Bonn Econ Discussion Papers

Discussion Paper 22/2002

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September 2002

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AN EXAMINATION OF THE EFFECTS OF PARAMETER MISSPECIFICATION ON THE DUPLICATION OF BONDS

ANTJE DUDENHAUSEN AND LUTZ SCHLÖGL

ABSTRACT. It is well-known that Gaussian hedging strategies are robust in the sense that they always lead to a cost process of bounded variation and that a superhedge is possible if upper bounds on the volatility of the relevant processes are available, cf. El Karoui, Jeanblanc-Picqué and Shreve (1998) and in particular for applications to fixed income derivatives Dudenhausen, Schlögl and Schlögl (1998). These results crucially depend on the choice of certain "natural" hedge instruments which are not always available in the market and fail to hold otherwise. In this paper, the problem of optimally selecting hedging instruments from a given set of traded assets, in particular of zero coupon bonds, is studied. Misspecified hedging strategies lead to a non-vanishing cost process, which in turn depends on the particular choice of instruments. The effect of this choice on the cost process is analyzed. Referring to bond markets, a thorough study of the implications of volatility mismatching is made and explicit results are stated for a broad range of volatility scenarios.

1. INTRODUCTION

For various reasons, models currently used for derivatives pricing imperfectly represent reality. This is particularly striking in the fixed income sector, where models are recalibrated on a daily basis to whichever yield curve is observed in the market. Therefore, the behaviour of a model under misspecification of either its parameters or the underlying asset dynamics is an important question and the subject of active research.

Among others, Ahn, Muni and Swindle (1997), and Lyons (1995) study the impact of uncertain volatility on option prices and hedging strategies. It is the existence of a self-financing replication strategy which lies at the heart of pricing contingent claims by the principle of no-arbitrage. Therefore, in our opinion, it is important to study the behaviour of hedging strategies under misspecification explicitly. It has been shown, cf. El Karoui, Jeanblanc-Picqué and Shreve (1998) that hedging strategies derived from Gaussian models are particularly robust with respect to misspecification, where Gaussian is to be understood in the sense that the dynamics of the process relevant for hedging are lognormal. An alternative derivation of this result as well as applications to fixed income securities are given in Dudenhausen, Schlögl and Schlögl (1998).

An important application of the robustness of Gaussian hedging strategies in an uncertain volatility environment is the fact that superhedging strategies can be implemented in the

Date. September 22, 2002.

Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 and the German Academic Exchange Service, DAAD HSP III at the University of Bonn is gratefully acknowledged. The authors would like to thank Philipp Schönbucher, Daniel Sommer, and Dieter Sondermann (at the University of Bonn), Rüdiger Frey (at ETH Zürich), as well as Erik Schlögl (at the University of Technology, Sydney) for fruitful discussions. The usual disclaimers apply.

natural hedge instruments if upper bounds on the volatility are given. In the case of an option, the natural hedge instruments are typically the underlying as well as the zero-coupon bond with the same maturity as the option. In general, there will not be a liquid market for this zero-coupon bond, so that it will not be available for hedging purposes. We see that the superhedging result in the natural hedge instruments is of limited practical applicability. This raises the question of how the natural hedge instruments might be synthesized by dynamic trading strategies.

When using a dynamically complete model, it is always theoretically possible to identically duplicate any asset by a self-financing trading strategy. The completeness or otherwise of the model is determined by the volatility structure of the assets traded. We clarify the condition on the volatility structure needed to ensure completeness without the usual assumption that one of the traded assets is a continuously rolled-over savings account. We derive the cost process L resulting from a misspecification of asset volatility and show that L necessarily has a non-trivial martingale component. The model assumed for hedging is assumed to be Gaussian in the sense that hedging strategies are constructed using deterministic asset volatilities. In contrast, the true volatilities can be stochastic and the number of factors driving the true dynamics is allowed to differ from the number assumed.

We perform concrete calculations within the framework of Gaussian term structure models. Given the price of a zero-coupon bond we consider trading strategies which identically replicate this asset. Under misspecification, any such strategy will lead to a non-vanishing cost process L which, as already stated, has a non-trivial martingale component. Using a mean-variance approach, we analyze its dependence on the choice of hedge instruments, which will be zero-coupon bonds of a different maturity.

The paper is organized as follows. In the next section, we introduce the probabilistic framework and review some previous results on misspecification. Then, we derive the dependence of the cost process on the choice of hedging instruments. Section 4 deals with model completeness and the cost process resulting from a misspecification of the volatility structure. The remainder of the paper focuses on the duplication of zero-coupon bonds. We give two criteria for the optimal choice of bond maturities. Calculations in the Vasicek model of the term structure are presented, some of which extend to the appendix. We close with some preliminary conclusions.

2. Probabilistic Framework

The purpose of this section is mainly to clarify the terms we will be using and to fix notation. We use the same terminology as in our previous paper Dudenhausen, Schlögl and Schlögl (1998).

All the stochastic processes we consider are defined on an underlying stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}, P)$, which satisfies the usual hypotheses. The probability measure P only serves to fix the equivalence class of the probability measures describing the expectations of agents in the economy. In particular, it is not assumed to be a martingale measure. Trading terminates at time $T^* > 0$. We assume that the price processes of underlying assets are described by strictly positive, continuous semimartingales.

DEFINITION 2.1 (Trading Strategy, Duplication). Let $Y^{(1)}, \ldots, Y^{(N)}$ denote the price processes of underlying assets. A trading strategy ϕ in these assets is given by an $\mathbb{R}^N-$valued, predictable process which is integrable with respect to Y. The value process $V(\phi)$ associated with ϕ is defined by

$$V(\phi) = \sum_{i=1}^{N} \phi^{(i)} Y^{(i)}$$

If C is a contingent claim with maturity T, then ϕ duplicates C iff

$$V_T(\phi) = C \quad P - \text{a.s.}$$

If X denotes the price process of an additional asset, then the strategy ϕ identically replicates X iff the two processes $V(\phi)$ and X are indistinguishable.

Note that trading strategies are not assumed to be self-financing. This is because even strategies which are self-financing within a given model will no longer be so under the effects of misspecification. The degree to which a strategy is not self-financing is captured by its cost process, which is defined as follows:

DEFINITION 2.2 (Cost Process). If ϕ is a trading strategy in the assets $Y^{(1)}, \ldots, Y^{(N)}$, the cost process $L(\phi)$ associated with ϕ is defined as follows:

$$L_t(\phi) := V_t(\phi) - V_0(\phi) - \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} \, dY_u^{(i)}$$

The infinitesimal increment $dL_t(\phi)$ is the incremental cost incurred at time t by rebalancing the portfolio as prescribed by the strategy ϕ. The portfolio strategies we consider in actual calculations will be continuous semimartingales themselves. If ϕ is a continuous semimartingale, the same is true of the value process $V(\phi)$ and the cost process $L(\phi)$. Itô's lemma then implies that we can calculate the increment of the cost process as follows

$$dL(\phi) = \sum_{i=1}^{N} \left(Y^{(i)} \, d\phi^{(i)} + d \left\langle Y^{(i)}, \, \phi^{(i)} \right\rangle \right) \tag{1}$$

The strategy ϕ is self-financing iff the cost process $L(\phi)$ is identically zero. In this case, the value process $V(\phi)$ is alway a continuous semimartingale because it can be represented as a stochastic integral.

In the presence of misspecification, the duplication of a contingent claim by a self-financing strategy may not be possible. An alternative concept is that of a superhedge.

DEFINITION 2.3 (Superhedge). Consider a contingent claim C maturing at time $T \in [0, T^*]$. A superhedge for C is a portfolio strategy ϕ which replicates C and for which the paths of the rebalancing cost process $L(\phi)$ are almost surely monotonically decreasing.

According to our definition, a strategy ϕ replicating a contingent claim C maturing at $T \in [0, T^*]$ is a superhedge iff at each time $t \in [0, T]$ the incremental cost $dL_t(\phi)$ of rebalancing the portfolio is non-positive, so that no funds need to be injected into the portfolio while still replicating the contingent claim at time T. Our definition is similar the one introduced in Föllmer and Sondermann (1986). In that paper, it is shown how one can arrive at a representation of the cost process as the continuous-time limit of the costs incurred when the portfolio is rebalanced discretely. The concept of a superhedge only refers to the local properties of the cost process. Therefore, it can be shown (cf. Dudenhausen, Schlögl and Schlögl (1998)) that discounting is not an issue. We content ourselves with mentioning that the discounted cost process $L^*(\phi)$ is given by

$$L_t^*(\phi) := V_t^*(\phi) - \sum_{i=1}^{N} \int_0^t \phi_u^{(i)}\, dY_u^{*(i)} \tag{2}$$

where $V^*(\phi)$ and $Y^{*(i)}$ denote the discounted value process and asset price processes, respectively.

For the convenience of the reader, we now summarize the results on the robustness of Gaussian hedges relevant for our present work. A more detailed exposition, complete with proofs, can be found in Dudenhausen, Schlögl and Schlögl (1998).

The crucial assumption which characterizes Gaussian hedges is that the stochastic dynamics of the process relevant for hedging are driven by a geometric Brownian motion. In particular, this implies that the volatility is deterministic. It follows that hedge ratios can be expressed in terms of the cumulative distribution function of the standard normal distribution, giving rise to the term "Gaussian hedge". We give a formal definition of lognormality.

DEFINITION 2.4 (Lognormal Process). We call a stochastic process Z *lognormal* iff it can be written in the form

$$dZ_t = Z_t\left(\mu_t dt + \tilde{\sigma}_Z(t) dW_t\right) \tag{3}$$

with *deterministic* dispersion coefficients $\tilde{\sigma}_Z : [0, T[\to \mathbb{R}_+^d$.

A convenient payoff structure to study the effects of misspecification is that of an exchange option. One has the following result, originally due to Margrabe (1978).

THEOREM 2.5. *Let X, Y be the price processes of two assets. Consider an option to exchange X for Y at the maturity date T, i.e. a European option with payoff $[X_T - Y_T]^+$. In a model where the quotient process $Z := \frac{X}{Y}$ is lognormal, it holds that*

(a) *The price process $C = (C_t)_{0 \le t \le T}$ of the exchange option is given by*

$$C_t = C(t, X_t, Y_t) := X_t \mathcal{N}(h^{(1)}(t, Z_t)) - Y_t \mathcal{N}(h^{(2)}(t, Z_t))$$

where $\mathcal{N}$ denotes the one-dimensional standard normal distribution function, $\tilde{\sigma}_Z$ is the deterministic volatility of Z, and where the functions $h^{(1)}$ and $h^{(2)}$ are given by

$$h^{(1)}(t, z) = \frac{\ln(z) + \frac{1}{2}\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}{\sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}} \tag{4}$$

$$h^{(2)}(t, z) = h^{(1)}(t, z) - \sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds} \tag{5}$$

(b) *The hedge portfolio* $\Phi = (\Phi)_{0 \le t \le T}$ *for this option in terms of the assets* X *and* Y *is given by*

$$\begin{array}{rrcll} & \phi_t^X & := & \mathcal{N}(h^{(1)}(t, Z_t)) & \textit{units of } X \\ \textit{and} & \phi_t^Y & := & -\mathcal{N}(h^{(2)}(t, Z_t)) & \textit{units of } Y \end{array}$$

The two assets X and Y are the so-called "natural" hedge instruments for the exchange option. This is so because the option can be hedged solely by taking positions in X and Y, regardless of the number of factors driving the stochastic dynamics.

It can be shown that the cost process of the strategy given in theorem 2.5 is always of bounded variation. If the true dynamics of Z are such that its martingale part can be written as

$$dZ^M = \sum_{i=1}^{n} Z\sigma^{(i)} dW^{(i)} \tag{6}$$

one obtains a more explicit representation of the cost process:

Proposition 2.6. *Given assumption (6) and a hedging strategy according to theorem 2.5, we have the following equation on* $[0, T[$*:*

$$dL_t = \frac{1}{2} X_t \mathcal{N}'(H_t^{(1)}) \frac{1}{\tilde{v}(t)} \left(\|\sigma_t\|^2 - \|\tilde{\sigma}_Z(t)\|^2 \right) dt \tag{7}$$

where $H_t^{(1)} = h^{(1}(t, Z_t)$*.*

In particular this implies

Corollary 2.7. *By equation (7), the strategy* Φ *is a superhedge for the exchange option iff for each* $t \in [0, T]$ *we have*

$$\|\sigma_t\| \le \|\tilde{\sigma}_Z(t)\| \quad P\text{-}a.s. \tag{8}$$

3. Lack Of "Natural" Hedge Instruments

The results reviewed in the previous section show that "Gaussian" hedges, if carried out in the "natural" hedging instruments, are robust in the sense that they imply a cost process of finite variation irrespective of the true dynamics of the underlying assets. If an upper bound for the volatility of the underlying is known, the Gaussian hedging strategy obtained for the maximum volatility superreplicates the option. At the same time a non-trivial superhedge is only possible given a minimum amount of information on the hedge instruments.

However, in many applications, in particular fixed income derivatives, the "natural" instruments leading to a cost process of finite variation are not always traded. As an example, consider even the plain vanilla European call option. The natural hedging instruments in this case are the underlying and the zero coupon bond maturing at option expiry. Typically, such a zero coupon bond will not be liquidly traded, and therefore not available for hedging purposes.

This means that we also need information on the correlation of the natural instruments with those available in the market. Given this information in a complete market, the natural hedging instruments can at least theoretically be synthesized by a dynamic hedging strategy. Thus we will now analyze the dependence of the cost process on the choice of hedge instruments.

Suppose that we are given a hedging strategy $\phi = (\phi^X, \phi^Y)$ for a contingent claim C with maturity T which involves positions in the underlying assets X and Y. Also, we assume that there are additional assets $Y^{(1)}, \ldots, Y^{(n)}$, where $Y^{(1)} = Y$. If we want to hedge C without using asset X, a natural way to proceed is to find a strategy $\tilde{\phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)$ involving positions in $Y^{(1)}, \ldots, Y^{(n)}$, which identically replicates X, so that

$$\forall t \in [0,T] : X_t = \sum_{i=1}^{n} \tilde{\phi}_t^i Y_t^{(i)} \tag{9}$$

This immediately gives a hedging strategy $\psi = (\psi^1, \ldots, \psi^n)$ for C_T in $Y^{(1)}, \ldots, Y^{(n)}$ by

$$\psi^1 = \phi^Y + \phi^X \tilde{\phi}^1 \quad , \quad \psi^i = \phi^X \tilde{\phi}^i \quad \forall i \geq 2$$

LEMMA 3.1. *The cost processes of ϕ and ψ are related as follows*

$$dL(\psi) = dL(\phi) + \phi^X dL(\tilde{\phi}) \tag{10}$$

PROOF: The cost process of ψ is given by

$$dL(\psi) = dV(\psi) - \sum_{i=1}^{n} \psi^i dY^{(i)}$$

By construction, $V(\psi) = V(\phi)$, so that

$$\begin{aligned} dL(\psi) &= dV(\phi) - \sum_{i=1}^{n} \psi^i dY^{(i)} \\ &= dL(\phi) + \phi^X dX + \phi^Y dY - \sum_{i=1}^{n} \psi^i dY^{(i)} \\ &= dL(\phi) + \phi^X \left\{ dX - \sum_{i=1}^{n} \tilde{\phi}^i dY^{(i)} \right\} \\ &= dL(\phi) + \phi^X dL(\tilde{\phi}) \end{aligned}$$

□

Suppose now that $\phi = (\phi^X, \phi^Y)$ is a superreplicating strategy for C_T, i.e. it holds that

$$dL(\phi) \leq 0$$

The condition for ψ to be superreplicating is

$$dL(\psi) = dL(\phi) + \phi^X dL(\tilde{\phi}) \leq 0$$

Without any information on the interdependence of the two cost processes $L(\phi)$ and $L(\tilde{\phi})$, the most obvious superhedging strategy would be to demand that each term in the equation above is non-positive. From the hedging formula presented in theorem 2.5, one can assume that ϕ^X does not change its sign. Therefore, $L(\tilde{\phi})$ must be monotonic, i.e. a sub- or superhedge for X. If ϕ^X is positive, this effectively means that the superhedge is constructed in two steps. First the missing asset X is superhedged with the available assets.

This superhedge is then used as an input for the original superhedging strategy. However, in the next section we show that, since $\tilde{\phi}$ is a strategy identically replicating the asset X, the cost process $L(\tilde{\phi})$ has a non-vanishing martingale part. This implies that a superhedge is not possible using such a two-step strategy.

4. Synthesizing Hedge Instruments

In this section, we study the case where the hedge instrument X is not liquidly traded in the market and a potential hedger must use other assets $Y^1, ..., Y^n$ to synthesize X. We place ourselves in a diffusion setting, i.e. the prices X, $Y^1, \dots, Y^n$ are given by Itô processes which are driven by a d-dimensional Brownian motion W defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$dX_t = X_t\{\mu_t^X dt + \sigma_t^X dW_t\} \tag{11}$$

$$dY_t^i = Y_t^i\{\mu_t^i dt + \sigma_t^i dW_t\} \tag{12}$$

where μ^X, σ^X and μ^i, σ^i are suitably integrable stochastic processes. We assume the prices X, $Y^1, \dots, Y^n$ are arbitrage-free. This implies that there is a "market price of risk" process φ such that for any $i \in \{1, \dots, n\}$:

$$\mu^X - \sigma^X \varphi = \mu^i - \sigma^i \varphi$$

Synthesizing X out of $Y^1, \dots, Y^n$ involves finding a self-financing strategy ϕ with a position of ϕ^i in asset Y^i for each $i \in \{1, \dots, n\}$ such that $X = \sum_{i=1}^n \phi^i Y^i$. The following proposition characterizes these strategies ϕ.

Proposition 4.1. *Suppose that $\lambda^1, \dots, \lambda^n$ are predictable processes satisfying the following to conditions:*

$$(i) \ \sum_{i=1}^n \lambda_t^i = 1 \qquad (ii) \ \sum_{i=1}^n \lambda_t^i \sigma_t^i = \sigma_t^X$$

For each $i \in \{1, \dots, n\}$, we set $\phi^i := \frac{X}{Y^i}\lambda^i$. Then ϕ is a self-financing strategy which identically duplicates X. In particular, any such strategy is of the form above.

Proof: Suppose that weights $\lambda^1, \dots, \lambda^n$ are given which satisfy conditions (i) and (ii) and that ϕ is the corresponding strategy. By condition (i), it is clear that $\sum_{i=1}^n \phi^i Y^i = X$. By the no-arbitrage condition and because of (ii) we have

$$\sum_{i=1}^n \lambda^i \mu^i = \sum_{i=1}^n \lambda^i \{\mu^X + \varphi(\sigma^i - \sigma^X)\} = \mu^X$$

From this we see that ϕ is also self-financing because

$$\sum_{i=1}^n \phi_t^i dY_t^i = X_t \sum_{i=1}^n \lambda_t^i \{\mu_t^i dt + \sigma_t^i dW_t\} = X_t\{\mu^X dt + \sigma_t^X dW_t\} = dX_t$$

Conversely, if ϕ is a self-financing strategy which identically duplicates X, then the weights $\lambda^1, \dots, \lambda^n$ determined by $\lambda^i := \frac{Y^i}{X}\phi^i$ will satisfy the two conditions.

□

The weights $\lambda^1, \dots, \lambda^n$ are to be interpreted as portfolio weights, i.e. λ^i is the proportion of total capital to be invested in asset Y^i.

A question that arises naturally is whether a duplicating strategy exists. This is only true irrespective of the concrete choice of X if the market determined by $Y^1, \dots, Y^n$ is dynamically complete. It is helpful to clarify the minimum asset structure necessary for dynamic completeness as we wish to focus on trading only a limited number of available assets. In particular, it is unrealistic to assume that one of these is a continuously compounded savings account, i.e. instantaneously risk-free. The characterization of completeness in this case is a slight generalization of the classical Black-Scholes framework, which has been well-studied in the literature (cf. chapter 6 of Duffie (1996)).

DEFINITION 4.2 (Completeness). The market determined by the assets $Y^{(1)}, ..., Y^{(n)}$ is complete iff any (suitably integrable) contingent T-claim is attainable; that is, if for any such claim C there exists a self-financing portfolio strategy ϕ in $Y^{(1)}, ..., Y^{(n)}$ such that $C = V_T(\phi)$. In the opposite case, the model is said to be *incomplete.*

PROPOSITION 4.3. *The market given by the assets $Y^1, \ldots, Y^n$ is complete iff, for $\lambda^1 \otimes P$-almost all $(t, \omega) \in [0, T^*] \times \Omega$, the affine subspace generated by $\sigma_t^1(\omega), \ldots, \sigma_t^n(\omega)$ has dimension d.*

The proof of this proposition is given in the appendix. It is a straight-forward generalization of the rank condition in the classical Black-Scholes setting. However, in our case it is the affine structure of the volatility that matters. This is obscured in the case where one asset is the continuously compounded savings account, because one only needs to consider the volatility structure of the remaining "risky" assets.

The rank condition also tells us that the minimal number of assets needed for dynamic completeness is $n = d + 1$. If the market determined by $Y^1, \ldots, Y^n$ is complete and $n = d + 1$, the volatility vectors $\sigma^1, \ldots, \sigma^n$ must be affine independent almost surely. This implies that the porfolio weights λ^i given in proposition 4.1 and therefore the replicating strategy for X are uniquely determined.

5. VOLATILITY MISSPECIFICATION

The strategy ϕ descibed in proposition 4.1 requires knowledge of the true asset volatilities. We now examine the case where the hedger does not have this information and constructs a strategy based on assumed volatilities $\tilde{\sigma}^X$, $\tilde{\sigma}^1, \ldots, \tilde{\sigma}^n$. We think of these assumed volatilities as having been obtained from a model which is fit to the prices X, $Y^1, \ldots, Y^n$. In other words, the weights $\lambda^1, \ldots, \lambda^n$ of the strategy are constructed so that

$$(i') \quad \sum_{i=1}^{n} \lambda_t^i = 1 \qquad (ii') \quad \sum_{i=1}^{n} \lambda_t^i \tilde{\sigma}_t^i = \tilde{\sigma}_t^X$$

In this case the strategy will still replicate X, but it will no longer be self-financing.

PROPOSITION 5.1. *The cost process of the strategy ϕ is given by*

$$dL_t = X_t \left(\sigma_t^X - \sum_{i=1}^{n} \lambda_t^i \sigma_t^i \right) (\varphi_t dt + dW_t) \tag{13}$$

If we use Y^n as numeraire, the discounted cost process L^* has the form

$$dL_t^* = X_t^* \left(\sigma_t^X - \sum_{i=1}^{n} \lambda_t^i \sigma_t^i \right) \{(\varphi_t - \sigma_t^n)dt + dW_t\} \tag{14}$$

PROOF: The costs of the strategy are given by $dL = dX - \sum_{i=1}^n \phi^i dY^i$. The definition of ϕ^i gives

$$dL_t = X_t \left\{ \left(\mu_t^X - \sum_{i=1}^n \lambda_t^i \mu_t^i \right) dt + \left(\sigma_t^X - \sum_{i=1}^n \lambda_t^i \sigma_t^i \right) dW_t \right\}$$

The no-arbitrage condition implies that

$$\mu^X - \sum_{i=1}^n \lambda^i \mu^i = \left(\sigma^X - \sum_{i=1}^n \lambda^i \sigma^i \right) \varphi$$

Therefore

$$dL_t = X_t \left(\sigma_t^X - \sum_{i=1}^n \lambda_t^i \sigma_t^i \right) (\varphi_t dt + dW_t)$$

The dicounted cost process is given by $dL^* = dX^* - \sum_{i=1}^n \phi^i dY^{i*}$. The dynamics of the asset prices are

$$dX_t^* = X_t^* (\sigma_t^X - \sigma_t^n) \{ (\varphi_t - \sigma_t^n) dt + dW_t \}$$
$$dY^{i*}_t = Y^{i*}_t (\sigma_t^i - \sigma_t^n) \{ (\varphi_t - \sigma_t^n) dt + dW_t \}$$

Using the properties of λ^i we get

$$dL_t^* = X_t^* \left(\sigma_t^X - \sum_{i=1}^n \lambda_t^i \sigma_t^i \right) \{ (\varphi_t - \sigma_t^n) dt + dW_t \}$$

□

The cost process vanishes iff $\sigma^X = \sum_{i=1}^n \lambda^i \sigma^i$. Generically, if the true volatilities are unknown, the weights λ^i chosen by the hedger will not satisfy this relation. We see then that the cost process necessarily has a non-vanishing martingale part. The size of this is determined by the expression

$$\Xi := \left\| \sigma^X - \sum_{i=1}^n \lambda^i \sigma^i \right\|, \tag{15}$$

which we call the volatility mismatch. In the next section, we analyze the problem of choosing the optimal instruments so as to minimize Ξ. Since selecting the best hedging instruments requires some knowledge about relationships between asset volatilities, we focus on fixed income securities, where some "stylized facts" about the term structure of volatility are available.

6. Duplication of Bonds

The term structure of volatility is particularly transparent for zero coupon bonds. This leads us to consider the problem of duplicating a zero-coupon bond with bonds of different maturities. For simplicity we restrict ourselves to the case of duplicating a zero coupon bond using only two other bonds. To be consistent, this implies that the hedger assumes a one-factor term structure model. For each maturity $T \in \mathbb{R}_+$ we denote the assumed lognormal bond price dynamics by

$$dB(t,T)^M = B(t,T)\tilde{\sigma}(t,T) dW_t \tag{16}$$

where $\tilde{\sigma}$ is a deterministic function which is monotonic in T and W is a one-dimensional Brownian motion. For maturities T_1 and T_2, Itô's lemma tells us that

$$d\left(\frac{B(t,T_2)}{B(t,T_1)}\right)^M = \frac{B(t,T_2)}{B(t,T_1)}\Big\{\tilde{\sigma}(t,T_2) - \tilde{\sigma}(t,T_1)\Big\}dW_t$$

With respect to the assumed model, the volatility of the forward price process of the bond T_2 with respect to T_1 is given by $\tilde{\sigma}(t,T_2,T_1) := \tilde{\sigma}(t,T_2) - \tilde{\sigma}(t,T_1)$. As we only consider bond price volatilities and volatilities of forward prices, we will refer to $\tilde{\sigma}(t,T_2,T_1)$ as the forward volatility and also use the shorter notation $\tilde{\sigma}^{T_2,T_1}(t)$.

LEMMA 6.1. *Let maturities $T, T_1, T_2 \in \mathbb{R}$ be given. In a one-factor term structure model as described above, there exists a unique self-financing strategy $\tilde{\phi} = \left(\tilde{\phi}^1, \tilde{\phi}^2\right)$ in the bonds T_1 and T_2 which identically replicates the bond T, i.e.*

$$B(.\,,T) = \tilde{\phi}^1 B(.\,,T_1) + \tilde{\phi}^2 B(.\,,T_2)$$

The trading strategy is given by

$$\tilde{\phi}^1_t = \frac{B(t,T)}{B(t,T_1)} \frac{\tilde{\sigma}(t,T_2,T)}{\tilde{\sigma}(t,T_2,T_1)}$$

$$\tilde{\phi}^2_t = \frac{B(t,T)}{B(t,T_2)} \frac{\tilde{\sigma}(t,T,T_1)}{\tilde{\sigma}(t,T_2,T_1)}$$

PROOF: Since the volatilities $\tilde{\sigma}(.,T_1)$, $\tilde{\sigma}(.,T_2)$ are not equal, they are affine independent and proposition 4.3 tells us that the market determined by the two bonds is complete. Therefore, there exists a self-financing strategy $\tilde{\phi}$ which identically replicates the bond T. In particular, for $i = 1, 2$

$$\tilde{\phi}^i_t = \frac{\lambda^i(t) B(t,T)}{B(t,T_i)}$$

where λ^1, λ^2 are determined by the equations

$$\tilde{\sigma}(t,T) = \lambda^1(t)\tilde{\sigma}(t,T_1) + \lambda^2(t)\tilde{\sigma}(t,T_2)$$
$$1 = \lambda^1(t) + \lambda^2(t)$$

This immediately gives

$$\lambda^1(t) = \frac{\tilde{\sigma}(t,T_2,T)}{\tilde{\sigma}(t,T_2,T_1)} \tag{17}$$

$$\lambda^2(t) = \frac{\tilde{\sigma}(t,T,T_1)}{\tilde{\sigma}(t,T_2,T_1)} \tag{18}$$

resulting in the claimed strategy.

□

There is no reason to assume a one-factor model perfectly reflects reality. So, in contrast to the hedger's assumptions, we let the real bond price dynamics be driven by an n–dimensional Brownian motion W. For each maturity $T \in [0, T^*]$, there is an $\mathbb{R}^n$-valued stochastic volatility process $\sigma^T := \sigma(.,T)$ such that

$$dB(t,T)^M = B(t,T)\sigma(t,T)dW_t$$

Once again, σ^{T_1,T_2} denotes the forward volatility process $\sigma^{T_1,T_2} := \sigma^{T_1} - \sigma^{T_2}$.

LEMMA 6.2. *Let maturities T, T_1, T_2 be given and let $\tilde{\phi}$ be the replication strategy given in the previous lemma. Then we have the following two representations for the martingale part of the cost process*

$$dL(\tilde{\phi})_t^M = B(t,T)\left\{\sigma_t^T - \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_1} - \frac{\tilde{\sigma}^{T,T_1}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2}\right\}dW_t \tag{19}$$

$$dL(\tilde{\phi})_t^M = B(t,T)\,\tilde{\sigma}^{T_2,T}(t)\left\{\frac{1}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2,T_1} - \frac{1}{\tilde{\sigma}^{T_2,T}(t)}\,\sigma_t^{T_2,T}\right\}dW_t \tag{20}$$

PROOF: Equation (19) can be obtained by inserting the values for λ^1, λ^2 above into the first equation of proposition 4.3. By definition, $\tilde{\sigma}^{T,T_1}(t) = \tilde{\sigma}^{T,T_2}(t) + \tilde{\sigma}^{T_2,T_1}(t)$.
Replacing $\tilde{\sigma}^{T,T_1}$ results in

$$dL(\tilde{\phi})_t^M = B(t,T)\left\{\sigma_t^{T,T_2} + \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2,T_1}\right\}dW_t$$

Simple regrouping now yields (20).

□

7. OPTIMAL SELECTION OF BOND MATURITIES

We give some optimality criteria for the choice of bond maturities when duplicating a zero-coupon bond. We use the instantaneous variance of the cost process to determine optimality. From equation (19) we see that

$$d\langle L(\tilde{\phi})\rangle_t = B(t,T)^2\left\|\sigma_t^T - \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_1} - \frac{\tilde{\sigma}^{T,T_1}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2}\right\|^2 dt$$

We introduce the following notation for the volatility mismatch.

$$K_t^1 := \left\|\sigma_t^T - \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_1} - \frac{\tilde{\sigma}^{T,T_1}(t)}{\tilde{\sigma}^{T_2,T_1}(t)}\,\sigma_t^{T_2}\right\| \tag{21}$$

A bond price model is determined by the bond volatilities. In particular, we can only find a self-financing replicating strategy for the bond with maturity T if it is possible to write its volatility as a linear combination of the hedge instruments' volatilities, (cf. proposition 4.3). Due to misspecification, it is no longer possible to match the bond volatility exactly. The choice of hedge instruments is determined by the attempt to make this mismatching as small as possible. Intuitively the best hedge instruments are those whose volatility structure is as close to that of the bond being hedged as possible, i.e. those bonds with the closest maturity dates. The next question which presents itself is whether to use longer or shorter bonds. Typically, bond volatility is increasing in the time to maturity. Therefore one might be tempted to prefer bonds with shorter maturities as hedge instruments due to their lower volatility. However, the effect of misspecification is determined by the relationship between true and assumed forward volatilies and not by the absolute value of the volatilities.

In practical applications one deals with an exogenously given finite set of bond maturities, so that the following definition can be used to determine the optimal pair of maturities for hedging.

Definition 7.1 (Robustness of hedge instruments). Let (T_1, T_2) and (T_1', T_2') be two pairs of bond maturities available for hedging the unavailable bond with maturity T. The pair (T_1, T_2) is *preferable* to (T_1', T_2') *iff*

$$\left\| \sigma_t^T - \frac{\tilde{\sigma}^{T_2,T}(t)}{\tilde{\sigma}^{T_2,T_1}(t)} \, \sigma_t^{T_1} - \frac{\tilde{\sigma}^{T,T_1}(t)}{\tilde{\sigma}^{T_2,T_1}(t)} \, \sigma_t^{T_2} \right\| \leq \left\| \sigma_t^T - \frac{\tilde{\sigma}^{T_2',T}(t)}{\tilde{\sigma}^{T_2',T_1'}(t)} \, \sigma_t^{T_1'} - \frac{\tilde{\sigma}^{T,T_1'}(t)}{\tilde{\sigma}^{T_2',T_1'}(t)} \, \sigma_t^{T_2'} \right\|$$

In this case it might be possible to find an optimal pair of maturities, given additional assumptions on the relationship between the true and the assumed volatility structure. A different approach, which we follow here, is to look at the behaviour of the cost process along the whole yield curve.

As before we want to duplicate a bond with maturity T. We fix the maturity of one bond and are interested in the effect of varying the maturity of the other bond. Since the problem is symmetric in T_1 and T_2 we choose to fix $T_2 \neq T$. From equation (20) we see that in this case we must study the process K^2 defined by:

$$K_t^2 := \left\| \frac{1}{\tilde{\sigma}^{T_2,T_1}(t)} \, \sigma_t^{T_2,T_1} - \frac{1}{\tilde{\sigma}^{T_2,T}(t)} \, \sigma_t^{T_2,T} \right\|$$

In the same manner as above, we compare different maturities as follows:

Definition 7.2. Let T and T_2 be fixed maturities of zero coupon bonds, T denotes the maturity of the bond to be duplicated and T_2 the maturity of one bond which we fix as a hedge instrument. We call the bond maturing at T_1 *preferable* to the one maturing at T_1' *iff* the instantaneous variance of the cost process is smaller,i.e.

$$\forall t \in [0, T] \quad \left\| \frac{1}{\tilde{\sigma}^{T_2,T_1}(t)} \, \sigma_t^{T_2,T_1} - \frac{1}{\tilde{\sigma}^{T_2,T}(t)} \, \sigma_t^{T_2,T} \right\| < \left\| \frac{1}{\tilde{\sigma}^{T_2,T_1'}(t)} \, \sigma_t^{T_2,T_1'} - \frac{1}{\tilde{\sigma}^{T_2,T}(t)} \, \sigma_t^{T_2,T} \right\|$$

A general solution of the problems stated above is not possible without further assumptions both on the set of available bond maturities as well as the true volatility structure. Therefore, further analysis requires a more specialized framework. The simplest case is that of the one-factor Vasicek model, which we consider next.

8. Examination of the Extended Vasicek Model

In this section we explicitly calculate the costs of bond duplication for the case of parameter misspecification in the extended Vasicek model. Under the spot martingale measure, the true dynamics of the continuously compounded spot rate are described by

$$dr_t = (\theta(t) - ar_t) \, dt + \sigma \, dW_t$$

Standard calculations show:

Proposition 8.1. *The volatility of a bond with maturity T is*

$$\sigma_t^T = \frac{\sigma}{a}(1 - e^{-a(T-t)})$$

and the forward volatility for T_2 with respect to T_1 is

$$\sigma_t^{T_2,T_1} = \frac{\sigma}{a}(e^{-a(T_1-t)} - e^{-a(T_2-t)})$$

The price of a European call-option with maturity T_1 on a bond with maturity T_2 is given by

$$
\begin{aligned}
C(t,T_2) &= B(t,T_2)N(h_1(t,T_1)) - KB(t,T_1)N(h_2(t,T_1)) \\
h_{1/2}(t,T_1) &= \frac{\ln\left(\frac{B(t,T_2)}{B(t,T_1)}\right) - \ln K \pm \frac{1}{2}v_{T_2}^2(t,T_1)}{v_{T_2}(t,T_1)} \\
v_{T_2}^2(t,T_1) &= \frac{\sigma^2}{2a^3}\left(1-e^{-2a(T_1-t)}\right)\left(1-e^{-a(T_2-t)}\right)^2
\end{aligned}
$$

In practice, the reversion level θ is used to fit the model to observed bond prices. However, it appears in neither the bond volatilities nor the call option price. "Explaining" the observed prices does not help to find the "right" hedging strategy, so that the hedger is still uncertain about the underlying parameters a and σ for which he assumes values of $\tilde{a}$ and $\tilde{\sigma}$. According to our definition, the best choice of hedge instruments for the duplication of a bond with maturity T minimizes the expression

$$
\left|\frac{\tilde{\sigma}_t^{T_2,T}}{\tilde{\sigma}_t^{T_2,T_1}}\sigma_t^{T_1} + \frac{\tilde{\sigma}_t^{T,T_1}}{\tilde{\sigma}_t^{T_2,T_1}}\sigma_t^{T_2} - \sigma_t^{T}\right| \tag{22}
$$

Due to the form of the bond volatility we can set $t = 0$ without loss of generality. The coefficients $\alpha_1 = \frac{\tilde{\sigma}_t^{T_2,T}}{\tilde{\sigma}_t^{T_2,T_1}}$ and $\alpha_2 = \frac{\tilde{\sigma}_t^{T,T_1}}{\tilde{\sigma}_t^{T_2,T_1}}$ appearing in (22) are independent of $\tilde{\sigma}$. This means that the misspecification of the short rate volatility does not influence the choice of hedge instruments. Therefore we assume $\sigma = \tilde{\sigma}$ and concentrate on the misspecification ofthe speed of mean reversion a.The bond price volatility is a decreasing function of a for every maturity so that the misspecification of a affects the bond volatilities in the same manner for all maturities.

Proposition 8.2.
Consider the case where $\tilde{a} < a$, i.e. an overestimation of all bond volatilities. Again, T denotes the maturity of the bond to be duplicated, T_2 the maturity of one bond which we fix as a hedge instrument.

(a) *Let T_1 and T_1' be two maturities, so that either $T_1, T_1' < T$ or $T_1, T_1' > T$. Then T_1 is preferable to T_1' in the sense of definition (7.2) iff $\mid T_1 - T \mid \leq \mid T_1' - T \mid$*

(b) *For every $\mu > 0$, $T_1 = T + \mu$ is preferable to $T_1' = T - \mu$*

In figure (8) we see that for fixed T_2 the volatility mismatch increases in the distance to T as stated in (a). If the distance to T is fixed, it is always better to hedge using the bond with the longer maturity. The distance of T_2 to T is equal in both plots. Comparing the two plots shows that the hedge costs are uniformly higher in T_1 if T_2 is shorter than T, once again illustrating part (b) of the proposition.

Proof of Proposition 8.2: Recall that we must minimize K_0^2, where

$$
K_0^2 = \left|\frac{1}{\tilde{\sigma}^{T_2,T_1}(0)}\,\sigma_0^{T_2,T_1} - \frac{1}{\tilde{\sigma}^{T_2,T}(0)}\,\sigma_0^{T_2,T}\right|
$$

We define the function v as

$$
v(T_1) := \frac{\sigma^{T_2,T_1}(0)}{\tilde{\sigma}^{T_2,T_1}(0)}
$$

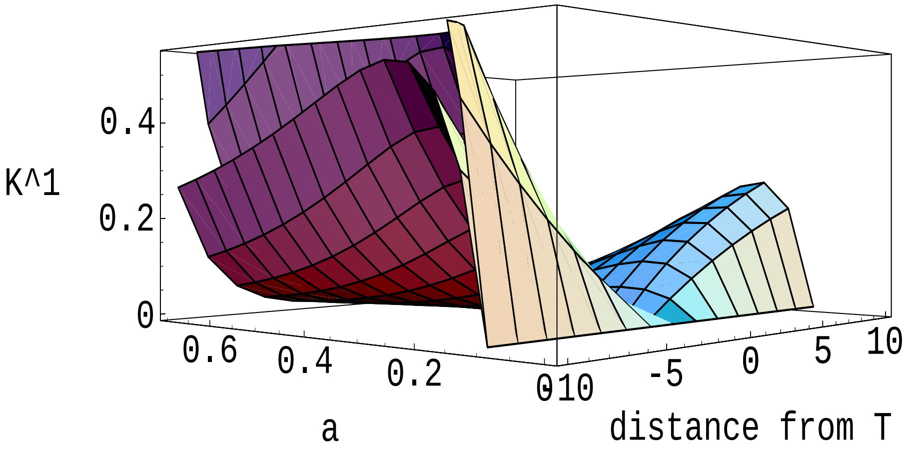

FIGURE 1. K^1 is plotted for $\tilde{a} = 0,1$, $T = 10$ and a fixed $T_2 = 5(15)$ with varying T_1 and a, $\quad (a > 0.1)$.

In the extended Vasicek model, v has the explicit form (recall that $\sigma = \tilde{\sigma}$)

$$v(T_1) = \frac{\tilde{a}}{a} \frac{e^{-aT_1} - e^{-aT_2}}{e^{-\tilde{a}T_1} - e^{-\tilde{a}T_2}}$$

The behaviour of K_0^2 is determined by v, because

$$K_0^2 = |v(T_1) - v(T)|$$

Recall that we are assuming $\tilde{a} < a$, i.e. that the hedger is overestimating bond volatility. As we show in the appendix, this implies that v is a strictly decreasing function of T_1. In particular, T is the only zero of K_0^2. This proves statement (a). Statement (b) follows from the fact that v is convex. The tedious proof of this is relegated to the appendix.

□

REMARK 8.3. Statement (b) of the proposition is only true if $\tilde{a} < a$. For $\tilde{a} > a$ the result is just the opposite, i.e. for every $\mu > 0$, $T - \mu$ is preferable to $T + \mu$.

9. CONCLUSIONS

In this paper, we have extended our research of Dudenhausen, Schlögl and Schlögl (1998) on the misspecification problem to the case where the "natural" instruments are not available for hedging purposes. We clarify the conditions on the volatility structure of traded assets needed to ensure market completeness in the case where none of the traded assets is a continuously rolled-over savings account. It turns out that the conditions are quite similar to those well-known from the classical Black/Scholes case. However, it is the affine structure of volatility which is decisive. This is due to the fact that the more general case can be reduced to the one where a locally risk-free asset exists by a change of numeraire.

We have analyzed the cost process obtained when identically replicating one asset with others and shown that it necessarily has a non-vanishing martingale part. On the basis of this result some criteria for the optimal choice of hedging instruments have been presented.

In particular, we have studied the problem of duplicating a zero-coupon bond with bonds of different maturities. Concrete calculations have been performed for the case of mis-specification of the mean-reversion parameter in the one-factor Vasicek model. From these some recommendations as to the optimal choice of maturity structure have been derived.

The extension of this analysis to other term-structure models and other types of mis-specification is the subject of on-going research.

10. Appendix

Proof of Proposition 4.3: 1. We choose Y^n as numeraire and denote the corresponding martingale measure by Q. We will start by showing that the rank condition is necessary. Let C_T be an attainable contingent claim settling at time T. Its price process C is given by

$$C_t := Y_t^n E^Q \left[\left. \frac{C_T}{Y_T^n} \right| \mathcal{F}_t \right] \tag{23}$$

The dynamics of the discounted price process $C^* := \frac{C}{Y^n}$ can be written as

$$dC^* = C^*(\sigma - \sigma^n)\, dW^*$$

where W^* is a d-dimensional Q-Brownian motion and σ is a d-dimensional predictable process. Since C_T is attainable, there is a self-financing portfolio ψ such that $C = \sum_{i=1}^n \psi^i Y^i$. The self-financing condition can be written as

$$dC^* = \sum_{i=1}^{n-1} \psi^i dY^{i*}$$

This is equivalent to

$$C^* \left(\sigma - \sigma^n\right)\, dW^* = \sum_{i=1}^{n-1} \psi^i Y^{i*} (\sigma^i - \sigma^n)\, dW^*$$

Therefore, it must be true that $\lambda^1 \otimes P$-almost surely

$$\sigma - \sigma^n = \sum_{i=1}^{n-1} \psi^i \frac{Y^i}{C} \left(\sigma^i - \sigma^n\right)$$

In particular, $\sigma_t(\omega)$ must lie in the affine subspace generated by $\sigma_t^1(\omega), \ldots, \sigma_t^n(\omega)$ for $\lambda^1 \otimes P$-almost all $(t, \omega) \in [0, T] \times \Omega$.

If the rank condition is not fulfilled, we can construct a process $\hat{\sigma}$ such that there is a set $F \subset [0, T] \times \Omega$ with $(\lambda^1 \otimes P)\,[F] > 0$ so that $\hat{\sigma}_t(\omega)$ does not lie in the affine subspace generated by $\sigma_t^1(\omega), \ldots, \sigma_t^n(\omega)$ for $(t, \omega) \in F$. We let the process $\hat{C}^*$ be a solution of the SDE

$$d\hat{C}^* = \hat{C}^*(\hat{\sigma} - \sigma^n) dW^*$$

If we define the claim $\hat{C}_T$ by $\hat{C}_T := Y_T^n \hat{C}_T^*$, then it is clear from the arguments above that $\hat{C}_T$ is not attainable.

2. We now show that the rank condition is sufficient. Again, let C_T be a (suitably integrable) claim settling at T and define C by (23). Because the rank condition is fulfilled, we can find predictable processes λ^i such that

$$\sigma - \sigma^n = \sum_{i=1}^{n-1} \lambda^i(\sigma^i - \sigma^n)$$

The process λ^n is defined by $\lambda^n := 1 - \sum_{i=1}^{n-1} \lambda^i$. We define ψ^i by $\psi^i := \frac{\lambda^i C}{Y^i}$ and show that ψ is a self-financing portfolio which replicates C. It is immediately clear that $\psi Y = C$. Furthermore

$$\begin{aligned} dC^* &= C^* \left(\sigma - \sigma^n\right) dW^* = \sum_{i=1}^{n-1} C^* \lambda^i \left(\sigma^i - \sigma^n\right) dW^* \\ &= \sum_{i=1}^{n-1} \psi^i Y^{i*} \left(\sigma^i - \sigma^n\right) dW^* = \sum_{i=1}^{n-1} \psi^i \, dY^{i*} \end{aligned} \tag{24}$$

Since the self-financing property is invariant under a change of numeraire and a change of measure, (24) shows that the portfolio ψ is indeed self-financing.

□

PROPOSITION 10.1. *Let $0 < \tilde{a} < a$ and $T_2 > 0$ be given. Then the function v defined by*

$$v(T_1) = \frac{\tilde{a}}{a} \frac{e^{-aT_1} - e^{-aT_2}}{e^{-\tilde{a}T_1} - e^{-\tilde{a}T_2}}$$

is strictly decreasing and convex.

PROOF: First we notice that the singularity of v at T_2 is removable with $v(T_2) = \frac{e^{-aT_2}}{e^{-\tilde{a}T_2}}$ The monotonicity of v is determined by

$$w := \frac{e^{-aT_1} - e^{-aT_2}}{e^{-\tilde{a}T_1} - e^{-\tilde{a}T_2}}$$

where

$$w = \frac{e^{-aT_2}}{e^{-\tilde{a}T_2}} \cdot \frac{e^{-a(T_1 - T_2)} - 1}{e^{-\tilde{a}(T_1 - T_2)} - 1}$$

Setting $x := T_1 - T_2$ we see that we have to analyse the function

$$f(x) := \frac{e^{-ax} - 1}{e^{-bx} - 1}$$

For clarity we have replaced $\tilde{a}$ by b. Straightforward calculations show that the first and second derivatives of f are

$$f'(x) = \frac{be^{bx} + ae^{2bx} - ae^{bx} - be^{(a+b)x}}{e^{ax}(e^{bx} - 1)^2}$$

$$f''(x) = \frac{e^{bx}}{e^{ax}(e^{bx} - 1)^3} \left(2ae^{bx}(a - b) + b^2(e^{ax} - e^{bx}) + e^{bx}(b^2 e^{ax} - a^2 e^{bx}) - (a - b)^2\right)$$

To prove the monotonicity of f we must consider the first derivative. It is enough to show that

$$b + ae^{bx} - be^{ax} \leq a$$

This is clearly true for $x = 0$, and it is easy to show that the left-hand side of the inequality is decreasing in x. For the convexity of f it suffices to show that

$$Z(x) := 2ae^{bx}(a-b) + b^2(e^{ax} - e^{bx}) + e^{bx}(b^2e^{ax} - a^2e^{bx}) - (a-b)^2$$

is non-negative. Again we have $Z(0) = 0$, so that we just need to demonstrate that Z is increasing. The derivative of Z is

$$Z'(x) = e^{bx}\{2ab(a-b) - b^3 - 2ba^2e^{bx} + ab^2e^{(a-b)x} + (a+b)b^2e^{ax}\} \tag{25}$$

Proceeding as before we have $Z'(0) = 0$ and again just have to show that $Z'(x)$ is increasing. Defining

$$u(x) = 2ab(a-b) - b^3 - 2ba^2e^{bx} + ab^2e^{(a-b)x} + (a+b)b^2e^{ax}$$

and we have

$$u'(x) = ab^2e^{bx}\{(a+b)e^{(a-b)x} + (a-b)e^{(a-2b)x} - 2a\}$$

As always we have $u'(0) = 0$, so that it is enough to show that u' is increasing. In fact we only need to prove that v is increasing, where v is defined by

$$v(x) := (a+b)e^{(a-b)x} + (a-b)e^{(a-2b)x} - 2a$$

The derivative of v is

$$v'(x) = (a^2 - b^2)e^{(a-b)x} + (a+b)(a-2b)e^{(a-2b)x}$$

So that

$$v'(x) \geq 0 \Leftrightarrow a^2 - b^2 + (a-b)(a-2b)e^{-bx} \geq 0$$

We define

$$w(x) := a^2 - b^2 + (a-b)(a-2b)e^{-bx}$$

The last step of the proof is to show that $w \geq 0$. This is immediate for $0 < b \leq \frac{a}{2}$, so that we can restrict ourselves to the case where $b \in [\frac{a}{2}, a[$. We have $w(0) > 0$ and

$$w'(x) = b(a-b)(2b-a)e^{-bx} \geq 0$$

It may be hard to believe, but this finally concludes the proof.

□

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