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Effectiveness of Hedging Strategies under Model Misspecification and Trading Restrictions

by

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EFFECTIVENESS OF HEDGING STRATEGIES UNDER MODEL MISSPECIFICATION AND TRADING RESTRICTIONS

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ABSTRACT. The following paper focuses on the incompleteness arising from model misspecification combined with trading restrictions. While asset price dynamics are assumed to be continuous time processes, the hedging of contingent claims occurs in discrete time. The trading strategies under consideration are understood to be self-financing with respect to an assumed model which may deviate from the "true" model, thus associating duplication costs with respect to a contingent claim to be hedged. Based on the robustness result of Gaussian hedging strategies, saying that a superhedge is achieved for convex payoff-functions if the "true" asset price volatility is dominated by the assumed one, the error of time discretising these strategies is analysed. It turns out that the time discretisation of Gaussian hedges gives rise to a duplication bias caused by asset price trends, which can be avoided by discretising the hedging model instead of discretising the hedging strategies. Additionally it is shown, that on the one hand binomial strategies incorporate similar robustness features as Gaussian hedges. On the other hand, the distribution of the cost process associated with the binomial hedge coincides with the distribution of the cost process associated with the Gaussian hedge in the limit. Together, the last results yield a strong argument in favour of discretising the hedge model instead of time discretising the strategies.

1. INTRODUCTION

Pricing by *No-Arbitrage* relies on the existence of self-financing and duplicating portfolio strategies which are specified on the basis of an assumed asset price dynamic which may of course deviate from the "true" asset price dynamic. The analysis of the implications of so called model misspecification to pricing and hedging contingent claims has achieved great acknowledgement in the scientific research. By assuming that the hedging strategies are carried out according to a model which differs from the true dynamic of market prices, the effectiveness of such strategies is analyzed in El Karoui, Jeanblanc-Piqué and Shreve (1998) and Dudenhausen, Schlögl and Schlögl (1998). The key result states that if the true volatility is locally bounded, then the hedging strategies implied by Black/Scholes-like models[1] corresponding to the upper volatility bound are robust with respect to convex payoff-functions. By using a strategy which is self-financing with respect to the "hedging model", the payoff of any contingent claim with convex payoff structure is dominated almost surely under any equivalent measure. The upper price bound is given by the initial investment into this hedging strategy. Similarly, a subhedging strategy is achieved by using

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[1]In particular, Black/Scholes-like models or Gaussian models are based on the assumption of a deterministic volatility structure such that the model is complete in the sense of Harrison and Pliska (1983) guaranteeing the existence of a self-financing trading strategy duplicating the payoff of the claim to be hedged.

the Black/Scholes model corresponding to the lower volatility bound. This strategy yields the lower arbitrage free price bound such that according to an uncertain volatility model there exists a whole interval of arbitrage free prices.

Additional to the effects of model misspecification on the effectiveness of continuos time trading strategies the effects of time discretising these strategies are analysed. In particular, using a trading strategy in discrete time gives rise to a non–vanishing cost process even if the strategy is self-financing if applied in continuous time. Obviously, non trivial[2] discrete time strategies are not able to perform in a self-financing way, while asset price dynamics are described by continuous time processes. The discrete application of a self-financing strategy may even lack to be self-financing in the mean unless the asset price processes are martingales under the objective probability measure, i.e. the drift of asset prices is unequal to zero.

Furthermore, if market incompleteness is not only due to sources of model and parameter misspecification but also to trading restrictions, a superhedge cannot be obtained even if volatility is bounded. Strategies, even if robust in continuous time, do not succeed if applied in discrete time, which is due to a discretisation error arising from applying a time continuous strategy only in discrete time. Combining the topic of model misspecification and trading restrictions, hedging strategies may be composed according to a continuous time hedging model but discretised in time, or they may be composed from a discretised hedging model. The first approach allows a decomposition of duplication costs into one part arising purely because of the deviation of assumed asset price dynamic and the "true" one as well as another part caused by the discretisation error.
For both approaches we study the case of a European option to exchange two assets, defining a suitably general payoff. At the same time discrete time hedging strategies can be calculated explicitely as follows: Either by assuming that the relevant dynamics are lognormal, receiving a time continuous strategy which is discretised in time afterwards or by assuming that the relevant dynamics are given by a binomial model yielding directly a discrete time trading strategy. The strategies under consideration are recalculated under the assumed model according to a discrete set of trading dates, given the market prices generated by the true dynamics. Therefore, we have inflows and/or outflows of funds from our hedging portfolio, defining a cost process along the lines of Föllmer and Sondermann (1986).

Considering the above outline of the analysis, the following paper is deeply related to several important issues of option pricing and hedging including topics like incomplete markets, model misspecification, option replication in discrete time and from discrete time to continuous time convergence. Obviously, the relevant literature is easily able to span several pages. We try to mention some of the most important and related works without postulating to give a complete summary of the existent literature. Listed according to the main topics we have:
Regarding the issue of *incomplete markets*, we refer to the papers of Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), Schweizer (1991-94), Delbaen and Schachermayer (1996), Delbaen, Monat, Schachermayer, Schweizer and Stricker (1997), Laurent and Pham (1999) as well as Pham, Rheinländer and Schweizer (1998).

[2]Non trivial in the sense of excluding static hedging.

The topic of *model misspecification* is studied in Avellaneda, Levy and Parás (1995), Lyons (1995), Bergman and Grundy (1996), El Karoui, Jeanblanc-Piqué and Shreve (1998), Hobson (1998)) or Dudenhausen, Schlögl and Schlögl (1998).
Transaction costs can naturally explain the purpose of discrete time hedging. Option replication in discrete time with transaction costs and its implication, conducted by Leland (1985), is also studied in Bensaid, Lesne and Scheinkman (1992), Boyle and Vorst (1992), in Avenalleda and Paras (1994), Grannan and Swindle (1996), Toft (1996).[3]
Discretely adjusted option hedges, firstly analysed in Boyle and Emanuel (1980) is as well analysed in Bertsimas, Kogan and Lo (1998).
For convergence results, i.e. *from discrete to continuous time finance* we refer to He (1990) and most importantly to Duffie and Protter (1992).

Including a short review of the robustness result of Gaussian hedges, the main results of this paper are: Gaussian hedging strategies are only robust under modelmisspecification with respect to convex payoff-functions iff applied in continuous time. Trading restrictions bias the effectiveness of a Gaussian strategy in the following way: If the asset price dynamic incorporates a non vanishing drift component under the objective probability measure, Gaussian strategies are less than self-financing on average. Binomial strategies proove out to incorporate similar robustness features as Gaussian hedges.[4] They can be suitably adjusted to asset price trends such that they are self-financing or even over-financing in the mean while the distribution of their cost process coincides with the distribution of the cost process associated with the Gaussian hedge if trading restrictions vanish.

The paper is organized as follows. The next section introduces the probabilistic setup. The well known robustness result of Gaussian hedges is summarised in section 3. Using the change of measure technique, the expected hedging costs with respect to continuous time trading are derived. We then proceed to formalise the pragmatic approach of assuming the dynamics of a suitable process to be lognormal for hedging purpose while nevertheless recalibrating to market prices in discrete time. In section 4 the duplication bias arrising if asset prices incorporate a drift component under the objective probability measure is analysed. Section 5 discusses the discretisation of the hedging model instead of the time discretisation of Gaussian strategies, yielding a similar robustness result for binomial hedging strategies compared to Gaussian strategies. The effectiveness of discrete time strategies coinciding only in its limiting cost processes are compared, explaining why binomial strategies are able to be self- or over-financing on average while the time discretised Gaussian hedges are not. Most of the theoretical findings are illustrated using Monte Carlo simulations. The last section concludes.

2. Probabilistic Framework

As already mentioned in the introduction, there are two reasons why the strategies under consideration are not self-financing. The continuous time trading strategies under consideration may deviate from being self-financing due to the effects of model misspecification or model incompleteness. The failure of these strategies is described by a continuous time

[3]In his paper, expected hedging costs are calculated in order to adjust the hedging volatility to transaction cotst.

[4]Binomial strategies achieve a superhedge for convex payoffs, if the returns of the relevant assets are dominated by the up and down-parameters of the assumed binomial process.

cost process. Obviously, discrete time strategies give rise to a discrete time cost process. If the discrete time strategy is given by a time discretisation of a continuous time strategy, the difference of the two cost processes is describing the costs of discretisation.
For these reasons, we adopt a more general definition of trading strategies which does not include the self-financing requirement. Associated with each strategy is a cost process resulting for its application in continuous time as well as the corresponding cost process if the strategies are applied in discrete time. The introduction of these processes which define the costs of discretisation is the main purpose of this section. We collect definitions and fix some terminology along the way.

All the stochastic processes we consider are defined on an underlying stochastic basis $(\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T^*]}, P)$, which satisfies the usual hypotheses. Trading terminates at time $T^* > 0$. We assume that the price processes of underlying assets are described by strictly positive, continuous semimartingales. By a contingent claim C with maturity $T \in [0, T^*]$, we simply mean a random payoff received at time T, which is described by the $\mathcal{F}_T$-measurable random variable C.

Definition 2.1 (Trading Strategy, Duplication). Let $S^{(1)}, \dots, S^{(N)}$ denote the price processes of underlying assets. A trading strategy ϕ in these assets is given by an $\mathbb{R}^N-$valued, predictable process which is integrable with respect to S. The value process $V(\phi)$ associated with ϕ is defined by

$$V(\phi) = \sum_{i=1}^{N} \phi^{(i)} S^{(i)}.$$

If C is a contingent claim with maturity T, then ϕ duplicates X iff

$$V_T(\phi) = C \quad P - \text{a.s.}$$

Definition 2.2 (Cost Process). If ϕ is a trading strategy in the assets $S^{(1)}, \dots, S^{(N)}$, the cost process $L(\phi)$ associated with ϕ is defined as follows:

$$L_t(\phi) := V_t(\phi) - V_0(\phi) - \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} \, dS_u^{(i)}.$$

In particular, Itô's lemma implies

$$L_t(\phi) := \sum_{i=1}^{N} \int_0^t S_u^{(i)} \, d\phi_u^{(i)} + \sum_{i=1}^{N} \int_0^t d\langle \phi^{(i)}, S^{(i)} \rangle_u.$$

Imagine now a trader who favors the strategy ϕ but places his trading decisions only in discrete time such that the restriction of the strategy ϕ to a discrete set of trading dates τ is to be analysed. In the following, the discrete application of ϕ is emphasized by the notation ϕ^τ, the value process of ϕ^τ is denoted by $V(\phi;\tau)$ (respectively $L(\phi;\tau)$ for the cost process). The trading dates τ are given by a sequence of refinements τ^n of the interval $[0, T]$, i.e.

$$\tau^n = \{t_0^n = 0 < t_1^n < \dots < t_n^n = T\}$$

with $\left|t_{k+1}^n - t_k^n\right| \to 0$ for $n \to \infty$ for all $i = 0, \dots, n$. The buy and sell decisions are carried out immediately after the prices are announced in discrete time and held constantly throughout the time period until the next decision happens. In particular, $\phi^{\tau^n}_{t^n_k}$ is t^n_{k-1}-measurable[5].

DEFINITION 2.3 (Discrete time version of trading strategies). The discrete time version of a trading strategy ϕ with respect to the refinement τ^n is defined for all $t \in [0,T]$ by:

$$\phi^{\tau^n}_t := \phi_{t^n_k} \text{ for } t \in]t^n_k, t^n_{k+1}].$$

The value process $V(\phi;\tau^n)$ associated with ϕ^{τ^n} is defined by setting $V_0(\phi;\tau^n) := V_0(\phi)$ and

$$V_t(\phi;\tau^n) = \sum_{i=1}^N \phi^{(i)}_{t^n_k} S^{(i)}_t \text{ for } t \in]t^n_k, t^n_{k+1}], \; ; \; 0 \le k \le n-1.$$

The cost process $L(\phi;\tau^n)$ associated with ϕ^{τ^n} is given by

$$L_t(\phi;\tau^n) \;:=\; V_t(\phi;\tau^n) - V_0(\phi) - \sum_{i=1}^N \left(\sum_{j=0}^{k-1} \phi^{(i)}_{t^n_j} \left(S^{(i)}_{t^n_{j+1}} - S^{(i)}_{t^n_j} \right) + \phi_{t^n_k}(S^{(i)}_t - S^{(i)}_{t^n_k}) \right)$$

for all $t \in]t^n_k, t^n_{k+1}]$, $0 \le k \le n-1$.

The strategy ϕ (ϕ^{τ^n}) is self-financing iff the cost process $L(\phi)$ ($L(\phi,\tau^n)$) is identically zero. However, if ϕ is self-financing this is not necessarily true for the discrete time version ϕ^{τ^n}. Notice that

$$L_t(\phi;\tau^n) \;=\; \sum_{i=1}^N \phi^{(i)}_{t^n_k} S^{(i)}_{t^n_k} - \left(V_0(\phi) + \sum_{i=1}^N \sum_{j=0}^{k-1} \phi^{(i)}_{t^n_j} \left(S^{(i)}_{t^n_{j+1}} - S^{(i)}_{t^n_j} \right) \right).$$

Assuming that ϕ is self-financing yields

$$\begin{aligned} L_t(\phi;\tau^n) &= V_0(\phi) + \sum_{i=1}^N \int_0^{t^n_k} \phi^{(i)}_u \, dS^{(i)}_u - \left(V_0(\phi) + \sum_{i=1}^N \sum_{j=0}^{k-1} \phi^{(i)}_{t^n_j} \left(S^{(i)}_{t^n_{j+1}} - S^{(i)}_{t^n_j} \right) \right) \\ &= \sum_{i=1}^N \sum_{j=0}^{k-1} \int_{t^n_j}^{t^n_{j+1}} \left(\phi^{(i)}_u - \phi^{(i)}_{t^n_j} \right) dS^{(i)}_u. \end{aligned}$$

Without additional assumptions it is even not clear if the strategy ϕ^{τ^n} is self-financing in expectation[6], i.e. if

$$E_P\left[L_t(\phi;\tau^n)\right] = 0,$$

where E_P denotes the expectation with respect to the objective measure P. Furthermore, the value of the discrete version ϕ^{τ^n} of ϕ is different to the value of the continuous strategy ϕ, unless the strategy ϕ itself is partwise constant. At time t, $t \in]t^n_k, t^n_{k+1}]$ it holds

$$V_t(\phi) - V_t(\phi;\tau^n) = \sum_{i=1}^N \left(\phi^{(i)}_t - \phi^{(i)}_{t^n_k} \right) S^{(i)}_t.$$

In particular, if ϕ is duplicating the claim C with maturity T, i.e. $V_T(\phi) = C_T$, this is not true for ϕ^{τ^n}, i.e. $V_T(\phi,\tau^n) = \phi_{t^n_{n-1}} S_T$. Using the strategy ϕ in order to hedge the payoff

[5]Using the agreement $\mathcal{F}_{t^n_{-1}} = \mathcal{F}_{t^n_0}$.

[6]A sufficient condition for ϕ^{τ^n} to be self-financing is given by assuming that the asset prices are martingales under the objective measure P.

C_T, the hedging error $L_T^C(\phi)$ equals the final costs $L_T(\phi)$. The hedging error of ϕ^{τ^n} is given by

$$\begin{aligned} L_T^C(\phi,\tau^n) &= L_T(\phi,\tau^n) + C_T - \sum_{i=1}^N \phi^{(i)}_{t^n_{n-1}} S^{(i)}_T \\ &= L_T(\phi,\tau^n) + V_T(\phi) - V_T(\phi,\tau^n) \\ &= \sum_{j=1}^n \left(V_{t^n_j}(\phi) - \sum_{i=1}^N \phi^{(i)}_{t^n_{j-1}} S^{(i)}_{t^n_j} \right). \end{aligned}$$

It is thus straightforward to interpret the cost of using the strategy ϕ only in discrete time as the costs of discretisation according to the difference of the cost processes corrected with the difference in value.

DEFINITION 2.4 (Costs of Discretization). If ϕ^{τ^n} is the discrete time version of the strategy ϕ in the assets $S^{(1)}, \dots, S^{(N)}$, the cost process $D(\phi,\tau^n)$ associated with the discrete application of ϕ is defined as follows:

$$D_t(\phi,\tau^n) := L_t(\phi,\tau^n) + V_t(\phi) - V_t(\phi,\tau^n) - L_t(\phi), \quad \text{for all } t \in]0,T].$$

The duplication costs of ϕ^{τ^n} with respect to the claim C maturing at T are given by

$$L_T^C(\phi,\tau^n) = D_T(\phi,\tau^n) + L_T(\phi).$$

In particular, the costs of discretisation at T are given by

$$D_T(\phi,\tau^n) = \sum_{i=1}^N \sum_{j=0}^{n-1} \int_{t^n_j}^{t^n_{j+1}} \left(\phi^{(i)}_u - \phi^{(i)}_{t^n_j} \right) dS^{(i)}_u .$$

However, if the trading strategies are continuous semimartingales themselves, we have

$$\begin{aligned} \lim_{n\to\infty} V_t(\phi;\tau^n) &= V_t(\phi) \\ \lim_{n\to\infty} \sum_{j=0}^{k-1} \phi^{(i)}_{t^n_j} \left(S^{(i)}_{t^n_{j+1}} - S^{(i)}_{t^n_j} \right) - \phi_{t^n_k}(S^{(i)}_t - S^{(i)}_{t^n_k}) &= \int_0^t \phi^{(i)}_u \, dS^{(i)}_u \end{aligned}$$

such that the costs of discretisation are vanishing in the limit, i.e.

$$\lim_{n\to\infty} D_t(\phi,\tau^n) = 0 \quad \text{and} \quad \lim_{n\to\infty} L_T^C(\phi,\tau^n) = L_T(\phi).$$

In particular, if ϕ is self-financing, this is true for the limit of the discrete time version of ϕ as well. If ϕ is superreplicating the payoff of the contingent claim C, i.e. $L(\phi) < 0$, this is a priori only true for the limit of the discrete time version as well. However, trading in discrete time may bias the outcome of the hedging strategy. Even the expectation with respect to P of the costs of discretisation may differ from zero. Generally, this will be the case unless the asset price dynamics are martingales under the (subjective) measure P.

Assuming that the probability space $(\Omega, \mathcal{F}, P)$ supports an d–dimensional Brownian motion W and that $\mathcal{F}$ is the augmented filtration generated by W we may without loss of generality write

$$dS^{(i)}_t = S^{(i)}_t \left(\mu^{(i)}_t \, dt + \sigma^{(i)}_t \, dW \right),$$

with $\sigma^{(i)} : [0,T[\to \mathbb{R}^d_+$. Furthermore we assume that the usual regularity conditions apply to $\mu^{(i)}$ and $\sigma^{(i)}$.

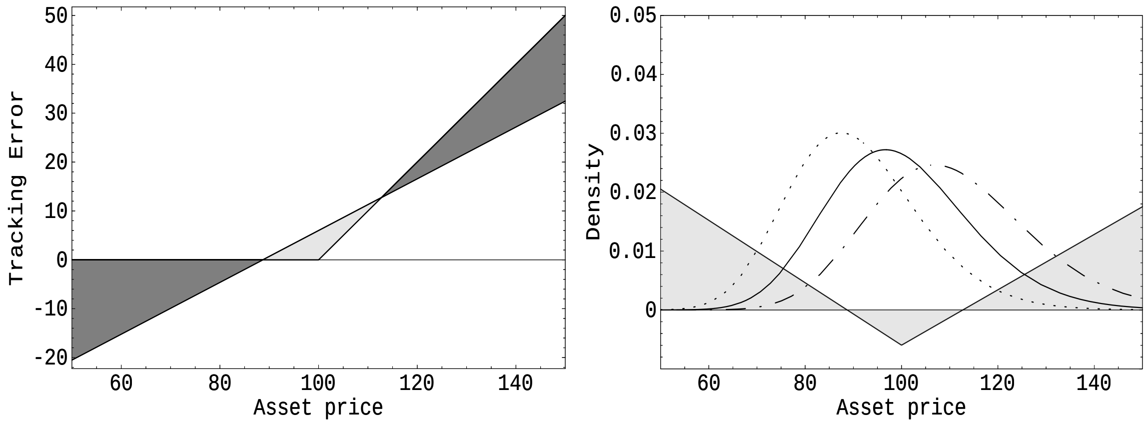

FIGURE 1. Duplication costs associated with a discrete time (static) Black-Scholes hedging strategy for an at the money call option maturing in three months. Given that the asset price follows a geometric Brownian Motion with volatility equal to the volatility of the Black-Scholes hedge, the tracking error vanishes in expectation if there is no drift in the underlying asset price process. Otherwise, the duplication costs are positively biased, i.e. the payoff is not sufficiently hedged in the mean.

PROPOSITION 2.5 (Expected Costs of Discretisation). *In a model where $\mu^{(i)}$ and $\sigma^{(i)}$ are only depending on time, the expected costs of applying the strategy ϕ according the discrete set τ^n of trading dates can be represented as follows:*

$$E_P\left[D_T(\phi,\tau^n)\right] = \sum_{i=1}^{N} S_0^{(i)} \sum_{j=0}^{n-1} \int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u) \exp\left\{\int_0^u \mu^{(i)}(s)\,ds\right\} E_{\hat{P}^{(i)}}\left[\int_{t_j^n}^{u} d\phi_s^{(i)}\right] du$$

$$= \sum_{i=1}^{N} S_0^{(i)} \sum_{j=0}^{n-1} E_{\hat{P}^{(i)}}\left[\int_{t_j^n}^{t_{j+1}^n} \left(\exp\left\{\int_0^{t_{j+1}^n} \mu^{(i)}(u)\,du\right\} - \exp\left\{\int_0^s \mu^{(i)}(u)\,du\right\}\right) d\phi_s^{(i)}\right],$$

where $d\hat{P}_t^{(i)} = Z_t^{(i)}\,dP_t$ *with* $Z_t^{(i)} := \exp\left\{\int_0^t \sigma^{(i)}(u)\,dW_u - \frac{1}{2}\int_0^t \|\sigma^{(i)}(u)\|^2\,du\right\}$

PROOF: The proof is given in the appendix.

In order to analyse the discretisation bias further we have to use additional assumptions about the strategies. The choice of the hedge model is based on the robustness result of Gaussian hedging strategies which is reviewed in the next section. Figure 1 already motivates the later finding that using a Gaussian hedge yields a positive duplication bias if there is an asset price trend (positive or negative) under the objective probability measure. In particular any convex payoff-function is subhedged on average.

3. DUPLICATION COSTS ASSOCIATED WITH CONTINUOUS TIME GAUSSIAN STRATEGIES DUE TO MODEL MISSPECIFICATION

According to the previous section, the cost process of a time discretisized trading strategy is viewed as a decomposition consisting of two parts. The costs purely due to model misspecification, i.e. assuming that continuous time hedging is possible, are considered on the one hand. On the other hand, there are additional costs arising from the time discrete application of the originally time continuos strategy, i.e. the discretisation error. Before putting things together, the costs are analysed by parts.

First, a short review of the robustness result according to Gaussian hedging strategies is given along the lines of Avellaneda, Levy and Parás (1995) and Dudenhausen, Schlögl and Schlögl (1998). Black/Scholes–like formulae for pricing derivatives follow from the assumption that the stochastic dynamics of the process relevant for hedging are driven by a geometric Brownian motion. In particular, this implies that the volatility is deterministic. Hedge ratios can then be expressed in terms of the cumulative distribution function of the standard normal distribution, therefore the term "Gaussian hedges".

DEFINITION 3.1 (Lognormal Process). We call a stochastic process Z *lognormal* iff it can be written in the form

$$dZ_t = Z_t\left(\mu_t dt + \tilde{\sigma}_Z(t) dW_t\right) \tag{1}$$

with *deterministic* dispersion coefficients $\tilde{\sigma}_Z : [0,T[\to \mathbb{R}^d_+$.

This lognormality assumption[7] allows the derivation of self-financing hedges if applied in continuous time:

PROPOSITION 3.2. *Let X, Y be the price processes of two assets. Consider an option to exchange X for Y at the maturity date T, i.e. an European option with payoff $[X_T - Y_T]^+$. In a model where the quotient process $Z := \frac{X}{Y}$ is lognormal, the hedge portfolio $\Phi = (\Phi)_{0\leq t\leq T}$ for this option in terms of the assets X and Y given by*

$$\begin{aligned} \phi_t^X &:= \mathcal{N}(h^{(1)}(t,Z_t)) \quad \text{units of } X \\ \text{and} \quad \phi_t^Y &:= -\mathcal{N}(h^{(2)}(t,Z_t)) \quad \text{units of } Y. \end{aligned}$$

$\mathcal{N}$ denotes the cumulative distribution function of the standard normal distribution and the functions $h^{(1)}$ and $h^{(2)}$ are given by

$$h^{(1)}(t,z) = \frac{\ln(z) + \frac{1}{2}\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}{\sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}} \tag{2}$$

$$h^{(2)}(t,z) = h^{(1)}(t,z) - \sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}. \tag{3}$$

In particular, the price process of the exchange option is given by

$$C(t,Z_t) = X_t\mathcal{N}(h^{(1)}(t,Z_t)) - Y_t\mathcal{N}(h^{(2)}(t,Z_t)).$$

PROOF: See Margrabe (1978) or Frey and Sommer (1996).

□

It is remarkable that the hedging strategy can be specified exclusively in X and Y, regardless of the dimension of the driving Wiener process. In particular, the pricing and hedging in theorem 3.2 is the same as for a model driven by a one-dimensional Wiener process

[7]For a succinct treatment of the significance of this assumption, see Rady (1997).

where

$$dZ_t^M = \tilde{v}_T(t)\, dW_t \quad \text{with} \quad \tilde{v}_T(t) := \sqrt{\int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds}.$$

dZ^M is denoting the martingale part of the Doob-Meyer decomposition of Z, the standard deviation v_T defining the volatility used to specify the strategy Φ is called hedge volatility. The trading strategy Φ given in (b) duplicates the exchange option at maturity even in the case of model misspecification if applied in continuous time. It is self-financing iff the true dynamics of X and Y are such that Z is lognormal and the true volatility of Z equals the assumed volatility. However, in the following paper it is not aquired that the true volatility σ matches the assumed volatility $\tilde{\sigma}$ such that the hedge volatility $\tilde{v}$ may be misspecified. Starting with the continuous time application of the strategy ϕ described in proposition 3.2 referring to the assets $(Z, 1)$ where $Z = \frac{X}{Y}$ instead of (X, Y) we have:

LEMMA 3.3. *In a model where X and Y are lognormal processes with volatilities $\tilde{\sigma}_X$ and $\tilde{\sigma}_Y$, respectively, Z is lognormal as well with*

$$\begin{aligned}\mu_Z(t) &= \mu_X(t) - \mu_Y(t) + \tilde{\sigma}_Y(t)\left(\tilde{\sigma}_Y(t) - \tilde{\sigma}_X(t)\right),\\ \tilde{\sigma}_Z(t) &= \tilde{\sigma}_X(t) - \tilde{\sigma}_Y(t).\end{aligned}$$

PROOF: Itô's chain rule implies

$$Z_t = Z_0 + \int_0^t X_s\, dY_s^{-1} + \int_0^t Y_s^{-1}\, dX_s + \int_0^t d\langle X, Y^{-1}\rangle_s.$$

Furthermore, we have

$$\begin{aligned}Y_t^{-1} &= Y_0^{-1} - \int_0^t Y_s^{-2}\, dY_s + \int_0^t Y_s^{-3}\, d\langle Y\rangle_s\\ &= Y_0^{-1} + \int_0^t Y_s^{-1}\left(\|\tilde{\sigma}_Y(s)\|^2 - \mu_Y(s)\right) ds - \int_0^t Y_s^{-1}\, d\widetilde{W}_s.\end{aligned}$$

Using $\langle X, Y^{-1}\rangle_t = -\int_0^t \frac{X_s}{Y_s}\, \tilde{\sigma}_X(s)\tilde{\sigma}_Y(s)\, ds$

$$\text{implies} \qquad Z_t = Z_0 + \int_0^t \left(\mu_X(s) - \mu_Y(s) + \tilde{\sigma}_Y(s)\right) ds + \int_0^t \left(\tilde{\sigma}_X(s) - \tilde{\sigma}_Y(s)\right) d\widetilde{W}_s.$$

□

In particular, the lognormality of X and Y is sufficient, but not necessary for the application of the strategy Φ. This remains valid in the degenerate cases where either X_T or Y_T is deterministic, so that proposition 3.2 can be applied to a standard put or call option on an asset with a lognormal price process.

PROPOSITION 3.4 (Costs of misspecification). *The discounted cost process $L^*(\phi)$ is given by*

$$L_t^*(\phi) = \int_0^t Z_u \frac{\mathcal{N}'(h^{(1)}(u, Z_u))}{2\tilde{v}_T(u)} \left(\|\sigma_Z(u)\|^2 - \|\tilde{\sigma}_Z(u)\|^2\right) du. \tag{4}$$

PROOF:

$$\begin{aligned} L_t^*(\phi) &= V_t^*(\phi) - \left(V_0^*(\phi) + \int_0^t \phi_u^X \, dZ_u\right) \\ &= \int_0^t Z_u \, d\mathcal{N}(h^{(1)}(u, Z_u)) - \int_0^t d\mathcal{N}(h^{(2)}(u, Z_u)) + \int_0^t d\langle Z, \mathcal{N}(h^{(1)})\rangle_u \end{aligned}$$

Since $h^{(2)}(t,z) = h^{(1)}(t,z) - \tilde{v}(t)$ and $\mathcal{N}'(h^{(2)}(t,z)) = z\mathcal{N}'(h^{(1)}(t,z))$, it holds

$$\begin{aligned} d\mathcal{N}(h^{(2)}(u, Z_u)) &= Z_u \, d\mathcal{N}(h^{(1)}(u, Z_u)) \\ &- \mathcal{N}'(h^{(1)}(u, Z_u)) Z_u \left(\tilde{v}'(u)\, dt + \frac{1}{2}\tilde{v}(u)(h_z^{(1)}(u, Z_u))^2 \, d\langle Z\rangle_u\right), \end{aligned}$$

where $h_z^{(1)}$ denotes the partial derivative given by $h_z^{(1)} = \frac{1}{z\tilde{v}}$. Assuming that the true dynamics of Z is given by

$$dZ_t = Z_t\left(\mu_t dt + \sigma_Z(t) dW_t\right)$$

yields

$$d\mathcal{N}(h^{(2)}(u, Z_u)) = Z_u \left(d\mathcal{N}(h^{(1)}(u, Z_u)) + \mathcal{N}'(h^{(1)}(u, Z_u)) \frac{\|\sigma(t)\|^2 - 2\tilde{v}(t)\tilde{v}'(t)}{2\tilde{v}(t)} \, du\right).$$

Finally, we have $-2\tilde{v}(t)\tilde{v}'(t) = \|\tilde{\sigma}(t)\|^2$ and $\langle Z, \mathcal{N}(h^{(1)})\rangle_u = Z_u \frac{\mathcal{N}'(h^{(1)}(u,Z_u))}{\tilde{v}(u)} \|\sigma(t)\|^2 \, du$.

Notice, that the above result holds true as well, if the true volatility is an arbitrary stochastic process and matches the result of El Karoui, Jeanblanc-Piqué and Shreve (1998) and Avellaneda, Levy and Parás (1995) that Gaussian hedging strategies for convex payoff–functions yield a superhedge, i.e. a decreasing cost process almost surely iff the true volatility is dominated by the assumed one. A detailed analysis of Gaussian hedging strategies including the application to fixed income products is given in Dudenhausen, Schlögl and Schlögl (1998).

Since option prices are increasing in volatility of the underlying, one would certainly expect that dominating the true volatility implies superreplication on average (under the equivalent martingale measure). What is remarkable, however, is that it implies superreplication with probability 1 under any equivalent measure, *including the real–world probabilities.* In particular, even if the underlying asset prices are driven by a drift component. Thus, if the purpose of hedging is the complete elimination of risk, given uncertainty about present and future volatility, one should hedge at the upper volatility bound. In cases where this upper bound is too high for this to be practicable, one could instead hedge at the upper bound of some confidence interval for the volatility, resulting in a superhedge as long as the realized volatility remains below this upper bound. The effectiveness in the sense of the cost distribution of a Gaussian superhedge, i.e. hedging at the upper volatility bound as well as the effectiveness of a confidence hedge are obviously depending on the asset price drift μ_Z.

This can easily be explained if one notes that (4) can alternatively be written in terms of

Distributions of hedging costs for dominated constant local volatilities under different drift scenarios

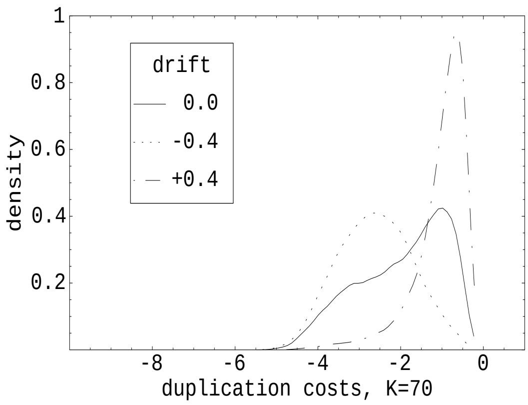

FIGURE 2. In–the–money option

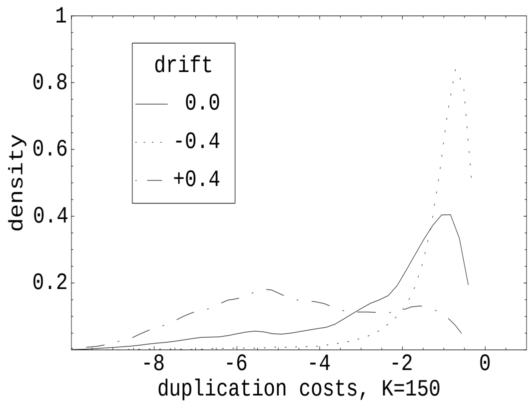

FIGURE 3. Out–of–the–money option

the Black/Scholes *gamma* or *theta*:

$$L_t^*(\phi) = \frac{1}{2}\int_0^t Z_u^2\, \tilde{\gamma}(u, Z_u)\left(\|\sigma_Z(u)\|^2 - \|\tilde{\sigma}_Z(u)\|^2\right) du$$
$$= -\int_0^t \tilde{\theta}(u, Z_u)\, \frac{\|\sigma_Z(u)\|^2 - \|\tilde{\sigma}_Z(u)\|^2}{\|\tilde{\sigma}_Z(u)\|^2}\, du$$

where

$$\tilde{\gamma}(u,z) = \mathcal{N}'(h^{(1)}(u,z))\frac{1}{z\tilde{v}(u)}, \quad \tilde{\theta}(u,z) = z\mathcal{N}'(h^{(1)}(u,z))\frac{\|\tilde{\sigma}_Z(u)\|^2}{2\tilde{v}(u)}.$$

Thus, as does the Black/Scholes gamma or theta, the cost process reacts more sensitively as the time to maturity decreases, in particular if the option is at the money. This serves on the one hand as an intuitive explanation why in order to obtain a superhedge it is not sufficient to dominate the total volatility[8]. On the other hand, it can be concluded that the costs process tends to be more sensitive if the asset price drift is given such that the asset price are close to the option strike immediately before maturity. While dominating the true volatility yields higher withdrawals (negative costs) from the duplicating portfolio if asset prices tend to be at the money immediately before maturity, this effect is reversed if the true volatility dominates the assumed volatility.

The effects mentioned above are illustrated in figure 2 and figure 3, resembling the cost distribution of Gaussian heding strategies composed according to $\tilde{\sigma}_Z = 0.4$ with respect to varying drift scenarios. Each cost distribution is generated by a Gauss–kernel density estimation from 50000 simulated hedging paths. The parameters for the underlying asset price process Z under the objective probability measure are given by $\sigma_Z = 0.3$ and $\mu_Z = 0.4$ (respectively -0.4 and zero). Time to maturity of the (plain vanilla call) option to be hedge is one year, the initial underlying price is 100 while the strike K of the options under consideration is chosen to be 70 and 150, the hedging frequency is 1000.
In particular, with respect to the in–the–money option and positive drift ($\mu > 0$), the

[8]Details and implications are given in Dudenhausen, Schlögl and Schlögl (1998)

expected withdrawals of hedge funds are higher under the objective measure P than under the risk neutral measure P^* corresponding to asset price simulation under $\mu = 0$. Obviously, this effect is reversed in case of the out–of–the–money option. This can also be justified theoretically by explicitely calculating the expected hedge costs. i.e.

PROPOSITION 3.5 (Expected costs of misspecification).

$$(5) \quad E_P\left[L_t^*(\phi)\right] = Z_0 \int_0^t e^{\int_0^u \mu_Z(s)\,ds} \, \frac{\mathcal{N}'(\bar{h}^{(1)}(u, e^{\int_0^u \mu_Z(s)\,ds} Z_0))}{2\bar{v}_T(u)} \left(\|\sigma_Z(u)\|^2 - \|\tilde{\sigma}_Z(u)\|^2\right) du.$$

where the functions $\bar{h}^{(1)}$, $\bar{h}^{(2)}$, *and* $\bar{v}_T$ *are given by*

$$(6) \qquad \bar{h}^{(1)}(t,z) = \frac{\ln(z) + \frac{1}{2}\bar{v}_T^2(t)}{\bar{v}_T(t)} \quad \bar{h}^{(2)}(t,z) = \bar{h}^{(1)}(t,z) - \bar{v}_T(t)$$

$$(7) \qquad \bar{v}_T(t) = \sqrt{\int_0^t \|\sigma_Z(s)\|^2 ds + \int_t^T \|\tilde{\sigma}_Z(s)\|^2 ds.}$$

PROOF: Using a change of measure yields (cf. proposition 2.5)

$$E_P\left[L_t^*(\phi)\right] = Z_0 \int_0^t \exp\left\{\int_0^u \mu_Z(s)\,ds\right\} \frac{E_{\hat{P}^Z}\left[\mathcal{N}'(h^{(1)}(u, Z_u))\right]}{2\tilde{v}_T(u)} \left(\|\sigma_Z(u)\|^2 - \|\tilde{\sigma}_Z(u)\|^2\right) du.$$

Furthermore, in appendix B (cf. lemma B.2) it is shown that

$$E_{\hat{P}^Z}\left[\mathcal{N}'(h^{(1)}(t, Z_t))\right] = \frac{\tilde{v}_T(t)}{\bar{v}_T(t)} \mathcal{N}'(\bar{h}^{(1)}(t, e^{\int_0^t \mu_Z(s)\,ds}\, Z_0)).$$

There are a few remarks necessary concerning proposition 3.5. Notice, that $\bar{v}_T^2(t)$ denotes the average variance from time 0 up to time T given the knowledge of the average "true" variance up to time t ($0 \le t \le T$). In particular, it holds

$$\bar{v}_T^2(0) = \int_0^T \|\tilde{\sigma}_Z(s)\|^2 ds, \quad \bar{v}_T^2(T) = \int_0^T \|\sigma_Z(s)\|^2 ds.$$

Let

$$\bar{c}(t,z) := z\mathcal{N}(\bar{h}^{(1)}(t,z)) - \mathcal{N}(\bar{h}^{(2)}(t,z)),$$

then $\bar{C}(t, F_t)$ can be understood as a "Gaussian" t_0–price of the exchange option under consideration given the "true" volatility until time t and the assumed volatility from time t up to time T which is evaluated at the asset price $F_t = e^{\int_0^t \mu_Z(s)\,ds} Z_0$. Analoguously to the above comments, equation 5 can again be written in terms of the *gamma* or θ of $\bar{C}$. Since the asset price dynamic of F is deterministic, the expected costs of misspecification can be expressed in closed form: With equation (5) and

$$\bar{C}_t(t,z) = z\mathcal{N}'(\bar{h}^{(1)}(t,z))\bar{v}'(t) = z\mathcal{N}'(\bar{h}^{(1)}(t,z))\frac{\|\sigma_Z(t)\|^2 - \|\tilde{\sigma}_Z(t)\|^2}{2\bar{v}_T(t)}$$

$$\bar{C}_z(t,z) = \mathcal{N}(\bar{h}^{(1)}(t,z))$$

it follows

$$E_P\left[L_t^*(\phi)\right] = \bar{C}(u, Z_0\, e^{\int_0^u \mu_Z(s)\,ds})\Big|_0^t - \int_0^t Z_0 \mu_Z(u) e^{\int_0^u \mu_Z(s)\,ds} \; \mathcal{N}(\bar{h}^{(1)}(u, e^{\int_0^u \mu_Z(s)\,ds} Z_0))\,du.$$

In particular, the expected duplication costs of the claim to be hedged are given by $E_P\left[L_T^*(\phi)\right]$ and can be expressed through the difference of $\bar{C}(T, Z_0\, e^{\int_0^T \mu_Z(s)\,ds})$ and $\bar{C}(0, Z_0)$ which is corrected with the value of rolling over the shares invested in Z according to a continuously money account with interest rate equal to μ_Z. On the one hand, $\bar{C}(T, Z_0\, e^{\int_0^T \mu_Z(s)\,ds})$ corresponds to the "true" t_0–price evaluated at the initial asset price Z_0 adjusted to the expected growth under P. On the other hand $\bar{C}(0, Z_0)$ equals the initial investment into the Gaussian hedge composed under the assumed volatility. In particular, the expected costs of misspecification are given by the difference of "true" and assumed price iff $\mu_Z = 0$. With respect to the risk neutral measure P^* it holds

$$E_{P^*}\left[L_t^*(\phi)\right] = \bar{C}(t, Z_0) - \bar{C}(0, Z_0).$$

COROLLARY 3.6 (Expected costs of misspecification).

$$E_P\left[L_t^*(\phi)\right] - E_{P^*}\left[L_t^*(\phi)\right] =$$
$$Z_0 \int_0^t \mu_Z(u) e^{\int_0^u \mu_Z(s)\,ds} \left(\mathcal{N}(\bar{h}^{(1)}(t, e^{\int_0^u \mu_Z(s)\,ds} Z_0)) - \mathcal{N}(\bar{h}^{(1)}(u, e^{\int_0^u \mu_Z(s)\,ds} Z_0))\,du\right).$$

PROOF: Corollary 3.6 is a direct consequence of the above remarks and

$$\begin{aligned}\bar{C}(t, Z_0\, e^{\int_0^t \mu_Z(s)\,ds}) &= \bar{C}(t, Z_0) + \int_{Z_0}^{Z_0\, e^{\int_0^t \mu_Z(s)\,ds}} \mathcal{N}(\bar{h}^{(1)}(t, x))\,dx \\ &= \bar{C}(t, Z_0) + \int_0^t \mu_Z(u) e^{\int_0^u \mu_Z(s)\,ds}\; \mathcal{N}(\bar{h}^{(1)}(t, e^{\int_0^u \mu_Z(s)\,ds} Z_0))\,du.\end{aligned}$$

4. BIAS ARRISING FROM TIME DISCRETISING A GAUSSIAN HEDGE

However, the above result does of course not carry through if the Gaussian strategy Φ is carried out with restriction to discrete time τ^n. Φ^{τ^n} neither duplicates the claim at maturity, Φ^{τ^n} is not self-financing even in the case without model misspecification, nor is it a superhedge if the true voaltility is dominated.

PROPOSITION 4.1 (Expected Costs of Discretisation). *If the quotient process Z is lognormal with μ_Z and σ_Z depending on time, i.e.*

$$dZ_t = Z_t\left(\mu_Z(t)dt + \sigma_Z(t)dW_t\right),$$

the expected costs of applying the strategy $\Phi = (\phi^X, \phi^Y)$ with hedging volatility $\tilde{v}_T$ in terms of the assets $Z = \frac{X}{Y}$ and 1 according the discrete set τ^n of trading dates, i.e.

$$\Phi_t^{\tau^n} = (\phi_{t_k^n}^X, \phi_{t_k^n}^Y) \quad \textit{for} \quad t \in]t_k^n, t_{k+1}^n],$$

are given by:

$$\begin{aligned}E_P\left[D_T^*(\phi, \tau^n)\right] &= Z_0 \sum_{j=0}^{n-1} \int_{t_j^n}^{t_{j+1}^n} \left(e^{\int_0^{t_{j+1}^n} \mu_Z(u)\,du} - e^{\int_0^s \mu_Z(u)\,du}\right) \\ &\quad \frac{E_{\hat{P}^Z}\left[\mathcal{N}'(h^{(1)}(s, Z_s))\right]}{\tilde{v}_T(s)} \left(\mu_Z(s) + \frac{1}{2}\left(\|\sigma_Z(s)\|^2 - \|\tilde{\sigma}_Z(s)\|^2\right)\right) \\ &- \frac{E_{\hat{P}^Z}\left[h^{(1)}(s, Z_s)\mathcal{N}'(h^{(1)}(s, Z_s))\right]}{2\tilde{v}_T^2(s)} \left(\|\sigma_Z(s)\|^2 - \|\tilde{\sigma}_Z(s)\|^2\right)\,ds.\end{aligned}$$

where $d\hat{P}_t^Z = D_t^Z \, dP_t$ *with* $D_t^Z := \exp\left\{\int_0^t \sigma_Z(u)\, dW_u - \frac{1}{2}\int_0^t \|\sigma_Z(u)\|^2\, du\right\}$.

PROOF: Notice, that with proposition 2.5 we have

$$\begin{aligned} E_P\left[D_T^*(\phi,\tau^n)\right] &= E_P\left[\sum_{j=1}^{n-1}\int_{t_j^n}^{t_{j+1}^n}\left(\mathcal{N}\left(h^{(1)}(u,Z_u)\right)-\mathcal{N}\left(h^{(1)}(t_j^n,Z_{t_j^n})\right)\right)dZ_u\right] \\ &= Z_0\sum_{j=0}^{n-1}E_{\hat{P}^Z}\left[\int_{t_j^n}^{t_{j+1}^n}\left(e^{\int_0^{t_{j+1}^n}\mu_Z(u)\,du}-e^{\int_0^s\mu_Z(u)\,du}\right)d\mathcal{N}\left(h^{(1)}(s,Z_s)\right)\right]. \end{aligned}$$

An application of Itô's lemma implies

$$\begin{aligned} &d\mathcal{N}(h^{(1)}(t,Z_t)) = \\ &\mathcal{N}'(h^{(1)}(t,Z_t))\left(h_t^{(1)}(t,Z_t)\,dt+h_z^{(1)}(t,Z_t)\,dZ_t+\frac{1}{2}\left(-h^{(1)}(t,Z_t)\left(h_z^{(1)}(t,Z_t)\right)^2+h_{zz}^{(1)}(t,Z_t)\right)d\langle Z\rangle_t\right), \end{aligned}$$

where h_t, h_z and h_{zz} denote partial derivatives, i.e. given

$$h^{(1)}(t,z)=\frac{\ln z+\frac{1}{2}\tilde{v}^2(t)}{\tilde{v}(t)},\ \tilde{v}(t)=\sqrt{\int_t^T\tilde{\sigma}^2(s)\,ds},$$

we have $h_t^{(1)}(t,z)=\tilde{v}'(t)-\frac{\tilde{v}'(t)}{\tilde{v}(t)}h^{(1)}(t,z)$; $h_z^{(1)}(t,z)=\frac{1}{\tilde{v}(t)z}$; $h_{zz}^{(1)}(t,z)=\frac{-1}{\tilde{v}(t)z^2}$, such that

$$\begin{aligned} d\mathcal{N}(h^{(1)}(t,Z_t))=\mathcal{N}'(h^{(1)}(t,Z_t)) \quad &\left(\frac{\mu_Z(t)+\tilde{v}'(t)\tilde{v}(t)-\frac{1}{2}\|\sigma_Z(t)\|^2}{\tilde{v}(t)}\,dt\right. \\ &-\frac{\tilde{v}'(t)\tilde{v}(t)+\frac{1}{2}\|\sigma_Z(t)\|^2}{\tilde{v}^2(t)}h^{(1)}(t,Z_t)\,dt \\ &\left.+\frac{\sigma_Z(t)}{\tilde{v}(t)}\,dW_t\right) \end{aligned}$$

Notice, that $\tilde{v}'(t)\tilde{v}(t)=-\frac{1}{2}\|\tilde{\sigma}_Z(t)\|^2$. Together with $dW_t=d\hat{W}_t+\sigma_Z\,dt$ it follows

$$\begin{aligned} d\mathcal{N}(h^{(1)}(t,Z_t)) &= \mathcal{N}'(h^{(1)}(t,Z_t))\left[\frac{\sigma_Z(t)}{\tilde{v}(t)}\,d\hat{W}_t\right. \\ &\left.\left(\frac{\mu_Z(t)+\frac{1}{2}\left(\|\sigma_Z(t)\|^2-\|\tilde{\sigma}_Z(t)\|^2\right)}{\tilde{v}(t)}-\frac{1}{2}\frac{\|\sigma_Z(t)\|^2-\|\tilde{\sigma}_Z(t)\|^2}{\tilde{v}^2(t)}h^{(1)}(t,Z_t)\right)dt\right]. \end{aligned}$$

COROLLARY 4.2. *If* $\sigma_Z(t)=\tilde{\sigma}_Z(t)$ *and either* $\mu_Z(t)>0$ *or* $\mu_Z(t)<0$ *for all* $t\in[0,T]$, *then the strategy* ϕ^{τ^n} *of proposition 4.1 is positively biased, i.e.*

$$E_P\left[D_T^*(\phi,\tau^n)\right]>0.$$

Considering a Gaussian hedge which is corresponding to the uniquely determined self-financing hedging strategy if carried out continuously, we have the following result if this hedge is applied in discrete time: besides being non-self-financing the time discretised hedging strategy is biased in the sense that it tends to subdominate the payoff of the exchange option if the drift μ_Z does not change its sign. This result has already been motivated in the last section, cf. figure (1) and is further illustrated in figure 4, 5 and 6

Distributions of hedging costs for discrete time gaussian hedging strategies under known volatility referring to different drift scenarios

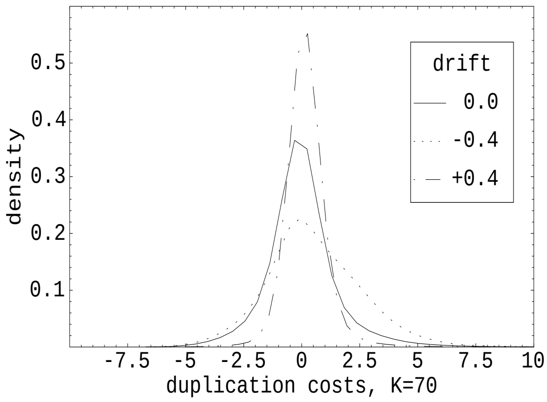

FIGURE 4. In–the–money option

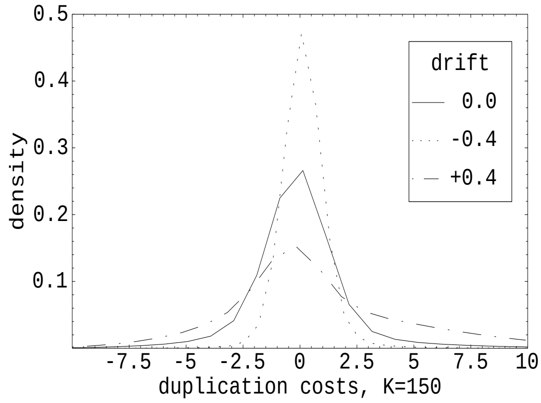

FIGURE 5. Out–of–the–money option

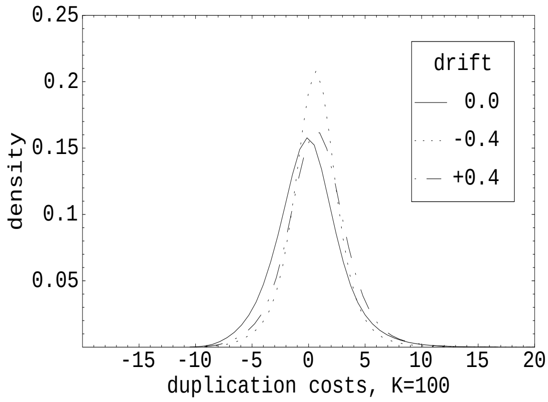

FIGURE 6. At–the–money option

showing the distribution of the discrete time Gaussian hedge under different drift scenarios. As before, each cost distribution is generated by a Gauss–kernel density estimation from 50000 simulated hedging paths. The parameters for the underlying asset price process Z under the objective probability measure are given by $\sigma_Z = 0.3$ and $\mu_Z = 0.4$ (respectively -0.4 and zero). Time to maturity of the (plain vanilla call) option to be hedged is one year, the initial underlying price is 100 while the strike K of the options under consideration is chosen to be 70, 100 and 150. The duplication portfolio is rebalanced only monthly.

Again, the expected costs can finally be expressed in closed form, i.e.

THEOREM 4.3. *The expected hedging costs under model misspecification and trading restrictions can be decomposed into two parts, associated to the misspecification error and associated with the trading restrictions, i.e.*

$$E_P\left[L_T^*(\phi;\tau^n)\right] = E_P\left[D_T^*(\phi;\tau^n)\right] + E_P\left[L_T^*(\phi)\right],$$

Distributions of hedging costs of discrete time Gaussian hedges for dominated constant local volatilities under different drift scenarios

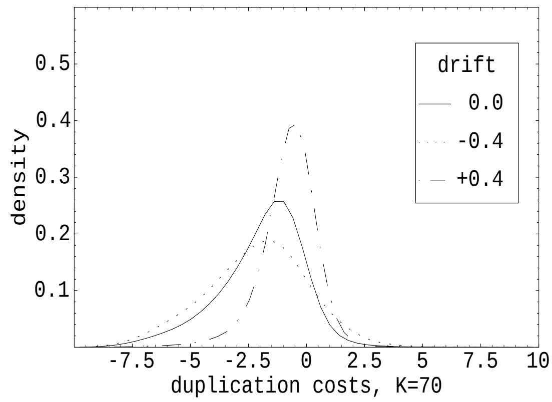

FIGURE 7. In–the–money option

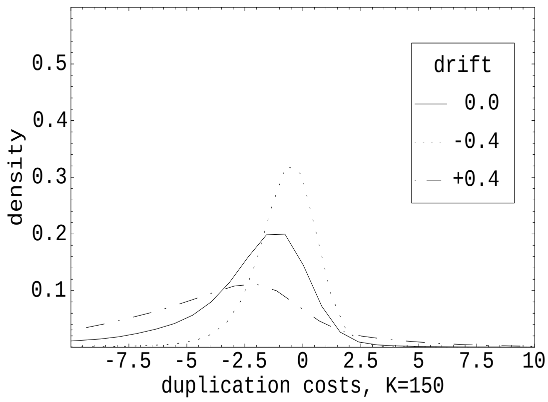

FIGURE 8. Out–of–the–money option

where $E_P\left[L_T^*(\phi)\right]$ *is already given in corollary 3.6 and* $E_P\left[D_T^*(\phi;\tau^n)\right]$ *denoting the expected error of time discretising the Gaussian hedge (concerning the exchange option) is given by*

$$E_P\left[D_T^*(\phi,\tau^n)\right] =$$

$$Z_0\sum_{j=0}^{n-1}\int_{t_j^n}^{t_{j+1}^n}\mu_Z(u)e^{\int_0^u\mu_Z(s)\,ds}\left(\mathcal{N}\left(\bar h^{(1)}(u,Z_0\,e^{\int_0^u\mu_Z(s)\,ds})\right)-\mathcal{N}\left(\bar h^{(1)}(t_j^n,Z_0\,e^{\int_0^{t_j^n}\mu_Z(s)\,ds})\right)\right)du.$$

PROOF: Using lemma B.1 and lemma B.2 of the appendix gives

$$\begin{aligned}E_{\hat P^Z}\left[\mathcal{N}'(h^{(1)}(t,Z_t))\right] &= \frac{\tilde v_T(t)}{\bar v_T(t)}\mathcal{N}'(\bar h^{(1)}(t,e^{\int_0^t\mu_Z(s)\,ds}\,Z_0)),\\ E_{\hat P^Z}\left[h^{(1)}(t,Z_t))\mathcal{N}'(h^{(1)}(t,Z_t))\right] &= \frac{\tilde v_T^2(t)}{\bar v_T^2(t)}\bar h^{(1)}(t,e^{\int_0^t\mu_Z(s)\,ds}\,Z_0)\mathcal{N}'(\bar h^{(1)}(t,e^{\int_0^t\mu_Z(s)\,ds}\,Z_0)).\end{aligned}$$

Inserting in the expression of proposition 4.1 and integrating by parts yields

$$\begin{aligned}E_P\left[D_T^*(\phi,\tau^n)\right] &= Z_0\sum_{j=0}^{n-1}\Bigg[\left(e^{\int_0^{t_{j+1}^n}\mu_Z(u)\,du}-e^{\int_0^s\mu_Z(u)\,du}\right)\mathcal{N}\left(\bar h^{(1)}(u,e^{\int_0^u\mu_Z(s)\,ds}\,Z_0)\right)\Big|_{t_j^n}^{t_{j+1}^n}\\ &\quad+\int_{t_j^n}^{t_{j+1}^n}\mu_Z(u)\,e^{\int_0^u\mu_Z(s)\,ds}\mathcal{N}\left(\bar h^{(1)}(u,e^{\int_0^u\mu_Z(s)\,ds}\,Z_0)\right)du\Bigg].\end{aligned}$$

5. DISCRETE TIME HEDGING MODEL

Of course, the incompleteness caused by proposing continuous time asset price dynamics while hedging in discrete time is non hedgeable. Until now, we focused on the discretisation error incurred by the inconsistency of assuming a continuous time hedging model while applying this strategy in discrete time. A Gaussian hedging strategy which is only applied according to discrete time implies a non vanishing cost process, even without assuming model misspecification. Furthermore, even if the "true" volatility is bounded from above, the time discretised version does not yield a superhedge for convex payoffs almost surely. It turns out that the costs of time discretising a strategy ment to be applied continuously may incorporate a duplication bias. For convex payoffs to be hedged, these strategies give rise to a subhedge on average, if the dynamic of the underlying price process incorporates a drift component with respect to the objective probability measure.

Notice, that the strategies under consideration are understood to be self-financing with respect to an assumed model called "hedging model" only. If the "true" model deviates from the assumed one, these strategies may fail to be effective. However, while facing the problem of model misspecification combined with trading restrictions, it is worth mentioning that the discrete time Gaussian hedge is in fact neither consistent with the hedging model nor with the true model.

Naturally, one may be tempted to discretise the hedging model instead of time discretising an originally continuous time trading strategy. Of course, there is still an inconsistency associated by using a discrete time hedging model while facing continuous time price processes (misspecified or not). On the other hand, strategies implied by model discretisation are at least compatible with its underlying hedging model. The analysis of the duplication costs arising from discretising the hedging model instead of discretising the trading strategies is carried out in this section. Naturally, we compare the effectiveness of the following hedging approach with the results of the last sections.

As before, the hedging of an exchange option is studied. W.l.o.g. we assume an equidistant set of trading dates τ^n with $t_k^n = \frac{kT}{n}$. Furthermore, for ease of notation we assume that the parameters defining the up- and downmovements u and d of the binomial hedging model, firstly motivated by Cox, Ross and Rubinstein (1979), are only depending on the degree of refinement, i.e. on n.

Definition 5.1. Let g_1^n and g_2^n be defined on τ^n as follows:

$$g_1^n(t_k^n, z) := \frac{C^n_{\mathrm{CRR}}(t^n_{k+1}, u_n z) - C^n_{\mathrm{CRR}}(t^n_{k+1}, d_n z)}{(u_n - d_n)z}, \tag{8}$$

$$g_2^n(t_k^n, z) := C^n_{\mathrm{CRR}}(t^n_k, d_n z) - g_1^n(t_k^n z) d_n z, \tag{9}$$

where $C^n_{\mathrm{CRR}}(t^n_n, z) = [z-1]^+$ and

$$C^n_{\mathrm{CRR}}(t^n_k, z) = \sum_{j=0}^{n-k} \binom{n-k}{j} \left(\frac{1-d_n}{u_n-d_n}\right)^j \left(\frac{u_n-1}{u_n-d_n}\right)^{n-k-j} \left[u_n^j d_n^{n-k-j} z - 1\right]^+, \quad k = 0, \ldots, n-1.$$

Lemma 5.2. *The duplication costs L_T^C for $C_T = [Z_T - 1]^+$ associated with the trading strategy $\Phi^n = \left(\phi_n^X, \phi_n^Y\right)$ in the assets $\left(\frac{X}{Y}, 1\right)$, where*

$$\Phi_t^n = \left(g_1^n(t_k^n, Z_{t_k^n}), g_2^n(t_k^n, Z_{t_k^n})\right) \text{ for } t \in]t_k^n, t_{k+1}^n]$$

and $Z_t = \frac{X_t}{Y_t}$ are given by

$$L_T^C(\Phi^n) = \sum_{j=1}^n C^n_{CRR}\left(t_j^n, Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n).$$

PROOF: It holds

$$\begin{aligned} L_T^C(\Phi^n) &= L_T(\Phi^n) + C_T - \left(g_1^n(t_{n-1}^n, Z_{t_{n-1}^n}) Z_{t_n^n} + g_2^n(t_{n-1}^n, Z_{t_{n-1}^n}) \right) \\ &= L_T(\Phi^n) + C_{\text{CRR}}^n\left(t_n^n, Z_{t_n^n}\right) - V_{t_n^n}(\Phi^n) \\ &= \sum_{j=1}^{n-1} \left(L_{t_{j+1}^n}(\Phi^n) - L_{t_j^n}(\Phi^n) \right) + C_{\text{CRR}}^n\left(t_n^n, Z_{t_n^n}\right) - V_{t_n^n}(\Phi^n). \end{aligned}$$

Notice, that

$$\begin{aligned} L_{t_{j+1}^n}(\Phi^n) - L_{t_j^n}(\Phi^n) &= V_{t_{j+1}^n}(\Phi^n) - V_{t_j^n}(\Phi^n) - g_1^n(t_j^n, Z_{t_j^n}) \left(Z_{t_{j+1}^n} - Z_{t_j^n} \right) \\ &= g_1^n(t_j^n, Z_{t_j^n}) Z_{t_{j+1}^n} + g_2^n(t_j^n, Z_{t_j^n}) - g_1^n(t_j^n, Z_{t_j^n}) \left(Z_{t_{j+1}^n} - Z_{t_j^n} \right) - V_{t_j^n}(\Phi^n) \\ &= g_1^n(t_j^n, Z_{t_j^n}) Z_{t_j^n} + g_2^n(t_j^n, Z_{t_j^n}) - V_{t_j^n}(\Phi^n) \\ &= \frac{1-d_n}{u_n-d_n} C_{\text{CRR}}^n\left(t_{j+1}^n, u_n Z_{t_j^n}\right) + \frac{u_n-1}{u_n-d_n} C_{\text{CRR}}^n\left(t_{j+1}^n, d_n Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n) \\ &= C_{\text{CRR}}^n\left(t_j^n, Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n). \end{aligned}$$

The incremental costs of rearranging the portfolio as prescribed by the binomial hedge coincide with the difference of the fair price proposed by the binomial model and the value of the strategy. Taking all withdrawls from and all infusion into the portfolio ocurring during the set of trading dates together, the duplication costs of the claim to be hedged are matched.

PROPOSITION 5.3. *If Φ^n is defined as before, it holds*

(a) $C_{CRR}^n\left(t_j^n, Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n) = 0$ *iff* $\frac{Z_{t_j^n}}{Z_{t_{j-1}^n}} \in \{d_n, u_n\}$ *P–a.s.,*

(b) $C_{CRR}^n\left(t_j^n, Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n) \le 0$ *iff* $\frac{Z_{t_j^n}}{Z_{t_{j-1}^n}} \in [d_n, u_n]$ *P–a.s.,*

(c) $C_{CRR}^n\left(t_j^n, Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n) \ge 0$ *iff* $\frac{Z_{t_j^n}}{Z_{t_{j-1}^n}} \in]\infty, d_n] \cup [u_n, \infty]$ *P–a.s..*

PROOF:

$$\begin{aligned} & C_{\text{CRR}}^n\left(t_j^n, Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n) \\ = \;& C_{\text{CRR}}^n\left(t_j^n, Z_{t_j^n}\right) - \left(g_1^n(t_{j-1}^n, Z_{t_{j-1}^n}) Z_{t_j^n} + g_2^n(t_{j-1}^n, Z_{t_{j-1}^n}) \right) \end{aligned}$$

Inserting g_1^n and g_2^n according to equation (8) and equation (9) and defining $x_{t_j^n} := \frac{Z_{t_j^n}}{Z_{t_{j-1}^n}}$ yields

$$= C_{\text{CRR}}^n\left(t_j^n, x_{t_j^n} Z_{t_{j-1}^n}\right) - \left[\frac{x_{t_j^n} - d_n}{u_n - d_n} C_{\text{CRR}}^n\left(t_j^n, u_n Z_{t_{j-1}^n}\right) + \frac{u_n - x_{t_j^n}}{u_n - d_n} C_{\text{CRR}}^n\left(t_j^n, d_n Z_{t_{j-1}^n}\right) \right].$$

It is easily shown that $C_{\text{CRR}}^n\left(t_j^n, z\right)$ is convex in z which gives the result.

COROLLARY 5.4. *A superhedge is achieved iff the returns of the underlying asset are within the range of the up and down parameters of the assumed binomial process for each trading* *period, i.e. iff*

$$\frac{Z_{t_j^n}}{Z_{t_{j-1}^n}} \in [d_n, u_n] \quad P\text{–}a.s. \text{ for all } j = 1, \ldots, n.$$

Obviously, a binomial hedging strategy is not able to dominate the payoff of the exchange option under consideration if the true asset price dynamic is driven by a geometric Brownian motion. The requirement, that the asset price stays in the interval defined by the assumed up- and downmovements of the binomial model, is stronger than the upper volatility bound required for the continuous time Gauss hedge. Superhedging according to a binomial strategy affords the incorporation of the "true" asset price drift. If superhedging is not possible, the drift can at least be used to assure that the binomial hedge is self-financing (respectively over-financing) in the mean. Specifying u_n and d_n depending on volatility and drift allows to avoid the discretisation error associated with a Gaussian hedge specified exclusively through the assumed volatility. A suitable specification of u_n and d_n is very straightforward and can easily be motivated as follows. For notational convenience assume that the P–dynamic of Z is given by

$$dZ_t = Z_t\left(\mu\, dt + \sigma\, dW_t\right).$$

Notice that the conditional expectation of the incremental costs of rearranging the binomial hedge are given by

$$\begin{aligned} & E_P\left[C^n_{\text{CRR}}\left(t_j^n, Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n) \mid Z_{t_{j-1}^n}\right] \\ = \; & E_P\left[C^n_{\text{CRR}}\left(t_j^n, Z_{t_{j-1}^n}\, \exp^{\left(\mu-\frac{1}{2}\sigma^2\right)(t_j^n - t_{j-1}^n) + \sigma\left(W_{t_j^n} - W_{t_{j-1}^n}\right)}\right)\right. \\ & - \left.\left(\frac{e^{\mu(t_j^n - t_{j-1}^n)} - d_n}{u_n - d_n}\, C^n_{\text{CRR}}\left(t_j^n, u_n Z_{t_{j-1}^n}\right) + \frac{u_n - e^{\mu(t_j^n - t_{j-1}^n)}}{u_n - d_n}\, C^n_{\text{CRR}}\left(t_j^n, d_n Z_{t_{j-1}^n}\right)\right) \right| Z_{t_{j-1}^n}\Bigg]. \end{aligned}$$

The convexity of C^n_{CRR} together with

$$u_n \frac{e^{\mu(t_j^n - t_{j-1}^n)} - d_n}{u_n - d_n} + d_n \frac{u_n - e^{\mu(t_j^n - t_{j-1}^n)}}{u_n - d_n} = e^{\mu(t_j^n - t_{j-1}^n)}$$

implies that specifying

$$u_n = e^{\left(\mu - \frac{1}{2}\sigma^2\right)\left(t_j^n - t_{j-1}^n\right) + \sigma\sqrt{t_j^n - t_{j-1}^n}}, \quad d_n = e^{\left(\mu - \frac{1}{2}\sigma^2\right)\left(t_j^n - t_{j-1}^n\right) - \sigma\sqrt{t_j^n - t_{j-1}^n}}$$

assures that

$$E_P\left[C^n_{\text{CRR}}\left(t_j^n, Z_{t_j^n}\right) - V_{t_j^n}(\Phi^n) \mid Z_{t_{j-1}^n}\right] \leq 0,$$

i.e. the binomial hedge is over-financing on average. Furthermore, specifying u_n and d_n suitably can additionally guarantee the convergence of the cost process associated with the binomial hedge to the one of a Gaussian hedge in distribution if the incompleteness arrising from trading restriction vanishes, i.e. if $n \to \infty$.

PROPOSITION 5.5. *Using the JR-specification of Jarrow and Rudd (1983) of the binomial parameters u and d, i.e. let*

$$u_n := \exp\left\{\frac{\mu_Z - \frac{1}{2}\|\tilde{\sigma}_Z\|^2}{n} + \frac{\tilde{\sigma}_Z}{\sqrt{n}}\right\}, \quad d_n := \exp\left\{\frac{\mu_Z - \frac{1}{2}\|\tilde{\sigma}_Z\|^2}{n} - \frac{\tilde{\sigma}_Z}{\sqrt{n}}\right\}$$

Distributions of hedging costs for JR–like hedging strategies under known volatility referring to different drift scenarios

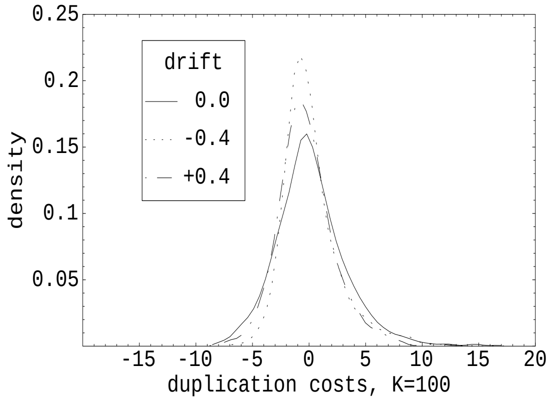

FIGURE 9. At–the–money option

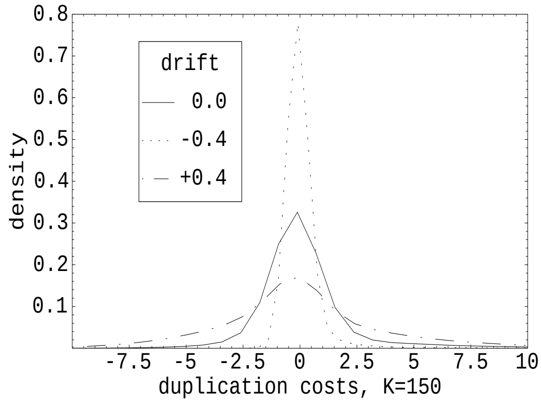

FIGURE 10. Out–of–the–money option

assures for $n \to \infty$ a convergence in distribution of the cost process associated with the CRR-strategy as defined in lemma 5.2 towards the cost process associated with the Gaussian strategy as defined in proposition 3.2.

PROOF: Using Berry-Esseen-type inequalities it can be shown that the sequence of g_1^n and g_2^n converges uniformly on the compact set [0,T[to the continuous functions specifying the Gaussian hedge, c.f. Dudenhausen (2001). Referring to corollary 5.2. of Duffie and Protter (1992) this ensures the convergence in distribution of the gains process which also implies the convergence in distribution of the cost process.

Once again, each cost distribution illustrated in figure 9 and figure 10 is generated by a Gauss–kernel density estimation from 50000 simulated hedging paths. The parameters for the underlying asset price process Z under the objective probability measure are given by $\sigma_Z = 0.3$ and $\mu_Z = 0.4$ (respectively -0.4 and zero). Time to maturity of the (plain vanilla call) option to be hedge is one year, the initial underlying price is 100 while the strike K of the options under consideration is chosen to be 100 and 150 while the duplication portfolio is rebalanced only monthly. This time, the strategies under consideration are given by a binomial model with a Jarrow Rudd specification as in proposition 5.5. Comparing these simulation results with the ones illustrated in figure 5 and figure 6 clearly favours the model discretisation in form of the binomial hedge, suitably adjusted to the asset price drift.

6. CONCLUSION

The results of this paper present a strong argument to discretise the hedging model instead of discretising the hedging strategies if the rebalancing of the portfolio is restricted to a set of discrete time trading dates. Black/Scholes-type strategies and binomial strategies to hedge derivatives with convex payoff-profiles can be understood to incorporate comparable robustness features in the sense of superhedging. Dominating the payoff of a contingent claim almost surely with respect to all equivalent measures is obviously independent of the drift of the underlying under the objective measure. However, if market incompleteness is not only due to sources of model and parameter misspecification but also to trading restrictions, a non-trivial superhedge cannot be obtained even if volatility is bounded. Therefore, it is adequate to allow the strategy to depend on more parameters than only the hedge volatility with vanishing influence, if the distance of trading dates converges to zero. This can easily be conducted with binomial-strategies but not with the discrete time version of Gaussian hedges. In comparison to a simple Black/Scholes strategy the advantage of the CRR-like hedging strategy is particularily transparent if the market is complete without the introduction of trading restrictions. While the Gaussian hedging strategy applied in discrete time subdominates the convex payoff to be hedged on average for positive as well as negative asset price trends, the binomial hedge which is suitably adjusted to the drift component is (almost) self-financing in the mean, tending to favour the outcome of the hedge. Since the costs processes coincide in the limit, there is nothing lost by using the binomial hedge instead of the Gaussian hedge if the trading frequency is increased. While employing (theoretically incompatible) lognormal models may be justified by volatility uncertainty in the specification of the "true" model, employing binomial models although assuming continuous time price processes may be justified by trading restrictions.

Appendix A. Proof of Proposition 2.5

PROOF: Notice, that the expected costs of discretisation are given by

$$E_P\left[D_T(\phi,\tau^n)\right] = \sum_{i=1}^{N} E_P\left[\sum_{j=0}^{n-1}\left(\int_{t_j^n}^{t_{j+1}^n} \phi_u^{(i)}\,dS_u^{(i)} - \phi_{t_j^n}^{(i)}\left(S_{t_{j+1}^n}^{(i)} - S_{t_j^n}^{(i)}\right)\right)\right].$$

Considering the expectation

$$\begin{aligned} & E_P\left[\int_{t_j^n}^{t_{j+1}^n} \phi_u^{(i)}\,dS_u^{(i)} - \phi_{t_j^n}^{(i)}\left(S_{t_{j+1}^n}^{(i)} - S_{t_j^n}^{(i)}\right)\right] \\ = & \int_{t_j^n}^{t_{j+1}^n} E_P\left[\phi_u^{(i)}\,dS_u^{(i)}\right] - E_P\left[E_P\left[\phi_{t_j^n}^{(i)}\left(S_{t_{j+1}^n}^{(i)} - S_{t_j^n}^{(i)}\right)\Big|\, S_{t_j^n}^{(i)}\right]\right] \\ = & \int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u) E_P\left[\phi_u^{(i)} S_u^{(i)}\right]\,du - \left(\exp\left\{\int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u)\,du\right\} - 1\right) E_P\left[\phi_{t_j^n}^{(i)} S_{t_j^n}^{(i)}\right] \end{aligned}$$

and using a change of measure together with an application of Bayes rule yields

$$= \int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u) E_{\hat{P}^{(i)}}\left[\phi_u^{(i)}\hat{D}_u^{(i)} S_u^{(i)}\right]\,du - \left(\exp\left\{\int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u)\,du\right\} - 1\right) E_{\hat{P}^{(i)}}\left[\phi_{t_j^n}^{(i)}\hat{D}_{t_j^n}^{(i)} S_{t_j^n}^{(i)}\right]$$

where $dP_t^{(i)} = \hat{D}_t^{(i)}\,d\hat{P}_t$ with $\hat{D}_t^{(i)} := \exp\left\{-\int_0^t \sigma_u^{(i)}\,d\hat{W}_u - \frac{1}{2}\int_0^t \|\sigma_u^{(i)}\|^2\,du\right\}$. Notice, that $\hat{W}$ with

$$\hat{W}_t = W_t - \int_0^t \sigma_u^{(i)}\,du$$

is a $\hat{P}^{(i)}$–Brownian motion. It holds

$$\begin{aligned} dS_t^{(i)} &= S_t^{(i)}\left(\mu^{(i)}(t)\,dt + \sigma^{(i)}(t)\,dW_t\right) \\ &= S_t^{(i)}\left[\left(\mu^{(i)}(t) + \|\sigma^{(i)}(t)\|^2\right)\,dt + \sigma^{(i)}(t)\,d\hat{W}_t\right] \\ \text{and}\ \hat{D}_t^{(i)} S_t^{(i)} &= S_0^{(i)} \exp\left\{-\int_0^t \sigma^{(i)}(u)\,d\hat{W}_u - \frac{1}{2}\int_0^t \|\sigma^{(i)}(u)\|^2\,du\right\} \\ &\quad \exp\left\{\int_0^t \sigma^{(i)}(u)\,d\hat{W}_u + \int_0^t \left(\mu^{(i)}(u) + \frac{1}{2}\|\sigma^{(i)}(u)\|^2\right)\,du\right\} \\ &= S_0 \exp\left\{\int_0^t \mu^{(i)}(u)\,du\right\} \end{aligned}$$

Inserting yields

$$\begin{aligned} & E_P\left[\int_{t_j^n}^{t_{j+1}^n} \phi_u^{(i)}\,dS_u^{(i)} - \phi_{t_j^n}^{(i)}\left(S_{t_{j+1}^n}^{(i)} - S_{t_j^n}^{(i)}\right)\right] \\ = \;& S_0^{(i)}\left[\int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u)\exp\left\{\int_0^u \mu^{(i)}(s)\,ds\right\}\,E_{\hat{P}^{(i)}}\left[\phi_u^{(i)}\right]\,du\right. \\ & \left.-\left(\exp\left\{\int_{t_j^n}^{t_{j+1}^n} \mu_u^{(i)}\,du\right\} - 1\right)\exp\left\{\int_0^{t_j^n} \mu^{(i)}(u)\,du\right\}\,E_{\hat{P}^{(i)}}\left[\phi_{t_j^n}^{(i)}\right]\right] \\ = \;& S_0^{(i)}\left[\int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u)\exp\left\{\int_0^u \mu^{(i)}(s)\,ds\right\}\,E_{\hat{P}^{(i)}}\left[\phi_u^{(i)}\right]\,du\right. \\ & \left.-\left(\exp\left\{\int_0^{t_{j+1}^n} \mu_u^{(i)}\,du\right\} - \exp\left\{\int_0^{t_j^n} \mu^{(i)}(u)\,du\right\}\right)\,E_{\hat{P}^{(i)}}\left[\phi_{t_j^n}^{(i)}\right]\right] \end{aligned}$$

Notice, that

$$E_{\hat{P}^{(i)}}\left[\phi_u^{(i)}\right] = E_{\hat{P}^{(i)}}\left[\phi_{t_j^n}^{(i)} + \int_{t_j^n}^u d\phi_s^{(i)}\right]$$

together with

$$\int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u)\exp\left\{\int_0^u \mu^{(i)}(s)\,ds\right\}\,du = \exp\left\{\int_0^{t_{j+1}^n} \mu_u^{(i)}\,du\right\} - \exp\left\{\int_0^{t_j^n} \mu^{(i)}(u)\,du\right\}$$

finally implies

$$\begin{aligned} & E_P\left[\int_{t_j^n}^{t_{j+1}^n} \phi_u^{(i)}\,dS_u^{(i)} - \phi_{t_j^n}^{(i)}\left(S_{t_{j+1}^n}^{(i)} - S_{t_j^n}^{(i)}\right)\right] \\ = \;& S_0^{(i)}\int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u)\exp\left\{\int_0^u \mu^{(i)}(s)\,ds\right\}\,E_{\hat{P}^{(i)}}\left[\int_{t_j^n}^u d\phi_s^{(i)}\right]\,du \\ = \;& S_0^{(i)}\int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u)\exp\left\{\int_0^u \mu^{(i)}(s)\,ds\right\}\,E_{\hat{P}^{(i)}}\left[\int_{t_j^n}^{t_{j+1}^n} 1_{\{t_j^n \le s \le u\}}\,d\phi_s^{(i)}\right]\,du. \end{aligned}$$

Changing the order of integration yields

$$
\begin{aligned}
& E_P\left[\int_{t_j^n}^{t_{j+1}^n} \phi_u^{(i)}\, dS_u^{(i)} - \phi_{t_j^n}^{(i)}\left(S_{t_{j+1}^n}^{(i)} - S_{t_j^n}^{(i)}\right)\right] \\
= \;& S_0^{(i)} E_{\hat{P}^{(i)}}\left[\int_{t_j^n}^{t_{j+1}^n}\int_{t_j^n}^{t_{j+1}^n} \mu^{(i)}(u)\exp\left\{\int_0^u \mu^{(i)}(s)\right\} 1_{\{t_j^n\le s\le u\}}\, du\;\, d\phi_s^{(i)}\right] \\
= \;& S_0^{(i)}\, E_{\hat{P}^{(i)}}\left[\int_{t_j^n}^{t_{j+1}^n}\left(\exp\left\{\int_0^{t_{j+1}^n}\mu^{(i)}(u)\, du\right\} - \exp\left\{\int_0^s \mu^{(i)}(u)\, du\right\}\right) d\phi_s^{(i)}\right].
\end{aligned}
$$

Appendix B. Useful Integrals

LEMMA B.1. *For $a, b \in I\!R$ $(b > -1)$ it holds*

$$
\begin{aligned}
&(i) \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\, e^{ax-\frac{1}{2}bx^2}\, dx &=& \quad \frac{1}{\sqrt{1+b}}\, e^{\frac{a^2}{2(1+b)}} \\
&(ii) \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x\, e^{-\frac{1}{2}x^2}\, e^{ax-\frac{1}{2}bx^2}\, dx &=& \quad \frac{a}{(1+b)^{\frac{3}{2}}}\, e^{\frac{a^2}{2(1+b)}}
\end{aligned}
$$

PROOF:
ad (i)

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\, e^{ax-\frac{1}{2}bx^2}\, dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\left(\sqrt{1+b}x-\frac{a}{\sqrt{1+b}}\right)^2-\frac{a^2}{1+b}\right)} \\
&= e^{\frac{a^2}{2(1+b)}}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{1+b}x-\frac{a}{\sqrt{1+b}}\right)^2}
\end{aligned}
$$

substitution of $z := \sqrt{1+b}x - \dfrac{a}{\sqrt{1+b}}$ yields

$$
\begin{aligned}
&= \frac{1}{\sqrt{1+b}}\, e^{\frac{a^2}{2(1+b)}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\, dz \\
&= \frac{1}{\sqrt{1+b}}\, e^{\frac{a^2}{2(1+b)}}\int_{-\infty}^{+\infty}\mathcal{N}'(z)\, dz.
\end{aligned}
$$

where $\mathcal{N}$ denotes the cumulative distribution function of the standard normal distribution.

ad (ii)

$$
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\, xe^{ax-\frac{1}{2}bx^2}\, dx = e^{\frac{a^2}{2(1+b)}}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} xe^{-\frac{1}{2}\left(\sqrt{1+b}x-\frac{a}{\sqrt{1+b}}\right)^2}
$$

substitution of $z := \sqrt{1+b}x - \dfrac{a}{\sqrt{1+b}}$ yields

$$
\begin{aligned}
&= \frac{1}{\sqrt{1+b}}\, e^{\frac{a^2}{2(1+b)}}\int_{-\infty}^{+\infty}\left(\frac{z}{\sqrt{1+b}}+\frac{a}{1+b}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\, dz \\
&= \frac{1}{\sqrt{1+b}}\, e^{\frac{a^2}{2(1+b)}}\int_{-\infty}^{+\infty}\left(\frac{z}{\sqrt{1+b}}+\frac{a}{1+b}\right)\mathcal{N}'(z)\, dz.
\end{aligned}
$$

LEMMA B.2. *If $\hat{W}$ is a Brownian Motion with respect to the probability measure $\hat{P}$ and*

$$Z_t = Z_0 \ \exp\left\{\int_0^t \left(\mu_Z(u) + \frac{1}{2}\|\sigma_Z(u)\|^2\right)\, du + \int_0^t \sigma_Z(u)\, d\hat{W}_u\right\},$$

then it holds

$$\begin{aligned}(i)\quad E_{\hat{P}^Z}\left[\mathcal{N}'(h^{(1)}(t,Z_t))\right] &= \frac{\tilde{v}_T(t)}{\bar{v}_T(t)}\mathcal{N}'(\bar{h}^{(1)}(t, e^{\int_0^t \mu_Z(s)\,ds}\, Z_0)),\\ (ii)\quad E_{\hat{P}^Z}\left[h^{(1)}(t,Z_t))\mathcal{N}'(h^{(1)}(t,Z_t))\right] &= \frac{\tilde{v}_T^2(t)}{\bar{v}_T^2(t)}\bar{h}^{(1)}(t, e^{\int_0^t \mu_Z(s)\,ds}\, Z_0)\mathcal{N}'(\bar{h}^{(1)}(t, e^{\int_0^t \mu_Z(s)\,ds}\, Z_0)).\end{aligned}$$

PROOF: Notice, that

$$\begin{aligned}&E_{\hat{P}^Z}\left[\mathcal{N}'(h^{(1)}(t,Z_t))\right]\\ &= \frac{1}{\sqrt{2\pi}}E_{\hat{P}^Z}\left[\exp\left\{-\frac{1}{2}\left(\frac{\bar{v}_T(t)}{\tilde{v}_T(t)}\bar{h}^{(1)}(t, e^{\int_0^t \mu_Z(s)\,ds}Z_0) + \frac{1}{\tilde{v}_T(t)}\int_0^t \sigma_Z(u)\, d\hat{W}_u\right)^2\right\}\right]\\ &= \frac{1}{2\pi}\ exp\left\{-\frac{1}{2}\left(\frac{\bar{v}_T^2(t)}{\tilde{v}_T^2(t)}\left(\bar{h}^{(1)}(t, e^{\int_0^t \mu_Z(s)\,ds}Z_0)\right)^2\right)\right\}\\ &\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\exp\left\{\frac{-\bar{v}_T(t)\bar{h}^{(1)}(t, e^{\int_0^t \mu_Z(s)\,ds}Z_0)\sqrt{\int_0^t \|\sigma_Z(u)\|^2\, du}}{\tilde{v}_T^2(t)}x - \frac{\int_0^t \|\sigma_Z(u)\|^2\, du}{\tilde{v}_T^2(t}x^2\right\}\, dx.\end{aligned}$$

Using lemma B.1 part (i) of the appendix gives

$$E_{\hat{P}^Z}\left[\mathcal{N}'(h^{(1)}(t,Z_t))\right] = \frac{\tilde{v}_T(t)}{\bar{v}_T(t)}\mathcal{N}'(\bar{h}^{(1)}(t, e^{\int_0^t \mu_Z(s)\,ds}\, Z_0)).$$

The second implication follows directly with lemma B.1 part (ii).

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