly compute all moments of the prudence and temperance lotteries. We show that distributions preferred by a prudent decision maker must have higher skewness *as defined by high odd moments of any order*, but they may or may not have higher kurtosis *as defined by higher even moments of any order*. We refer to this as the *kurtosis robustness feature* of prudence. That is, the preference for skewness of a prudent decision maker must not be disturbed by differences in kurtosis. In particular, prudence does not only determine preference between distributions that purely differ in skewness. To best of our knowledge, this has not been discussed in any paper discussing prudence as being related to skewness preference. Whether the prudent lottery choice has the smaller or larger kurtosis solely depends on the skewness of the risk that has to be apportioned. Therefore, what makes the preference a strong preference is that asymmetric risks consistently have to be apportioned in the same way. This also helps to explain recent experimental evidence of Ebert and Wiesen (2009) who find a significant difference in the number of prudent decisions for adversely skewed zero-mean risks. Likewise, though not as clear-cut, we show that temperance implies a preference for distributions with small kurtosis as defined by small even moments and which is robust towards variation in the odd moments. This is referred to as the *skewness robustness feature of temperance.*

In section 4, we generalize these results and investigate all moments of the proper risk apportionment lotteries of all orders. We show how all higher-order risk preferences of odd and even order (not only prudence and temperance), respectively, are related to skewness preference and kurtosis aversion in a complementary way. This should raise more interest in these concepts which are generally viewed as rather abstract. In both sections 3 and 4, we will discuss how our results relate to those of Roger (forthcoming) and which of his results are particular to the symmetry of the zero-mean risks.

Section 5 concludes. All proofs are given in the appendix.

2 Proper risk apportionment, skewness, kurtosis and moments

We first define the lotteries of Eeckhoudt and Schlesinger (2006) and explain the importance of proper risk apportionment. Let X be Bernoulli distributed with parameter 0.5. Let $k > 0$ such that the amount $-k$ can be interpreted as a sure reduction in wealth. For all $n \in \mathbb{N}$ let ϵ_n be a zero-mean risk (i.e. $E[\epsilon_n] = 0$) with finite moments. The lotteries for monotonicity and risk aversion, respectively, are given by $A_1 = -k,\ B_1 = 0$ and $A_2 = \epsilon_1,\ B_2 = 0$. For the first two so-called higher-order preferences, prudence and temperance, the lotteries are

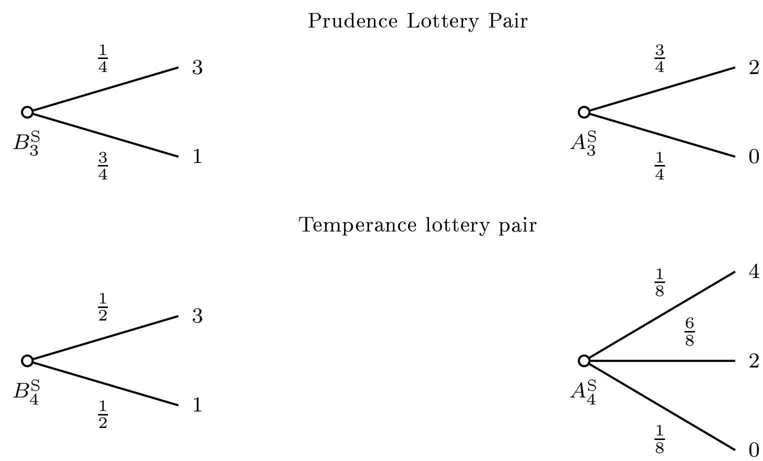
$$A_3 = X \cdot 0 + (1 - X)\,(\epsilon_1 - k) = XB_1 + (1 - X)(A_1 + \epsilon_1)$$
$$B_3 = X(-k) + (1 - X)\epsilon_1 = XA_1 + (1 - X)(B_1 + \epsilon_1)$$

and

$$A_4 = X \cdot 0 + (1 - X)(\epsilon_1 + \epsilon_2) = XB_2 + (1 - X)(A_2 + \epsilon_2)$$
$$B_4 = X\epsilon_1 + (1 - X)\epsilon_2 = XA_2 + (1 - X)(B_2 + \epsilon_2).$$

Figure 1 illustrates examples of these lotteries where outcomes have been aggregated. For higher orders,

Fig. 1 Examples of a prudence and a temperance lottery pair with symmetric (S) zero-mean risks



The prudence lotteries are constructed with initial wealth $x = 2$, fixed loss $-k = -1$ and the zero-mean risk ϵ yields 1 or -1 with equal probability. For the temperance lotteries, initial wealth is $x = 2$ and the zero-mean risks $\tilde{\epsilon_1}$ and $\tilde{\epsilon_2}$ both yield 1 or -1 with equal probability.

proper risk apportionment of order n is defined iteratively by continuing the previously illustrated nesting process, i.e.

$$A_n = XB_{n-2} + (1 - X)\left(A_{n-2} + \epsilon_{\llcorner n/2 \lrcorner}\right)$$
$$B_n = XA_{n-2} + (1 - X)\left(B_{n-2} + \epsilon_{\llcorner n/2 \lrcorner}\right)$$

where $\llcorner n/2 \lrcorner$ is the largest integer smaller than or equal to $n/2$. An agent exhibits *proper risk apportionment of order* n, if she prefers B_n over A_n for all wealth levels x, for all sure losses k and, in particular, for all zero-mean risks ϵ. A prudent decision maker, for example, will prefer to disaggregate the sure loss $-k$ and the zero-mean risk ϵ. That is, she prefers to have the two items in different rather than in the same of two equally likely states of nature. In other words, she disaggregates the two "harms" of a sure loss and a zero-mean risk.[4] A financial economist might speak of a preference for diversification. An equivalent interpretation is that the additional risk is preferred when wealth is higher. These numerous interpretations already illustrate the implicit generality of the preference. Moreover, preference between the proper risk apportionment lotteries has strong implications within the EU framework as shown by Eeckhoudt and Schlesinger (2006).

Theorem 1 *Within the EU paradigm with differentiable utility function u, proper risk apportionment of order n is equivalent to the condition $sgn(u^{(n)}) = (-1)^{n+1}$.*

Thus, the lottery preference of B_2 over A_2, for example, is equivalent to a concave utility function within the differentiable EU framework, i.e. to risk aversion. While none of the results in this paper are based on EU, the above theorem tells us how to interpret them under the assumption of EU.

Next we review the qualitative definitions of skewness and kurtosis, respectively. For the purpose of this paper, it will be particularly insightful to discuss how skewness and kurtosis are reflected in discrete (lottery) distributions. This will be done with reference to Figure 1.

Generally, a distribution is *right-skewed* if it has a longer right tail. This is true for lottery B_3^{S} in Figure 1 because the low outcome 1 has a small distance to the mean of 1.5, whereas the high outcome 3 has a large distance to the mean. In general, any binary lottery is right-skewed if and only if the high outcome occurs with the smaller probability. Formally, this is a consequence of Theorem 1 in Ebert and Wiesen (2009) and Theorem 2 in this paper. Thus, lottery A_3^{S} in Figure 1 (which also has mean 1.5) is left-skewed. The particular lottery pair $(A_3^{\mathrm{S}}, B_3^{\mathrm{S}})$ has been introduced in Mao (1970) and motivated the definition of downside risk aversion in Menezes et al. (1980). A downside risk-averse decision maker will prefer B_3^{S} over A_3^{S}. She rather goes for the smaller outcome 1 most of the time such that she is safe with respect to the worst outcome 0 that can occur when taking A_3^{S} instead. Choice B_3^{S} also implies a small chance of winning the high prize (outcome 3).

Now we consider the lotteries B_4^{S} and A_4^{S} in Figure 1 to discuss kurtosis. Generally, high kurtosis of a distribution implies peakedness and fat tails. Peakedness means that there is a high probability (a "peak" in the frequency distribution) of outcomes close to the mean. Fat tails mean that there is a chance of extreme outcomes (compared to the mean) to occur, i.e. such outcomes have a high probability mass. This is true for lottery A_4, which has a probability peak of 6/8 at its mean, which is 2. Lottery B_4^{S}, in contrast, has no probability mass at its mean (which is also 2) and its outcomes are also less extreme compared to those of lottery B_4^{S}. Thus, lottery A_4^{S} has a higher kurtosis than lottery B_4^{S}.

Now we discuss statistical moments and how they relate to skewness and kurtosis. We denote the pth (non-standardized) central moment of a random variable Z by

$$M_p(Z) = E[(Z - E[Z])^p].$$

[4] This interpretation from Eeckhoudt and Schlesinger (2006) requires the decision maker to be risk-averse such that a zero-mean risk indeed constitutes a harm.

When speaking of moments we always mean (non-standardized) central moments. It is important to note that in this paper skewness and kurtosis do not refer to the third and fourth moment, respectively. If not noted otherwise, they refer to the qualitative features discussed above. One reason is that the third and fourth moment, respectively, might fail to indicate that a distribution is more skewed or leptokurtic than another one.[5] On the other hand, all higher odd and even moments share reasonable properties of a skewness and kurtosis measure, respectively; see van Zwet (1964). In general, the link between any *finite* number of moments and preference is flawed. For example, for any utility function u with $u' > 0$ and $u'' < 0$, there exist random variables X and Y such that X has the higher mean and the lower variance, but u prefers Y to X; see Brockett and Kahane (1992) and Brockett and Garven (1998) for explicit examples. Therefore, a more reliable requirement for a distribution to be more skewed is that *all* odd moments are at least as high as the corresponding moments of the distribution in comparison. Likewise, for a distribution to be more leptokurtic, *all* its even moments are required to be higher. The results in our discussion of higher-order risk preferences, skewness preference and kurtosis aversion can be based on these strong notions of skewness and kurtosis.[6]

The following theorem shows that in the case of binary risks any single odd moment is an appropriate measure of skewness, as the sign of all other odd moments is redundant. It also provides additional intuition to the recent result of Chiu (forthcoming), who shows that all binary risks are *generalized skewness comparable* implying that third-order moment preferences over such risks are consistent with EU maximization. As binary risks are widely employed in experiments the result could be useful to test for skewness preference.

Theorem 2 *Consider a binary lottery $B = Xy_1 + (1 - X)y_0$ with X being Bernoulli distributed with parameter p and without loss of generality let $y_1 > y_0$. Then*

$$\exists\, n \geq 3 \text{ odd: } M_n(B) < 0 \Longrightarrow M_n(B) < 0 \;\forall\, n \geq 3 \text{ odd}$$

where the relation $<$ may be replaced by $>$ or $=$.

3 Prudence, Temperance and Moments

In this section, we present the statistical characterizations of prudence and temperance in terms of moments. The following Propositions 1-4 generalize Propositions 1-4 in Roger (forthcoming) to arbitrary zero-mean risks. Proposition 1 is a generalization of Proposition 3 in Ebert and Wiesen (2009) to moments of order higher than 4. Propositions 1-4 are also generalizations of results in Ekern (1980) in that they consider all moments rather than only moments $1, 2, \ldots, n$ where n is the considered degree of risk aversion.

We start with Proposition 1 which presents a statistical characterization of prudence in terms of moments.

5 We give such an example for the third moment in Figure 3.

6 For more on moments and other measures of skewness see, e.g., MacGillivray (1986).

Proposition 1 *(All moments of the prudence lotteries.)*
For $p \in \mathbb{N}$ we have

(1)
$$M_p\left(A_3\right) = \begin{cases} \left(\frac{k}{2}\right)^p + \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j} & , \ p \text{ even} \\ \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j} & , \ p \text{ odd} \end{cases}$$

(2)
$$M_p\left(B_3\right) = \begin{cases} \left(\frac{k}{2}\right)^p + \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j} & , \ p \text{ even} \\ \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j} & , \ p \text{ odd} \end{cases}$$

(3)
$$M_p\left(B_3\right) - M_p\left(A_3\right) = \begin{cases} \frac{1}{2}\sum_{\substack{j=2 \\ j \text{ odd}}}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j} & , \ p \text{ even} \\ \frac{1}{2}\sum_{\substack{j=2 \\ j \text{ even}}}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j} & , \ p \text{ odd.} \end{cases}$$

Further, the difference $M_p(B_3) - M_p(A_3)$ is strictly positive for all p odd. For all p even, it can be positive, negative or zero.

From Menezes et al. (1980) we already knew that the prudence lotteries have equal mean and variance and that B_3 has a higher third moment. These results are recovered from part (3) in Proposition 1 by considering $p = 1, 2, 3$. Firstly, let us discuss the implication from part (3) stating that *all* odd moments for the prudent lottery choice B_3 are strictly larger than those of the corresponding imprudent lottery A_3. This shows that the prudent lottery choice B_3 is indeed more skewed to the right (not only in an approximate third-order sense), for all possible zero-mean risks. Secondly, part (3) implies that the even moments may not be identical as proven for symmetric zero-mean risks ϵ_1 in Roger (forthcoming). Roger's result is obtained as a special case from part (3), as symmetry of a random variable implies all its odd moments to be zero. Proposition 1 shows that in that case, and only in that case, lotteries A_3 and B_3 have equal kurtosis. This can also be seen qualitatively from our sample lottery pair in Figure 1. Both lotteries A_3^{S} and B_3^{S} have a 3/4-probability peak at an outcome close to the mean (distance of 0.5) which are 2 and 1, respecitively. The "extreme" outcomes of lotteries A_3^{S} and B_3^{S} are 0 and 3, respectively. Both have a distance of 1.5 from the mean and occur with equal probability.

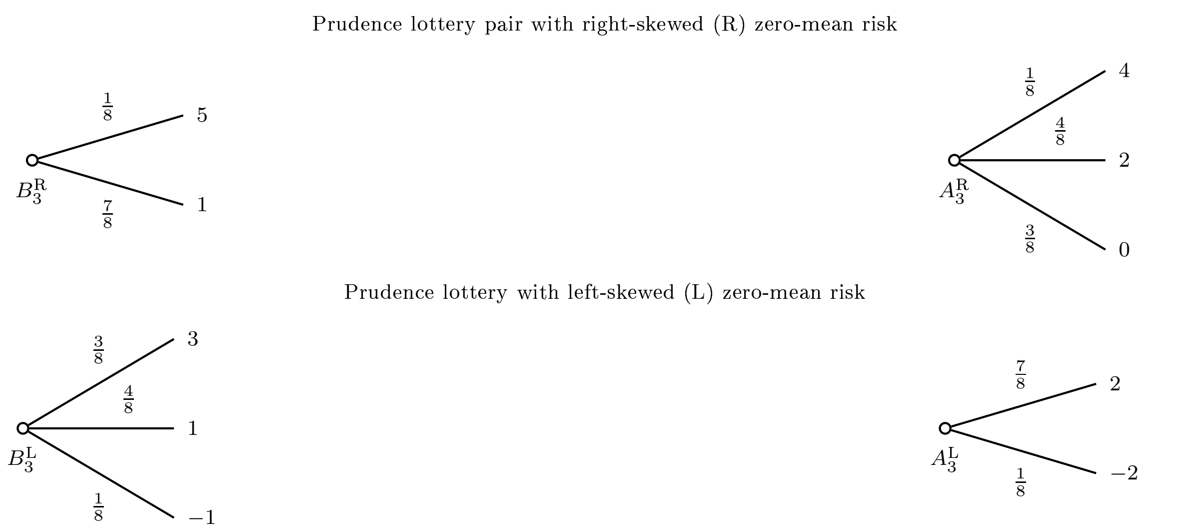
In the general case, the even moments of the prudent choice can be larger or smaller than those of the imprudent choice. They are larger (smaller) if and only if the zero-mean risks to be apportioned are right-skewed (left-skewed), in the sense that *most* of its odd moments are positive (negative). An example is given in Figure 2. Lottery B_3^{R} has a 7/8 probability peak at 1 which is close to the mean of 1.5. It also has a very extreme outcome 5. Lottery A_3^{R} in contrast has only a 4/8 probability peak at the outcome 2 which is close to the mean and both remaining outcomes 0 and 4 are less extreme than 5 as their distance to the mean of 1.5 is smaller. Analogous arguments apply to lottery pair $(A_3^{\mathrm{L}}, B_3^{\mathrm{L}})$ where the zero-mean risk is left-skewed and thus A_3^{L} has the higher kurtosis.

In general, prudence must be understood as a preference for high skewness (i.e. high odd moments of all orders) that is robust towards variation in kurtosis (i.e. differences in high even moments of all orders).

We refer to this as the *kurtosis robustness feature of prudence.* That is, prudence not only determines preference between distributions that purely differ in their skewness. Prudence implies preference for distributions with higher skewness independent of whether they have the higher or smaller kurtosis. To best of our knowledge, this has not yet been pointed out in any discussion of prudence and skewness preference.

Thus, the restriction to symmetric zero-mean risks in the proper risk apportionment model of Eeckhoudt and Schlesinger (2006) is rather severe from a statistical point of view. It reduces prudence to "pure" *skewness seeking* (distributions with higher odd moments are preferred) and neglects the kurtosis robustness feature. Empirical support for the kurtosis robustness feature has been found in the experiment of Ebert and Wiesen (2009) who conclude that there is more to prudence than skewness seeking. A prudent decision is made more frequently when the zero-mean risk is left-skewed, i.e. the even moments are higher for the imprudent choice. An interpretation is that when the risk is left-skewed, for a prudent decision maker it constitutes a greater harm such that there is a higher necessity to be prudent. Proposition 1 is a generalization of their Proposition 3 which puts their result on a sound theoretical basis that is not based on an approximate fourth-order analysis. Next we present a characterization of temperance in terms of moments.

Fig. 2 Prudence lottery pairs with skewed zero-mean risks



This Figure shows a prudence lottery pair $(A_3^{\mathrm{L}}, B_3^{\mathrm{L}})$ where the zero-mean risk ϵ_1 is left-skewed and a prudence lottery pair $(A_3^{\mathrm{R}}, B_3^{\mathrm{R}})$ where the zero-mean risk ϵ_1 is right-skewed. Both B_3^{L} and B_3^{R} are, respectively, more skewed to the right than A_3^{L} and A_3^{R}. However, whereas B_3^{R} has a higher kurtosis than A_3^{R}, B_3^{L} has a smaller kurtosis than A_3^{L}. This is in accordance with the result on moments proven in Proposition 1. The prudence lotteries with the right-skewed zero-mean risk are constructed with initial wealth $x = 2$, loss $-k = -1$ and the zero-mean risk ϵ_1 yields 3 with probability 1/4 and -1 with probability 3/4. For the prudence lotteries with the left-skewed zero-mean risk, initial wealth is $x = 2$, the loss is $-k = -1$ and the zero-mean risk ϵ_1 yields -3 with probability 1/4 and 1 with probability 3/4.

Proposition 2 *(All moments of the temperance lotteries.) For $p \in \mathbb{N}$*

$$M_p(A_4) = \frac{1}{2}\sum_{j=0}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]E\left[\epsilon_1^{p-j}\right] \tag{1}$$

$$M_p(B_4) = \frac{1}{2}\left(E\left[\epsilon_2^p\right] + E\left[\epsilon_1^p\right]\right) \tag{2}$$

$$M_p(B_4) - M_p(A_4) = -\frac{1}{2}\left(\sum_{j=2}^{p-1}\binom{p}{j}E\left[\epsilon_1^j\right]E\left[\epsilon_2^{p-j}\right]\right). \tag{3}$$

Further, for $p > 4$ odd the difference $M_p(B_3) - M_p(A_3)$ can be positive, negative or zero.

Roger (forthcoming) further shows that in the case of symmetric zero-mean risks

$$M_p(A_n) = M_p(B_n) = 0 \ \forall \ p \ odd.$$

For illustrative purposes, consider the case of $p = 5$ and $n = 4$. Using equation (5), we have

$$\begin{aligned} M_5(B_4) - M_5(A_4) &= -\frac{1}{2}\left(\sum_{j=2}^{5-1}\binom{5}{j}E\left[\epsilon_1^j\right]E\left[\epsilon_2^{5-j}\right]\right) \\ &= -\frac{1}{2}\left(\binom{5}{2}E\left[\epsilon_1^2\right]E\left[\epsilon_2^3\right] + \binom{5}{3}E\left[\epsilon_1^3\right]E\left[\epsilon_2^2\right] + 0\right) \end{aligned} \tag{1}$$

which can be positive, negative or zero, depending on the third moments of the zero-mean risks. The proof in the appendix essentially generalizes this example to all odd moments. We interpret the last statement of Proposition 2 as the *skewness robustness feature* of temperance. Roger also shows that $M_p(B_4) - M_p(A_4) < 0$ holds for all $p > n$ even. This we cannot prove in the general case. To see the reason why, in equation (5) set $p = 6$ and $n = 4$, i.e.

$$\begin{aligned} M_6(B_4) - M_6(A_4) &= \frac{1}{2}\left(\sum_{j=2}^{5}\binom{6}{j}E\left[\epsilon_1^j\right]E\left[\epsilon_2^{6-j}\right]\right) \\ &= -\frac{1}{2}\left(0 + \binom{6}{2}E\left[\epsilon_1^2\right]E\left[\epsilon_2^4\right] + \binom{6}{3}E\left[\epsilon_1^3\right]E\left[\epsilon_2^3\right] + \binom{6}{4}E\left[\epsilon_1^4\right]E\left[\epsilon_2^2\right] + 0\right). \end{aligned} \tag{2}$$

This expression might become positive if the middle term is negative which could happen if and only if the two zero-mean risks are adversely skewed. However, we could conjecture that for all random variables ϵ_1 and ϵ_2 this is not possible. Using Proposition 2, part (3), the conjecture can be validated or dismissed for any risks specifically considered. Evidently, it is true if both zero-mean risks are symmetric or skewed in the same direction. For prudence, we obtained the clear statement that proper risk apportionment implies preference for large odd moments of all orders that is robust towards variation in the even moments. Analogously, we find some evidence that temperance is a preference for small even moments (kurtosis aversion) that is robust towards variation in the odd moments (skewness robustness).

4 Higher-order generalizations

In this section, we generalize the results from the previous section to risk apportionment of orders higher than 4. Lemma 1 presents recursive formulae that can be used to compute any moment of a proper risk apportionment lottery of any order and thus completes our moment characterization of higher-order risk preferences.

Lemma 1 *For $n \geq 3$ (even or odd) we have the following recursive formulae*

$$M_p(A_n) = \frac{1}{2}\left(M_p(B_{n-2}) + M_p(A_{n-2}) + \sum_{j=2}^{p}\binom{p}{j} E\left[\epsilon^j_{\llcorner n/2 \lrcorner}\right] M_{p-j}(A_{n-2})\right) \tag{3}$$

$$M_p(B_n) = \frac{1}{2}\left(M_p(A_{n-2}) + M_p(B_{n-2}) + \sum_{j=2}^{p}\binom{p}{j} E\left[\epsilon^j_{\llcorner n/2 \lrcorner}\right] M_{p-j}(B_{n-2})\right) \tag{4}$$

$$M_p\left(B_n\right) - M_p\left(A_n\right) = \frac{1}{2}\left(\sum_{j=2}^{p}\binom{p}{j} E\left[\epsilon^j_{\llcorner n/2 \lrcorner}\right]\left(M_{p-j}(B_{n-2}) - M_{p-j}(A_{n-2})\right)\right). \tag{5}$$

We now investigate how our Proposition 2 and Roger's Proposition 3 generalize to higher even orders.

Proposition 3 *Let $n \geq 4$.*

(1) $M_p\left(A_n\right) - M_p\left(B_n\right) = 0, \ for \ 1 \leq p < n$

(2) $M_n\left(A_n\right) > M_n\left(B_n\right), \quad for \ p = n.$

Further, for $p > n$ odd the difference $M_p\left(B_3\right) - M_p\left(A_3\right)$ can be positive, negative or zero.

The last statement says that all higher-order risk preferences of even order have a skewness robustness feature, i.e. the preferred lottery may or may not have higher odd moments of any order. As Roger showed for symmetric zero-mean risks, we also conjecture (as we did in the case of temperance) that in the general case $M_p\left(A_n\right) > M_p\left(B_n\right)$ for $p \geq n$ even is true. For any lotteries specifically considered, this can be checked using equation (5) in Lemma 1. The next Proposition generalizes Roger's Proposition 4.

Proposition 4 *$n \geq 3$ odd.*

(1) $M_p\left(A_n\right) = M_p\left(B_n\right)$ *for $p < n$*

(2) $M_p\left(B_n\right) - M_p\left(A_n\right) > 0$ *for $p = n$.*

Further, for $p > n$ even the difference $M_p\left(B_3\right) - M_p\left(A_3\right)$ can be positive, negative or zero.

The last statement says that all higher-order risk preferences of odd order have a kurtosis robustness feature. Under the symmetry assumption, for $n \geq 3$ odd Roger (forthcoming) shows

$$\begin{aligned}
&(1a) && M_p(A_n) = M_p(B_n) = 0 && \forall\ p < n \text{ odd},\\
&(2') && M_p(A_n) = -M_p(B_n) < 0 && \forall\ p\ \geq n \text{ odd},\\
&(3) && M_p(A_n) = M_p(B_n) && \forall\ p > n \text{ even}.
\end{aligned}$$

While (1a) trivially holds for prudence, in general only the first equality is true. The following is a counterexample for the second inequality. For $n = 5$ and $p = 3$ the recursive formula derived in Lemma 1, equation (5) gives

$$M_3(A_5) = \frac{1}{2}\left(M_3(B_3) + M_3(A_3) + \sum_{j=2}^{3}\binom{3}{j}E[\epsilon_2^j]M_{3-j}(A_3)\right).$$

From Proposition 1, $M_3(B_3) = \frac{1}{2}\left(\binom{3}{2}E[\epsilon_1^2]\frac{k}{2} + E[\epsilon_1^3]\right)$, $M_3(A_3) = \frac{1}{2}\left(\binom{3}{2}E[\epsilon_1^2]\left(-\frac{k}{2}\right) + E[\epsilon_1^3]\right)$ and $M_1(A_3) = 0$ such that

$$M_3(A_5) = 2E[\epsilon_1^3]$$

which can be negative, positive or zero, depending on the asymmetry of the zero-mean risks.
A counterexample for (3) is given by the fourth central moment of the prudence lotteries, i.e. $n = 3$ and $p = 4$, as discussed subsequent to Proposition 1.
The equality in (2') is not true and a counterexample is given by the third moment of the prudence lotteries; see parts (1) and (2) of Proposition 1. Also, in the general case we cannot prove the inequality $M_p(B_n) - M_p(A_n) > 0$ for p odd, which is redundant from (2'). To see the reason why, take $n = 5$ and $p = 7$ in (5) and impute the expressions for the moments of the prudence lotteries stated in Proposition 1. We get

$$\begin{aligned}
M_7(B_5) - M_7(A_5) = \frac{1}{2}\Bigg\{ &\sum_{j=2,\ j \text{ even}}^{7}\binom{7}{j}E\left[\epsilon_5^j\right]\left(\sum_{l=2,\ l \text{ even}}^{7-j}\binom{7-j}{l}E\left[\epsilon_2^l\right]\left(\frac{k}{2}\right)^{7-j-l}\right)\\
&+ \sum_{j=2,\ j \text{ odd}}^{7}\binom{7}{j}E\left[\epsilon_5^j\right]\left(\sum_{l=2,\ l \text{ odd}}^{7-j}\binom{7-j}{l}E\left[\epsilon_3^l\right]\left(\frac{k}{2}\right)^{7-j-l}\right)\Bigg\}.
\end{aligned}$$

The second sum of the above expression can be computed as

$$\begin{aligned}
&\binom{7}{3}E\left[\epsilon_5^3\right]\left(+\binom{4}{3}E\left[\epsilon_3^3\right]\left(\frac{k}{2}\right)^1\right) + \binom{7}{5}E\left[\epsilon_5^5\right]\cdot\left(\binom{2}{1}\cdot 0\right)\\
&= 4E\left[\epsilon_3^3\right]\left(\frac{k}{2}\right),
\end{aligned}$$

which might be negative such that the whole expression might be negative. However, we again conjecture that this is not possible. For the prudence lotteries $M_p(B_n) - M_p(A_n) > 0$ for $p \geq 3$ odd is true (see

Proposition 1). It is also true for the example of the edginess lottery pair depicted in Figure 3. Clearly, B_5^S is skewed to the right as it has a long right tail due to outcome 4 being far right of the mean of 1.5 and occuring with small probability. The right tail is shorter as outcome 0 is closer to the mean and has heavier probability mass. Analogous arguments imply that A_5^S is left-skewed. As all zero-mean risks used in the construction of (A_5^S, B_5^S) are symmetric, all even moments of the two lotteries are equal, i.e. they have the same kurtosis. Likewise, B_5^S has higher odd moments of order 5 and higher which indicates that it is more skewed to the right, although the third moments of the lotteries are the same.

The previous example shows two important points. Firstly, it illustrates why the third moment of a distribution can fail as a measure of skewness. Secondly, prudence does not exhaustively describe skewness preference. The right-skewed lottery B_5^S is preferred to the left-skewed lottery A_5^S if and only if the decision maker exhibits edginess. This illustrates that higher-order risk preferences of any order are important in modeling skewness preference. Analogous arguments show that all higher-order risk preferences of even order imply kurtosis aversion in a complementary way.

Fig. 3 Edginess lottery pair with symmetric (S) zero-mean risks

This figure shows an edginess lottery pair (A_5^S, B_5^S) where both zero-mean risks are symmetric. B_5^S is more skewed to the right than A_5^S, although the lotteries do not differ in their third moment. B_5^S has higher odd moments of all orders higher than three. Initial wealth is $x = 2$, loss $-k = -1$ and the zero-mean risks $\tilde{\epsilon_1}$ and $\tilde{\epsilon_2}$ both yield 1 or -1 with equal probability. Thus, the nested prudence lotteries used in the construction are A_3^S and B_3^S displayed in Figure 1.

5 Conclusion

In this paper, we presented a characterization of higher-order risk preferences in terms of statistical moments. This characterization provides a better understanding of how higher-order risk preferences are related to skewness preference and kurtosis aversion. Further, moments are well understood such that our results should be easily accessible to a wide audience in economics and finance.

Prudence is shown to be a preference for high odd moments (skewness seeking) that is robust towards variation in the even moments (kurtosis robustness). In particular, prudence does not only determine preference between distributions that purely differ in their skewness. However, restriction to symmetric zero-mean risks in the proper risk apportionment model of Eeckhoudt and Schlesinger reduces prudence to “pure” skewness seeking. Thus, our theoretical results are in line with experimental evidence by Ebert and Wiesen (2009), who find that there is more to prudence than skewness seeking. Analogous results

in the present paper relate temperance to preference for small even moments (kurtosis aversion) that is robust towards variation in the odd moments (skewness robustness).
Moreover, we showed that not only prudence and temperance, but all higher-order risk preferences of odd and even order, respectively, are related to skewness preference and kurtosis aversion in a complementary way. This highlights the importance of these concepts which are generally viewed as rather abstract and thus have not received that much attention in the literature yet.

Appendix (Proofs)

Proof of Theorem 2. We first show that $M_3(B) < 0$ implies $M_n(B) < 0 \; \forall \; n > \; 3$ odd. Using translation invariance we can write the nth central moment of B as

$$M_n(B) = M_n(B - y_0) = M_n\left(X(y_1 - y_0)\right) = E\left[(X(y_1 - y_0) - p(y_1 - y_0))^n\right]$$

which for the Bernoulli distribution can be computed explicitly as

$$M_n(B) = p\left((y_1 - y_0) - p(y_1 - y_0)\right)^n + (1-p)\left(-p(y_1 - y_0)\right)^n.$$

Using that n is odd this is easily simplified to

$$M_n(B) = (y_1 - y_0)^n \cdot \left(p\,(1-p)^n - (1-p)p^n\right).$$

It is easily seen that $(p(1-p)^n - (1-p)p^n) < 0 \Longleftrightarrow p > 0.5$, and since $(y_1 - y_0)^n > 0$ by definition we have

$$M_n(B) < 0 \Longleftrightarrow p > 0.5. \tag{6}$$

From Theorem 1 in Ebert and Wiesen (2009) we have that $p > 0.5$ if and only if the third central moment of B is strictly negative. Thus the claim is proved for $n = 3$ by the necessity of the equivalence in (6). Now suppose that for some (arbitrary) n we have $M_n(B) < 0$. Then by the sufficiency in (6) we have $p > 0.5$ which implies $M_3(B) < 0$ from which the claim follows as just demonstrated. The statements for the other relations are obtained analogously. □

The following lemma is proven in Roger (forthcoming) and will be used several times in our proofs.

Lemma 2 *(Roger's Lemma) Let X be Bernoulli distributed with parameter* 0.5 *and be independent from Y_1 and Y_2. Then:*

$$E\left[(XY_1 + (1-X)Y_2)^p\right] = \frac{1}{2}\left(E\left[Y_1^p\right] + E\left[Y_2^p\right]\right).$$

If $E[Y_1] = E[Y_2]$, then

$$M_p\left[XY_1 + (1-X)Y_2\right] = \frac{1}{2}\left(M_p\left[X\right] + M_p\left[Y\right]\right).$$

Proof of Proposition 1. We first define auxiliary lotteries

$$\hat{A}_3 := A_3 + \frac{k}{2} = X \cdot \frac{k}{2} + (1-X)\left(-\frac{k}{2} + \epsilon_1\right)$$
$$\hat{B}_3 := B_3 + \frac{k}{2} = X\left(-\frac{k}{2}\right) + (1-X)\left(\epsilon_1 + \frac{k}{2}\right).$$

These lotteries can be understood as the prudence lotteries shifted such that they have mean zero. Because the operator $M_p(\cdot)$ is translation invariant we have

$$M_p(A_3) = M_p(\hat{A}_3) = E[\hat{A}_3^p] \tag{7}$$

which analogously holds for B_3. Thus it suffices to focus on the computation of the non-central moments $E[\hat{A}_3^p]$ and $E[\hat{B}_3^p]$. In the second equality below we apply Roger's Lemma and obtain

$$\begin{aligned}
M_p(A_3) &= E\left[\left\{X \cdot \frac{k}{2} + (1-X)\left(\epsilon_1 - \frac{k}{2}\right)\right\}^p\right] \\
&= \frac{1}{2}E\left[\left(\epsilon_1 + \left(-\frac{k}{2}\right)\right)^p\right] + \frac{1}{2}\left(\frac{k}{2}\right)^p \\
&= \frac{1}{2}E\left[\sum_{j=0}^{p}\binom{p}{j}\epsilon_1^j\left(-\frac{k}{2}\right)^{p-j}\right] + \frac{1}{2}\left(\frac{k}{2}\right)^p \\
&= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\left(-\frac{k}{2}\right)^{p-j} E\left[\epsilon_1^j\right] + \frac{1}{2}\left(\left(-\frac{k}{2}\right)^p + \left(\frac{k}{2}\right)^p\right).
\end{aligned} \tag{8}$$

where we used that the summand for $j=1$ is zero since $E[\epsilon_1]=0$. This argument will be used several times in the proofs of this paper. Similarly, for B_3 we get

$$M_p(B_3) = \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}\left(\frac{k}{2}\right)^{p-j} E\left[\epsilon_1^j\right] + \frac{1}{2}\left(\left(-\frac{k}{2}\right)^p + \left(\frac{k}{2}\right)^p\right). \tag{9}$$

To prove (1) and (2), if p is odd we have

$$\begin{aligned}
M_p(A_3) &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j} + \frac{1}{2}\left(-\left(\frac{k}{2}\right)^p + \left(\frac{k}{2}\right)^p\right) \\
&= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j}
\end{aligned}$$

and analogously

$$M_p(B_3) = \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j}.$$

If p is even

$$\begin{aligned}M_p(A_3) &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j}+\frac{1}{2}\left(\left(\frac{k}{2}\right)^p+\left(\frac{k}{2}\right)^p\right)\\ &= \left(\frac{k}{2}\right)^p+\frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(-\frac{k}{2}\right)^{p-j}\end{aligned}$$

and analogously

$$M_p(B_3) = \left(\frac{k}{2}\right)^p+\frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j}.$$

Proof of part (3). For odd p, using the expressions proven in (1) and (2), this difference can be computed as

$$\begin{aligned}M_p(B_3)-M_p(A_3) &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\underbrace{\left(\left(\frac{k}{2}\right)^{p-j}-\left(-\frac{k}{2}\right)^{p-j}\right)}_{=\begin{cases}2\left(\frac{k}{2}\right)^{p-j} & ,\ p-j \text{ odd} \Leftrightarrow j \text{ even}\\ 0 & ,\ \text{o.w.}\end{cases}}\\ &= \sum_{j=2,\ j\text{ even}}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j}.\end{aligned}$$

Similarly, for even p we obtain

$$\begin{aligned}M_p(B_3)-M_p(A_3) &= \frac{1}{2}\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\underbrace{\left(\left(\frac{k}{2}\right)^{p-j}-\left(-\frac{k}{2}\right)^{p-j}\right)}_{=\begin{cases}2\left(\frac{k}{2}\right)^{p-j} & ,\ p-j \text{ odd} \Leftrightarrow j \text{ odd}\\ 0 & ,\ \text{o.w.}\end{cases}}\\ &= \sum_{j=2,\ j\text{ odd}}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]\left(\frac{k}{2}\right)^{p-j}.\end{aligned}$$

The claims on the sign of $M_p(B_3)-M_p(A_3)$ are evident from the expressions proved. □

Proof of Proposition 2. The proof essentially follows that of Roger for the symmetric case. By application of Roger's Lemma we have

$$\begin{aligned}M_p(A_4) &= E\left[(1-X)^p(\epsilon_1+\epsilon_2)^p\right]=\frac{1}{2}E\left[(\epsilon_1+\epsilon_2)^p\right]\\ &= \frac{1}{2}\sum_{j=0}^{p}\binom{p}{j}E\left[\epsilon_1^j\right]E\left[\epsilon_2^{p-j}\right]\end{aligned}$$

and

$$M_p\left(B_4\right)=\frac{1}{2}\left(E\left[\epsilon_2^p\right]+E\left[\epsilon_1^p\right]\right).$$

Claim (3) follows immediately by substraction. For the last statement, consider the products $E\left[\epsilon_1^j\right]\cdot E\left[\epsilon_2^{p-j}\right]$. Suppose both zero-mean risks are binary and recall the result of Theorem 2. Obviously, if both zero-mean risks are right-skewed, then all these products are positive such that $M_p\left(B_4\right)-M_p\left(A_4\right)<0$. If both zero-mean risks are symmetric, we have that the difference is zero (as shown by Roger). Finally, as p is odd, $p-j$ is odd if and only if j is even. Thus, if both zero-mean risks are left-skewed, we have that $(B_4)-M_p\left(A_4\right)>0$. □

Proof of Lemma 1. First, let n be even. By Roger's Lemma we have

$$\begin{aligned}
M_p\left(A_n\right)=E\left[A_n^p\right] &= E\left[\left(XB_{n-2}+(1-X)\left(A_{n-2}+\epsilon_{\llcorner n/2\lrcorner}\right)\right)^p\right]\\
&=\frac{1}{2}\left(E\left[B_{n-2}^p\right]+E\left[\left(A_{n-2}+\epsilon_{\llcorner n/2\lrcorner}\right)^p\right]\right)\\
&=\frac{1}{2}\left(E\left[B_{n-2}^p\right]+E\left[A_{n-2}^p\right]+\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_{\llcorner n/2\lrcorner}^j\right]E\left[A_{n-2}^{p-j}\right]\right)\\
&=\frac{1}{2}\left(E\left[B_{n-2}^p\right]+E\left[A_{n-2}^p\right]+\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_{\llcorner n/2\lrcorner}^j\right]M_{p-j}(A_{n-2})\right)
\end{aligned}\tag{10}$$

and similarly

$$\begin{aligned}
M_p\left(B_n\right) &= \frac{1}{2}\left(E\left[A_{n-2}^p\right]+E\left[\left(B_{n-2}+\epsilon_{\llcorner n/2\lrcorner}\right)^p\right]\right)\\
&=\frac{1}{2}\left(E\left[B_{n-2}^p\right]+E\left[A_{n-2}^p\right]+\sum_{j=2}^{p}\binom{p}{j}E\left[\epsilon_{\llcorner n/2\lrcorner}^j\right]M_{p-j}(B_{n-2})\right).
\end{aligned}\tag{11}$$

Thus we get

$$M_p\left(B_n\right)-M_p\left(A_n\right)=\frac{1}{2}\left(\sum_{j=2}^{p}\binom{p}{j}\left[\epsilon_{\llcorner n/2\lrcorner}^j\right]\left(M_{p-j}(B_{n-2})-M_{p-j}(A_{n-2})\right)\right)$$

which is equation (5). Now assume n is odd. Like in the proof of Proposition 1 define $\hat{A}_3=A_3+\frac{k}{2}$ and $\hat{B}_3=B_3+\frac{k}{2}$. For $n\geq 5$ we naturally extend this definition, i.e. let

$$\begin{aligned}
\hat{A}_n &= X\hat{B}_{n-2}+(1-X)\left(\epsilon_{\llcorner n/2\lrcorner}+\hat{A}_{n-2}\right)\\
\hat{B}_n &= X\hat{A}_{n-2}+(1-X)\left(\epsilon_{\llcorner n/2\lrcorner}+\hat{B}_{n-2}\right).
\end{aligned}$$

Then, like in the proof of Proposition 1, we have

$$\begin{aligned} M_p\left(A_n\right) = M_p\left(\hat{A}_n\right) = E\left[\hat{A}_n^p\right] &= \frac{1}{2}\left(E\left[\hat{B}_{n-2}^p\right] + E\left[\left(\epsilon_{\llcorner n/2 \lrcorner} + \hat{A}_{n-2}\right)^p\right]\right) \\ &= \frac{1}{2}\left(E\left[\hat{B}_{n-2}^p\right] + \sum_{j=0}^{p}\binom{p}{j} E\left[\epsilon_{\llcorner n/2 \lrcorner}^j\right] E\left[\hat{A}_{n-2}^{p-j}\right]\right) \\ &= \frac{1}{2}\left(E\left[\hat{B}_{n-2}^p\right] + E\left[\hat{A}_{n-2}^p\right] + \sum_{j=2}^{p}\binom{p}{j} E\left[\epsilon_{\llcorner n/2 \lrcorner}^j\right] E\left[\hat{A}_{n-2}^{p-j}\right]\right) \\ &= \frac{1}{2}\left(M_p(B_{n-2}) + M_p(A_{n-2}) + \sum_{j=2}^{p}\binom{p}{j} E\left[\epsilon_{\llcorner n/2 \lrcorner}^j\right] M_{p-j}(A_{n-2})\right) \end{aligned} \tag{12}$$

and analogously

$$M_p\left(B_n\right) = \frac{1}{2}\left(M_p(A_{n-2}) + M_p(B_{n-2}) + \sum_{j=2}^{p}\binom{p}{j} E\left[\epsilon_{\llcorner n/2 \lrcorner}^j\right] M_{p-j}(B_{n-2})\right). \tag{13}$$

Equations (12) and (13), respectively, are identical to equations (10) and (11). Thus the subtraction of equation (12) from equation (13) is also given by equation (5). □

Proof of Proposition 3. We prove (1) by induction. For $n = 4$ we show that for $p = 1, 2, 3$ the summands in the equation of Proposition 2, part (3), are zero. The only effective summand is for $p = 3$ which is zero because $E[\epsilon_2^{3-2}] = 0$. Now assume the claim is true for $n-2$. Let $p < n$. For $j = 2, 3, ..., n$ we have $p-j < n-j \leq n-2$, thus $M_{p-j}(B_{n-2}) - M_{p-j}(A_{n-2}) = 0$ by the induction assumption. Then the claim directly follows from equation (5) in Lemma 1. Also (2) is proven by induction. For $n = 4$ the claim can easily be inferred from Proposition 2, part (3). Now assume the claim is true for $n-2$. Equation (5) for $p = n$ is

$$M_n(B_n) - M_n(A_n) = \frac{1}{2}\left(\sum_{j=2}^{n}\binom{n}{j} E[\epsilon_{\llcorner n/2 \lrcorner}^j]\left(M_{n-j}(B_{n-2}) - M_{n-j}(A_{n-2})\right)\right). \tag{14}$$

For $j = 2$ we have $M_{n-2}(B_{n-2}) - M_{n-2}(A_{n-2}) > 0$ by the induction assumption, further $E[\epsilon_{\llcorner n/2 \lrcorner}^2] > 0$ and thus this summand is strictly positive. For $j > 2$ all summands are zero by Proposition part (1) of this Proposition and the claim follows. To prove the last statement, suppose that $\epsilon_1, \epsilon_2, \ldots, \epsilon_{\llcorner n/2 \lrcorner - 1}$ are symmetric. Then from Roger (forthcoming), Proposition 3, we have that $M_k\left(B_{n-2}\right) - M_k\left(A_{n-2}\right)$ is strictly positive for $k \geq n$ even and zero otherwise. We want to show that for $p > n$ odd $M_p\left(B_n\right) - M_p\left(A_n\right)$ can be positive, negative, or zero. In order to do this, we consider the summands in Equation (5) in Lemma 1 and start with those summands for which j is even. As p is odd, $p - j$ is odd and thus $M_{p-j}\left(B_{n-2}\right) - M_{p-j}\left(A_{n-2}\right)$ is zero always. If j is odd, then $p - j$ is even and thus $M_{p-j}\left(B_{n-2}\right) - M_{p-j}\left(A_{n-2}\right)$ is zero if $p - j < n - 2$ and strictly positive otherwise. Now, if $\epsilon_{\llcorner n/2 \lrcorner}$ is symmetric, i.e. $E[\epsilon_{\llcorner n/2 \lrcorner}^j] = 0$ for all j odd, all summands are zero and we have (as proven by Roger) that $M_p\left(B_n\right) - M_p\left(A_n\right) = 0$. If $\epsilon_{\llcorner n/2 \lrcorner}$ is right-skewed and binary (see Theorem 2), then all summands are positive and as $p > n$ at least one summand is strictly positive, such that $M_p(B_n) - M_p(A_n) > 0$. Similarly, if $\epsilon_{\llcorner n/2 \lrcorner}$ is left-skewed and binary, we obtain that $M_p(B_n) - M_p(A_n) < 0$. □

Proof of Proposition 4. By induction. For prudence, i.e. $n = 3$, both claims (1) and (2) could be verified using part (3) of Proposition 1. However, the results are also given in Crainich and Eeckhoudt (2006) and Ebert and Wiesen (2009). Suppose the claim is true for $n-2$. For part (1), the induction assumption is that $p < n-2$ implies that $M_p(B_{n-2}) - M_p(A_{n-2}) = 0$. If $p < n$, then for $j = 2, 3, \ldots, p$ we have $p - j < n - j \leq n - 2$. Thus $M_p(B_{n-2}) - M_p(A_{n-2}) = 0$ for $j = 2, 3, \ldots, p$ such that each summand on the right hand side of equation (5) in Lemma 1 is zero. For part (2), the induction assumption is $M_{n-2}(B_{n-2}) - M_{n-2}(A_{n-2}) > 0$. Consider equation (14) which likewise holds for n odd. The summand for $j = 2$ is strictly positive by the induction assumption and all other summands are zero by part (1) of this Proposition and the claim follows. To prove the last statement, suppose that $\epsilon_1, \epsilon_2, \ldots, \epsilon_{\llcorner n/2 \lrcorner - 1}$ are symmetric. Then from Roger (forthcoming), Proposition 4, we have that $M_k(B_{n-2}) - M_k(A_{n-2})$ is strictly positive for $k \geq n$ odd and zero otherwise. We want to show that for $p > n$ even $M_p(B_n) - M_p(A_n)$ can be positive, negative, or zero. In order to do this, we consider the summands in Equation (5) in Lemma 1 and start with those summands for which j is even. As p is even, $p - j$ is even and thus $M_{p-j}(B_{n-2}) - M_{p-j}(A_{n-2})$ is zero always. If j is odd, then $p - j$ is odd and thus $M_{p-j}(B_{n-2}) - M_{p-j}(A_{n-2})$ is zero if $p - j < n - 2$ and strictly positive otherwise. Now, if $\epsilon_{\llcorner n/2 \lrcorner}$ is symmetric, all summands are zero and we have (as proven by Roger) that $M_p(B_n) - M_p(A_n) = 0$. If $\epsilon_{\llcorner n/2 \lrcorner}$ is right-skewed and binary (see Theorem 2), then all summands are positive and as $p > n$ at least one summand is strictly positive, such that $M_p(B_n) - M_p(A_n) > 0$. Similarly, if $\epsilon_{\llcorner n/2 \lrcorner}$ is left-skewed and binary, we obtain that $M_p(B_n) - M_p(A_n) < 0$. □

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