CONTINUOUS-TIME TERM STRUCTURE MODELS*

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The HJM term structure methodology (see Heath *et al.* [15]) is based on an arbitrage-free dynamics of the instantaneous continuously compounded forward rates. The existence of such rates, however, requires a certain degree of smoothness with respect to the tenor of the bond prices and their volatilities, and thus working with such models may be inconvenient. For example, Brace *et al.* [4] parametrize their version of the lognormal forward LIBOR model introduced by Miltersen, Sandmannn and Sondermann (cf., [20],[23]) with a piecewise constant volatility function but need to consider smooth volatility functions in order to analyse the model in the HJM framework.

In the present paper the problem of continuous-time modelling of term structure of interest rates is considered in a general manner. We describe certain properties which are valid for wide classes of term structure models, so that a basis for the discussion of any specific model is developed. Three such special systems are put forward, and their properties are discussed (we refer to them as to the setups (BP), (FP) and (LR) in what follows). The paper proceeds as follows. In Section 1, we deal with the question of existence and uniqueness of a savings account implied by a given (weakly) arbitrage-free continuous-time family of bond prices. Section 2 is devoted to the problem of construction of an arbitrage-free family of bond prices given a family of stochastic volatilities of forward rates and an initial term structure. In Section 3, a construction of a lognormal model of forward LIBOR rates is presented. The next section deals with a rather peculiar approach to the modelling of the term structure which was proposed recently by Flesaker and Hughston [11]. Finally, in the last two sections, we review the valuation results for the interest rate derivatives (for the sake of brevity, we focus on *caps* and *swaptions*) which were obtained for the models we consider here.

Let us comment briefly on the existence of a short-term rate of interest. In the traditional models, in which the instantaneous continuously compounded short rate r_t is well defined, the *savings account* process, $\tilde{B}$ say, satisfies

$$\tilde{B}_t = \exp\left(\int_0^t r_u \, du\right) \tag{1}$$

so that it represents the amount generated at time t by continuously reinvesting \$1 in the short rate r_t (in such a framework, the absence of arbitrage is related to the non-negativity of the short rate). Though we will deal sometimes with an (implied) savings account, we will not assume that its sample paths are absolutely continuous with respect to the Lebesgue measure, therefore, models in which the instantaneous rate of interest is not well defined will be also covered by our subsequent analysis. In this more general setting, the absence of arbitrage between bonds of different maturities implies the existence of a savings account which follows a process of finite variation. If, in addition, the absence of arbitrage between bonds and cash is assumed, the savings account is shown to follow an increasing process.

1 Bond price models

We start by introducing some notation which we shall need in the sequel. Given a probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$ endowed with the filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ which satisfies the usual conditions, we denote by $\mathcal{M}_{loc}(\mathbf{P})$ and $\mathcal{M}(\mathbf{P})$ the class of all real-valued local martingales and the class of all real-valued martingales, respectively. The subscript $_c$ will indicate that we consider processes with continuous sample paths, and the superscript $^+$ will denote the collection of all strictly positive processes which belong to a given class of processes. For instance, $\mathcal{M}_c^+(\mathbf{P})$ stands for the class of strictly positive $\mathbf{P}$-martingales with continuous sample paths. We denote by $\mathcal{V}$ (by $\mathcal{A}$, resp.) the class of all real-valued adapted (predictable, resp.) processes of finite variation. We write $\mathcal{S}_p(\mathbf{P})$ to denote the class of all real-valued special semimartingales, that is, those processes X which admit a decomposition $X_t = X_0 + M_t + A_t$, where $M \in \mathcal{M}_{loc}(\mathbf{P})$ and $A \in \mathcal{A}$. Abusing slightly our convention, we denote by $\mathcal{S}_p^+(\mathbf{P})$ the class of those special semimartingales $X \in \mathcal{S}_p(\mathbf{P})$ which are strictly positive, and such that, in addition, the process of left hand limits,[1] X_{t-}, is also strictly positive. Notice that $\mathcal{S}_p(\mathbf{P})$ (as well as $\mathcal{S}_p^+(\mathbf{P})$) is invariant with respect to an equivalent change of the underlying probability measure. More precisely, $\mathcal{S}_p(\mathbf{P}) = \mathcal{S}_p(\mathbf{Q})$ and $\mathcal{S}_p^+(\mathbf{P}) = \mathcal{S}_p^+(\mathbf{Q})$ if $\mathbf{P}$ and $\mathbf{Q}$ are mutually equivalent[2] probability measures on $(\Omega, \mathcal{F}_{T^*})$ such that the Radon-Nikodým density

$$\Lambda_t = \frac{d\mathbf{Q}}{d\mathbf{P}}_{\,|\,\mathcal{F}_t}, \quad \forall\, t \in [0, T^*],$$

follows a locally bounded process (see, Dellacherie and Meyer [8], p.258). Since we shall assume throughout that $(\mathcal{F}_t)_{t\in[0,T^*]}$ is a natural filtration of a Wiener process, the Radon-Nikodým density Λ will always follow a continuous exponential martingale, hence, a locally bounded process. Therefore, we may and do write simply $\mathcal{S}_p$ and $\mathcal{S}_p^+$ in what follows.

By a default-free zero-coupon bond of maturity T (a *bond,* for short) we mean a financial security which pays to its holder one monetary unit at time T. Let us fix a strictly positive horizon date $T^* > 0$, and let $B(t,T)$ stand for the price at time $t \le T$ of bond which matures at time $T \le T^*$. By a *family of bond prices* we mean an arbitrary family of strictly positive real-valued adapted processes $B(t,T)$, $t \in [0,T]$, with $B(T,T) = 1$ for every $T \in [0,T^*]$. It should be stressed that a bond price process $B(\cdot,T)$ is not necessarily a semimartingale. The following set of assumptions is subsequently referred to as (BP),

Assumptions (BP)
(BP.1) W is a d-dimensional standard Wiener process defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$. The filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ coincides with the natural filtration of W.
(BP.2) For any fixed maturity date $T \in [0,T^*]$, the bond price process $B(t,T)$, $t \in [0,T]$, belongs to the class $\mathcal{S}_p^+$.
(BP.3) For any fixed $T \in [0,T^*]$, the *forward process* $F_B(t,T,T^*) \stackrel{\text{def}}{=} B(t,T)/B(t,T^*)$ follows a martingale under $\mathbf{P}$, or equivalently,

$$B(t,T) = \mathbf{E_P}(B(t,T^*)/B(T,T^*)\,|\,\mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{2}$$

By virtue of assumptions (BP.1)-(BP.3), the process $F_B(t,T,T^*)$, $t \in [0,T]$, follows a strictly positive continuous $\mathbf{P}$-martingale with respect to the filtration of a Wiener process, that is, $F_B(\cdot,T,T^*) \in \mathcal{M}_c^+(\mathbf{P})$. Hence, for any fixed $T \in [0,T^*]$ there exists a $\mathbf{R}^d$-valued predictable process $\gamma(t,T,T^*)$, $t \in [0,T]$, integrable with respect to the Wiener process W, and such that[3]

$$F_B(t,T,T^*) = F_B(0,T,T^*)\,\mathcal{E}_t\left(\int_0^\cdot \gamma(u,T,T^*)\cdot dW_u\right), \tag{3}$$

where $\cdot$ stands for the usual inner product in $\mathbf{R}^d$. In other words, for any fixed maturity $T \in [0,T^*]$ the dynamics of $F_B(t,T,T^*)$ is given by the expression

$$dF_B(t,T,T^*) = F_B(t,T,T^*)\gamma(t,T,T^*)\cdot dW_t. \tag{4}$$

Let us now consider two arbitrary maturities $T,U \in [0,T^*]$. We define the forward process $F_B(t,T,U)$ by setting

$$F_B(t,T,U) \stackrel{\text{def}}{=} \frac{F_B(t,T,T^*)}{F_B(t,U,T^*)} = \frac{B(t,T)}{B(t,U)}, \quad \forall\, t \in [0, T\wedge U]. \tag{5}$$

[1] All processes are assumed to be càdlàg, that is, almost all sample paths are right continuous functions with finite left hand limits,

[2] We write $\mathbf{P} \sim \mathbf{Q}$ to denote that two probability measures $\mathbf{P}$ and $\mathbf{Q}$ are mutually equivalent.

[3] We write $\mathcal{E}_t(X)$, $t \in [0,T^*]$, to denote the Doléans exponential of a semimartingale X.

Suppose first that $U > T$; then the amount

$$f_B(t,T,U) = (U-T)^{-1}(F_B(t,T,U)-1) \tag{6}$$

is the *simple (annualized) forward rate* over the time interval $[T,U]$ prevailing at time t. On the other hand, if $U < T$ then $F_B(t,T,U)$ represents the value at time t of the *forward price* for the settlement date U of a T-maturity bond. The following lemma is a straightforward consequence of the Itô formula.

Lemma 1.1 *For any maturities $T,U \in [0,T^*]$, the dynamics under $\mathbf{P}$ of the forward process is given by the following expression*

$$dF_B(t,T,U) = F_B(t,T,U)\,\gamma(t,T,U)\cdot(dW_t - \gamma(t,U,T^*)\,dt), \tag{7}$$

where

$$\gamma(t,T,U) = \gamma(t,T,T^*) - \gamma(t,U,T^*), \quad \forall\, t \in [0, U\wedge T]. \tag{8}$$

Combining Lemma 1.1 with Girsanov's theorem we obtain

$$dF_B(t,T,U) = F_B(t,T,U)\,\gamma(t,T,U)\cdot dW_t^U, \tag{9}$$

where

$$W_t^U = W_t - \int_0^t \gamma(u,U,T^*)du, \quad \forall t \in [0,U]. \tag{10}$$

The process W^U is a standard Wiener process on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,U]}, \mathbf{P}^U)$, with the probability measure $\mathbf{P}^U \sim \mathbf{P}$ defined on $(\Omega, \mathcal{F}_U)$ by means of the Radon-Nikodým derivative

$$\frac{d\mathbf{P}^U}{d\mathbf{P}} = \mathcal{E}_U\left(\int_0^\cdot \gamma(u,U,T^*)\cdot dW_u\right), \quad \mathbf{P}-\text{a.s.} \tag{11}$$

We conclude that the forward process $F_B(t,T,U)$ follows an exponential local martingale under the "forward" probability measure $\mathbf{P}^U$, because (9) yields

$$F_B(t,T,U) = F_B(0,T,U)\,\mathcal{E}_t\left(\int_0^\cdot \gamma(u,T,U)\cdot dW_u^U\right), \quad \forall\, t\in[0,U\wedge T]. \tag{12}$$

Observe also that we have $\mathbf{P}^{T^*} = \mathbf{P}$ and $W^{T^*} = W$; readers familiar with the technique of arbitrage pricing under stochastic interest rates will easily recognize the underlying probability measure $\mathbf{P}$ as a "forward" probability measure (cf., Jamshidian [17], Geman [12], El Karoui and Rochet [10], El Karoui *et al.* [9]) associated with the horizon date T^* rather than the "spot" martingale measure.

1.1 Forward measures and spot measures

The concept of a *forward probability measure* appears to be more ambiguous in the present framework than in the traditional setup. We define it in terms of the behaviour of relative bond prices.

Definition 1.1 Let U be a fixed maturity date. A probability measure $\mathbf{Q}^U \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_U)$ is called a *forward probability measure* for the date U if for any maturity $T \in [0,T^*]$ the forward process $F_B(t,T,U), t\in[0,T\wedge U]$, follows a local martingale under $\mathbf{Q}^U$.

By virtue of (BP.3), the underlying probability measure $\mathbf{P}$ is a forward probability measure for the date T^* in the sense of Definition 1.1. Let us now introduce the notion of a spot probability measure within our present framework. Intuitively speaking, a spot measure is a forward measure[4] associated with the initial date $T = 0$. Its formal definition relates to a very specific kind of discounting, however. It should be stressed that neither a forward measure for the date T^*, nor a spot measure are uniquely defined (therefore, it would be inappropriate to write $\mathbf{P} = \mathbf{Q}^{T^*}$).

Definition 1.2 A *spot probability measure* for the setup (BP) is an arbitrary probability measure $\tilde{\mathbf{P}} \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ and such that the family $B(t,T)$ satisfies

$$B(t,T) = \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T], \ \forall\, T \in [0,T^*], \tag{13}$$

for some process $\tilde{B} \in \mathcal{A}^+$, with $\tilde{B}_0 = 1$.

1.2 Arbitrage-free properties

We shall study two forms of absence of arbitrage. The first, weaker notion refers to a pure bond market. The second form assumes, in addition, that *cash* is also present. By cash we mean money which can be carried over at no cost, rather then a savings account yielding a positive interest. In both cases, self-financing trading strategies can be introduced by making reference to continuous trading in any finite family of zero-coupon bonds with different maturities. Also the notion of an arbitrage opportunity can be introduced as in the classical Harrison and Pliska [13] framework. For these reasons we do not go into details here. We formulate instead a sufficient condition for the absence of arbitrage between bonds with different maturities (as well as between bonds and cash).

Definition 1.3 A family $B(t,T)$ of bond prices is said to satisfy a *weak no-arbitrage condition* if and only if there exists a probability measure $\mathbf{Q} \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ such that for any maturity $T < T^*$ the forward process $F_B(t,T,T^*) = B(t,T)/B(t,T^*)$ belongs to $\mathcal{M}_{loc}(\mathbf{Q})$. We say that the family $B(t,T)$ fulfills a *no-arbitrage condition* if, in addition, inequality $B(T,U) \leq 1$ holds for arbitrary maturities $T, U \in [0,T^*]$ such that $T \leq U$.

Assumption (BP.3) is, of course, sufficient for the family $B(t,T)$ to satisfy a weak no-arbitrage condition as we may take $\mathbf{Q} = \mathbf{P}$. As already mentioned, if a family $B(t,T)$ satisfies a weak no-arbitrage condition then it is possible to construct a model of the securities market with absence of arbitrage across bonds with different maturities.[5] We shall now focus on the absence of arbitrage between all bonds and cash. Inequality $B(T,U) \leq 1$ which is equivalent to $F_B(T,U,T) \leq 1$ gives immediately

$$F_B(t,U,T) = \mathbf{E}_{\mathbf{P}}(F_B(T,U,T) \,|\, \mathcal{F}_t) \leq 1, \quad \forall\, t \in [0,T]. \tag{14}$$

Since the forward process $F_B(t,U,T)$ has continuous sample paths, we may reformulate this condition as follows.

(BP.4) For arbitrary two maturities $T \leq U$, the following inequality holds with probability 1

$$B(t,U) \leq B(t,T), \quad \forall\, t \in [0,T]. \tag{15}$$

Suppose, on the contrary, that $B(t,U) > B(t,T)$ for certain maturities $U > T$. In such a case, by issuing at time t a bond of maturity U, and purchasing a T-maturity bond, one could have locked

[4] For brevity, we shall write *forward measure* (*spot measure,* resp.) instead of *forward probability measure* (*spot probability measure,* resp.) in what follows.

[5] Let us stress that no explicit reference to the presence of *cash* or a *savings account* is made in the definition of the weak no-arbitrage condition.

in a riskless profit if, in addition, cash were present in the economy. Indeed, to meet the liability at time U it would be enough to carry over the period $[T, U]$ (at no cost) one unit of cash received at time T. Observe that the following three conditions are equivalent: (i) the bond price $B(t, T)$ is a nonincreasing function of maturity T, (ii) the forward process $F_B(t, T, U)$ is never less than one, and (iii) the bond price $B(T, U)$ is never strictly greater than 1, where $T \leq U$ are arbitrary maturities. Not surprisingly, the problem of absence of arbitrage between bonds and cash appears to be closely related to the question of existence of an increasing savings account implied by the family $B(t, T)$. A formal definition of an *implied savings account* reads as follows.

Definition 1.4 A *savings account implied* by the family $B(t, T)$ of bond prices is an arbitrary process $\tilde{B}$, with $\tilde{B}_0 = 1$, satisfying the following conditions: (i) B belongs to $\mathcal{A}^+$, (ii) there exists a probability measure $\tilde{\mathbf{P}}$ on $(\Omega, \mathcal{F}_{T^*})$ equivalent to $\mathbf{P}$ and such that for every $T \in [0, T^*]$ the relative bond price

$$\tilde{Z}(t, T) \stackrel{\text{def}}{=} B(t, T)/\tilde{B}_t, \quad \forall\, t \in [0, T], \tag{16}$$

follows a $\tilde{\mathbf{P}}$-martingale.

It is clear that $\tilde{Z}(t, T)$ is a $\tilde{\mathbf{P}}$-martingale if for any maturity T the bond price $B(t, T)$ satisfies

$$B(t, T) = \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T]. \tag{17}$$

Notice also that by virtue of condition (ii) we have, in particular,

$$B(0, T) = \mathbf{E}_{\tilde{\mathbf{P}}}(1/\tilde{B}_T), \quad \forall\, t \in [0, T^*], \tag{18}$$

so that an implied savings account $\tilde{B}$ fits also the initial term structure $B(0, T)$, $T \in [0, T^*]$. It is also clear that the probability measure $\tilde{\mathbf{P}}$ of Definition 1.4 is a spot probability measure for the family $B(t, T)$, in the sense of Definition 1.2.

One might wonder if the normalized bond price $B^*_t \stackrel{\text{def}}{=} B(t, T^*)/B(0, T^*)$ would be a plausible choice of an implied savings account (corresponding, of course, to $\tilde{\mathbf{P}} = \mathbf{P}$). It follows immediately from (BP.3) that

$$\mathbf{E}_{\mathbf{P}}(B^*_t/B^*_T \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{P}}(B(t, T^*)/B(T, T^*) \,|\, \mathcal{F}_t) = B(t, T), \quad \forall\, t \in [0, T]. \tag{19}$$

Hence, the only property of an implied savings account which is not necessarily satisfied by the process B^* is condition (i) of Definition 1.4. It appears that in most typical continuous-time models of term structure the bond price $B(t, T^*)$ has sample paths which are of infinite variation. Let $\tilde{B}$ stand for any savings account implied by a family $B(t, T)$ which satisfies (BP.1)-(BP.4). Combining (BP.4) with (17) we find that

$$B(t, U) = \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_U \mid \mathcal{F}_t) \leq \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathcal{F}_t) = B(t, T),$$

if $t \leq T \leq U$. Upon choosing $T = t$ this yields $\mathbf{E}_{\tilde{\mathbf{P}}}(1/\tilde{B}_U \,|\, \mathcal{F}_t) \leq 1/\tilde{B}_t$ for $U \geq t$ so that $1/\tilde{B}$ follows a strictly positive supermartingale under $\tilde{\mathbf{P}}$. Since $\tilde{B}$ was assumed to be a process of finite variation, the martingale part of $1/\tilde{B}$ vanishes, and thus the sample paths of $1/\tilde{B}$ are decreasing functions. Conversely, if a family $B(t, T)$ admits an increasing implied savings account $\tilde{B}$ then, of course, property (BP.4) holds. We conclude that the following simple lemma is valid.

Lemma 1.2 *Suppose a family $B(t, T)$ of bond prices satisfies assumptions* (BP.1)-(BP.3). *Then the following are equivalent:* (i) *condition* (BP.4) *holds, and* (ii) *a savings account implied by the family $B(t, T)$, if it exists, follows an increasing process.*

1.3 Existence of an implied savings account

In the present section we shall establish existence of an increasing process $\tilde{B}$ which represents an implied savings account for a family $B(t,T)$ of bond prices satisfying assumptions (BP.1)-(BP.4). We start with an auxiliary result which deals with the behaviour of the *terminal discount factor* $D_t = B^{-1}(t,T^*)$, $t \in [0,T^*]$ (note that D belongs to the class $\mathcal{S}_p^+$ since $B(\cdot,T^*)$ does).

Lemma 1.3 *Under hypotheses* (BP.1)-(BP.4)*, the terminal discount factor D follows a strictly positive supermartingale under the forward probability measure* $\mathbf{P}$.

Proof. Combining (2) with (15) we get

$$\mathbf{E_P}(B(t,T^*)/B(U,T^*)\,|\,\mathcal{F}_t) \le \mathbf{E_P}(B(t,T^*)/B(T,T^*)\,|\,\mathcal{F}_t),$$

so that $\mathbf{E_P}(D_U\,|\mathcal{F}_t) \le \mathbf{E_P}(D_T\,|\mathcal{F}_t)$ for $t \le T \le U \le T^*$. By setting $t = T$ in the last inequality we find that

$$\mathbf{E_P}(D_U\,|\mathcal{F}_T) \le \mathbf{E_P}(D_T\,|\mathcal{F}_T) = D_T$$

for every $T \le U \le T^*$. □

We shall make use of the following classical result (see, for instance, Theorem 6.19 in Jacod [16]).

Proposition 1.1 *Suppose that X belongs to the class $\mathcal{S}_p^+$, with $X_0 = 1$. Then there exists a unique pair (M,A) of processes such that:* (i) $X_t = M_t A_t$ *for every* $t \in [0,T^*]$, (ii) M *belongs to* $\mathcal{M}_{loc}^+(\mathbf{P})$, *and* $M_0 = 1$, (iii) A *belongs to* $\mathcal{A}^+$, *and* $A_0 = 1$. *If, in addition, X is a supermartingale then A is a decreasing process.*

Remark 1.1 It is well-known that if a strictly positive special semimartingale X follows a supermartingale then the process of left hand limits X_{t-} is also strictly positive (see Proposition 6.20 in Jacod [16]), hence, X automatically belongs to the class $\mathcal{S}_p^+$. Furthermore, it is clear that if the process M in the decomposition above has continuous sample paths (this holds in our case, since the underlying filtration is generated by a Wiener process), then A necessarily belongs to the class $\mathcal{S}_p^+$.

We find convenient to identify the implied savings account through a multiplicative decomposition of the terminal discount factor D. Since the underlying filtration is generated by a Wiener process, Proposition 1.1 can be restated as follows.

Corollary 1.1 *Under hypotheses* (BP.1)-(BP.3)*, there exists a predictable process ξ integrable with respect to the Wiener process W, and such that the terminal discount factor D admits the unique decomposition*

$$D_t = D_0 \tilde{A}_t \tilde{M}_t = D_0\, \tilde{A}_t\, \mathcal{E}_t \left(\int_0^{\cdot} \xi_u \cdot dW_u \right), \quad \forall\, t \in [0,T^*], \tag{20}$$

where $\tilde{M}$ is in $\mathcal{M}_{c,loc}^+(\mathbf{P})$ and $\tilde{A}$ belongs to $\mathcal{A}^+$, with $\tilde{A}_0 = \tilde{M}_0 = 1$. If, in addition, assumption (BP.4) *is satisfied then $\tilde{A}$ is a decreasing process.*

Proof. All assertions are immediate consequences of Lemma 1.3 combined with Proposition 1.1 and the representation theorem for strictly positive martingales with respect to the natural filtration of a Wiener process. □

For our further purposes, it is convenient to rewrite (20) as follows

$$\tilde{B}_t \stackrel{\text{def}}{=} 1/\tilde{A}_t = \frac{B(t,T^*)}{B(0,T^*)}\,\mathcal{E}_t\left(\int_0^{\cdot} \xi_u \cdot dW_u\right), \quad \forall\, t \in [0,T^*]. \tag{21}$$

We can now concentrate on the issue of existence of an implied savings account. To show existence, it is enough to check that the process $\tilde{B}$ given by (21) satisfies Definition 1.4. We formulate the next result under the hypotheses (BP.1)-(BP.4). It is clear, however, that under (BP.1)-(BP.3) all assertions of Proposition 1.2 remain valid, except for the property that $\tilde{B}$ is an increasing process.

Proposition 1.2 *Assume that hypotheses* (BP.1)-(BP.4) *are satisfied and the process $\tilde{M}$ defined by decomposition* (20) *belongs to $\mathcal{M}_c^+(\mathbf{P})$. Let $\tilde{B} = 1/\tilde{A}$ be an increasing predictable process uniquely determined by the multiplicative decomposition* (20) *of the terminal discount factor D under the forward measure $\mathbf{P}$. Then $\tilde{B}$ represents a savings account implied by the family $B(t,T)$ of bond prices. Savings account $\tilde{B}$ is associated with the spot probability measure $\tilde{\mathbf{P}}$ given by the formula*

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} \stackrel{\text{def}}{=} \tilde{M}_{T^*} = \tilde{B}_{T^*}\,B(0,T^*), \quad \mathbf{P}-\text{a.s.} \tag{22}$$

Moreover, the process $\tilde{B}_t/B(t,T^*)$ follows a martingale under the forward probability measure $\mathbf{P}$.

Proof. Let $\tilde{\mathbf{P}}$ be an arbitrary probability measure on $(\Omega, \mathcal{F}_{T^*})$ equivalent to $\mathbf{P}$. Then the Radon-Nikodým density of $\tilde{\mathbf{P}}$ with respect to $\mathbf{P}$ restricted to the σ-field $\mathcal{F}_t$ equals

$$\Gamma_t \stackrel{\text{def}}{=} \frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}_{\,|\,\mathcal{F}_t} = \mathcal{E}_t\left(\int_0^{\cdot} \tilde{\eta}_u \cdot dW_u\right), \quad \mathbf{P}-\text{a.s.}, \tag{23}$$

for some predictable process $\tilde{\eta}$. We start by considering a zero-coupon bond of maturity T^*. By virtue of (20) the dynamics of the relative bond price $\tilde{Z}(t,T^*) = B(t,T^*)/\tilde{B}_t$ under $\mathbf{P}$ is

$$\tilde{Z}(t,T^*) = B(0,T^*)/M_t = B(0,T^*)\,\mathcal{E}_t^{-1}\left(\int_0^{\cdot} \xi_u \cdot dW_u\right) \tag{24}$$

for every $t \in [0,T^*]$, hence, under $\tilde{\mathbf{P}}$ we have

$$\tilde{Z}(t,T^*) = B(0,T^*)\exp\left(-\int_0^t \xi_u \cdot d\tilde{W}_u - \frac{1}{2}\int_0^t \xi_u \cdot (2\tilde{\eta}_u - \xi_u)\,du\right), \tag{25}$$

with $\tilde{W}_t = W_t - \int_0^t \tilde{\eta}_u\,du$ following a Wiener process under $\tilde{\mathbf{P}}$. It is thus evident that the relative bond price $\tilde{Z}(t,T^*)$ is a local martingale under $\tilde{\mathbf{P}}$ provided that $\tilde{\eta} = \xi$. Under this assumption we have

$$\tilde{Z}(t,T^*) = B(0,T^*)\,\mathcal{E}_t\left(-\int_0^{\cdot} \xi_u \cdot d\tilde{W}_u\right). \tag{26}$$

By setting $\tilde{\eta} = \xi$ in (22) we define a candidate for a spot probability measure $\tilde{\mathbf{P}}$. It remains to check that condition (ii) of Definition 1.4 is satisfied, that is, for any maturity $T < T^*$ the relative bond price $\tilde{Z}(t,T) = B(t,T)/\tilde{B}_t$ also follows a martingale under $\tilde{\mathbf{P}}$. To this end, observe that equality $\tilde{\eta} = \xi$ combined with (23)-(24) gives

$$\Gamma_t = \frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}_{\,|\,\mathcal{F}_t} = \frac{B(0,T^*)}{\tilde{Z}(t,T^*)} = \frac{\tilde{B}_t\,B(0,T^*)}{B(t,T^*)}, \quad \forall\, t \in [0,T^*]. \tag{27}$$

We wish to show that for any maturity $T < T^*$ we have

$$B(t,T) = \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{28}$$

Using the abstract Bayes rule we get

$$I_t \stackrel{\mathrm{def}}{=} \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{P}}(\tilde{B}_t\Gamma_T/(\tilde{B}_T\Gamma_t) \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{P}}(B(t,T^*)/B(T,T^*) \,|\, \mathcal{F}_t),$$

where the last equality is a consequence of (27). Using (2) we find that $I_t = B(t,T)$, as expected. Thus, we have shown that the process $\tilde{B} = 1/\tilde{A}$ satisfies the definition of an implied savings account, with $\tilde{\mathbf{P}}$ being the corresponding spot probability measure. The last statetement follows immediately from (27). □

Remark 1.2 The probability measure $\tilde{\mathbf{P}}$ given by (22) plays the role of the spot probability measure associated with $\mathbf{P}$. It is clear that the forward probability measure $\mathbf{P}$ and its associated spot probability measure are related to each other by means of the formula

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = \tilde{B}_{T^*}\, B(0,T^*), \quad \mathbf{P}-\text{a.s.} \tag{29}$$

Hence, both these measures coincide if and only if $\tilde{B}_{T^*}$ is a degenerate random variable. It was already mentioned that the uniqueness of a spot and forward measure is not a universal property. On the other hand, it can be checked that for any forward measure $\mathbf{Q}$ for the date T^*, the probability measure $\tilde{\mathbf{Q}}$ given by

$$\frac{d\tilde{\mathbf{Q}}}{d\mathbf{Q}} = \tilde{B}_{T^*}\, B(0,T^*), \quad \mathbf{Q}-\text{a.s.}, \tag{30}$$

represents a spot measure for the family $B(t,T)$. Conversely, if $\tilde{\mathbf{Q}}$ is a spot probability measure then the probability $\mathbf{Q}$ given by (30) is a forward measure for the date T^*.

1.4 Uniqueness of an implied savings account

The aim of this section is to establish uniqueness of an implied savings account. We start by an auxiliary result.

Proposition 1.3 *Let $\tilde{B}$ and $\hat{B}$ be two processes from $\mathcal{A}^+$ such that for every $T \in [0,T^*]$*

$$\mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathcal{F}_t) = \mathbf{E}_{\hat{\mathbf{P}}}(\hat{B}_t/\hat{B}_T \mid \mathcal{F}_t), \quad \forall\, t \in [0,T], \tag{31}$$

where $\tilde{\mathbf{P}} \sim \hat{\mathbf{P}}$ are two probability measures on $(\Omega, \mathcal{F}_{T^*})$. If $\tilde{B}_0 = \hat{B}_0$ then $\tilde{B} = \hat{B}$.

Before we proceed to the proof of Proposition 1.3, let us quote the following result from Dellacherie and Meyer [8] (p.231).

Lemma 1.4 *Let A be an increasing process*[6] *such that the random variable A_{T^*} is $\mathbf{P}$-integrable. Denote by A^p the dual predictable projection of A. Then*

$$A_t^p = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathbf{E}_{\mathbf{P}}(A_{(k+1)2^{-n}t} - A_{k2^{-n}t} \,|\, \mathcal{F}_{k2^{-n}t}), \quad \forall\, t \in [0,T^*],$$

where the convergence is in the sense of the weak L^1 norm. If A has no predictable jumps then the convergence is in the sense of (strong) L^1 norm. Moreover, for any bounded predictable process H we have

$$\int_0^t H_{u-}\, dA_u^p = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} H_{k2^{-n}t}\, \mathbf{E}_{\mathbf{P}}(A_{(k+1)2^{-n}t} - A_{k2^{-n}t} \,|\, \mathcal{F}_{k2^{-n}t}), \quad \forall\, t \in [0,T^*].$$

[6]Process A is defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$ which satisfies the usual conditions.

Proof of Proposition 1.3. We introduce predictable processes of finite variation $\tilde{A} = 1/\tilde{B}$ and $\hat{A} = 1/\hat{B}$. Assume first that $\tilde{\mathbf{P}} = \hat{\mathbf{P}}$, so that we have

$$Y_t \stackrel{\text{def}}{=} \hat{A}_t \, \mathbf{E}_{\hat{\mathbf{P}}}(\tilde{A}_{T^*} \,|\, \mathcal{F}_t) = \tilde{A}_t \, \mathbf{E}_{\tilde{\mathbf{P}}}(\hat{A}_{T^*} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T^*].$$

Equality, $\hat{A} = \tilde{A}$ follows immediately from the uniqueness of a multiplicative decomposition of strictly positive semimartingale Y (it is clear that Y belongs to $\mathcal{S}_p^+$). We now consider the general case. Since $\tilde{\mathbf{P}} \sim \hat{\mathbf{P}}$, the process Λ defined by

$$\Lambda_t = \frac{d\hat{\mathbf{P}}}{d\tilde{\mathbf{P}}}_{\,|\,\mathcal{F}_t} = \mathbf{E}_{\tilde{\mathbf{P}}}(\Lambda_{T^*} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T^*],$$

follows a strictly positive continuous (hence predictable) martingale under $\tilde{\mathbf{P}}$. Equality (31) combined with the Bayes rule yields

$$\mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{A}_T/\tilde{A}_t \,|\, \mathcal{F}_t) = \mathbf{E}_{\hat{\mathbf{P}}}(\hat{A}_T/\hat{A}_t \,|\, \mathcal{F}_t) = \mathbf{E}_{\tilde{\mathbf{P}}}(\Lambda_T \hat{A}_T/(\Lambda_t \hat{A}_t) \,|\, \mathcal{F}_t)$$

for every $T \in [0, T^*]$, and thus

$$\mathbf{E}_{\tilde{\mathbf{P}}}\left(\Lambda_t(\tilde{A}_T - \tilde{A}_t)/\tilde{A}_t \,\middle|\, \mathcal{F}_t\right) = \mathbf{E}_{\tilde{\mathbf{P}}}\left(\Lambda_T(\hat{A}_T - \hat{A}_t)/\hat{A}_t \,\middle|\, \mathcal{F}_t\right) \tag{32}$$

for every $t \in [0, T]$. We wish to show that processes $\tilde{A}$ and $\hat{A}$ admit the same dual predictable projection, and thus coincide. Let us fix an arbitrary $t \in [0, T^*]$. By virtue of (32) we have

$$\mathbf{E}_{\tilde{\mathbf{P}}}\left(\Lambda_{t_k^n}(\tilde{A}_{t_{k+1}^n} - \tilde{A}_{t_k^n})/\tilde{A}_{t_k^n} \,\middle|\, \mathcal{F}_{t_k^n}\right) = \mathbf{E}_{\tilde{\mathbf{P}}}\left(\Lambda_{t_{k+1}^n}(\hat{A}_{t_{k+1}^n} - \hat{A}_{t_k^n})/\hat{A}_{t_k^n} \,|\, \mathcal{F}_{t_k^n}\right),$$

where for every natural n and every $k = 0, \ldots, 2^n - 1$, we set $t_k^n = k2^{-n}t$. By virtue of Lemma 1.4 for the left hand side of the last equality we get

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Lambda_{t_k^n} \tilde{A}_{t_k^n}^{-1} \, \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{A}_{t_{k+1}^n} - \tilde{A}_{t_k^n} \,|\, \mathcal{F}_{t_k^n}) = \int_0^t \Lambda_u \tilde{A}_{u-}^{-1} \, d\tilde{A}_u,$$

since the process $H_t = \Lambda_t/\tilde{A}_{t-}$ is predictable, and manifestly $\tilde{A}^p = \tilde{A}$. To show that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{A}_{t_k^n}^{-1} \, \mathbf{E}_{\tilde{\mathbf{P}}}\left(\Lambda_{t_{k+1}^n}(\hat{A}_{t_{k+1}^n} - \hat{A}_{t_k^n}) \,\middle|\, \mathcal{F}_{t_k^n}\right) = \int_0^t \Lambda_u \hat{A}_{u-}^{-1} \, d\hat{A}_u, \quad \forall\, t \in [0, T^*],$$

it is enough to verify that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{A}_{t_k^n}^{-1} \, \mathbf{E}_{\tilde{\mathbf{P}}}\left((\Lambda_{t_{k+1}^n} - \Lambda_{t_k^n})(\hat{A}_{t_{k+1}^n} - \hat{A}_{t_k^n}) \,\middle|\, \mathcal{F}_{t_k^n}\right) = \int_0^t \hat{A}_{u-}^{-1} \, d\,\langle \Lambda, \hat{A} \rangle_u = 0.$$

The last equality follows from the fact that the predictable quadratic covariation $\langle \Lambda, \hat{A} \rangle$ vanishes ($\hat{A}$ being a predictable process of finite variation has null continuous martingale component). □

The following corollary to Proposition 1.3 establishes the uniqueness of an implied savings account.

Corollary 1.2 *Uniqueness of an implied savings account holds under the hypotheses* (BP.1)-(BP.3).

Proof. Let $\tilde{B}$ and $\hat{B}$ be two arbitrary savings accounts implied by the family $B(t,T)$. Condition (ii) of Definition 1.4 yields for every $T \in [0,T^*]$

$$\mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathcal{F}_t) = B(t,T) = \mathbf{E}_{\hat{\mathbf{P}}}(\hat{B}_t/\hat{B}_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T],$$

where $\tilde{\mathbf{P}}$ and $\hat{\mathbf{P}}$ are mutually equivalent probability measures on $(\Omega, \mathcal{F}_{T^*})$. Also, $\tilde{B}$ and $\hat{B}$ are predictable processes of finite variation, hence, equality $\tilde{B} = \hat{B}$ is a straightforward consequence of Proposition 1.3. □

The next result examines a relationship between the class of spot probability measures and forward probability measures (for the proof, see Musiela and Rutkowski [21]).

Proposition 1.4 *Under the hypotheses* (BP.1)-(BP.3), *the class of forward measures for the date* T^* *and the class of spot measures admit a common element if and only if the implied savings account satisfies* $\tilde{B}_{T^*} = \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_{T^*})$, *that is, if the random variable* $\tilde{B}_{T^*}$ *is degenerate.*

1.5 Bond price volatility

Throughout this section we assume that a family $B(t,T)$ of bond prices satisfies (BP.1)-(BP.4). We introduce formally the notion of a bond price volatility as follows.

Definition 1.5 An $\mathbf{R}^d$-valued adapted process $b(t,T)$ is called a *bond price volatility* for maturity T if the bond price $B(t,T)$ admits the representation

$$dB(t,T) = B(t,T)b(t,T) \cdot dW_t + dC_t^T, \tag{33}$$

where C^T is a predictable process of finite variation.

Under (BP.1)-(BP.2), the existence and uniqueness of bond price volatility $b(t,T)$ for any maturity T is a simple consequence of the canonical decomposition of the special semimartingale $B(\cdot,T) \in \mathcal{S}_p^+$, combined with the predictable representation theorem. It is not hard to check that the bond price volatility as defined above is in fact invariant with respect to an equivalent change of probability measure, that is, under an arbitrary probability measure $\hat{\mathbf{P}} \sim \mathbf{P}$ we have

$$dB(t,T) = B(t,T)b(t,T) \cdot d\hat{W}_t + d\hat{C}_t^T \tag{34}$$

for some predictable process of finite variation $\hat{C}_t^T$, where $\hat{W}$ is a Wiener process under $\hat{\mathbf{P}}$. Since (BP.3)-(BP.4) are also satisfied there exists a unique savings account $\tilde{B}$ associated with a spot probability measure $\tilde{\mathbf{P}}$. For any maturity T the relative bond price $\tilde{Z}(t,T) = B(t,T)/\tilde{B}$ follows a local martingale under $\tilde{\mathbf{P}}$ so that

$$\tilde{Z}(t,T) = B(0,T)\, \mathcal{E}_t \left(\int_0^{\cdot} b(t,T) \cdot d\tilde{W}_u \right). \tag{35}$$

By comparing the last equality with (26) we find that $b(t,T^*) = -\xi_t$, i.e., the volatility of a T^*-maturity bond is determined by the multiplicative decomposition (20). On the other hand, upon setting $T = t$ in (35), we get the following representation for a savings account $\tilde{B}$ in terms of bond price volatilities

$$\tilde{B}_t = B^{-1}(0,t) \exp\left(-\int_0^t b(u,t) \cdot d\tilde{W}_u + \frac{1}{2} \int_0^t \| b(u,t) \|^2 \, du \right), \quad \forall\, t \in [0,T^*]. \tag{36}$$

Remark 1.3 Note that for any maturities $T, U \in [0, T^*]$ we have

$$\gamma(t,T,U) = b(t,T) - b(t,U), \quad \forall\, t \in [0, T \wedge U], \tag{37}$$

where $\gamma(t,T,U)$ is the volatility of the forward process $F_B(t,T,U)$. Hence, the *forward volatilities* $\gamma(t,T,U)$ are uniquely specified by the bond price volatilities $b(t,T)$. It is thus natural to ask if the converse implication holds, that is, if the bond price volatilities are uniquely specified by forward volatilities.

Example 1.1 Let us focus on a special case when processes C^T are absolutely continuous, that is, when for any maturity $T \le T^*$ we have

$$\frac{dB(t,T)}{B(t,T)} = a(t,T)\,dt + b(t,T) \cdot dW_t \tag{38}$$

for a certain adapted process $a(t,T)$ with integrable sample paths. It is well-known that such a form of the bond price dynamics arises naturally in the Heath-Jarrow-Morton [15] framework. Our goal is to show that (38) combined with the weak no-arbitrage condition implies the existence of an absolutely continuous savings account. It leads also, under mild additional assumptions, to the existence of continuously compounded forward rates. Note that forward process $F_B(t,T,T^*)$ follows under $\mathbf{P}$

$$dF_B(t,T,T^*) = F_B(t,T,T^*)\Big(\big(c(t,T) - c(t,T^*)\big)\,dt + \big(b(t,T) - b(t,T^*)\big) \cdot dW_t\Big), \tag{39}$$

where

$$c(t,T) \stackrel{\text{def}}{=} a(t,T) - b(t,T) \cdot b(t,T^*), \quad \forall\, t \in [0,T]. \tag{40}$$

Suppose that the family $B(t,T)$ satisfies the weak no-arbitrage condition. To be more specific, we assume here that the forward processes $F_B(t,T,T^*)$ are martingales under an equivalent probability measure[7] $\mathbf{Q}$, so that in particular, the expectation $\mathbf{E_Q}(B^{-1}(T,T^*))$ is finite for every $T \le T^*$. Then there exists an adapted process, h say, such that $\mathbf{Q}$ satisfies

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}_{T^*}\Big(\int_0^{\cdot} h_u \cdot dW_u\Big), \quad \mathbf{P}-\text{a.s.},$$

and for every $T \le T^*$

$$c(t,T) - c(t,T^*) + h_t \cdot \big(b(t,T) - b(t,T^*)\big) = 0, \quad \forall\, t \in [0,T]. \tag{41}$$

This implies that the process $c(t,T) + h_t \cdot b(t,T)$ is independent of T, meaning that for any maturity $T \le T^*$ we have

$$r_t \stackrel{\text{def}}{=} c(t,T^*) + h_t \cdot b(t,T^*) = c(t,T) + h_t \cdot b(t,T), \quad \forall\, t \in [0,T],$$

where r_t, $t \in [0,T^*]$, is an adapted process whose sample paths are integrable, with probability 1. Furthermore, the bond price satisfies under $\mathbf{Q}$

$$\frac{dB(t,T)}{B(t,T)} = \Big(r_t + b(t,T) \cdot b(t,T^*)\Big)\,dt + b(t,T) \cdot d\hat{W}_t, \tag{42}$$

where $\hat{W}_t = W_t - \int_0^t h_u\,du$ for every $t \in [0,T^*]$. We set

$$\eta_t = \mathcal{E}_t\Big(-\int_0^{\cdot} b(t,T^*) \cdot d\hat{W}_t\Big), \quad \forall\, t \in [0,T^*],$$

[7] We do not assume here that the underlying probability $\mathbf{P}$ is the forward measure for the date T^*.

and we assume that $\mathbf{E_Q}(\eta_{T^*}) = 1$. Let us define an adapted continuous process of finite variation $\tilde{B}$ by formula (1). It is easily seen that the process $Y_t = \tilde{B}_t / B(t,T^*)$ also follows a martingale under $\mathbf{Q}$, since $dY_t = -Y_t\, b(t,T^*) \cdot d\hat{W}_t$, with $Y_0 = 1/B(0,T^*)$, and thus $Y_t = \eta_t / B(0,T^*)$ for $t \in [0,T^*]$. It is useful to observe that we have

$$\eta_t = \tilde{B}_t B^{-1}(t,T^*) B(0,T^*), \quad \forall\, t \in [0,T^*]. \tag{43}$$

Let us define a probability measure $\tilde{\mathbf{P}} \sim \mathbf{Q}$ by setting $d\tilde{\mathbf{P}} = \eta_{T^*}\, d\mathbf{Q}$. In view of (43) we obtain

$$\mathbf{E_Q}(1/B(T,T^*) \,|\, \mathcal{F}_t) = \eta_t\, \mathbf{E}_{\tilde{\mathbf{P}}}(\eta_T^{-1} B^{-1}(T,T^*) \,|\, \mathcal{F}_t) = \tilde{B}_t B^{-1}(t,T^*)\, \mathbf{E}_{\tilde{\mathbf{P}}}(1/\tilde{B}_T \,|\, \mathcal{F}_t), \tag{44}$$

and thus

$$B(t,T) = B(t,T^*)\, \mathbf{E_Q}(F_B(T,T,T^*) \,|\, \mathcal{F}_t) = \tilde{B}_t\, \mathbf{E}_{\tilde{\mathbf{P}}}(1/\tilde{B}_T \,|\, \mathcal{F}_t), \tag{45}$$

where the first equality is a consequence of the martingale property of $F_B(t,T,T^*)$ under $\mathbf{Q}$, and the second one is a consequence of (44). In view of (45) it is clear that for any maturity T the discounted process $\tilde{Z}(t,T) = B(t,T)/\tilde{B}_t$ is a martingale under $\tilde{\mathbf{P}}$. We conclude that $\tilde{B}$ is the unique savings account implied by the family $B(t,T)$.

To show the existence of instantaneous forward rates $f(t,T)$ we shall follow Baxter [1]. We assume, in addition, that

$$\mathbf{E}_{\tilde{\mathbf{P}}}\left(\int_0^{T^*} |r_t| B_t^{-1}\, dt \right) < \infty,$$

and we denote by $G(t,u)$ the jointly measurable version of the martingale[8]

$$G(t,u) = \mathbf{E}_{\tilde{\mathbf{P}}}(r_u / B_u \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,u].$$

The conditional version of Fubini's theorem yields

$$\int_0^T G(t,u)\, du = \mathbf{E}_{\tilde{\mathbf{P}}}\left(\int_0^T r_u B_u^{-1}\, du \,\Big|\, \mathcal{F}_t \right) = 1 - \mathbf{E}_{\tilde{\mathbf{P}}}(1/B_T \,|\, \mathcal{F}_t) \tag{46}$$

since $dB_t^{-1} = -r_t B_t^{-1}\, dt$. By combining (45) with (46) we get

$$B(t,T) = B_t \left(1 - \int_0^T G(t,u)\, du \right). \tag{47}$$

It follows immediately from (47) that $B(t,T)$ is differentiable in T. Furthermore, for any fixed $T \le T^*$ the implied instantaneous forward interest rate $f(t,T)$ equals

$$f(t,T) = -\frac{\partial \log B(t,T)}{\partial T} = B_t B^{-1}(t,T) G(t,T), \quad \forall\, t \in [0,T], \tag{48}$$

or equivalently,

$$f(t,T) = B_t B^{-1}(t,T)\, \mathbf{E}_{\tilde{\mathbf{P}}}(r_T / B_T \mid \mathcal{F}_t).$$

It is now easy to check that $f(t,T^*) = \mathbf{E_Q}(r_{T^*} \,|\, \mathcal{F}_t)$ for every $t \in [0,T^*]$, so that the forward rate $f(\cdot,T^*)$ is a martingale under the forward measure $\mathbf{Q}$. It is thus clear that for any fixed maturity T, the process $f(\cdot,T)$ follows a continuous semimartingale with an absolutely continuous component of finite variation. More explicitly, we have

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\, du + \int_0^t \sigma(u,T) \cdot d\tilde{W}_u, \quad \forall\, t \in [0,T], \tag{49}$$

[8] We refer to Baxter [1] for the existence of such a version of $G(t,u)$.

for some adapted processes α and σ, where $\tilde{W}$ is a Wiener process under $\tilde{\mathbf{P}}$, given by the formula $\tilde{W}_t = \hat{W}_t - \int_0^t b(u,T^*)\,du$, and for any $T \le T^*$

$$\sigma(t,T) = -\frac{\partial \log b(t,T)}{\partial T}, \quad \alpha(t,T) = -\sigma(t,T)\cdot b(t,T). \tag{50}$$

To check the first equality in (50) it is enough to show that the bond price volatilities are absolutely continuous with respect to T, or more exactly, that for any maturity T the bond price volatility $b(t,T)$ satisfies

$$b(t,T) = -\int_t^T \sigma(t,u)\,du, \quad \forall\, t \in [0,T].$$

This can be done by a straightforward application of Fubini's theorem to the formula

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\,du\Big), \quad \forall\, t \in [0,T].$$

The second equality in (50) now follows from Girsanov's theorem. For $T = T^*$, it is enough to examine first the martingale $f(t,T^*)$ under the forward probability measure $\mathbf{Q}$, and then to derive the dynamics of $f(t,T^*)$ under the spot probability measure $\tilde{\mathbf{P}}$.

2 Forward processes

In this section, we examine a method of bond price modelling based on the exogeneous specification of forward volatilities, that is, the volatilities of forward processes. It should be stressed that we no longer assume to be given a family of bond prices. We assume instead the following setup, referred to as (FP) in what follows.

Assumptions (FP)
(FP.1) W is a standard d-dimensional Wiener process defined on an underlying filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$. The filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ is the natural filtration of W.
(FP.2) For any $T \in [0,T^*)$ we are given an adapted $\mathbf{R}^d$-valued process $\gamma(t,T,T^*)$, $t \in [0,T]$, such that

$$\mathbf{P}\left(\int_0^T \|\gamma(u,T,T^*)\|^2\,du < +\infty\right) = 1. \tag{51}$$

By convention, $\gamma(t,T^*,T^*) = 0 \in \mathbf{R}^d$ for every $t \in [0,T^*]$.
(FP.3) A deterministic function $P(0,T)$, $T \in [0,T^*]$, with $P(0,0) = 1$, which represents an initial term structure of interest rates[9] is prespecified.

We introduce first the notion of a family of forward processes implied by setup (FP).

Definition 2.1 Given setup (FP), for any maturity $T \in [0,T^*]$ we define the *forward process* $F(t,T,T^*)$, $t \in [0,T]$, by specifying its dynamics under $\mathbf{P}$

$$dF(t,T,T^*) = F(t,T,T^*)\,\gamma(t,T,T^*)\cdot dW_t, \quad \forall\, t \in [0,T], \tag{52}$$

and the initial condition

$$F(0,T,T^*) = P(0,T)/P(0,T^*), \quad \forall\, T \in [0,T^*]. \tag{53}$$

[9] To avoid confusion, we have changed the notation as $P(0,T)$ is now an exogeneous initial term structure, which needs to be matched by a family $B(t,T)$ of bond prices we are aiming to specify.

For any $T \le T^*$, the unique solution to (52) is given by the exponential formula

$$F(t,T,T^*) = \frac{P(0,T)}{P(0,T^*)}\, \mathcal{E}_t \left(\int_0^{\cdot} \gamma(u,T,T^*) \cdot dW_u \right), \quad \forall\, t \in [0,T]. \tag{54}$$

We postulate that the process $F(t,T,T^*)$ has a financial interpretation as the ratio of bond prices

$$F(t,T,T^*) = B(t,T)/B(t,T^*), \quad \forall\, t \in [0,T], \tag{55}$$

it should made clear that the bond prices $B(t,T)$ are yet unspecified, however. Our goal is to construct a family $B(t,T)$ which is consistent with a prespecified dynamics (54) of forward processes, and fits the initial term structure $P(0,T)$, that is, $B(0,T) = P(0,T)$ for any maturity $T \le T^*$. Note that in this section a bond price $B(t,T)$ is not required to be a semimartingale, in general. Nevertheless, in some circumstances we shall make reference to the volatility of a bond price, which is defined only for a bond price which actually follows a semimartingale.

In view of the postulated relationship (55), it is clear that given a setup (FP), in order to construct a family of bond prices it is sufficient to specify a bond price for maturity T^*. When looking for a suitable candidate for $B(t,T^*)$ one has to take into account the terminal condition $B(T^*,T^*) = 1$ and the initial condition $B(0,T^*) = P(0,T^*)$. A family $B(t,T)$ is then defined by setting

$$B(t,T) \stackrel{\text{def}}{=} F(t,T,T^*)B(t,T^*), \quad \forall\, t \in [0,T]. \tag{56}$$

Such a family of bond prices always fits the prespecified initial term structure, the terminal condition $B(T,T) = 1$ is not necessarily satisfied for maturities shorter than T^*, however, unless the bond price $B(t,T^*)$ is chosen in a judicious way. Before we analyse this problem, let us introduce the counterpart of condition (BP.4) of Section 1. For this purpose, it is convenient to define the family of processes $F(t,T,U)$ by setting

$$F(t,T,U) \stackrel{\text{def}}{=} F(t,T,T^*)/F(t,U,T^*), \quad \forall\, t \in [0, T \wedge U]. \tag{57}$$

(FP.4) For any maturities $T, U \in [0,T^*]$ such that $T \le U$ we have

$$F(t,T,U) \ge 1, \quad \forall\, t \in [0,T]. \tag{58}$$

Notice that (FP.4) implies, in particular, that $P(0,U) \le P(0,T)$ whenever $T \le U$ (hence, all forward rates prevailing at time 0 are non-negative). A family of bond prices associated with the setup (FP) is formally defined as follows.

Definition 2.2 We say that a family $B(t,T)$ of bond prices *is associated with* (FP) if: (i) processes $F(t,T,T^*)$ given by (54) coincide with processes $F_B(t,T,T^*)$ which equal

$$F_B(t,T,T^*) \stackrel{\text{def}}{=} B(t,T)/B(t,T^*), \quad \forall\, t \in [0,T^*], \tag{59}$$

(ii) $B(0,T) = P(0,T)$ for every $T \in [0,T^*]$.

To show that any family of forward processes $F(t,T,T^*)$ admits an associated family $B(t,T)$ of bond prices we will make use of the notion of a savings account implied by the setup (FP). Formally, by a *savings account implied by the setup* (FP) we mean any process which represents an implied savings account for some family $B(t,T)$ of bond prices associated with (FP).

2.1 Existence of bond prices associated with (FP)

For arbitrary maturity $T \in [0, T^*]$ we can represent the volatility $\gamma(t, T, T^*)$ as follows

$$\hat{b}(t,T) = \gamma(t,T,T^*) + \hat{b}(t,T^*), \quad \forall\, t \in [0,T]. \tag{60}$$

Given a family of forward volatilities $\gamma(t,T,T^*)$, in order to determine uniquely all processes $\hat{b}(t,T)$ it is sufficient to specify the process $\hat{b}(t,T^*)$. The bond price volatilities $b(t,T)$ of any associated family $B(t,T)$ (if they exist, that is, if $B(\cdot,T)$ are semimartingales) do necessarily satisfy relationship (60), that is, for any maturity $T \le T^*$ we have

$$b(t,T) = \gamma(t,T,T^*) + b(t,T^*), \quad \forall\, t \in [0,T], \tag{61}$$

Of course, this does not mean that any family of processes $\hat{b}(t,T)$ which satisfies (60) can represent price volatilities of some family $B(t,T)$ of bond prices associated with (FP). On the other hand, it follows immediately from (60)-(61) that for an arbitrary choice of the process $\hat{b}(t,T^*)$, there exists a unique process ψ such that the "true" bond price volatilities $b(t,T)$ should satisfy for any maturity $T \le T^*$

$$b(t,T) = \hat{b}(t,T) + \psi_t, \quad \forall\, t \in [0,T].$$

In fact, it is enough to set $\psi_t = b(t,T^*) - \hat{b}(t,T^*)$ for every $t \in [0,T^*]$. For the sake of expositional simplicity, we shall assume from now on that the forward volatilities $\gamma(t,T,T^*)$ are bounded. To provide an explicit construction of a family of bond prices associated with the setup (FP), we start with an arbitrarily chosen adapted bounded $\mathbf{R}^d$-valued process $\hat{b}(t,T^*)$. We define a probability measure $\hat{\mathbf{P}} \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ by setting

$$\frac{d\hat{\mathbf{P}}}{d\mathbf{P}} = \mathcal{E}_{T^*}\left(-\int_0^{\cdot} \hat{b}(u,T^*)\cdot dW_u\right), \quad \mathbf{P}-\text{a.s.}, \tag{62}$$

so that the process $\hat{W}_t$ given by the formula $\hat{W}_t = W_t + \int_0^t \hat{b}(u,T^*)\,du$ is a Wiener process under $\hat{\mathbf{P}}$. We now define a candidate for an implied savings account[10] process $\hat{B}_t$ by setting

$$\hat{B}_t = P^{-1}(0,t)\,\exp\left(-\int_0^t \hat{b}(u,t)\cdot d\hat{W}_u + \frac{1}{2}\int_0^t \|\hat{b}(u,t)\|^2\,du\right), \tag{63}$$

where $\hat{b}(t,T)$ is defined by (60). We are in a position to introduce a family $B(t,T)$ of processes by setting

$$B(t,T) = P(0,T)\hat{B}_t\,\mathcal{E}_t\left(\int_0^{\cdot} \hat{b}(u,T)\cdot d\hat{W}_u\right), \quad \forall\, t \in [0,T], \tag{64}$$

for any maturity $T \in [0,T^*]$. We claim that $B(t,T)$ is a family of bond prices associated with (FP). To check this, we analyse the forward process $F_B(t,T,T^*)$ associated with the family $B(t,T)$. It is clear that

$$F_B(t,T,T^*) = \frac{P(0,T)}{P(0,T^*)}\exp\left(\int_0^t \gamma(u,T,T^*)\cdot d\hat{W}_u - \frac{1}{2}\int_0^t (\|\hat{b}(u,T)\|^2 - \|\hat{b}(u,T^*)\|^2)\,du\right).$$

To show that condition (i) of Definition 2.2 is met it is enough to observe that making use of equalities (60) and (64) after simple manipulations we find that

$$F_B(t,T,T^*) = \frac{P(0,T)}{P(0,T^*)}\,\mathcal{E}_t\left(\int_0^{\cdot} \gamma(u,T,T^*)\cdot dW_u^*\right).$$

[10] It should be emphasized that it is not known a priori if the process $\hat{B}$ is of finite variation (or even if it is a semimartingale). It appears that $\hat{B}$ is of finite variation if and only if it represents an implied savings account for a family $B(t,T)$ of bond prices defined by formula (64) below. In the opposite case, neither $\hat{B}$ nor the bond prices are semimartingales.

The second condition of Definition 2.2 is an immediate consequence of formulae (63)-(64). Family $B(t,T)$ of bond prices constructed above satisfies the weak no-arbitrage condition, moreover, under assumption (FP.4) the no-arbitrage condition holds.

Proposition 2.1 *Assume that the volatilities $\gamma(t,T,T^*)$ of forward processes are bounded. Under hypotheses* (FP.1)-(FP.3), *for an arbitrary bounded adapted process $\hat{b}(t,T^*)$, processes $B(t,T)$ defined by* (62)-(64) *represent a family of bond prices associated with the setup* (FP). *The family $B(t,T)$ fits the initial term structure $P(0,T)$ and satisfies the weak no-arbitrage condition. If, in addition, assumption* (FP.4) *holds then the family $B(t,T)$ satisfies the no-arbitrage condition. Furthermore, the process $\hat{B}$ given by* (63) *represents a savings account implied by the family $B(t,T)$ if and only if it follows a predictable process of finite variation.*

Proof. Only the last statement needs a proof. The "only if" clause follows directly from the definition of a savings account. The "if" clause is a consequence of results of the previous section. Actually, for any maturity T the relative process

$$\hat{Z}(t,T) \stackrel{\text{def}}{=} B(t,T)/\hat{B}_t = P(0,T)\,\mathcal{E}_t\left(\int_0^{\cdot} \hat{b}(u,T)\cdot d\hat{W}_u\right), \quad \forall\, t\in[0,T],$$

is evidently in $\mathcal{M}_{loc}(\hat{\mathbf{P}})$. Note that if the volatility of a T^*-maturity bond equals $\hat{b}(u,T^*)$ then, of course, the process $\hat{b}(t,T)$ given by (60) represents the bond price volatility for maturity T. To conclude, it is enough to compare formulae (63) and (36). □

Remark 2.1 Generally speaking, there is no reason to expect that the process $\hat{B}_t$ will always follow a semimartingale, hence, also a bond price $B(t,T)$ need not to be a semimartingale. Though general criteria for the semimartingale property of $\hat{B}$ are hardly expected, in particular circumstances such a property can be verified directly.

Example 2.1 Let us now consider a simple example (we take $d=1$, for convenience). Assume that the forward volatilities $\gamma(t,T,T^*)$ are constant, more precisely, there exists a non-zero real γ such that $\gamma(t,T,T^*)=\gamma$ for every $T\in[0,T^*)$ and $t\in[0,T]$. Furthermore, we have as usual $\gamma(t,T^*,T^*)=0$ for every $t\in[0,T^*]$. This implies that for any maturity $T\in[0,T^*)$

$$F(t,T,T^*) = \frac{P(0,T)}{P(0,T^*)}\exp\Big(\gamma W_t - \frac{1}{2}\gamma^2 t\Big), \quad \forall\, t\in[0,T]. \tag{65}$$

On the other hand, we assume that the deterministic function $P(0,t)$ representing the initial term structure belongs to $\mathcal{S}_p^+$. Let us choose $\hat{b}(t,T^*)=0$ for every $t\in[0,T^*]$ so that for any maturity $T\in[0,T^*)$ we have (cf., (60))

$$\hat{b}(t,T) = \gamma(t,T,T^*) = \gamma, \quad \forall\, t\in[0,T]. \tag{66}$$

Notice also that the probability measure $\hat{\mathbf{P}}$ defined by (62) satisfies $\hat{\mathbf{P}}=\mathbf{P}$, so that $\hat{W}=W$. Hence, the process $\hat{B}$ given by (63) equals[11]

$$\hat{B}_t = P^{-1}(0,t)\exp\Big(-\gamma W_t + \frac{1}{2}\gamma^2 t\Big), \quad \forall\, t\in[0,T^*), \tag{67}$$

and $\hat{B}_{T^*}=P^{-1}(0,T^*)$. Let us first find the bond price $B(t,T^*)$. By virtue of (64) it is clear that $B(t,T^*)=P(0,T^*)\hat{B}_t$ for every $t\in[0,T^*]$. More explicitly,

$$B(t,T^*) = \frac{P(0,T^*)}{P(0,t)}\exp\left(-\gamma W_t + \frac{1}{2}\gamma^2 t\right), \quad \forall\, t\in[0,T^*),$$

[11] Notice that $\hat{B}$ is predictable, since any optional process with respect to a filtration of a Wiener process is predictable. On the other hand, $\hat{B}$, being obviously a semimartingale, does not follow a process of finite variation as it admits a non-zero continuous martingale component.

and $B(T^*, T^*) = 1$. Let us now consider a bond of maturity $T < T^*$. In view of (64) we have

$$B(t,T) = P(0,T)\hat{B}_t \exp\left(\int_0^t \hat{b}(u,T)\, dW_u - \frac{1}{2}\int_0^t \hat{b}^2(u,T)\, du\right), \quad \forall\, t \in [0,T].$$

Combining (66) with (67) we find that for any maturity $T < T^*$ we have $B(t,T) = P(0,T)/P(0,t)$ for every $t \in [0,T]$. This completes the construction of a family $B(t,T)$ associated with the setup (FP). Let us now investigate basic properties of this family. First, observe that for any maturity $T < T^*$ we have

$$F_B(t,T,T^*) \stackrel{\text{def}}{=} \frac{B(t,T)}{B(t,T^*)} = \frac{P(0,T)}{P(0,T^*)} \exp\left(\gamma W_t - \frac{1}{2}\gamma^2 t\right), \quad \forall\, t \in [0,T],$$

so that the forward processes $F_B(t,T,T^*)$ and $F(t,T,T^*)$ coincide. We shall now check that the process $\tilde{B}$ which equals $\tilde{B}_t = P^{-1}(0,t)$ for $t \in [0,T^*)$, and

$$\tilde{B}_{T^*} = P^{-1}(0,T^*) \exp\Big(\gamma W_{T^*} - \frac{1}{2}\gamma^2 T^*\Big),$$

is the unique implied savings account for the family $B(t,T)$. It is clear that $\tilde{B}$ belongs to $\mathcal{A}^+$; it is thus enough to check that all relative bond prices $\tilde{Z}(t,T) = B(t,T)/\tilde{B}_t$ follow local martingales under some probability measure $\tilde{\mathbf{P}} \sim \mathbf{P}$. For any maturity $T < T^*$ we have $\tilde{Z}(t,T) = P(0,T)$ for every $t \in [0,T]$, hence, $\tilde{Z}(t,T)$ follows trivially a martingale under any probability measure equivalent to $\mathbf{P}$. For T^* we have

$$\tilde{Z}(t,T^*) = P(0,T^*) \exp\Big(-\gamma\,(W_t - \gamma t) - \frac{1}{2}\gamma^2 t\Big), \quad \forall\, t \in [0,T^*],$$

so that $\tilde{Z}(t,T^*)$ is a martingale under the probability measure $\tilde{\mathbf{P}} \sim \mathbf{P}$ which is given by the formula

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = \exp\Big(\gamma W_{T^*} - \frac{1}{2}\gamma^2 T^*\Big), \quad \mathbf{P} - \text{a.s.}$$

It is worthwhile to observe that due to the jump at time T^*, the savings account $\tilde{B}$ is not increasing, even if the initial term structure $P(0,t)$ is a strictly decreasing function. Let us now determine the bond price volatilities. It is apparent that $b(t,T) = 0$ for any maturity $T < T^*$, while $b(t,T^*) = -\gamma$ (it seems interesting to compare this with our initial guess: $\hat{b}(t,T) = \gamma$ for $T < T^*$, and $\hat{b}(t,T^*) = 0$). This example, though rather simplistic, provides some insight into the features of the proposed procedure. Firstly, the process $\hat{B}_t$ given by (63) does not necessarily represent the savings account implied by the family $B(t,T)$. Secondly, the implied savings account may follow a discontinuous process; in our example, this feature is related to the fact that the forward volatilites $\gamma(t,T,U)$ are discontinuous in U.

2.2 Uniqueness of bond prices associated with (FP)

Since an arbitrary family $B(t,T)$ of bond prices associated with the setup (FP) satisfies

$$F(t,T,T^*) = F_B(t,T,T^*) = \frac{B(t,T)}{B(t,T^*)}, \quad \forall\, t \in [0,T],$$

we have (see, formula (57))

$$B(t,T) = \frac{B(t,T)}{B(t,t)} = \frac{F(t,T,T^*)}{F(t,t,T^*)} = F(t,T,t), \quad \forall\, t \in [0,T]. \tag{68}$$

We conclude easily that the family of bond prices associated with a given family $F(t,T,T^*)$ of forward processes is uniquely determined. This means also that there is uniqueness of a savings account implied by an arbitrary family of forward processes. On the other hand, it is worthwhile to notice that formula (68) is not as explicit as it may appear at the first glance, since the dynamics of the process $F(t,T,t)$ is not available, in general.

3 Forward LIBOR rates model

To introduce the notion of a *forward LIBOR rate,* it will be convenient to place ourselves within the setup (BP) (alternatively, one could have started from the setup (FP)). This means that we are given a family $B(t,T)$ of bond prices, or at least the family $F_B(t,T,U)$ of the corresponding forward processes. A strictly positive real number $\delta < T^*$ is fixed throughout. By definition, the forward δ-LIBOR rate[12] $L(t,T)$ for the future date $T \le T^* - \delta$ prevailing at time t is given by the conventional market formula

$$1 + \delta L(t,T) = F_B(t,T,T+\delta), \quad \forall\, t \in [0,T]. \tag{69}$$

Comparing (69) with (6) we find that $L(t,T) = f_B(t,T,T+\delta)$ so that the forward LIBOR rate $L(t,T)$ represents in fact the forward-forward rate prevailing at time t over the future time interval $[T,T+\delta]$. We can also re-express $L(t,T)$ directly in terms of bond prices as for any $T \in [0,T^*-\delta]$ we have

$$1 + \delta L(t,T) = B(t,T)/B(t,T+\delta), \quad \forall\, t \in [0,T]. \tag{70}$$

In particular, the initial term structure of forward LIBOR rates satisfies

$$L(0,T) = f_B(0,T,T+\delta) = \delta^{-1}\left(\frac{B(0,T)}{B(0,T+\delta)} - 1\right) = \delta^{-1}(F_B(0,T,T+\delta) - 1). \tag{71}$$

Given a family $F_B(t,T,T^*)$ of forward processes, it is not hard to derive the dynamics of the associated family of forward LIBOR rates. For instance, one finds that under the forward measure $\mathbf{P}^{T+\delta}$ we have[13]

$$dL(t,T) = \delta^{-1} dF_B(t,T,T+\delta) = \delta^{-1} F_B(t,T,T+\delta)\, \gamma(t,T,T+\delta) \cdot dW_t^{T+\delta}, \tag{72}$$

where $W_t^{T+\delta}$ and $\mathbf{P}^{T+\delta}$ are defined by (10) and (11), respectively. This means that $L(\cdot,T)$ solves the equation

$$dL(t,T) = \delta^{-1}(1 + \delta L(t,T))\, \gamma(t,T,T+\delta) \cdot dW_t^{T+\delta} \tag{73}$$

subject to the initial condition (71). Assume, in addition, that the forward LIBOR rates $L(t,T)$ follow strictly positive processes. Then formula (73) can be rewritten as follows

$$dL(t,T) = L(t,T)\, \lambda(t,T) \cdot dW_t^{T+\delta}, \tag{74}$$

where

$$\lambda(t,T) = \frac{1 + \delta L(t,T)}{\delta L(t,T)}\, \gamma(t,T,T+\delta), \quad \forall\, t \in [0,T]. \tag{75}$$

The above shows, that the family of forward processes, as defined in Section 1, specifies uniquely the associated family of forward LIBOR rates. To construct lognormal model of forward LIBOR rates it is more convenient to go the other way round, however. We shall provide first a construction of lognormal model of forward LIBOR rates in a discrete-tenor setting, and then, under additional assumptions, also in a fully continuous-time framework.

Assumptions (LR)
(LR.1) W is a standard d-dimensional Wiener process defined on an underlying filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$, where the filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ is the natural filtration of W.
(LR.2) A family of volatility coefficients of forward LIBOR rates is prespecified: for any maturity

[12] In market practice, several types of LIBOR rates occur, for instance, 3-month LIBOR rate, 6-month LIBOR rate, etc. For the ease of exposition, we consider here a fixed maturity δ only. Therefore, we shall frequently write LIBOR instead of δ-LIBOR in what follows.

[13] Note that since the LIBOR rate considered relates to a fixed time interval of length δ, the overall time range of $L(t,T)$ is $T+\delta$ rather than T. This explains why the dynamics of the forward LIBOR rate $L(t,T)$ refers to a Wiener process associated with the date $T+\delta$.

$T \leq T^* - \delta$ we are given an $\mathbf{R}^d$-valued bounded deterministic function[14] $\lambda(\cdot, T)$ which represents the volatility of the forward LIBOR rate $L(\cdot, T)$.
(LR.3) We assume a strictly decreasing and strictly positive initial term structure $P(0,T), T \in [0, T^*]$. In other words, we are given an initial term structure $L(0,T)$ of forward LIBOR rates

$$L(0,T) = \delta^{-1} \left(\frac{P(0,T)}{P(0,T+\delta)} - 1 \right), \quad \forall\, T \in [0, T^* - \delta]. \tag{76}$$

3.1 Discrete-tenor case

In this section we restrict ourselves to the study of a *discrete-tenor* version of a lognormal model of forward LIBOR rates. It should be stressed that a so-called discrete-tenor model still possesses certain continuous-time features, for instance, the forward LIBOR rates follow continuous-time processes. For the sake of notational simplicity, we shall assume from now on that the horizon date T^* is a multiple of δ, say, $T^* = M\delta$ for a natural M. We shall focus on a finite number of dates, $T^*_{m\delta} = T^* - m\delta$ for $m = 1, \ldots, M-1$. The construction is based on backward induction, therefore, we start by defining the forward LIBOR rate with the longest maturity, $L(t, T^*_\delta)$. The dynamics of $L(t, T^*_\delta)$ under the (forward) probability measure $\mathbf{P}$ is assumed to be given by the following expression (cf., (74))

$$dL(t, T^*_\delta) = L(t, T^*_\delta)\, \lambda(t, T^*_\delta) \cdot dW_t \tag{77}$$

with

$$L(0, T^*_\delta) = \delta^{-1} \left(\frac{P(0, T^*_\delta)}{P(0, T^*)} - 1 \right), \tag{78}$$

or explicitly

$$L(t, T^*_\delta) = \delta^{-1} \left(\frac{P(0, T^*_\delta)}{P(0, T^*)} - 1 \right) \mathcal{E}_t \left(\int_0^\cdot \lambda(u, T^*_\delta) \cdot dW_u \right), \quad \forall\, t \in [0, T^*_\delta]. \tag{79}$$

Since $P(0, T^*_\delta) > P(0, T^*)$ it is clear that $L(t, T^*_\delta)$ is in $\mathcal{M}^+_c(\mathbf{P})$. Also, for any fixed $t \leq T^* - \delta$ the random variable $L(t, T^*_\delta)$ has lognormal distribution under $\mathbf{P}$. The next step is to define the forward LIBOR rate for the date $T^*_{2\delta}$. To this end, we make use of relationship (75) for $T = T^*_\delta$, that is,

$$\gamma(t, T^*_\delta, T^*) = \frac{\delta L(t, T^*_\delta)}{1 + \delta L(t, T^*_\delta)}\, \lambda(t, T^*_\delta), \quad \forall\, t \in [0, T^* - \delta]. \tag{80}$$

As soon as the forward volatility $\gamma(t, T^*_\delta, T^*)$ is determined by (80), it is easy to find the forward process $F_B(t, T^*_\delta, T^*)$, since its dynamics under $\mathbf{P}$ equalsfootnoteNote that our construction provides not only the family of forward LIBOR rates but also the family of forward processes.

$$dF_B(t, T^*_\delta, T^*) = F_B(t, T^*_\delta, T^*)\, \gamma(t, T^*_\delta, T^*) \cdot dW_t \tag{81}$$

and initial condition is $F_B(0, T^*_\delta, T^*) = P(0, T^*_\delta)/P(0, T^*)$. It is apparent that the forward process $F_B(t, T^*_\delta, T^*)$ belongs to $\mathcal{M}^+_c(\mathbf{P})$ as the volatility $\gamma(t, T^*_\delta, T^*)$ is easily seen to follow a bounded process. We introduce a d-dimensional Wiener process $W^{T^*_\delta}_t$ which corresponds to the date T^*_δ by setting

$$W^{T^*_\delta}_t = W_t - \int_0^t \gamma(u, T^*_\delta, T^*)\, du, \quad \forall\, t \in [0, T^*_\delta]. \tag{82}$$

By virtue of the boundedness of the process $\gamma(t, T^*_\delta, T^*)$, the existence of the process W^{T^*} and of the associated probability measure $\mathbf{P}^{T^*_\delta} \sim \mathbf{P}$ given by the formula

$$\frac{d\mathbf{P}^{T^*_\delta}}{d\mathbf{P}} = \mathcal{E}_{T^*_\delta} \left(\int_0^\cdot \gamma(u, T^*_\delta, T^*) \cdot dW_u \right), \quad \mathbf{P} - \text{a.s.}, \tag{83}$$

[14] Actually, λ could follow a stochastic process; we deliberately focus here on a so-called lognormal model of forward LIBOR rates.

is obvious. We are now in a position to specify the dynamics of the forward LIBOR rate for the date $T^*_{2\delta}$ under the forward probability measure $\mathbf{P}^{T^*_\delta}$. Analogously to (77) we set

$$dL(t,T^*_{2\delta}) = L(t,T^*_{2\delta})\,\lambda(t,T^*_{2\delta})\cdot dW_t^{T^*_\delta}, \tag{84}$$

subject to the initial condition

$$L(0,T^*_{2\delta}) = \delta^{-1}\left(\frac{P(0,T^*_{2\delta})}{P(0,T^*_\delta)} - 1\right). \tag{85}$$

Solving equation (84) and comparing with (74) for $T = T^*_{2\delta}$, we arrive at the following expression for the forward volatility $\gamma(t,T^*_{2\delta},T^*_\delta)$

$$\gamma(t,T^*_{2\delta},T^*_\delta) = \frac{\delta L(t,T^*_{2\delta})}{1+\delta L(t,T^*_{2\delta})}\,\lambda(t,T^*_{2\delta}), \quad \forall\, t\in[0,T^*_{2\delta}]. \tag{86}$$

Now, in order to find $\gamma(t,T^*_{2\delta},T^*)$ we make use of the relationship (cf., (8))

$$\gamma(t,T^*_{2\delta},T^*_\delta) = \gamma(t,T^*_{2\delta},T^*) - \gamma(t,T^*_\delta,T^*), \quad \forall\, t\in[0,T^*_{2\delta}]. \tag{87}$$

Given the process $\gamma(t,T^*_{2\delta},T^*_\delta)$, we can define the pair $(W^{T^*_{2\delta}},\mathbf{P}^{T^*_{2\delta}})$ corresponding to the date $T^*_{2\delta}$ and so forth. It is clear that by working backward in time up to the first relevant date, $T^*_{(M-1)\delta} = \delta$, we can construct a family of forward LIBOR rates $L(t,T^*_{m\delta})$, $m = 1,\dots,M-1$, in such a way that the lognormal distribution of every process $L(t,T^*_{m\delta})$ under the corresponding forward probability measure $\mathbf{P}^{T^*_{(m-1)\delta}}$ is ensured. To be more specific, for any $m = 1,\dots,M-1$ we have

$$dL(t,T^*_{m\delta}) = L(t,T^*_{m\delta})\,\lambda(t,T^*_{m\delta})\cdot dW_t^{T^*_{(m-1)\delta}}, \tag{88}$$

where $W^{T^*_{(m-1)\delta}}$ follows a standard Wiener process under the forward measure $\mathbf{P}^{T^*_{(m-1)\delta}}$. This completes the derivation of a lognormal model of forward LIBOR rates in a discrete-tenor framework.

In this section, the implied savings account is seen as a discrete-time process, $\tilde{B}_t$, $t = 0,\delta,\dots,M\delta$. Intuitively, the value $\tilde{B}$ of a savings account at time t can be interpreted as cash amount accumulated up to time t by rolling over a series of zero-coupon bonds with the shortest maturities available. Since we work here in a discrete-tenor framework, in order to find the discrete-time process $\tilde{B}$ it is not necessary to specify the bond prices. Indeed, the knowledge of forward bond prices is sufficient for this. To justify the last statement, observe that by virtue of (5) we have

$$F_B(t,m\delta,(m+1)\delta) = \frac{F_B(t,m\delta,T^*)}{F_B(t,(m+1)\delta,T^*)} = \frac{B(t,m\delta)}{B(t,(m+1)\delta)}, \tag{89}$$

which in turn yields, upon setting $t = m\delta$,

$$F_B(m\delta,m\delta,(m+1)\delta) = 1/B(m\delta,(m+1)\delta). \tag{90}$$

It follows directly from (90) that the one-period bond price $B(m\delta,(m+1)\delta)$ is uniquely specified by the model. Although the bond maturing at time $m\delta$ does not exist physically after this date, it is reasonable to consider $F_B(m\delta,m\delta,(m+1)\delta)$ as its forward value at time $m\delta$ for the next future date $(m+1)\delta$. Put another way, the cash value at time $(m+1)\delta$ of one unit of cash received at time $m\delta$ equals $B^{-1}(m\delta,(m+1)\delta)$. This means that the discrete-time savings account $\tilde{B}$ is given by the formula

$$\tilde{B}_{m\delta} = \prod_{j=0}^{m} F_B((j-1)\delta,(j-1)\delta,j\delta) = \Big(\prod_{j=1}^{m} B((j-1)\delta,j\delta)\Big)^{-1}, \quad \forall\, m = 0,\dots,M-1, \tag{91}$$

as by convention $\tilde{B}_0 = 1$. Note that since

$$F_B(m\delta, m\delta, (m+1)\delta) = 1 + \delta L(m\delta, (m+1)\delta) > 1, \quad \forall\, m = 1, \ldots, M-1, \tag{92}$$

and

$$\tilde{B}_{(m+1)\delta} = F_B(m\delta, m\delta, (m+1)\delta)\, \tilde{B}_{m\delta}, \tag{93}$$

we have $\tilde{B}_{(m+1)\delta} > \tilde{B}_{m\delta}$ for every $m = 1, \ldots, M-1$, that is, the implied savings account $\tilde{B}_{m\delta}$ follows a strictly increasing process. It should be stressed that a bond price $B(l\delta, m\delta)$ remains yet unspecified (except for all bonds with one-period to maturity, that is, with $l = m - 1$). One might now wonder if it is plausible to define a bond price $B(l\delta, m\delta)$ by setting

$$B(l\delta, m\delta) = \mathbf{E}_{\mathbf{P}}(\tilde{B}_{l\delta}/\tilde{B}_{m\delta} \,|\, \mathcal{F}_{l\delta}) \tag{94}$$

for $l \le m \le M$. Such an approach would mean that we assume that the forward probability measure $\mathbf{P}$ is in the same time a spot probability measure, that is, a bond price discounted by the savings account $\tilde{B}$ follows a martingale under $\mathbf{P}$. It can be verified that specification (94) is not compatible with properties of the family $F_B(t, m\delta, T^*)$ of forward processes, in general (we refer to [21] for details). On the other hand, we may define the probability measure $\tilde{\mathbf{P}} \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ by setting (cf., formula (29))

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = B_{T^*} P(0, T^*), \quad \mathbf{P} - \text{a.s.} \tag{95}$$

The probability measure $\tilde{\mathbf{P}}$ is a plausible choice of a spot measure, indeed, if we set

$$B(l\delta, m\delta) = \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_{l\delta}/\tilde{B}_{m\delta} \,|\, \mathcal{F}_{l\delta}) \tag{96}$$

for every $l \le m \le M$, then for $l = m - 1$ equality (96) coincides with (90). It should be stressed that it is impossible to uniquely determine the continuous-time dynamics of a bond price $B(t, T^*_{m\delta})$ within the framework of the discrete-tenor model of forward LIBOR rates presented above (for this purpose, the knowledge of forward LIBOR rates for all maturities is necessary).

3.2 Continuous-tenor case

By a *continuous-tenor model* we mean a model in which the forward LIBOR rate $L(t, T)$ has lognormal distribution for any maturity $T \in [0, T^*]$. We start from a discrete-tenor skeleton constructed in the previous section. To produce a fully continuous-tenor model it is thus sufficient to fill the gaps between the discrete dates. A fully continuous-time model of forward LIBOR rates was achieved, be different means, in Brace *et al.* [4]. It should be stressed, however, that arbitrage-free features of the model proposed in [4], as well as of the model presented below, are not apparent. On the contrary, as we shall see in what follows, it may happen, in general, that the bond price $B(T, U)$ implied by the model is greater than 1 with positive probability.[15] To construct a model in which all forward LIBOR rates $L(t, T)$ follow lognormal processes under judiciously chosen probability measures we shall also proceed by backward induction. To make such an approach feasible, we need to impose additional assumptions which will allow to specify first the forward process and the corresponding probability measure for any date belonging to the interval $[T^*_\delta, T^*]$.

First step. We start by constructing a discrete-tenor model using the procedure described in the previous section.

Second step. In the second step, we construct forward rates and forward measures for maturities $T \in (T^*_\delta, T^*)$. In this case we do not have to take into account the forward LIBOR rates $L(t, T)$

[15] This corresponds to negative values of a forward-forward interest rate over the interval $[T, U]$ for some dates T, U which satisfy $U - T \ne \delta$.

(such rates do not exist in our model after the date T^*_δ). From the previous step, we are given the values $\tilde{B}_{T^*_\delta}$ and $\tilde{B}_{T^*}$ of a savings account. It is important to observe that not only $\tilde{B}_{T^*_\delta}$, but also $\tilde{B}_{T^*}$ is a $\mathcal{F}_{T^*_\delta}$-measurable random variable. We start by defining, through formula (95), a spot probability measure $\tilde{\mathbf{P}}$ associated with the discrete-tenor model.[16] Now, since the model needs to fit the initial term structure, we look for an increasing function $\alpha : [T^*_\delta, T^*] \to [0,1]$ such that $\alpha(T^*_\delta) = 0$, $\alpha(T^*) = 1$, and the process

$$\log \tilde{B}_t = (1-\alpha(t)) \log \tilde{B}_{T^*_\delta} + \alpha(t) \log \tilde{B}_{T^*}, \quad \forall\, t \in [T^*_\delta, T^*],$$

satisfies $P(0,t) = \mathbf{E}_{\tilde{\mathbf{P}}}(1/\tilde{B}_t)$ for every $t \in [T^*_\delta, T^*]$. Since we have $0 < \tilde{B}_{T^*_\delta} < \tilde{B}_{T^*}$, and $P(0,t)$, $t \in [T^*_\delta, T^*]$, is a strictly decreasing function, a function α with required properties exists and is unique.

Remark 3.1 The second step in our construction corresponds, in a sense, to the specific choice of of bond price volatility σ made in [4]. It is assumed in [4] that for every $T \in [0, T^*]$ the bond price volatility $b(t,T)$ vanishes for every $t \in [(T-\delta) \vee 0, T]$. It follows from this that the implied savings account equals $P^{-1}(0,t)$ for every $t \in [0,\delta]$. The construction of a continuous-tenor model in [4] relies on the forward induction, as opposed to the backward induction method which is used here.

Third step. In the previous step we have constructed the savings account $\tilde{B}_t$ for every $t \in [T^*_\delta, T^*]$. Hence the forward measure for any date $T \in (T^*_\delta, T^*)$ can be defined by the formula

$$\frac{d\mathbf{P}^T}{d\tilde{\mathbf{P}}} = \frac{1}{B_T P(0,T)}, \quad \tilde{\mathbf{P}} - \text{a.s.} \tag{97}$$

Combining (97) with (95) we get

$$\frac{d\mathbf{P}^T}{d\mathbf{P}} = \frac{d\mathbf{P}^T}{d\tilde{\mathbf{P}}} \frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = \frac{B_{T^*} P(0,T^*)}{B_T P(0,T)}, \quad \mathbf{P} - \text{a.s.},$$

for every $T \in [T^*_\delta, T^*]$, so that

$$\frac{d\mathbf{P}^T}{d\mathbf{P}}_{\,|\mathcal{F}_t} = \mathbf{E}_{\mathbf{P}} \left(\frac{B_{T^*} P(0,T^*)}{B_T P(0,T)} \,\Big|\, \mathcal{F}_t \right), \quad \forall\, t \in [0,T].$$

Exponential representation of the above martingale, that is, the formula

$$\frac{d\mathbf{P}^T}{d\mathbf{P}}_{\,|\mathcal{F}_t} = \frac{P(0,T^*)}{P(0,T)} \mathcal{E}_t \left(\int_0^{\cdot} \gamma(u,T,T^*) \cdot dW^*_u \right), \quad \forall\, t \in [0,T],$$

yields the forward volatility $\gamma(t,T,T^*)$ for any maturity $T \in (T^*_\delta, T^*)$. This in turn allows to define also the associated $\mathbf{P}^T$-Wiener process W^T_t. Given the forward probability measure $\mathbf{P}^T$ and the associated Wiener process W^T_t, we define the forward LIBOR rate process $L(t, T-\delta)$ for arbitrary $T \in (T^*_\delta, T^*)$ by setting (cf., (77)-(78))

$$dL(t,T_\delta) = L(t,T_\delta)\, \lambda(t,T_\delta) \cdot dW^T_t,$$

where $T_\delta = T - \delta$, with initial condition

$$L(0,T_\delta) = \delta^{-1} \left(\frac{P(0,T_\delta)}{P(0,T)} - 1 \right).$$

Finally, we set (cf., (80))

$$\gamma(t, T^*_\delta, T^*) = \frac{\delta L(t,T^*_\delta)}{1+\delta L(t,T^*_\delta)}\, \lambda(t,T^*_\delta), \quad \forall\, t \in [0, T^*_\delta],$$

[16] The uniqueness of a spot probability measure is not an issue here.

hence, we are in the position to define[17] also the forward measure $\mathbf{P}^T$ for the date $T = T_\delta$. To define forward probability measures $\mathbf{P}^U$ and the corresponding Wiener processes W_t^U for maturities $U \in (T^*_{2\delta}, T^*_\delta)$ we set

$$\gamma(t,U,T) = \gamma(t,T_\delta,T) = \frac{\delta L(t,T_\delta)}{1+\delta L(t,T_\delta)}\,\lambda(t,T_\delta), \quad \forall\, t \in [0,T_\delta],$$

where $U = T_\delta$ so that $T = U + \delta$ belongs to (T^*_δ, T^*). To determine $\gamma(t,U,T^*)$ we use the relationship

$$\gamma(t,U,T^*) = \gamma(t,U,T) - \gamma(t,T,T^*), \quad \forall\, t \in [0,U].$$

It is thus clear that proceeding by backward induction we are able to specify a fully continuous-time family $L(t,T)$ of forward LIBOR rates with desired properties. Moreover, since we determine also a family of forward volatilities $\gamma(t,T,T^*)$, $T \in (0,T^*)$, we construct in the same time a family $F(t,T,T^*)$ of forward processes, namely, $F(t,T,T^*)$ is given a solution to the equation

$$dF(t,T,T^*) = F(t,T,T^*)\gamma(t,T,T^*)\cdot dW_t.$$

From the preceding section we know that such a family admits an associated family $B(t,T)$ of bond prices, which can be formally defined by setting $B(t,T) = F(t,T,t)$. Bond prices $B(t,T)$ will always satisfy the weak no-arbitrage condtion, the no-arbitrage with cash property ($F_B(T,T,U) \geq 1$ for $U \geq T$) may fail to hold, in general, as the following counter-example shows.

Counter-example. Assume, for the sake of expositional simplicity,[18] that $P(0,T^*_\delta) = P(0,T^*)$, or equivalently, that $\tilde{B}_{T^*_\delta} = \tilde{B}_{T^*}$. This means that in our construction we put $\tilde{B}_t = \tilde{B}_{T^*_\delta} = \tilde{B}_{T^*}$ for $t \in [T^*_\delta, T^*]$. Consequently, for every $T \in [T^*_\delta, T^*]$ the forward measure $\mathbf{P}^T$ coincides with $\mathbf{P}$. Moreover, $F_B(t,T,U) = 1$ for any $T,U \in [T^*_\delta, T^*]$ and every $t \in [0, T\wedge U]$. It is not difficult to check that the following recurrent relationship holds

$$F_B(t,T,U) = \frac{1+\delta L(t,T)}{1+\delta L(t,U)}\,F_B(t,T+\delta,U+\delta), \quad \forall t \in [0,T], \tag{98}$$

provided that maturities $T \leq U$ belong to the same interval $((m-1)\delta, m\delta)$ for some $m = 1,\dots,M$. In our case, (98) yields for $m = 1$ and $t = T$

$$F_B(T,T,U) = \frac{P(0,T^*) + \delta P(0,T)\,\mathcal{E}_T\left(\int_0^\cdot \lambda(u,T)\,dW_u\right)}{P(0,T^*) + \delta P(0,U)\,\mathcal{E}_T\left(\int_0^\cdot \lambda(u,U)\,dW_u\right)}.$$

Let us take $d = 1$, and let us assume that for some maturities $T,U \in (T^*_{2\delta}, T^*_\delta)$ we have $\lambda(u,T) = \lambda_1$ and $\lambda(u,U) = \lambda_2$ for some strictly positive real numbers $\lambda_1 < \lambda_2$. Then

$$F_B(T,T,U) = 1/B(T,U) = \frac{P(0,T^*) + \delta P(0,T)\exp(\lambda_1 W_T - \frac{1}{2}\lambda_1^2 T)}{P(0,T^*) + \delta P(0,U)\exp(\lambda_2 W_T - \frac{1}{2}\lambda_2^2 T)}. \tag{99}$$

It follows easily from (99) that $\mathbf{P}(F_B(T,T,U) < 1) = \mathbf{P}(B(T,U) > 1) > 0$. This inequality violates, of course, the absence of arbitrage between bonds and cash. It is thus apparent that the arbitrage-free features of the continuous-time model of forward LIBOR rates depend essentially on the choice of volatilities $\lambda(t,T)$.

4 Flesaker-Hughston model

The aim of this section is to analyse the paper by Flesaker and Hughston [11] within the framework of general theory of continuous-time term structure models. It appears that their approach fits

[17] Our procedure is, of course, consistent with what was done in a discrete-tenor step.

[18] Although equality $P(0,T^*_\delta) = P(0,T^*)$ contradicts our general assumption that the initial term structure $P(0,T)$ is strictly decreasing, this simplification is not an essential in our counter-example and thus may be relaxed.

well our general framework presented in Sections 1-2. Indeed, from the theoretical viewpoint it appears to be very close to the classical methodology which hinges on the exogenous specification of the short-term interest rate. The interesting feature of their approach is the fact that they have managed to provide examples of bond price models in which closed-form valuation results for both caps and swaptions of all maturities are available. It should be pointed out that the short-term rate process, r say, which is shown to be strictly positive, follows a rather involved process of Itô type. Therefore, one could not expect to arrive at similar results by means of the exogenous specification of the short-term rate.

The unique input of the model is a strictly positive supermartingale, A say, defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$, where $T^* \le \infty$ is a fixed horizon date. Let $B(t,T)$ denote the price at time t of a zero-coupon bond which matures at time T. Given a process A, we define the bond price $B(t,T)$ by means of the following *pricing formula* (using the terminology of Flesaker and Hughston [11]):

$$B(t,T) \stackrel{\text{def}}{=} \mathbf{E_P}(A_T \,|\, \mathcal{F}_t)/A_t, \quad \forall\, t \in [0,T], \tag{100}$$

for any maturity $T \in [0,T^*]$. As usual, the σ-field $\mathcal{F}_0$ is assumed to be trivial, so that A_0 is a positive number. It is thus evident that we may assume, without loss of generality, that $A_0 = 1$. The following statements are easy to check:
(i) $B(T,T) = 1$ for any maturity T,
(ii) $B(t,U) \le B(t,T)$ for every $t \in [0,T]$, and all maturities U,T such that $U \ge T$.
Furthermore, in order to fit the initial term structure, we have to specify A in such a way that

$$P(0,T) = \mathbf{E_P}(A_T), \quad \forall\, T \in [0,T^*], \tag{101}$$

where $P(0,T)$ is a prespecified initial term structure.

Remark 4.1 Equivalently, by denoting $D = 1/A$ we get

$$B(t,T) = D_t\, \mathbf{E_P}(1/D_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T], \tag{102}$$

for any T, where D is a strictly positive submartingale under $\mathbf{P}$ (if D is an increasing process then it represents , of course, a savings account).

4.1 Pricing probability versus risk-neutral probability

Our goal is to show that we may do an equivalent change of probability measure $\mathbf{P}$ to get the standard formula

$$B(t,T) = \tilde{B}_t\, \mathbf{E}_{\tilde{\mathbf{P}}}(1/\tilde{B}_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T], \tag{103}$$

where $\tilde{B}$ is an increasing process (typically, $\tilde{B}$ will possess absolutely continuous sample paths). In this case, we will have also

$$\tilde{B}_t\, \mathbf{E}_{\tilde{\mathbf{P}}}(X/\tilde{B}_T \,|\, \mathcal{F}_t) = \mathbf{E_P}(XA_T \,|\, \mathcal{F}_t)/A_t, \quad \forall\, t \in [0,T],$$

for any (European) contingent claim X which settles at time T (that is, for any $\mathcal{F}_T$-measurable random variable satisfying suitable integrability assumptions). For any probability measure $\tilde{\mathbf{P}} \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ we denote by η_{T^*} the Radon-Nikodým density

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = \eta_{T^*}, \quad \mathbf{P}\text{-a.s.}, \tag{104}$$

so that

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}_{\,|\,\mathcal{F}_t} = \eta_t = \mathbf{E_P}(\eta_{T^*} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T^*].$$

Then we have the following auxiliary lemma (let us stress that a strictly positive supermartingale A is fixed throughout).

Lemma 4.1 *Let $\tilde{\mathbf{P}}$ be an arbitrary probability measure equivalent to $\mathbf{P}$. Define the process $\tilde{B}$ by setting*

$$\tilde{B}_t = \eta_t D_t = \eta_t / A_t, \quad \forall\, t \in [0, T^*]. \tag{105}$$

Then for any maturity $T \in [0, T^*]$ we have

$$B(t,T) \stackrel{\text{def}}{=} \mathbf{E_P}(A_T \,|\, \mathcal{F}_t)/A_t = \tilde{B}_t \, \mathbf{E}_{\tilde{\mathbf{P}}}(1/\tilde{B}_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T^*]. \tag{106}$$

Proof. Formula (106) is a straightforward consequence of the abstract Bayes rule. Indeed, we have

$$\tilde{B}_t \, \mathbf{E}_{\tilde{\mathbf{P}}}(1/\tilde{B}_T \,|\, \mathcal{F}_t) = \tilde{B}_t \, \eta_t^{-1} \, \mathbf{E_P}(\eta_T / \tilde{B}_T \,|\, \mathcal{F}_t) = \mathbf{E_P}(A_T \,|\, \mathcal{F}_t)/A_t = B(t,T),$$

where the first equality follows from the Bayes rule, and the second one is a consequence of (105). □

We shall now focus on the absence of arbitrage between bonds with different maturities. In view of our preceding analysis, it seems justified to claim that the model is arbitrage-free if and only if there exists a probability measure $\tilde{\mathbf{P}} \sim \mathbf{P}$ such that the process $\tilde{B}$ given by (105) is of finite variation (actually, an increasing process). In this case, $\tilde{B}$ can be identified as an implied savings account, and we are back in the traditional framework.

Remark 4.2 If $T^* < \infty$, one may alternatively take $B(t, T^*)$ as a numéraire. On the other hand, it follows immediately from (100) that all "discounted" processes

$$\tilde{B}(t,T) = B(t,T)/A_t = \mathbf{E_P}(A_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T],$$

are martingales under $\mathbf{P}$. This property does not imply immediately that the model is arbitrage-free, however, as we do not assume that A represents the price process of a tradable asset.

Proposition 4.1 *Let A be a strictly positive supermartingale. Then there exists a (unique) strictly positive martingale η with $\eta_0 = 1$ such that the process $\tilde{B}_t = \eta_t / A_t$ is an increasing process. The family $B(t,T)$ of bond prices defined by* (106) *satisfies the no-arbitrage condition, and the arbitrage price $\pi_t(X)$ at time t of any claim*[19] *X which settles at time T equals*

$$\pi_t(X) = \tilde{B}_t \, \mathbf{E}_{\tilde{\mathbf{P}}}(X/\tilde{B}_T \,|\, \mathcal{F}_t) = \mathbf{E_P}(X A_T \,|\, \mathcal{F}_t)/A_t, \quad \forall\, t \in [0, T], \tag{107}$$

where the probability measure $\tilde{\mathbf{P}}$ is defined by (104).

Proof. The first assertion follows from the multiplicative decomposition of A. The first equality in (107) is standard. The second one is an immediate consequence of Bayes rule. □

It follows from Proposition 4.1 that the construction of a bond price which is based on (100) for some A is equivalent to the construction which assumes the existence of an increasing savings account B. From the theoretical viewpoint the Flesaker-Hughston approach should thus be seen as a variant of the traditional methodology based on the concept of a savings account. It appears, however, that in some circumstances (for instance, if one wishes to value caps or swaptions) for the sake of computational convenience, it may be more convenient to start with (100).

Remark 4.3 The second equality in (107) explains the name of a *pricing measure* attributed by Flesaker nad Hugston to the underlying probability measure $\mathbf{P}$. It should be made clear, however, that any probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ can be seen as a *pricing measure,* associated with

[19] More precisely, of any attainable and integrable claim X.

a certain supermartingale $\tilde{A}$ under $\mathbf{Q}$. In this sense, the terminology proposed by Flesaker and Hughston seems to be slightly misleading, as it suggests an exceptional role played by the underlying probability measure $\mathbf{P}$. To be more formal, for any probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ there exists a $\mathbf{Q}$-supermartingale $\tilde{A}$ such that

$$B(t,T) = \mathbf{E}_{\mathbf{Q}}(\tilde{A}_T \,|\, \mathcal{F}_t)/\tilde{A}_t, \quad \forall\, t \in [0,T],$$

for any $T \in [0,T^*]$, and

$$\pi_t(X) = \mathbf{E}_{\mathbf{Q}}(X\tilde{A}_T \,|\, \mathcal{F}_t)/\tilde{A}_t, \quad \forall\, t \in [0,T],$$

for any contingent claim X (under appropriate integrability conditions).

4.2 Special cases

In view of the discussion above, the importance of the new approach lies mostly in new examples of explicit constructions of the bond price. Therefore, we present below two specific cases of bond price models, due to Flesaker and Hughston [11], which are based on formula (100).

Example 4.1 The first example assumes that the supermartingale A is given by the formula

$$A_t \stackrel{\text{def}}{=} f(t) + g(t)M_t, \quad \forall\, t \in [0,T^*], \tag{108}$$

where $f,\, g\, :\, [0,T^*] \to \mathbf{R}_+$ are strictly positive decreasing functions, and M is a strictly positive martingale defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$, with $M_0 = 1$. It follows immediately from (100) that for any maturity $T \in [0,T^*]$ the bond price $B(t,T)$ equals

$$B(t,T) = \frac{f(T) + g(T)M_t}{f(t) + g(t)M_t}, \quad \forall\, t \in [0,T]. \tag{109}$$

To the best of our knowledge, the main reason for this specific choice of a supermartingale A lies in the fact that this model gives relatively simple valuation formulae simultaneously for all caplets and swaptions (see Sections 5-6 below). Moreover, the model easily fits the initial yield curve. Indeed, it is sufficient to choose strictly positive decreasing functions f and g in such a way that the equality

$$P(0,T) = \frac{f(T) + g(T)}{f(0) + g(0)}, \quad \forall\, T \in [0,T^*].$$

is satisfied. In order to get more explicit valuation formulae we assume, in addition, that M solves the stochastic differential equation

$$dM_t = \sigma_t M_t\, dW_t, \quad M_0 = 1, \tag{110}$$

for some process $\sigma\, :\, [0,T^*] \to \mathbf{R}$, where W is a Wiener process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$ where, as usual, the filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ is generated by W. Solving (110) we get

$$M_t = \mathcal{E}_t \left(\int_0^{\cdot} \sigma_u\, dW_u \right), \quad \forall\, t \in [0,T^*].$$

For any probability measure $\tilde{\mathbf{P}} \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ we have

$$\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = \eta_{T^*} = \mathcal{E}_{T^*} \exp \left(\int_0^{\cdot} \gamma_u\, dW_u \right), \quad \mathbf{P}\text{-a.s.}, \tag{111}$$

for a certain process γ. Suppose, in addition, that f and g are differentiable functions. Then it is not difficult to check, using Itô's formula and Girsanov's theorem, that the process γ which equals

$$\gamma_t = \frac{\sigma_t g(t) M_t}{f(t) + g(t)M_t}, \quad \forall\, t \in [0,T^*],$$

is the unique right choice of $\tilde{\mathbf{P}}$ in the sense of Proposition 4.1. Indeed, for such a choice of $\tilde{\mathbf{P}}$ the process $\tilde{B}_t = \eta_t / A_t$ satisfies

$$d\tilde{B}_t = r_t \tilde{B}_t \, dt,$$

where r stands for the following process[20]

$$r_t = -\frac{f'(t) + g'(t) M_t}{f(t) + g(t) M_t} \geq 0, \quad \forall\, t \in [0, T^*]. \tag{112}$$

This shows that the model is in fact based on the non-negative short-term interest rate process and thus is arbitrage-free. Furthermore, it is not hard to check that $r_t = f(t,t)$, where

$$f(t,T) = -\frac{\partial \log B(t,T)}{\partial T}$$

and $B(t,T)$ is given by (109). This provides an additional evidence which supports our claim that B represent the savings account.

Example 4.2 The second example, less useful, assumes that $T^* = \infty$ and the supermartingale A equals

$$A_t \stackrel{\text{def}}{=} \int_t^\infty \phi(s) M(t,s) \, ds, \quad \forall\, t \in \mathbf{R}_+,$$

where for every $s > 0$ the process $M(t,s)$, $t \leq s$, is assumed to follow a martingale on $(\Omega, (\mathcal{F}_t), \mathbf{P})$. Furthermore, $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ is a strictly positive deterministic function. For instance

$$\phi(s) = -\frac{\partial B(0,s)}{\partial s}.$$

In this case we have

$$B(t,T) = \frac{\int_T^\infty \phi(s) M(t,s) \, ds}{\int_t^\infty \phi(s) M(t,s) \, ds}, \quad \forall\, t \in [0,T],$$

for any maturity T.

5 Valuation of caps

Consider a fixed-for-floating payer swap settled in arrears, with notional amount 1. The floating rate $L(T_{j-1})$ received at time T_j, $j = 1, \ldots, n$, is set at time T_{j-1} by the reference to the price of a zero coupon over that period, for instance, the LIBOR rate $L(T_j)$. The LIBOR rate $L(T_{j-1})$ is set at time T_{j-1} and satisfies

$$B(T_{j-1}, T_j)^{-1} = 1 + L(T_{j-1})(T_j - T_{j-1}). \tag{113}$$

For notational convenience, we assume that $T_j - T_{j-1} = \delta > 0$ for every $j = 1, \ldots, n$. Recall that the forward LIBOR rate $L(t,T)$ for the period of length δ is given by the formula

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t, T, T+\delta), \quad \forall\, t \in [0,T]. \tag{114}$$

The swap cash flows at times $T_j, j = 1, \ldots, n$, where $T_j - T_{j-1} = \delta$, $T_0 = T$, are $L(T_{j-1})\delta$ and $-\kappa\delta$, where κ is the prespecified fixed level of interest rate. The value at time t of a forward swap with fixed rate κ, denoted by $\mathbf{FS}_t$ or $\mathbf{FS}_t(\kappa)$, is

$$\mathbf{FS}_t = \mathbf{E}_{\tilde{\mathbf{P}}} \Big(\sum_{j=1}^n \frac{B_t}{B_{T_j}} (L(T_{j-1}) - \kappa)\delta \,\Big|\, \mathcal{F}_t \Big) = \sum_{j=1}^n \mathbf{E}_{\tilde{\mathbf{P}}} \Big[\frac{B_t}{B_{T_j}} \Big(B(T_{j-1}, T_j)^{-1} - (1 + \kappa\delta) \Big) \,\Big|\, \mathcal{F}_t \Big],$$

[20] It is instructive to compare the dynamics of the short-term rate r under both measures, $\mathbf{P}$ and $\tilde{\mathbf{P}}$.

where $\tilde{\mathbf{P}}$ is the spot martingale measure. It is not hard to check that

$$\mathbf{FS}_t = B(t,T_0) - \sum_{j=1}^{n} c_j B(t,T_j), \quad \forall\, t \in [0,T], \tag{115}$$

where $c_j = \kappa\delta$ for $j = 1, \ldots, n-1$, and $c_n = 1 + \kappa\delta$. An *interest cap* (known also as a *ceiling rate agreement*) is a contractual arrangement where the grantor (seller) has an obligation to pay cash to the holder (buyer) if a particular interest rate exceeds a mutually agreed level at some future date or dates. Similarly, in an *interest floor* the grantor has an obligation to pay cash to the holder if the interest rate is below a prespecified level. Let us denote by κ and by δ the cap rate and the length of a caplet (i.e., one leg of a cap). Similarly to swap agreements, caps and floors may be settled either *in arrears* or *in advance.* In a forward cap and floor on principal 1 settled in arrears at times T_j, $j = 1, \ldots, n$, where $T_j - T_{j-1} = \delta$, $T_0 = T$, the cash flows at times T_j are $(L(T_{j-1}) - \kappa)^+\delta$ and $(\kappa - L(T_{j-1}))^+\delta$, respectively. The arbitrage price $\mathbf{FC}_t$ at time $t \le T_0$ of a *forward cap* with strike κ is

$$\mathbf{FC}_t = \sum_{j=1}^{n} \mathbf{E}_{\tilde{\mathbf{P}}} \Big(\frac{B_t}{B_{T_j}} (L(T_{j-1}) - \kappa)^+ \delta \, \Big| \, \mathcal{F}_t \Big). \tag{116}$$

One may check that the cap price $\mathbf{FC}_t$ satisfies

$$\mathbf{FC}_t = \sum_{j=1}^{n} \mathbf{E}_{\tilde{\mathbf{P}}} \Big[\frac{B_t}{B_{T_j}} \Big(1 - (1+\kappa\delta) B(T_{j-1}, T_j) \Big)^+ \, \Big| \, \mathcal{F}_t \Big], \tag{117}$$

hence, a cap can also be interpreted as a specific portfolio of put options on zero-coupon bonds.

5.1 Market caps valuation formula

We shall now describe how the "market" prices caps. The valuation formulae commonly used by market practitioners assume that the underlying asset follows a geometric Wiener process under some probability measure. Since the formal definition of this probability measure is not available, we shall refer to is as the *market probability*. Let us consider an interest rate cap with expiration date T and fixed strike level κ. Market practice is to price the option assuming that the underlying forward interest rate process is lognormally distributed with zero drift. Let us first consider a caplet, that is, one leg of a cap. Assume that the forward LIBOR rate $L(t,T)$, $t \in [0,T]$, for the period of length δ follows a geometric Wiener process under the "market probability" $\mathbf{Q}$, more specifically,

$$dL(t,T) = \sigma L(t,T)\, dW_t, \tag{118}$$

for a one-dimensional Wiener process W and a strictly positive real number σ. The initial condition is derived from the initial yield curve $Y(0,T)$, namely, we have

$$1 + \delta L(0,T) = \frac{P(0,T)}{P(0,T+\delta)} = \exp\Big((T+\delta) Y(0,T+\delta) - TY(0,T) \Big) \tag{119}$$

for any maturity T. The unique solution of (118) is, of course,

$$L(t,T) = L(0,T) \exp\Big(\sigma W_t - \frac{1}{2}\,\sigma^2 t^2 \Big), \quad \forall\, t \in [0,T]. \tag{120}$$

The market price at time t of a caplet with expiration date T and strike level κ is calculated by the formula

$$\mathbf{Cpl}_t = \delta B(t, T+\delta)\, \mathbf{E}_{\mathbf{Q}} \Big((L(T) - \kappa)^+ \, \Big| \, \mathcal{F}_t \Big), \quad \forall\, t \in [0,T], \tag{121}$$

where we write briefly $L(T)$ to denote $L(T,T)$. More explicitly, we have

$$\mathbf{Cpl}_t = \delta B(t,T+\delta)\Big[\, L(t,T)N\Big(\hat{e}_0(t)\Big) - \kappa N\Big(\hat{e}_0(t) - \hat{v}_0(t)\Big)\,\Big], \quad \forall\, t \in [0,T], \tag{122}$$

where

$$\hat{e}_0(t) = \frac{\log L(t,T) - \log\kappa + \frac{1}{2}\,\hat{v}_0^2(t)}{\hat{v}_0(t)}, \quad \forall\, t \in [0,T],$$

and

$$\hat{v}_0^2(t) = \sigma^2(T-t), \quad \forall\, t \in [0,T].$$

This means that market practitioners price caplets using the Black futures formula, with discount from the settlement date $T+\delta$. More generally, the cap settled in arrears at times $T_j,\ j=1,\dots,n$, where $T_j - T_{j-1} = \delta,\ T_0 = T,$ is also priced by means of formula (122), that is,

$$\mathbf{FC}_t = \delta \sum_{j=0}^{n-1} B(t,T_{j+1})\Big[\, L(t,T_j)N\Big(\hat{e}_j(t)\Big) - \kappa N\Big(\hat{e}_j(t) - \hat{v}_j(t)\Big)\,\Big], \tag{123}$$

where $\hat{e}_j(t)$ and $\hat{v}_j(t)$ are given by the formulae

$$\hat{e}_j(t) = \frac{\log L(t,T_j) - \log\kappa + \frac{1}{2}\,\hat{v}_j^2(t)}{\hat{v}_j(t)}, \quad \forall\, t \in [0,T_j], \tag{124}$$

and

$$\hat{v}_j^2(t) = \sigma_j^2(T_j - t), \quad \forall\, t \in [0,T_j]. \tag{125}$$

for some strictly positive constants $\sigma_j,\ j=0,\dots,n-1$. As usual, by convention the principal of the cap is set to be equal to 1. This means that the market assumes that for any maturity $T_j,\ j=0,\dots,n-1$, the corresponding forward LIBOR rate follows lognormal probability law under the inuitive "market probability". As we shall see in the sequel, the cap valuation formula obtained within the framework of lognormal model of forward LIBOR rates agrees with the market practice.

5.2 Caps valuation - Gaussian HJM model

In the case of the Gaussian HJM model (that is, in the HJM model in which bond price volatilities are assumed to be deterministic) we have the following well-known result which can be established by means of the forward measure technique (see Brace and Musiela [5]-[6]).

Proposition 5.1 *Suppose that the term structure of interest rates is described by the HJM model with deterministic bond price volatilities $b(t,T)$, $t \in [0,T]$, $T \in [0,T^*]$.. Then the price at time t of a forward cap with strike rate κ equals*

$$\mathbf{FC}_t = \sum_{j=1}^{n}\Big(B(t,T_{j-1})N(e_j(t)) - (1+\kappa\delta)B(t,T_j)N(e_j(t)-v_j(t))\Big),$$

where

$$e_j(t) = \frac{\log B(t,T_{j-1}) - \log B(t,T_j) - \log(1+\kappa\delta) + \frac{1}{2}v_j^2(t,)}{v_j(t)} \tag{126}$$

and

$$v_j^2(t) = \int_t^{T_{j-1}} ||\, b(u,T_{j-1}) - b(u,T_j)\,||^2\, du. \tag{127}$$

for every $j = 1,\dots,n$.

5.3 Caps valuation - Forward LIBOR rate model

Recall that the price of a *forward cap* equals (see formula (116))

$$\mathbf{FC}_t = \sum_{j=1}^{n} \mathbf{E}_{\tilde{\mathbf{P}}}\Big(\frac{B_t}{B_{T_j}}(L(T_{j-1})-\kappa)^+\delta \,\Big|\, \mathcal{F}_t\Big) = \sum_{j=1}^{n} \mathbf{Cpl}^j_t, \tag{128}$$

where

$$\mathbf{Cpl}^j_t = \mathbf{E}_{\tilde{\mathbf{P}}}\Big(\frac{B_t}{B_{T_j}}(L(T_{j-1})-\kappa)^+\delta \,\Big|\, \mathcal{F}_t\Big) = B(t,T_j)\,\mathbf{E}_{\mathbf{P}_{T_j}}\Big((L(T_{j-1},T_{j-1})-\kappa)^+\delta \,\Big|\, \mathcal{F}_t\Big)$$

for every $j=1,\dots,n$. On the other hand, in the case of the forward LIBOR rate model the dynamics of $L(t,T_{j-1})$, $t\in[0,T_{j-1}]$, under the forward probability measure $\mathbf{P}^{T_j}$ is

$$dL(t,T_{j-1}) = L(t,T_{j-1})\,\lambda(t,T_{j-1})\cdot dW_t^{T_j}, \tag{129}$$

where W^{T_j} follows a d-dimensional Wiener process under the probability measure $\mathbf{P}^{T_j}$ and $\lambda(\cdot,T_{j-1}) : [0,T_{j-1}] \to \mathbf{R}^d$ is a deterministic function. It is thus clear that

$$L(t,T_{j-1}) = \mathcal{E}_t\left(\int_0^{\cdot} \lambda(u,T_{j-1})\cdot dW_u^{T_j}\right), \quad \forall\, t\in[0,T_{j-1}]. \tag{130}$$

Let us first consider a caplet with expiry date T and strike rate κ. The proof of the next result is standard, hence it is left to the reader.

Lemma 5.1 *Assume the forward LIBOR rate model of Section* 3. *The price at time t of the caplet with strike κ maturing at $T=T_0$ equals*

$$\mathbf{Cpl}_t = \delta B(t,T+\delta)\Big(L(t,T)N(\tilde{e}_0(t)) - \kappa\,N(\tilde{e}_0(t)-\tilde{v}_0(t))\Big), \quad \forall\, t\in[0,T], \tag{131}$$

where

$$\tilde{e}_0(t) = \frac{\log L(t,T) - \log\kappa + \frac{1}{2}\,\tilde{v}_0^2(t)}{\tilde{v}_0(t)}, \quad \forall\, t\in[0,T], \tag{132}$$

and

$$\tilde{v}_0^2(t) = \int_t^T \|\,\lambda(u,T)\,\|^2\,du, \quad \forall\, t\in[0,T]. \tag{133}$$

The following proposition is an immediate consequence of Lemma 5.1 and formula (130). The valuation formula (134) was first obtained by different means in Miltersen *et al.* [20], and then derived by the method presented here by Brace *et al.* [4].

Proposition 5.2 *Consider an interest rate cap with expiration date T_0 and strike level κ. The price at time $t\in[0,T_0]$ of a cap in the forward LIBOR rate model equals*

$$\mathbf{FC}_t = \delta\sum_{j=0}^{n-1} B(t,T_{j+1})\Big(L(t,T_j)N(\tilde{e}_j(t)) - \kappa\,N(\tilde{e}_j(t)-\tilde{v}_j(t))\Big), \quad \forall\, t\in[0,T], \tag{134}$$

where $\tilde{e}_j(t)$ and $\tilde{v}_j(t)$ are given by the formulae

$$\tilde{e}_j(t) = \frac{\log L(t,T_j) - \log\kappa + \frac{1}{2}\,\tilde{v}_j^2(t)}{\tilde{v}_j(t)}, \quad \forall\, t\in[0,T_j], \tag{135}$$

and

$$\tilde{v}_j^2(t) = \int_t^{T_j} \|\,\lambda(u,T_j)\,\|^2\,du, \quad \forall\, t\in[0,T_j]. \tag{136}$$

5.4 Caps valuation - Flesaker-Hughston model

As usual, it is enough to find the value of a particular caplet. Let us fix the caplet's expiration date T and the settlement date $T+\delta$ where $\delta > 0$ is a fixed number. We have

$$B(T, T+\delta) = \frac{1}{1+\delta L(T)}, \tag{137}$$

where $L(T)$ represents, by definition, the LIBOR rate at time T. Equivalently, we have

$$L(T) \stackrel{\text{def}}{=} \delta^{-1}\big(B^{-1}(T, T+\delta) - 1\big). \tag{138}$$

The caplet pays $X = (L(T) - \kappa)^+$ at the settlement $T+\delta$, or equivalently the payoff

$$Y = B(T, T+\delta)X = B(T, T+\delta)(L(T) - \kappa)^+$$

at the expiration date T. Easy calculations show that

$$Y = \big(1 - (1+\kappa\delta)B(T, T+\delta)\big)^+ = \big(1 - (1+\kappa\delta)A_T^{-1}\,\mathbf{E_P}(A_{T+\delta}\,|\,\mathcal{F}_T)\big)^+.$$

Therefore, the arbitrage price $\mathbf{Cpl}_t$ of a caplet which equals

$$\mathbf{Cpl}_t = \pi_t(Y) = A_t^{-1}\,\mathbf{E_P}(YA_T\,|\,\mathcal{F}_t), \quad \forall\, t \in [0, T],$$

satisfies

$$\mathbf{Cpl}_t = A_t^{-1}\,\mathbf{E_P}\Big(\big(A_T - (1+\kappa\delta)\mathbf{E_P}(A_{T+\delta}\,|\,\mathcal{F}_T)\big)^+\,\Big|\,\mathcal{F}_t\Big). \tag{139}$$

Let us now consider a (forward) cap, that is, a portfolio of caplets, each of which pays $X_j = (L(T_j) - \kappa)^+$ at the settlement date $T_j + \delta$, where $j = 0, \ldots, n-1$. To price a cap it is sufficient to sum up the values of the underlying caplets. Therefore, for every $t \in [0, T]$ the arbitrage price $\mathbf{FC}_t$ of a cap equals

$$\mathbf{FC}_t = A_t^{-1}\sum_{j=1}^{n}\mathbf{E_P}\Big(\big(A_{T_{j-1}} - (1+\kappa\delta)\mathbf{E_P}(A_{T_j}\,|\,\mathcal{F}_{T_{j-1}})\big)^+\,\Big|\,\mathcal{F}_t\Big), \tag{140}$$

where $T_j = T + j\delta$ for $j = 0, \ldots, n$. From now on we focus on the particular model presented in Example 4.1. Recall that we have

$$A_T = f(T) + g(T)M_T, \quad \forall\, T \in [0, T^*],$$

so that

$$\mathbf{E_P}(A_{T+\delta}\,|\,\mathcal{F}_T) = f(T+\delta) + g(T+\delta)M_T, \quad \forall\, T \in [0, T^*].$$

Substituting into (139) we obtain

$$\mathbf{Cpl}_t = (f(t) + g(t)M_t)^{-1}\,\mathbf{E_P}\Big(\big(f(T) - cf(T+\delta) - (cg(T+\delta) - g(T))M_T\big)^+\,\Big|\,\mathcal{F}_t\Big),$$

where $c = 1 + \kappa\delta$, or equivalently,

$$\mathbf{Cpl}_t = (f(t) + g(t)M_t)^{-1}\,\mathbf{E_P}\big((a_0 - b_0M_T)^+\,|\,\mathcal{F}_t\big), \tag{141}$$

where

$$a_0 = f(T) - (1+\kappa\delta)f(T+\delta), \qquad b_0 = (1+\kappa\delta)g(T+\delta) - g(T). \tag{142}$$

More generally, the price of a cap equals (cf., (140))

$$\mathbf{FC}_t = (f(t) + g(t)M_t)^{-1}\sum_{j=0}^{n-1}\mathbf{E_P}\big((a_j - b_jM_{T_j})^+\,|\,\mathcal{F}_t\big), \quad \forall\, t \in [0, T], \tag{143}$$

where

$$a_j = f(T_j) - (1+\kappa\delta) f(T_{j+1}), \qquad b_j = (1+\kappa\delta) g(T_{j+1}) - g(T_j) \tag{144}$$

and $T_j = T + j\delta$ for $j = 0, \dots, n$. Our goal is to calculate explicitly the price $\mathbf{FC}_t$ of the cap in terms of the value at time t of the underlying martingale M, and thus also in terms of the bond price $B(t,T)$ or in terms of the forward LIBOR rate $L(t,T)$. It follows from (109) that

$$M_t = \frac{f(t)B(t,T) - f(T)}{g(T) - g(t)B(t,T)}, \quad \forall\, t \in [0,T].$$

On the other hand, the forward LIBOR rate $L(t,T)$ is known to be given by the formula

$$L(t,T) = \delta^{-1}\Big(\frac{B(t,T)}{B(t,T+\delta)} - 1\Big), \quad \forall\, t \in [0,T],$$

Combining the last formula with (109) and solving for M_t we find

$$M_t = \frac{(1+\delta L(t,T)) f(T+\delta) - f(T)}{g(T) - (1+\delta L(t,T)) g(T+\delta)}, \quad \forall\, t \in [0,T].$$

We conclude that it is sufficient to express the prices of interest rate derivatives in terms of M_t.

Remark 5.1 It is interesting to observe that for small values of κ (all other variables being fixed) we have $a_0 > 0$ and $b_0 < 0$. This means that the caplet is always exercised (i.e., $L(T) \geq \kappa$ with probability 1) and thus

$$\mathbf{Cpl}_t = (f(t) + g(t)M_t)^{-1}\, \mathbf{E_P}\big(a - bM_T \,|\, \mathcal{F}_t\big) = (f(t) + g(t)M_t)^{-1}(a - bM_t),$$

since M is a $\mathbf{P}$-martingale. On the other hand, if the strike level κ is large enough, then $a_0 < 0$ and $b_0 > 0$. Therefore, the caplet is never exercised, and its price is equal to zero. This suggests that the δ-LIBOR rate specified by the model is bounded from above and from below. Indeed, it is not difficult to find the lower and upper bounds for the LIBOR rate at time T predicted by the model. Firstly, combining (109) with (138) we obtain

$$L(T) = \delta^{-1}\, \frac{f(T) - f(T+\delta) + (g(T) - g(T+\delta))M_T}{f(T+\delta) + g(T+\delta)M_T} = \delta^{-1}\frac{c_1 + c_2 M_T}{c_3 + c_4 M_T}$$

for strictly positive reals c_i, $i = 1, \dots, 4$. Since M_T is a strictly positive random variable, this yields

$$\frac{f(T) - f(T+\delta)}{f(T+\delta)} = \frac{c_1}{c_3} < \delta L(T) < \frac{c_2}{c_4} = \frac{g(T) - g(T+\delta)}{g(T+\delta)}$$

if $c_1/c_3 < c_2/c_4$, and

$$\frac{f(T) - f(T+\delta)}{f(T+\delta)} = \frac{c_1}{c_3} > \delta L(T) > \frac{c_2}{c_4} = \frac{g(T) - g(T+\delta)}{g(T+\delta)}.$$

if $c_1/c_3 > c_2/c_4$. To avoid trivialities, we shall assume from now on that the strike level κ belongs to the interval $(\delta\kappa_{min}, \delta\kappa_{max})$, where

$$\kappa_{min} \stackrel{\text{def}}{=} \frac{f(T) - f(T+\delta)}{f(T+\delta)} \wedge \frac{g(T) - g(T+\delta)}{g(T+\delta)} \tag{145}$$

and

$$\kappa_{max} \stackrel{\text{def}}{=} \frac{f(T) - f(T+\delta)}{f(T+\delta)} \vee \frac{g(T) - g(T+\delta)}{g(T+\delta)}. \tag{146}$$

Notice that this means that either (A): $a_0 > 0$ and $b_0 > 0$, or (B): $a_0 < 0$ and $b_0 < 0$. We shall focus on case (A) in what follows; the second case can be dealt with along the same lines.

The following result provides cap valuation formula.

Proposition 5.3 *Assume that the volatility coefficient σ in (110) is a deterministic function. If $a > 0$ and $b > 0$ then the caplet's price at time 0 equals*

$$\mathbf{Cpl}_0 = (f(0) + g(0))^{-1} \big(a_0 N(d_1(0,T)) - b_0 N(d_2(0,T))\big), \tag{147}$$

where

$$d_1(0,T) = \frac{\log c_0 + \frac{1}{2} v^2(0,T)}{v(0,T)}, \tag{148}$$

with

$$c_0 = \frac{a_0}{b_0} = \frac{f(T) - (1+\kappa\delta) f(T+\delta)}{(1+\kappa\delta) g(T+\delta) - g(T)},$$

$d_2(0,T) = d_1(0,T) - v(0,T)$, and

$$v^2(0,T) = \int_0^T \sigma_u^2 \, du.$$

More generally, for arbitrary $t \in [0,T]$ we have

$$\mathbf{Cpl}_t = (f(t) + g(t)M_t)^{-1} \big(a_0 N(d_1(t,T)) - b_0 M_t N(d_2(t,T))\big), \tag{149}$$

where

$$d_1(t,T) = \frac{\log c_t + \frac{1}{2} v^2(t,T)}{v(t,T)}, \tag{150}$$

with

$$c_t = M_t^{-1} \frac{a_0}{b_0} = M_t^{-1} \frac{f(T) - (1+\kappa\delta) f(T+\delta)}{(1+\kappa\delta) g(T+\delta) - g(T)},$$

$d_2(t,T) = d_1(t,T) - v(t,T)$, and

$$v^2(t,T) = \int_t^T \sigma_u^2 \, du.$$

Proof. Let as first assume that $t = 0$. In view of (110) we have $M_T = e^{\xi - \frac{1}{2} v^2(0,T)}$, where ξ is a Gaussian random variable with zero mean and variance $v^2(0,T)$. Applying Lemma 5.2 to the valuation formula the equality

$$\mathbf{Cpl}_0 = (f(0) + g(0))^{-1} \, \mathbf{E_P}((a_0 - b_0 M_T)^+)$$

and substituting (142) we obtain (147). For $t > 0$, it is sufficient to observe that

$$\mathbf{E_P}\big((a_0 - b_0 M_T)^+ \,|\, \mathcal{F}_t\big) = \mathbf{E_P}\big((a_0 - b_0 M_t \zeta)^+ \,|\, \mathcal{F}_t\big)$$

where ζ equals

$$\zeta = \exp\left(\int_t^T \sigma_u \, dW_u - \frac{1}{2} \int_t^T \sigma_u^2 \, du \right).$$

Since ζ is independent of the σ-field $\mathcal{F}_t$ the assertion easily follows. □

Lemma 5.2 *Let ξ be a zero mean Gaussian random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ with variance σ^2. Then for any strictly positive reals a and b we have*

$$\mathbf{E_P}\Big((a - b e^{\xi - \frac{1}{2}\sigma^2})^+\Big) = aN(h) - bN(h - \sigma), \tag{151}$$

where

$$h = \frac{\log a - \log b + \frac{1}{2}\sigma^2}{\sigma}.$$

Remark 5.2 In order to price a cap it is sufficient to sum the prices of underlying caplets. To this end, one needs to determine first the signs of a_j's and b_j's. As already mentioned, it is straightforward to express the cap price in terms of the bond price $B(t,T)$ or the forward LIBOR rate $L(t,T)$. This remark applies also to the swaption's price, which is given by formula (163) below.

6 Valuation of swaptions

Let us denote by $\kappa(t,T,n)$ the *forward swap rate* at time t for the future date T. For every $t \leq T$ we have

$$\kappa(t,T,n) = (B(t,T) - B(t,T_n)) \Big(\delta \sum_{j=1}^{n} B(t,T_j)\Big)^{-1},$$

so that the swap rate $\kappa(T,T,n)$ equals

$$\kappa(T,T,n) = (1 - B(T,T_n)) \Big(\delta \sum_{j=1}^{n} B(T,T_j)\Big)^{-1}. \tag{152}$$

A *payer's swaption* is an option on the underlying payer swap with strike level 0. Therefore, the price $\mathbf{PS}_t$ at time t of the payer swaption equals

$$\mathbf{PS}_t = \mathbf{E}_{\tilde{\mathbf{P}}}\Big[\frac{B_t}{B_T}\Big(FS_T(\kappa)\Big)^+ \Big| \mathcal{F}_t\Big],, \quad \forall\, t \in [0,T],$$

where as usual $\tilde{\mathbf{P}}$ denotes the spot martingale measure, or more explicitly,

$$\mathbf{PS}_t = \mathbf{E}_{\tilde{\mathbf{P}}}\Big[\frac{B_t}{B_T}\Big(\mathbf{E}_{\tilde{\mathbf{P}}}\Big(\sum_{j=1}^{n} \frac{B_T}{B_{T_j}}(L(T_{j-1}) - \kappa)\delta \,\Big|\, \mathcal{F}_T\Big)\Big)^+ \Big| \mathcal{F}_t\Big], \quad \forall\, t \in [0,T]. \tag{153}$$

Similarly, for the *receiver's swaption* (that is, an option on a receiver swap) we have

$$\mathbf{RS}_t = \mathbf{E}_{\tilde{\mathbf{P}}}\Big[\frac{B_t}{B_T}\Big(-FS_T(\kappa)\Big)^+ \Big| \mathcal{F}_t\Big], \quad \forall\, t \in [0,T].$$

It should be made clear that a swaption may be exercised only at its maturity date T. However, if the swaption is exercised, it gives rise to a sequence of cash flows at prespecified future dates. Therefore, a payer's (receiver s, resp.) swaption can be also viewed as a sequence of call (put, resp.) options on a swap rate which are not allowed to be exercised separately. Formally, at time T the owner of a payer's swaption receives the value of a sequence of cash flows, discounted from time $T_j,\ j = 1,\dots,n$, to T, defined by

$$K_j(\kappa) = \delta\,(\kappa(T,T,n) - \kappa)^+, \quad \forall\, j = 1,\dots,n.$$

Remark 6.1 Since the relationship $\mathbf{PS}_t - \mathbf{RS}_t = \mathbf{FS}_t$ is valid for any model of stochastic term structure, and the value of a forward swap is always (cf., (115))

$$\mathbf{FS}_t = B(t,T_0) - \sum_{j=1}^{n} c_j B(t,T_j), \quad \forall\, t \in [0,T],$$

it is sufficient to examine the case of the payer's swaption, that is, call option on the swap rate.

6.1 Market swaptions valuation formula

The market formula for pricing payer's swaptions, based on the assumption that the underlying swap rate is lognormally distributed under the "market probability", is

$$\mathbf{PS}_t = \delta \sum_{j=1}^{n} B(t,T_j)\Big[\kappa(t,T,n)N\Big(h(t,T)\Big) - \kappa N\Big(h(t,T)-\gamma\sqrt{T-t}\Big)\Big], \quad \forall\, t \in [0,T], \tag{154}$$

where

$$h(t,T) = \frac{\log\kappa(t,T,n) - \log\kappa + \frac{1}{2}\gamma^2(T-t)}{\gamma\sqrt{T-t}}, \quad \forall\, t \in [0,T].$$

For simplicity, let us consider the special case of $t = 0$. Then we have

$$\mathbf{PS}_0 = \delta\, \mathbf{E}_{\tilde{\mathbf{P}}}\Big(\sum_{j=1}^{n} \frac{1}{B_{T_j}}\,(\kappa(T,T,n)-\kappa)^+\Big) = \delta \sum_{j=1}^{n} B(0,T_j)\,\mathbf{E}_{\mathbf{P}_{T_j}}\Big((\kappa(T,T,n)-\kappa)^+\Big).$$

Intuitively, the market seems to identify the forward probability measures $\mathbf{P}^{T_j}$, $j = 1,\dots,n$, with $\mathbf{P}^T$, and assumes lognormality under $\mathbf{P}^T$ of the forward swap rate process $\kappa(t,T,n)$, $t \in [0,T]$, The swaptions valuation formula obtained below within the the forward LIBOR rate framework is more complex, it reduces to market formula (154) - commonly referred to as Black swaptions formula - only under very special circumstances. Jamshidian [18] (see also [19]) proposed recently a model of a *forward swap rate* which yields the conventional market formula for prices of certain swaptions.

6.2 Swaptions valuation - Gaussian HJM model

Since we may rewrite (153) in the following way

$$\mathbf{PS}_t = \mathbf{E}_{\tilde{\mathbf{P}}}\Big[\frac{B_t}{B_T}\Big(1-\sum_{j=1}^{n} c_j B(T,T_j)\Big)^+ \,\Big|\, \mathcal{F}_t\Big], \quad \forall\, t \in [0,T],$$

it is clear that a payer's swaption can be seen as a put option on a coupon-bearing bond with the coupon rate κ. This makes the problem of swaption valaution relatively easy in a term structure model model which assumes deterministic volatilities of bond prices. The following result (cf., Brace and Musiela [5]-[6]) provides a quasi-explicit formula for the arbitrage price of the swaption in the Gaussian HJM model.

Proposition 6.1 *Assume the Gaussian HJM model of the term structure of interest rates. For every $t \in [0,T]$ the arbitrage price of the payer's swaption equals*

$$\int_{\mathbf{R}^k} \Big(B(t,T)n_k(x) - \sum_{i=1}^{n} c_i B(t,T_i)n_k(x+\gamma_i)\Big)^+ dx$$

where n_k is the standard d-dimensional Gaussian density function

$$n_k(x) = (2\pi)^{-k/2}\, e^{-\|x\|^2/2}, \quad \forall\, x \in \mathbf{R}^k.$$

Moreover, $\gamma_1,\dots,\gamma_n$ are vectors in $\mathbf{R}^k$ such that for every $i,j = 1,\dots,n$ we have

$$\gamma_i \cdot \gamma_j = \int_t^T (b(u,T^i)-b(u,T))\cdot(b(u,T^j)-b(u,T))\,du. \tag{155}$$

6.3 Swaptions valuation - Forward LIBOR rate model

Let us now establish the swaptions valuation formula for the forward LIBOR rate model of Section 3. In a general HJM framework the price of a payer's swaption with expiry date $T = T_0$ and strike level κ equals (cf., [5]-[6])

$$\mathbf{PS}_t = \mathbf{E}_{\tilde{\mathbf{P}}}\Big[\,\frac{B_t}{B_T}\Big(\mathbf{E}_{\tilde{\mathbf{P}}}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(L(T_{j-1})-\kappa)\delta\,\Big|\,\mathcal{F}_T\Big)\Big)^{+}\,\Big|\,\mathcal{F}_t\Big],$$

or equivalently,

$$\mathbf{PS}_t = \mathbf{E}_{\tilde{\mathbf{P}}}\Big[\,\frac{B_t}{B_T}\mathbf{E}_{\tilde{\mathbf{P}}}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(\kappa(T,T,n)-\kappa)^{+}\delta\,\Big|\,\mathcal{F}_T\Big)\,\Big|\,\mathcal{F}_t\Big]$$

for every $t \in [0,T]$. Let us denote by A the exercise set, that is,

$$A = \{\omega\in\Omega \,|\, (\kappa(T,T,n)-\kappa)^{+} > 0\} = \{\omega\in\Omega \,|\, \sum_{j=1}^{n} c_j B(T,T_j) < 1\},$$

where $c_j = \kappa\delta$ for $j = 1,\dots,n-1$, and $c_n = 1+\kappa\delta$. The option's exercise set A is manifestly $\mathcal{F}_T$-measurable, hence, the following result.

Lemma 6.1 *The following equality holds*

$$\mathbf{PS}_t = \delta\sum_{j=1}^{n} B(t,T_j)\,\mathbf{E}_{\mathbf{P}_{T_j}}\Big((L(T,T_{j-1})-\kappa)\,\mathrm{I}_A\,\Big|\,\mathcal{F}_t\Big), \quad \forall\, t\in[0,T]. \tag{156}$$

Proof. Since

$$\mathbf{PS}_t = \mathbf{E}_{\tilde{\mathbf{P}}}\Big[\,\frac{B_t}{B_T}\,\mathrm{I}_A\,\mathbf{E}_{\tilde{\mathbf{P}}}\Big(\sum_{j=1}^{n}\frac{B_T}{B_{T_j}}(L(T_{j-1})-\kappa)\delta\,\Big|\,\mathcal{F}_T\Big)\Big)\,\Big|\,\mathcal{F}_t\Big],$$

we have

$$\begin{aligned}
\mathbf{PS}_t &= \delta\,\mathbf{E}_{\tilde{\mathbf{P}}}\Big[\,\mathbf{E}_{\tilde{\mathbf{P}}}\Big(\sum_{j=1}^{n}\frac{B_t}{B_{T_j}}(L(T_{j-1})-\kappa)\,\mathrm{I}_A\,\Big|\,\mathcal{F}_T\Big)\Big)\,\Big|\,\mathcal{F}_t\Big]\\
&= \delta\,\mathbf{E}_{\tilde{\mathbf{P}}}\Big(\sum_{j=1}^{n}\frac{B_t}{B_{T_j}}(L(T_{j-1})-\kappa)\,\mathrm{I}_A\,\Big|\,\mathcal{F}_t\Big)\\
&= \delta\sum_{j=1}^{n} B(t,T_j)\,\mathbf{E}_{\mathbf{P}_{T_j}}\Big((L(T_{j-1})-\kappa)\,\mathrm{I}_A\,\Big|\,\mathcal{F}_t\Big).
\end{aligned}$$

Furthermore, for every $j = 1,\dots,n$, we have

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}_{T_j}}\Big((L(T_{j-1})-\kappa)\,\mathrm{I}_A\,\Big|\,\mathcal{F}_t\Big) &= \mathbf{E}_{\mathbf{P}_{T_j}}\Big(\,\mathbf{E}_{\mathbf{P}_{T_j}}(L(T_{j-1},T_{j-1})-\kappa\,|\,\mathcal{F}_T)\,\mathrm{I}_A\,\Big|\,\mathcal{F}_t\Big)\\
&= \mathbf{E}_{\mathbf{P}_{T_j}}\Big((L(T,T_{j-1})-\kappa)\,\mathrm{I}_A\,\Big|\,\mathcal{F}_t\Big)
\end{aligned}$$

since $\mathcal{F}_t \subset \mathcal{F}_T$, and the forward LIBOR rate process $L(t,T_{j-1})$ follows a martingale under the forward probability measure $\mathbf{P}^{T_j}$. □

For arbitrary $j = 1,\dots,n$, we denote by G_j the joint probability distribution function of n-dimensional random variable $(X_1(0),\dots,X_n(0))$ under the forward probability measure $\mathbf{P}^{T_j}$, where for every $k = 1,\dots,n$, the random variable $X_k(t)$ is given by the formula

$$X_k(t) = \int_t^T \lambda(u,T_{k-1})\cdot dW_u^{T_k}, \quad \forall\, t\in[0,T]. \tag{157}$$

We write also

$$\lambda_k^2(t) = \int_t^T \|\lambda(u,T_{k-1})\|^2\,du, \quad \forall\, t \in [0,T],$$

for every $k = 1,\dots,n$. Notice that we have

$$L(T,T_{j-1}) = L(t,T_{j-1})\,\exp\Big(X_j(t) - \frac{1}{2}\,\lambda_j^2(t)\Big), \quad \forall\, t \in [0,T].$$

For the sake of completeness, we quote the following result from [4].

Proposition 6.2 *Assume the forward LIBOR rate model of Section 3. Then the price at time 0 of a payer's swaption with expiration date $T = T_0$ and strike level κ equals*

$$\mathbf{PS}_0 = \delta \sum_{j=1}^n B(0,T_j) \int_{\mathbf{R}^n} \Big(L(0,T_{j-1})\,e^{y_j - \frac{1}{2}\lambda_j^2(0)} - \kappa\Big)\,\mathrm{I}_D(y_1,\dots,y_n)\,dG_j(y_1,\dots,y_n), \tag{158}$$

where D stands for the set

$$D = \Big\{(y_1,\dots,y_n) \in \mathbf{R}^n \,\Big|\, \sum_{j=1}^n c_j \prod_{k=0}^{j-1} \Big(1 + \delta\,L(0,T_k)\,e^{y_{k+1} - \frac{1}{2}\,\lambda_k^2(0)}\Big)^{-1} < 1\Big\}.$$

Proof. Recall that the dynamics of the forward LIBOR rate is (see (74))

$$dL(t,T_{j-1}) = L(t,T_{j-1})\,\lambda(t,T_{j-1}) \cdot dW_t^{T_j},$$

for every $j = 1,\dots n$. Furthermore, the bond price $B(t,T_j)$ admits the following representation

$$B(t,T_j) = \frac{B(t,T_j)}{B(t,T)} = \prod_{k=1}^j \frac{B(t,T_k)}{B(t,T_{k-1})} = \prod_{k=1}^j (F_B(t,T_{k-1},T_k))^{-1}.$$

Hence, in view of (70) for $t = T$ we have

$$B(T,T_j) = \prod_{k=1}^j \Big(1 + \delta L(T,T_{k-1})\Big)^{-1}.$$

Consequently, the exercise set A can be re-expressed in terms of forward LIBOR rates, namely,

$$A = \Big\{\omega \in \Omega \,\Big|\, \sum_{j=1}^n c_j \prod_{k=0}^{j-1} \Big(1 + \delta L(T,T_k)\Big)^{-1} < 1\Big\},$$

or more explicitly,

$$A = \Big\{\omega \in \Omega \,\Big|\, \sum_{j=1}^n c_j \prod_{k=0}^{j-1} \Big[1 + \delta\,L(t,T_k)\,\mathcal{E}_t^T\Big(\int_0^{\cdot} \lambda(u,T_k) \cdot dW_u^{T_{k+1}}\Big)\Big]^{-1} < 1\Big\},$$

where for arbitrary $t \in [0,T]$ we denote

$$\begin{aligned}
\mathcal{E}_t^T\Big(\int_0^{\cdot} \lambda(u,T_k) \cdot dW_u^{T_{k+1}}\Big) &= \exp\Big(\int_t^T \lambda(u,T_k) \cdot dW_u^{T_{k+1}} - \frac{1}{2}\int_t^T \|\lambda(u,T_k)\|^2\,du\Big) \\
&= \exp\Big(X_k(t) - \frac{1}{2}\,\lambda_k^2(t)\Big).
\end{aligned}$$

Furthermore, the forward Wiener processes W^{T_k}, $k = 0, \dots, n$, satisfy the following relationship

$$W_t^{T_{k+1}} = W_t^{T_k} + \int_0^t \frac{\delta L(u, T_k)}{1 + \delta L(u, T_k)} \lambda(u, T_k)\, du, \quad \forall\, t \in [0, T_k].$$

In view of Lemma 6.1 we conclude that in order to find an explicit valuation formula for the swaption, it is sufficient to determine the joint conditional law under each forward measure $\mathbf{P}^{T_j}$ of the n-dimensional random variable $(X_1(0), \dots, X_n(0))$, where $X_1(0), \dots, X_n(0)$ are given by (157). Note that for $t = 0$ we have

$$A = \Big\{ \omega \in \Omega \,\Big|\, \sum_{j=1}^n c_j \prod_{k=0}^{j-1} \Big(1 + \delta\, L(0, T_k)\, e^{\, X_{k+1}(0) - \frac{1}{2} \lambda_k^2(0)} \Big)^{-1} < 1 \Big\}.$$

This completes the proof of the asserted formula. An extension of the valuation formula (158) to the case of arbitrary $t \in [0, T]$ is straightforward. □

6.4 Swaptions valuation - Flesaker-Hughston model

Recall that a payer's swaption pays $X_j = \delta(\kappa(T, T, n) - \kappa)^+$ at the future date $T_j = T + j\delta$ for every $j = 1, \dots, n$. By discounting these payments to the date T we conclude that the swaption is essentially equivalent to the contingent claim

$$X = \delta\, (\kappa(T, T, n) - \kappa)^+ \sum_{j=1}^n B(T, T_j) = \Big(1 - B(T, T_n) - \kappa\delta \sum_{j=1}^n B(T, T_j) \Big)^+$$

which settles at time T. In terms of the supermartingale A we get

$$X = \Big(1 - A_T^{-1}\, \mathbf{E_P}(A_{T_n} \,|\, \mathcal{F}_T) - \kappa\delta \sum_{j=1}^n A_T^{-1}\, \mathbf{E_P}(A_{T_j} \,|\, \mathcal{F}_T) \Big)^+. \tag{159}$$

Therefore, the arbitrage price of a payer's swaption equals

$$\mathbf{PS}_t = A_t^{-1}\, \mathbf{E_P} \Big[\Big(A_T - \mathbf{E_P}(A_{T_n} \,|\, \mathcal{F}_T) - \kappa\delta \sum_{j=1}^n \mathbf{E_P}(A_{T_j} \,|\, \mathcal{F}_T) \Big)^+ \Big| \mathcal{F}_t \Big] \tag{160}$$

for every time t before the expiration date T. In the case of the model introduced in Example 4.1 this yields

$$\mathbf{PS}_t = (f(t) + g(t) M_t)^{-1}\, \mathbf{E_P} \Big[\Big(f(T) + g(T) M_T - f(T_n) - g(T_n) M_T - \kappa\delta \big(\sum_{j=1}^n f(T_j) + g(T_j) M_T \big) \Big)^+ \Big| \mathcal{F}_t \Big]$$

for every $t \in [0, T]$. Put another way,

$$\mathbf{PS}_t = (f(t) + g(t) M_t)^{-1}\, \mathbf{E_P} \big((a - b M_T)^+ \,|\, \mathcal{F}_t \big), \tag{161}$$

where

$$a = f(T) - f(T_n) - \kappa\delta \sum_{j=1}^n f(T_j), \quad b = g(T) - g(T_n) - \kappa\delta \sum_{j=1}^n g(T_j). \tag{162}$$

Proposition 6.3 *Assume that the volatility coefficient σ in formula* (110) *is a deterministic function. Suppose that the coefficients a and b given by* (162) *are strictly positive. Then the arbitrage price $\mathbf{PS}_t$ at time $t \in [0, T]$ of a payer's swaption with expiration date T and strike level κ equals*

$$\mathbf{PS}_t = (f(t) + g(t) M_t)^{-1} \big(a N(\tilde{d}_1(t, T)) - b M_t N(\tilde{d}_2(t, T)) \big), \tag{163}$$

where

$$\tilde{d}_1(t,T) = \frac{\log \tilde{c}_t + \frac{1}{2}v^2(t,T)}{v(t,T)}, \tag{164}$$

with

$$\tilde{c}_t = M_t^{-1}\,\frac{a}{b} = M_t^{-1}\,\frac{f(T) - f(T_n) - \kappa\delta\,\sum_{j=1}^n f(T_j)}{g(T) - g(T_n) - \kappa\delta\,\sum_{j=1}^n g(T_j)},$$

$\tilde{d}_2(t,T) = \tilde{d}_1(t,T) - v(t,T)$, *and*

$$v^2(t,T) = \int_t^T \sigma_u^2\,du.$$

Proof. The asserted formula is an immediate consequence of equalities (161)-(162) and Lemma 5.2. □

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