Discussion Paper B–398

Log-Normal Interest Rate Models: Stability and Methodology*

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October 1996. This version: January 10, 1997

Abstract

The lognormal distribution assumption for the term structure of interest is the most natural way to exclude negative spot and forward rates. However, imposing this assumption on the continuously compounded interest rate has a serious drawback: rates explode and expected rollover returns are infinite even if the rollover period is arbitrarily short. As a consequence such models cannot price one of the most widely used hedging instrument on the Euromoney market, namely the Eurodollar future contract.

The purpose of this paper is twofold: First to show that the problems with lognormal models result from modelling the wrong rate, namely the continuously compounded rate. If instead one models the effective annual rate these problems disappear. Second to give a survey on recent work on lognormal term structure models for effective or nominal forward rates.

Keywords: Term Structure Models, Lognormal Interest Rate, Eurodollar Futures

1 Introduction

The problems with a lognormal volatility structure for interest rate models are well-known. For a HJM-type model (Heath, Jarrow, and Morton (1992)) a lognormal volatility structure of the form $\sigma(t,T,f) := f(t,T)\gamma(t,T)$, where $f(t,T)$ denotes the instantaneous forward rate valid at time t for the infinitesimal time interval $[T, T+dt]$ and $\gamma(t,T)$ is deterministic, is inconsistent. It was already shown by Morton (1988) that the resulting rates become infinite with positive probability. This implies zero prices for contingent claims with positive payoff and hence arbitrage opportunities. For lognormal short rate models as proposed by Black, Derman, and Toy (1990) and Black and Karasinski (1991) it was shown by Hogan and Weintraub (1993) that for any positive time interval expected rollovers at the resulting short rate are infinite, a rather undesirable property, since the rollover account serves as the numeraire in these models. One consequence of this property is that these models cannot price Eurodollar interest rate futures.

It was observed by Sandmann and Sondermann (1993a) that the problems with lognormal interest rate models result from modelling instantaneous rates which have an infinitisimal accrual period. Assuming that the instantaneous rate is lognormal results in "double exponential" expressions, i.e. expressions

*The authors are grateful to David Heath, Ian Hofman, Bjarne Astrup Jensen, Kristian Miltersen, Marek Musiela and Jorgen Aase Nielsen for helpful discussions. Special thanks are due to Ian Hofman for drawing our attention to the problem of pricing Eurodollar futures and the Hogan-Weintraub results. Financial support from the Deutsche Forschungsgemeinschaft Sonderforschungsbereich 303 at the University of Bonn by both authors, is gratefully acknowledged.

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where the exponential function is itself an argument of an exponential, thus giving rise to infinite expectations under the martingale measure. The problem disappears if instead one models rates with a finite accrual period, e.g. 3-month-Libor rates or effective per annum rates. Contrary to continuously compounded rates these rates are directly observable in the market place, and most interest rate derivatives like Libor and swap derivatives are based on rates quoted for finite accrual periods like three, six or twelve months. Thus using simple or effective rates as the primary process for a term structure model seems natural, rather than as a secondary process derived from instantaneous rates (cp. also Jamshidian (1996a)).

It seems that the binomial short rate model of Sandmann and Sondermann (1989, 92, 93b) was the first model of effective interest rates. For this model there exists a general uniqueness and existence result (Theorem 2.1). Although this discrete model looks very similar to the Black, Derman, and Toy (1990) (BDT) model, its limit behaviour is quite different. In the limit of the BDT–model the continuously compounded instantaneous rate $r_c(t)$ follows a lognormal diffusion. In the limit of the Sandmann–Sondermann model $r_c(t)$ follows a diffusion which is neither normal nor lognormal, but a dynamic combination of both with the following properties (cp. Theorem 3.1): for small values of $r_c(t)$ the diffusion process approaches a lognormal diffusion [1], thus generating positive rates, whereas for large values of $r_c(t)$ the diffusion approaches a normal diffusion process, generating stable finite expected returns and futures values. It thus combines –in a very simple and straightforward setup – the strengths of the normal and lognormal model and avoids their shortcomings. As a consequence this model exhibits stable limit behavior under refinements of the discrete time interval, and allows also the pricing of Eurodollar futures. This crucial difference between the stability of lognormal continuously compounded rates and lognormal effective rates was pointed out in Sandmann and Sondermann (1993a).

It was rather obvious that the same stability structure as obtained for short rates also holds for forward rates. This stimulated further research on lognormal term structure models by a number of authors, with a shift from short to forward rate models. Within short time existence and pricing results (e.g. the "market" caplet formula) were obtained for lognormal interest rate models based on effective or simple rates which seemed inaccesible in the traditional instantaneous rate models. A survey of these results is given in section 4.

2 From continuously compounded to annual interest rates

Since the motivation of effective annual rates is natural in a discrete time model we start our discussion on the basis of the discrete binomial model proposed by Sandmann and Sondermann (1993b).

Denote by $\underline{\underline{T}} = \{0 = t_0 < t_1 < ... < t_N = T\}$ a set of trading dates which form an equidistant discretisation of the time axis. Suppose that the (annual) effective rate follows a path independent binomial process $r : \underline{\underline{T}} \times \Omega \to \mathrm{R}_{\geq 0}$ such that $r_e(i, n)$ is the (annual) effective rate at time $t_i \in \underline{\underline{T}}$ and state $n = 0, ..., i$. Let us assume that at time t_0 the prices of default free zero coupon bonds $B_0(t_i) = B(t_0, t_i)$ with maturity $t_i \in \underline{\underline{T}} \setminus \{t_0\}$ are known. The problem is to compute from these initial term structure the process of the effective rate in an arbitrage free way, such that negative effective rates are excluded and the initial term structure is consistent with the model. The initial effective rate $r_e(0, 0)$ at time t_0 is determined by the zero bond with $B(t_0, t_1) = \left(\frac{1}{1+r_e(1,0)}\right)^{\Delta t}$ and $\Delta t = t_1 - t_0 = t_{i+1} - t_i \quad \forall\, i = 0, ..., N-1.$

Let P^* be a fixed probability measure defined by the (possibly time and state dependent) transition probabilities $p(i, n) \in \,]0, 1[$ in each knot (i, n), $i = 0, \ldots, N-1$, $n = 0, \ldots, i$. Suppose that at time $t_i \in \underline{\underline{T}} \setminus \{t_N\}$ the effective rate is $r_e(i, n)$, such that either $r_e(i + 1, n + 1)$ (with probability $p(i, n)$) or

[1] Indeed, if the continuously compounded rate $r_c(t)$ becomes infinitesimal small, i.e. $r_c(t) = 0(dt)$, then the two dynamics coincide.

$r_e(i+1,n)$ (with probability $1-p(i,n)$) are the effective rates of the next period. Define as the local risk measure of the model the conditional variance of the logarithmic effective rate:

$$\begin{aligned}
\sigma^2(i,n) &= V_{p(i,n)}\left[\log r_e(i+1,.)|r_e(i,n)\right] \\
\Leftrightarrow \quad r_e(i+1,n+1) &= \exp\left\{\frac{\sigma(i,n)}{\sqrt{p(i,n)(1-p(i,n))}}\right\} \cdot r_e(i+1,n) \qquad (2.1)\\
&=: g(\sigma(i,n),p(i,n)) \cdot r_e(i+1,n) \geq r_e(i+1,n)
\end{aligned}$$

Thus future effective rates are proportional to each other and by induction we have:

$$r_e(i+1,n+1) = \left(\prod_{j=0}^{n} g(\sigma(i,j),p(i,j))\right) r_e(i+1,0) \qquad (2.2)$$

At each period t_i, $r(i,0)$ denotes the lowest possible effective rate and all other rates at that period are related to this rate by (2.2). The idea is now to construct the process $\{r_e\}$ in such a way that the zero bond prices discounted by the process $\{r_e\}$ become martingales under P^*. This is equivalent to choosing the accumulation factor

$$\beta(t_i,\cdot) = \prod_{j=0}^{i-1}(1+r_e(j,\cdot))^{\Delta t} \qquad t_i \in \underline{\underline{T}} \setminus \{t_0\} \qquad (2.3)$$

as numeraire and constructing the short rate process $\{r_e\}$ in such a way that, for any maturity $\tau \in \underline{\underline{T}}\setminus\{t_0\}$, the process $\{B(t_i,\tau)/\beta(t_i) : 0 \leq t_i \leq \tau\}$ is a P^*-martingale[2]. Note that if e.g. $\Delta t = .25$ then $\{r_e\}$ is the 3-month-Libor spot rate process, and the measure P^* of our model is what Jamshidian (1996a) calls the "spot Libor measure". The existence, construction and uniqueness of this spot Libor interest rate process $\{r_e\}$ is given by the following theorem:

Theorem 2.1 *Suppose $B(t_0,t_1) > B(t_0,t_2) > \ldots > B(t_0,t_N)$ are given zero bond prices. Then for any measure P^* defined by a time and state dependent specification of the transition probabilities $p(i,n) \in]0,1[$ and for any time and state dependent specification of the volatility structure $\sigma(i,n) > 0$ there exists a unique and positive binomial process of the effective rate $\{r_e(t_i)\}_{t_i \in \underline{\underline{T}}}$ such that the discounted price processes of the zero bonds are martingales under P^*.*

Proof: See the Appendix

3 On the Stability of Lognormal Short Rate Models

By construction the process $\{r_e\}$ depends on both the specification of the volatility and the risk neutral or spot Libor measure. To study the stability of our discrete model we consider the limit distribution of

[2] Instead of the (risk neutral) probability P^* we could also start with an arbitrary (subjective) probability Q and assume some given risk premium. Following e.g. Ingersoll jr. (1987) the absence of arbitrage opportunities implies that the excess return per unit risk $\lambda(i,n)$ defined as

$$\lambda(i,n) := \frac{E_Q\left[B(t_{i+1},\tau)|B(t_i,\tau)\right] - (1+r(i,n))^{\Delta t} B(t_i,\tau)}{\sqrt{V_Q\left[B(t_{i+1},\tau)|B(t_i,\tau)\right]}}$$

is independent of $\tau > t_i, \tau \in \underline{\underline{T}}$. If $\lambda(i,n) \equiv 0$, Q is the risk neutral measure. If instead $\lambda(i,n) \neq 0$ under the measure Q, it is easy to show that the following shift in the measure in terms of the discrete transition probabilities

$$p(i,n) := q(i,n) + \sqrt{q(i,n)(1-q(i,n))} \cdot \lambda(i,n)$$

determines the risk neutral measure P^*.

the effective rate process for $\Delta t = \frac{T}{n}$ and $n \to \infty$ in the following special situation. Suppose that, for any n, the risk neutral probability measure P_n^* is characterized by a constant transition probability $p \in]0,1[$ and the volatility $\sigma : \underline{\underline{T}} \to \mathrm{R}_{>0}$ is at most a function of time, such that $\sigma^2(t_i)$ is proportional to the length of the time interval with $\sigma^2(t_i) = h(t_i) \cdot \Delta t$, where $h(.)$ converges to a bounded function on $[0,T]$.

Let P^* be the weak limit of the measures P_n^* on a suitable common probability space[3]. Then, given the binomial structure of the model, it is not difficult to see that the limit model is of the form

$$\frac{dr_e(t)}{r_e(t)} = \mu(t)dt + \sigma(t)dW(t), \tag{3.1}$$

where $\mu(t)$ is some function of the inital zero bonds and the volatility specification and $W(t)$ is a standard Wiener process under P^*.

Theorem 3.1 *For the limit model (3.1) the continuously compounded rate $r_c(t)$ follows the following diffusion process:*

$$dr_c(t) = (1 - e^{-r_c(t)}) \left\{ \left(\mu(t) - \frac{1}{2}(1 - e^{-r_c(t)})\sigma^2(t) \right) dt + \sigma(t)dW(t) \right\} \tag{3.2}$$

Proof: The connection between the (annual) continuously compounded rate $r_c(t)$ and the (annual) effective rate $r_e(t)$ is $r_c(t) = \ln x(t)$ with $x(t) = 1 + r_e(t)$. Hence, the dynamics 3.1 and Ito's Lemma imply

$$\begin{aligned} dr_c &= \frac{1}{x}dx - \frac{1}{2}\frac{1}{x^2}d\langle x \rangle \\ &= \frac{1}{1+r_e}\left(\mu r_e dt + \sigma r_e dW(t)\right) - \frac{1}{2}\frac{1}{(1+r_e)^2}\sigma^2 r_e^2 dt \end{aligned}$$

Using the relation $r_e/(1+r_e) = 1 - e^{-r_c}$ gives (3.2). □

Remark

(i) For $r_c(t) \to \infty$ the dynamics (3.2) converges to the normal diffusion

$$dr_c(t) = \left(\mu(t) - \frac{1}{2}\sigma^2(t) \right) dt + \sigma(t)dW(t)$$

(ii) For $r_c(t) = o(dt)$ it follows from $1 - e^{-r_c(t)} = r_c(t) + o(dt^2)$ and $r_c^2(t) = o(dt^2)$ that (3.2) becomes

$$\frac{dr_c(t)}{r_c(t)} = \mu(t)dt + \sigma(t)dW(t)$$

Hence only for infinitesimal small values the continuously compounded rate $r_c(t)$ follows a lognormal diffusion with same drift and volatility as the effective rate $r_e(t)$. But $r_c(t)$ is the annual continuously compounded rate and for most paths far away from $o(dt)$.

In the following let $\{W(t)\}_{0 \le t \le T}$ be a standard Wiener process on the probability space $(\Omega, (\mathbb{F}_t), P)$ with the filtration induced by $W(t)$ and $E_t[\cdot]$ denote the conditional expectation under P w.r.t $\mathbb{F}_t$. The following result is due to Hogan and Weintraub (1993). Consider the following two processes for the continuously compounded short rate r_c:

$$dr_c = \alpha r_c dt + \sigma r_c dW(t) \tag{3.3}$$

[3] E.g. let $\Omega = D[0,T]$ be the set of real cadlag-functions on $[0,T]$. Let $\left(\underline{\underline{T}}(n)\right)_{n=1,2,...}$ with $\underline{\underline{T}}(n) := \{0 = t_0 < t_1 < ... < t_n = T \mid t_i = i \cdot \frac{T}{n}\}$ be a sequence of equidistant partitions of $[0,T]$. Embed, for any n, the discrete process r_e^n into $D[0,T]$ by $\tilde{r}_e^n = \sum_{t_i \in \underline{\underline{T}}(n)} r_e^n \cdot 1_{[t_i, t_{i+1}[}(t)$. Let $(\mathbb{F}_t^n)$ be the filtration on $D[0,T]$ generated by $\tilde{r}_e^n(t)$.

$$d\ln r = \kappa(\theta - \log r)dt + \sigma dW(t) \tag{3.4}$$

where κ and θ are non–negative. Then, for any $0 \le \tau < t < T$,

$$E_\tau[B(t,T)^{-1}] = \infty$$

Equation (3.3) is the model of Dothan (1978) and the limit of the Black, Derman, and Toy (1990) model. Equation (3.4) is the model of Black and Karasinski (1991)

This result has two consequences

(1^0) Expected accumulation factors over any positive time interval are infinite

(2^0) The models (3.3) and (3.4) attach negative infinite values to Eurodollar future contracts.

To see (1^0) observe that by Jensen's inequality

$$B(t,T)^{-1} = E_t\left[\exp\left\{-\int_t^T r_c(s)ds\right\}\right]^{-1} < E_t[\beta(t,T)].$$

Hence, for any $\tau < t$, $E_\tau[\beta(t,T)] > E_\tau[B(t,T)^{-1}] = \infty$.

For (2^0) observe how Eurodollar futures are quoted and settled. E.g. a quotation of 90% on a 3–month instrument implies an interest of 2.5% and a price of 97.5% times the face value of the contract[4]. At time t the contract is settled at price $100 - .25 \times r_L(t)$ percent, where $r_L(t)$ is the 3–month LIBOR rate valid at t. More generally, a Eurodollar future contract for the period $[t, T = t+\delta]$ is settled at t at the %–price

$$X = 100 - \delta r_L(t,\delta)\ , \tag{3.5}$$

where $r_L(t,\delta)$ is the δ–Libor rate ($\delta \le 1$) valid at t. Cox, Ingersoll, and Ross (1981) have shown that with continuous resettlement the future price $F_\tau(t,T)$ at time $\tau < t$ of a contract which settles at the amount X_t at time t is $E_\tau[X_t]$. Since at t

$$B(t,T) = (1 + \delta \cdot r_L(t,\delta)/100)^{-1} \tag{3.6}$$

(3.5) and (3.6) imply

$$X = 100 \times (2 - B(t,T)^{-1}) \tag{3.7}$$

and hence $F_\tau(t,T) = E_\tau[X] = -\infty$.

Theorem 3.2 *Let the annual effective rate r_e follow the lognormal diffusion*

$$\frac{dr_e(t)}{r_e(t)} = \mu(t)dt + \sigma(t)dW(t)$$

where $\mu(\cdot)$ and $\sigma^2(\cdot)$ are integrable and bounded functions on $[0,T]$. Then, for any $0 \le \tau < t < T$,

$$E_\tau[B(t,T)^{-1}] \quad \text{and} \quad E_\tau[\beta(t,T)].$$

are finite.

[4] See e.g. Hull (1993, p. 99)

Proof: Since by Jensen's inequality

$$B(t,T)^{-1} = E_\tau^{-1}\left[\exp\left\{-\int_t^T r_c(s)ds\right\}\right] < E_\tau\left[\exp\left\{\int_t^T r_c(s)ds\right\}\right]$$

it suffices to prove that the second expression is finite. Again Jensen's inequality and the relation $r_c(t) = \ln(1+r_e(t)) \geq 0$ imply

$$\begin{aligned}
E_\tau[\beta(t,T)] &= E_\tau\left[\exp\left\{\int_t^T \ln(1+r_e(s))ds\right\}\right] \\
&= E_\tau\left[\exp\left\{\int_t^T \frac{1}{T-t}\ln(1+r_e(s))^{T-t}ds\right\}\right] \\
&\leq E_\tau\left[\exp\left\{\ln\int_t^T \frac{1}{T-t}(1+r_e(s))^{T-t}ds\right\}\right] \\
&= \frac{1}{T-t}\int_t^T E_\tau\left[(1+r_e(s))^{T-t}\right]ds \\
&\leq \frac{1}{T-t}\int_t^T E_\tau\left[(1+r_e(s))^{k}\right]ds \quad \text{for } k=\min\{i\in\mathbb{N}|i\geq T-t\}
\end{aligned}$$

which is finite, since the above assumption implies in particular that $\forall i \leq k$

$$\int_t^T \exp\left\{\int_\tau^s \left(\mu(u)+\frac{i}{2}\sigma^2(u)\right)du\right\}^i ds \quad < \infty \quad .$$

□

Hence with lognormal effective short rates both expected returns and Eurodollar future prices are finite. Note that for $B(t,T) < \frac{1}{2}$ equation (3.7) becomes negative and hence also $F_\tau(t,T)$. But this is in accordance with the quotation convention of futures. E.g. $B(t,T) < \frac{1}{2}$ occurs for $\delta = 1$ and r_L greater 100%, but then future quotes will also be negative.

4 On the Stability of Lognormal Forward Rate Models

As already mentioned in the Introduction, it was quickly recognized that the existence and stability problems in the traditional lognormal term structure models could also be overcome by a shift of the volatility specification from continuously compounded rates to effective or simple rates.

Goldys, Musiela, and Sondermann (1994) shift the volatility specification from the rates $r(t,x)$ to the rates $j(t,x)$ which satisfy

$$\exp\{r(t,x)\} = 1 + j(t,x).$$

Here $r(t,x)$ resp. $j(t,x)$ denote the continuously compounded resp. effective annual rate prevailing at time t over the time interval $[t+x, t+x+dx]$. For any volatility specification $\sigma(t,x)$ for the rates $r(t,x)$ it is known (e.g. Brace and Musiela (1994b), Musiela and Sondermann (1993)) that the arbitrage free dynamics of the process $\{r(t,x) : t, x > 0\}$ satisfies

$$dr(t,x) = \left(\frac{\partial}{\partial x}r(t,x) + \sigma(t,x)\cdot\int_0^x \sigma(t,u)du\right)dt + \sigma(t,x)\cdot dW(t), \tag{4.1}$$

where $W(t)$ is a d-dimensional Brownian motion and $\cdot$ denotes the inner product in $\mathbb{R}^d$. But for a lognormal volatility structure of the continuously compounded forward rate of the form $\sigma(t,x) = r(t,x)\gamma(t,x)$ the resulting rate explodes (assume infinite value) with positive probability (as shown by Morton (1988)). Assuming instead a lognormal volatility structure on the effective rates $j(t,x)$, that is

$$dj(t,x) = ...dt + j(t,x)\gamma(t,x) \cdot dW(t), \tag{4.2}$$

where the volatility of the annual effective rates $\gamma(t,x)$ is deterministic, it is easy to see (cp. Theorem 3.1) that

$$\sigma(t,x) = \left(1 - e^{-r(t,x)}\right)\gamma(t,x).$$

Again the dynamics of $r(t,x)$ has now the same qualitative features as in Section 3, yielding both both positive rates and stability. Goldys, Musiela, and Sondermann (1994) show among other results that the system 4.2 with the appropiate arbitrage free drift has a unique positive solution (under some mild technical assumption on the process $\gamma(t,x)$). Furthermore they show that the Eurodollar futures price is well defined in their model.

Basically the same results were obtainted by Musiela (1994) for a lognormal volatility specification of simple forward rates $j(t,x)$ compounded m-times per year, i.e. rates quoted for an accrual period of length $\delta = m^{-1}$, defined by

$$(1 + \delta j(t,x))^m = e^{r(t,x)}.$$

In this case the volatility process takes the form

$$\sigma(t,x) = \delta^{-1}\left(1 - e^{-\delta r(t,x)}\right) \cdot \gamma(t,x).$$

Again for $\delta > 0$ the model is stable, i.e. has a unique positive solution and no explosion occurs. Note that for $\delta = 0$ one obtaines the volatility $\sigma(t,x) = r(t,x)\gamma(t,x)$ with exploding forward rates. This shows the importance of modelling effective or simple rates with a finite accrual period.

Sandmann, Sondermann, and Miltersen (1995) developed a lognormal term structure model for effective annual forward rates $f(t,x,\delta)$ defined by

$$(1 + f(t,x,\delta))^{\delta} = \exp\left\{\int_x^{x+\delta} r(t,u)du\right\}. \tag{4.3}$$

They derived the following relation between the Heath-Jarrow-Morton volatilities $\sigma(t,x)$ and the effective forward volatilities $\gamma(t,x,\delta)$:

$$\sigma(t,x+\delta) - \sigma(t,x) = \delta\left(1 - \exp\left\{\delta^{-1}\int_x^{x+\delta} r(t,u)du\right\}\right) \cdot \gamma(t,x,\delta).$$

For $\delta = 1$ they furthermore derive closed form solutions for options on zero bonds and caps and floors with annual payment dates, where the latter corresponds to the market practice of valuing caplets by Black's formula, based on the volatilities $\gamma(t,x,\delta)$. In a subsequent paper Miltersen, Sandmann, and Sondermann (1995) studied a lognormal model of simple forward rates $f(t,T,\delta)$ prevailing at time t for the time period $[T, T+\delta]$ defined by

$$1 + \delta \cdot f(t,T,\delta) = \exp\left\{\int_T^{T+\delta} f(t,u,0)du\right\},$$

where $f(t,u,0) = r(t,u-t)$ is the Heath-Jarrow-Morton continuously compounded forward rate and arbitrary $\delta > 0$. Also for this model they obtain closed form solutions for options on zero bonds and caps and floors based on δ-simple rates. Again the caplet formula coincides with the Black market formula, which shows that the market practice for pricing caps is consistent with an arbitrage free HJM term structure model based on lognormal simple rates.

Brace, Gatarek, and Musiela (1995) developed and implemented a model based on forward Libor rates $L(t,x)$ defined by

$$1 + \delta L(t,x) = \exp\left\{\int_x^{x+\delta} r(t,u)du\right\}. \tag{4.4}$$

They show that the corresponding HJM-model is arbitrage free by proving the existence of an equivalent martingale measure, a problem left open in Miltersen, Sandmann, and Sondermann (1995).

An alternative construction of an arbitrage free family of forward Libor processes was given by Musiela and Rutkowski (1995). In their approach instantaneous rates and the risk neutral measure are not needed at all. Instead they work with the "terminal" measure, i.e. the forward measure with respect to the final payment date. They show how in this measure forward Libor processes can be constructed by "backward induction" from the specification of the lognormal forward Libor volatility function.

Finally Jamshidian (1996a) extends the previous studies to swap markets. Specifying a lognormal volatility structure for forward swap rates he obtaines the industry standard Black-Scholes formula for European swaptions.

5 Conclusion

The simple observation of the crucial difference in modelling instantaneous and effective rates (first made by Sandmann and Sondermann (1993a)) has opened a new line of research on lognormal interest rate models. Traditional models start with modelling the dynamics of instantaneous spot or forward rates. In order to fit these models to observable market quotes for interest rate derivatives like caps and swaptions the model parameters have to be calibrated by a numerical root search algorithm. The resulting processes for market rates like forward Libor or swap rates are analyically complex and in general show no resemblance to the lognormal volatility structure implicitely assumed by the market (cp. Jamshidian (1996a)).

Many traditional models assume Gaussian volatility structures in order to ease analytical tractibility (e.g. Brace and Musiela (1994a, b), El Karoui et al. (1991, 92), Jamshidian (1989, 91), Hull and White (1990)). Beside the problem of negative interest rates – a problem underestimated as recently shown by Rogers (1996) – the closed form solutions obtained e.g. in Gaussian HJM-type models again bear no resemblance to market practice.

The shift to nominal rates as the primary process rather than as a secondary process derived from instantaneous rates has led to interest models which seem to better reflect market practice. Another important step in this development was the discovery of the consistency of the market caplet formula in such models (see Section 4). Instead of calibration by a root search procedure the model can be fitted directly to basic market segments like the cap market or the swaption market by specifying (implicite) observable lognormal market volatilities (see also Reed (1995a,b) and Jamshidian (1996b)).

Appendix

Proof of Theorem 2.1

Proof: The proof is done by induction. Suppose the binomial effective rate process $\{r_e\}$ is already constructed for all $t_i \leq t_n \in \underline{\underline{T}}$. The martingale property implies that the initial value of any zero coupon bond is equal to its expected discounted payoff under P^*. Define $p(i,n) \in]0,1[$ as the transition probabilities determined by the risk measure P^*. In particular this yields

$$B(t_0,t_n) = \left(\frac{1}{1+r(0,0)}\right)^{\Delta t} \cdot \sum_{i\in\kappa(n-1)} \left[\prod_{j=0}^{n-2} p(j,s(j,i))^{i_{j+1}}(1-p(j,s(j,i)))^{1-i_{j+1}} \left(\frac{1}{1+r_e(j+1,s(j+1,i))}\right)^{\Delta t}\right]$$

$$\begin{aligned}
\text{where} \quad \kappa(n-1) &= \{i=(0,i_1,...,i_{n-1}) \in \{0\}\times\{0,1\}^{n-1}\} \\
&\quad \text{is the set of all possible paths} \\
\text{and} \quad s(j,i) &:= \sum_{k=0}^{j} i_k \;\hat{=}\; \text{number of up–movements of the path } i\in\kappa(n-1) \\
&\quad \text{at time } t_j \le t_i
\end{aligned}$$

Define $x = r_e(n,0)$ and

$$\begin{aligned}
u(n+1,x) := \Bigg[\sum_{i\in\kappa(n-1)} &\left(\prod_{j=0}^{n-2} p(j,s(j,i))^{i_{j+1}}\left(1-p(j,s(j,i))\right)^{1-i_{j+1}}\right. \\
&\left.\cdot\left(\frac{1}{1+r_e(j+1,s(j+1,i))}\right)^{\Delta t}\right)\cdot\left(\frac{1}{1+r_e(0,0)}\right)^{\Delta t} \\
&\cdot\left(\frac{1}{1+x\cdot\prod_{k=0}^{s(n,i)} g(\sigma(n,k),p(n,k))}\right)^{\Delta t}\Bigg] - B(t_0,t_{n+1})
\end{aligned}$$

Since $g(\ ,\ ) \ge 1$ the function $u(n+1,.)$ is strictly decreasing for $x \ge 0$ and

a) $u(n+1,0) = B(t_0,t_n) - B(t_0,t_{n+1}) \quad > 0$ by assumption

b) $\lim_{x\to+\infty} u(n+1,x) = -B(t_0,t_{n+1}) \quad < 0$

Thus there exist a unique solution $x^* > 0$ such that $u(n+1,x^*) = 0$ □

References

Black, F. and P. Karasinski (1991): "Bond and Option Pricing when Short Rates are Lognormal," *Financial Analysts Journal,* 47, 52–59.

Black, F., E. Derman, and W. Toy (1990): "A One–Factor Model of Interest Rates and Its Application to Treasury Bond Options," *Financial Analysts Journal,* 46, 33–39.

Brace, A. and M. Musiela (1994a): "A Multifactor Gauss Markov Implementation of Heath, Jarrow, and Morton," *Mathematical Finance,* 4, 259–283.

Brace, A. and M. Musiela (1994b): "Swap Derivatives in a Gaussian HJM Framework," University of New South Wales, working paper.

Brace, A., D. Gatarek, and M. Musiela (1995): "The Market Model of Interest Rate Dynamics," to appear in *Mathematical Finance,* 1997.

Cox, J. C., J. E. Ingersoll jr., and S. A. Ross (1981): "The Relation Between Forward Prices and Futures Prices," *Journal of Financial Economics,* 9, 321–346.

Dothan, L. (1978): "On the Term Structure of Interest Rates," *Journal of Financial Economics,* 6, 59–69.

El Karoui, N., C. Lepage, R. Myneni, N. Roseau, and R. Viswanathan (1991): "The Valuation and Hedging of Contingent Claims with Markovian Interest Rate," Université de Paris 6, working paper.

El Karoui, N., R. Myneni, and R. Viswanathan (1992): "Arbitrage Pricing and Hedging of Interest Rate Claims with State Variables: I Theory," Université de Paris VI, working paper.

Goldys, B., M. Musiela, and D. Sondermann (1994): "Log-Normality of Rates and Term Structure Models," University of New South Wales, Australia, working paper.

Heath, D., R. Jarrow, and A. Morton (1992): "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica,* 60, 77–105.

Hogan, M. and K. Weintraub (1993): "The Lognormal Interest Rate Model and Eurodollar Futures," Citibank, New York, working paper .

Hull, J. (1993): *Options, Futures, and other Derivative Securities*, 2 edn, Prentice-Hall, Inc., New Jersey, USA.

Hull, J. and A. White (1990): "Pricing Interest–Rate Derivative Securities," *The Review of Financial Studies,* 3, 573–592.

Ingersoll jr., J. E. (1987): *Theory of Financial Decision Making*, Rowman & Littlefield Publishers, Inc., New Jersey, USA, chapter 18, pp. 387–401.

Jamshidian, F. (1989): "The Multifactor Gaussian Interest Rate Model and Implementation," working paper.

Jamshidian, F. (1991): "Bond and Option Evaluation in the Gausian Interest Rate Model," *Research in Finance,* 9, 131–170.

Jamshidian, F. (1996a): "Libor and Swap Market Models and Measures," Sakura Global Capital, working paper, forthcoming in *Finance and Stochastics.*

Jamshidian, F. (1996b): "Sorting out Swaptions," *Risk Magazine*, 9(3).

Miltersen, K. R., K. Sandmann, and D. Sondermann (1995): "Closed Form Solutions for Term Structure Derivatives with Log–Normal Interest Rates," to appear in *The Journal of Finance,* March 1997.

Morton, A. J. (1988): "Arbitrage and Martingales," *Technical Report 821*, Cornell University, New York.

Musiela, M. (1994): "Nominal Annual Rates and Lognormal Volatility Structure," University of New South Wales, working paper.

Musiela, M. and M. Rutkowski (1995): "Continuous-Time Term Structure Models," University of New South Wales, working paper, to appear in *Finance and Stochastics,* 1997.

Musiela, M. and D. Sondermann (1993): "Different Dynamical Specifications of the Term Structure of Interest Rates and their Implications," Universität Bonn, working paper B–260.

Reed, N. (1995a): "If the Cap Fits," *Risk Magazine,* 8(8), 34–35.

Reed, N. (1995b): "If the Cap Fits," *Risk Magazine,* 8(9), 19.

Rogers, C. (1996): "Gaussian Errors" *Risk Magazine,* 9(1), 42–45.

Sandmann, K. and D. Sondermann (1989): "A Term Structure Model and the Pricing of Interest Rate Options," Universität Bonn, working paper B–129.

Sandmann, K. and D. Sondermann (1992): "Interest Rate Options," in *Geld, Banken, Versicherungen,* eds. W. Heilmann (ed.), VVW, Karlsruhe, Germany, pp. 739–760.

Sandmann, K. and D. Sondermann (1993a): "On the Stability of Lognormal Interest Rate Models," Universität Bonn, working paper B–263.

Sandmann, K. and D. Sondermann (1993b): "A Term Structure Model and the Pricing of Interest Rate Options," *The Review of Futures Markets,* 12(2), 391–423.

Sandmann, K., D. Sondermann, and K. R. Miltersen (1995): "Closed Form Term Structure Derivatives in a Heath–Jarrow–Morton Model with Log–normal Annually Compounded Interest Rates," *Proceedings of the Seventh Annual European Research Symposium, Bonn, September 1994, Chicago Board of Trade* pp. 145–164.