DISCUSSION PAPER NO. B–308

CLOSED FORM SOLUTIONS FOR TERM STRUCTURE DERIVATIVES WITH LOG-NORMAL INTEREST RATES

KRISTIAN R. MILTERSEN, KLAUS SANDMANN, AND DIETER SONDERMANN

Abstract. We derive a unified model which gives closed form solutions for caps and floors written on interest rates as well as puts and calls written on zero-coupon bonds. The crucial assumption is that forward rates with a compounding period that matches the contract, which we want to price, is log-normally distributed. Moreover, this assumption is shown to be consistent with the Heath-Jarrow-Morton model for a specific choice of volatility.

1. Introduction

Closed-form solutions for interest rate derivatives, in particular caps, bond options, and swaptions, have been obtained by a number of authors for Markovian term structure models with normally distributed interest rates or alternatively log-normally distributed bond prices, see, e.g., El Karoui and Rochet (1989), Jamshidian (1990), Heath, Jarrow, and Morton (1992), and Brace and Musiela (1993). These models support Black-Scholes type formulas most frequently used by practitioners for pricing caps, bond options, and swaptions. Unfortunately, these models imply negative interest rates with positive probabilities, and hence they are not arbitrage free in an economy with opportunities for riskless and costless storage of money. Alternatively, modelling log-normally distributed interest rates avoids these problems of negative interest rates. However, as shown by Morton (1988) and Hogan and Weintraub (1993) these rates explode with positive probability implying zero prices for bonds and hence also arbitrage opportunities. Furthermore, so far, no closed form solutions are known for these models.

As has been observed by Sandmann and Sondermann (1994) the problems of exploding interest rates result from an unfortunate choice of compounding period of the interest rates modelled, namely the continuously compounded rate. Assuming that the continuously compounded interest rate is log-normally distributed results in "double exponential" expressions, i.e., the exponential function is itself an argument of an exponential function, thus giving rise to infinite expectations of the accumulation factor and of inverse bond prices under the martingale measure. The problem disappears if, instead of assuming that the continuously compounded interest rates are log-normally distributed, one assumes that rates with a strictly positive compounding period are log-normally distributed. In practice, interest rates, both spot and forward, are quoted as rates per annum, that is on a yearly basis, even if the compounding period is different from one year, e.g., three months. Moreover, effective annual rates[1] are calculated and used as

Date. March 1994. This version: March 2, 1995.

JEL Classification. G13.

Key words and phrases. Log-normal, α-compounding rates, Heath-Jarrow-Morton model.

An earlier version of this paper, Sandmann, Sondermann, and Miltersen (1994), was presented at *the Seventh Annual European Futures Research Symposium* in Bonn, Germany and at Norwegian School of Economics and Business Administration, Bergen, Norway. We are grateful to Peter Ove Christensen, Darrell Duffie, Bjørn Nybo Jørgensen, Claus Munk, Sven Rady, Gunnar Stensland, and the two CBOT discussants Chris Veld and Frans de Roon for comments and assistance. Financial support from the Danish Natural and Social Science Research Councils by the first author and from the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at Rheinische Friedrich-Wilhelms-Universität Bonn by all three authors is gratefully acknowledged. Document typeset in LaTeX.

[1]By effective annual rates we mean the annually compounded rate which yields the same return as the original rate compounded appropriately.

the benchmark for comparing nominal rates with different compounding periods. Hence, interest rates with strictly positive compounding periods are directly observable in the market place and form a natural starting point for modelling the term structure, see also Goldys, Musiela, and Sondermann (1994). Assuming that interest rates with a given strictly positive compounding period are log-normally distributed implies that the continuously compounded short rate follows a diffusion which is neither normal nor log-normal—but, in a sense, a dynamic combination of both models with the following properties: for small values the diffusion process of the continuously compounded rate approaches a log-normal diffusion[2], thus generating positive continuously compounded rates, whereas for large values the diffusion process of the continuously compounded rates approaches a normal diffusion process, generating stable finite expected returns and forward values. This model thus combines—in a very simple and straightforward setup—the strengths of the normal and log-normal model and avoids their shortcomings. This type of dynamics of the continuously compounded rates has been supported by an independent empirical study by Miltersen (1993). We are aware of two alternative approaches which are similar to our approach and which also avoid the problem of exploding rates: (i) Musiela (1994) models instantaneous forward rates with non-continuous compounding as log-normal and finds the corresponding dynamics of the continuously compounding rates. (ii) Ho et al. (1994) model "bankers discount" rates as log-normal. However, this latter approach implies negative bond prices with positive probability.

The main result of this paper is to derive a unified model which provides closed form solutions for interest rate caps and floors as well as puts and calls written on zero-coupon bonds within the context of a log-normal interest rate model. These solutions coincide with modifications of the Black-Scholes formula. In particular, for caps and floors with payment periods of the same length as the compounding period of the underlying interest rate we obtain the same Black formula as often used by market practitioners, however, *without* making the unrealistic assumption that forward rates are independent of the accumulation process. Thus, in this case our model supports market practice. For the case of put and call options, our derived closed form solution matches the formula derived in Käsler (1991).[3] In Käsler (1991) the formula is derived using no arbitrage arguments of bond prices, in this paper we contribute with an underlying interest rates model. Moreover, the log-normal assumption is shown to be consistent with the Heath-Jarrow-Morton model for a specific choice of volatility of the Heath-Jarrow-Morton model. Since the model implies non-negative interest rates with probability one, the model is arbitrage free in an economy with opportunities for risk- and costless storage of money.

The paper is organized as follows. Section 2 contains the model. Then the solutions for interest rate derivatives are derived in Section 3. The relation to the Heath-Jarrow-Morton model is found in Section 4. Finally, some proofs are placed in the Appendix.

2. A Model for α-Compounding Forward Rates

Let $P(t,T)$ denote the price, at date t, of a (default free) zero-coupon bond that pays out \$1 at maturity date T. For ease of notation we will term this a *bond*. Let $f(t,T,\alpha)$ denote the interest rate quoted on a yearly basis[4] with compounding period of length α prevailing at date t for the future time interval $[T, T+\alpha]$. That is,

$$P(t,T+\alpha) = P(t,T)\frac{1}{1+\alpha f(t,T,\alpha)}. \tag{1}$$

[2] Indeed, if the continuously compounded rate becomes infinitesimally close to zero, then the two dynamics coincide, for further details see Sandmann and Sondermann (1994).

[3] This formula is published in Käsler's Ph.D.-dissertation written in German. The formula appears in the English manuscript Rady and Sandmann (1994) which is a comparative study of different bond based no arbitrage models.

[4] We have chosen yearly basis because this is (by implicit assumption) the unit of the time line. That is, rates should be quoted in one time unit basis.

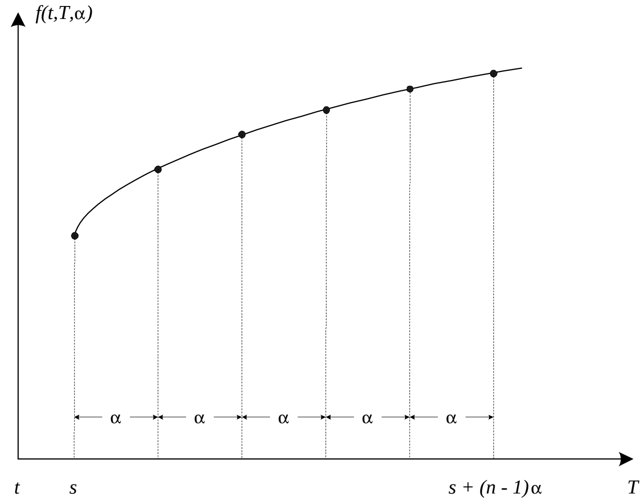

FIGURE 1. Forward rate curve for the α-compounding forward rates.

$f(t,T,\alpha)$ will be termed the α-compounding forward rate. $\alpha = 0$ corresponds to continuously compounding interest rates. This rate has to be treated as a special case in the following way

$$f(t,T,0) := \lim_{\alpha \to 0} f(t,T,\alpha),$$

deduced using l'Hospital's rule from the following definition of a continuously compounded forward rate

$$f(t,T,0) = -\frac{\frac{\partial}{\partial T} P(t,T)}{P(t,T)}.$$

Let $F(t,T,\alpha)$ denote the forward price, at date t, of a zero coupon bond for delivery at date T which pays \$1 at date $T+\alpha$. No arbitrage implies

$$F(t,T,\alpha) = \frac{P(t,T+\alpha)}{P(t,T)} = \frac{1}{1+\alpha f(t,T,\alpha)}. \tag{2}$$

Note that

$$\begin{aligned} P(t,s+n\alpha) &= P(t,s) \prod_{i=0}^{n-1} \frac{P(t,s+(i+1)\alpha,\alpha)}{P(t,s+i\alpha,\alpha)} \\ &= P(t,s) \prod_{i=0}^{n-1} F(t,s+i\alpha,\alpha) \\ &= P(t,s) \prod_{i=0}^{n-1} \frac{1}{1+\alpha f(t,s+i\alpha,\alpha)}, \qquad n = 1,\ldots, \quad s \in [t,t+\alpha). \end{aligned} \tag{3}$$

Especially, for $T-t$ a multiple of α

$$P(t,t+n\alpha) = \prod_{i=0}^{n-1} \frac{1}{1+\alpha f(t,t+i\alpha,\alpha)}, \qquad n = 1,\ldots.$$

Figure 1 shows the points on the forward rate curve used to price the bond with maturity $s+n\alpha$.

In our model, we will take as given a compounding period α. We then model the α-compounding forward rates as

$$df(\cdot,T,\alpha)_t = \mu(t,T) f(t,T,\alpha)dt + \gamma(t,T) f(t,T,\alpha)dW_t, \tag{4}$$

for $t \in [0,T]$, $T \in [0,\tau]$, where τ is the time horizon. $\mu(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are continuously differentiable non-stochastic functions of the time parameter set. $\{f(\cdot,T,\alpha)_t\}$ is initiated using the given term structure of interest rates observable at date zero

$$f(0,T,\alpha) = \frac{P(0,T)}{P(0,T+\alpha)} - 1.$$

The question of existence of such a stochastic process with a two-dimensional time parameter set (under suitable regularity conditions) is dealt with in Heath, Jarrow, and Morton (1992) and Morton (1988).[5] In the Heath-Jarrow-Morton model they interprete the process as the continuously compounded forward rate which is different from the interpretation in this paper, but this does not influence the existence part of the solution to the SDE (4).

This model does not only specify the stochastic model for the α-compounding forward rates, it determines simultaneously the stochastic model for all rates with any compounding period through the bond prices. That is, bond prices are calculated using Equation (3). Given the bond prices, forward rates can be calculated with any different compounding period than the chosen α using Equation (1). However, note that the domain of this stochastic description is only the time interval $[t+\alpha,\infty)$, cf. Figure 1. That is, for rates with shorter compounding periods, β, than α, i.e. $\beta < \alpha$, we have not determined the stochastic model of β-compounding forward rates (including the continuously compounded rates, i.e. $\beta = 0$) in the time interval $[t, t+\alpha-\beta]$.

Using Itô's lemma on Equation (2) gives that[6]

$$\text{(5)} \qquad \text{vol}\big(dF(\cdot,T,\alpha)_t\big) = -F(t,T,\alpha)^2\alpha\gamma(t,T)f(t,T,\alpha)dW_t = -F(t,T,\alpha)\big(1-F(t,T,\alpha)\big)\gamma(t,T)dW_t.$$

3. Closed Form Solutions for Interest Rate Derivatives

In this section, we focus on the arbitrage price of interest rate derivatives. More precisely, we consider two special interest rate derivatives: interest rate caps and floors and European debt options where the underlying security is a zero coupon bond. Since the construction of the underlying term structure model is very closely related to the Black-Scholes model, we should expect similar pricing formulas for these derivatives within our model.

Caps and floors are special types of options where a nominal interest rate is the underlying security. The underlying interest rate could be, for example, the three or six months LIBOR. A cap is an insurance against upward movements in the interest rate and a floor is an insurance against downward movements in the interest rate. Let $\{r_t\}$ be a nominal interest rate process with compounding period α, e.g., for $\alpha = \frac{1}{4}$ the process $\{r_t\}$ is the quoted three months LIBOR.[7]

---

[5] In fact, the existence proof can be generalized to stochastic functions μ and γ as shown in Miltersen (1994).

[6] We are only calculating the diffusion part of the Itô processes in this paper, since we know from, e.g., Harrison and Kreps (1979) and Harrison and Pliska (1981) that the drift part will not play any role for the pricing of contingent claims. For that purpose, we have introduced the obvious notation vol. That is, for the Itô process

$$dX_t = \xi(X_t,t)dt + \delta(X_t,t)dW_t$$

we introduce the notion of vol as

$$\text{vol}(dX_t) = \delta(X_t,t)dW_t.$$

[7] It is an assumption of the model that the underlying interest rate, $f(\cdot,\cdot,\alpha)$, is default-free since it is used for pricing default-free bonds. The LIBOR is based on a "replenished" AA rate and, hence, not default-free. However, (i) assuming that the short position of the cap or floor contract has the same credit quality as the one on which LIBOR is based and (ii) modelling the default risk as in Duffie and Singleton (1994) and Duffie, Schroder, and Skiadas (1994) the same formulas apply with the volatility process adjusted to include the default spread on LIBOR. As it is shown in Duffie (1994), the volatility of the credit spread and of the default free rate simply adds together to give the volatility of the defaultable rate. This result also applies to our model, SDE (4), with an appropriate dynamics of the default risk. Duffie and Singleton (1994) then shows that options etc. written on defaultable interest rates can be priced using standard option pricing techniques, such as valuing expectations under an equivalent martingale measure and solving PDEs, by (i) simply substituting the

Let $\mathbb{T} = \{t_1 < \cdots < t_N\}$ be a set of dates such that $\alpha = t_{i+1} - t_i$, for $i = 1, \ldots, N-1$. Then a cap contract with level L, face value V, underlying nominal interest rate process $\{r_t\}$, and payment dates $\mathbb{T}$ is defined by the payoff at all dates $t_i \in \mathbb{T}$

$$V\alpha[r_{t_{i-1}} - L]^+ = V\alpha \max\{r_{t_{i-1}} - L, 0\},$$

where r_t denotes the nominal interest rate valid at date t. Note that all payments are payed in arrear. A cap with only one payment date is called a caplet. Clearly all caps are portfolios of caplets. Therefore, we will concentrate on pricing a caplet. Let us have a closer look at the caplet with payment date t_i. Clearly,

$$r_t = f(t, t, \alpha).$$

Since the rate is known at date t_{i-1}, the payoff at time t_i is also known at time t_{i-1}, hence the present value of this payoff at date t_{i-1} is equal to

$$\begin{aligned} P(t_{i-1}, t_i)V\alpha\big[f(t_{i-1}, t_{i-1}, \alpha) - L\big]^+ &= \frac{V}{1 + \alpha f(t_{i-1}, t_{i-1}, \alpha)}\big[1 + \alpha f(t_{i-1}, t_{i-1}, \alpha) - (1 + \alpha L)\big]^+ \\ &= V\big[1 - \frac{1 + \alpha L}{1 + \alpha f(t_{i-1}, t_{i-1}, \alpha)}\big]^+ \\ &= V(1 + \alpha L)\big[\frac{1}{1 + \alpha L} - P(t_{i-1}, t_i)\big]^+ \\ &= V(1 + \alpha L)\big[\frac{1}{1 + \alpha L} - F(t_{i-1}, t_{i-1}, \alpha)\big]^+. \end{aligned} \tag{6}$$

The floor is just the opposite contract. At each date $t_i \in \mathbb{T}$ the payoff is defined by

$$V\alpha[L - r_{t_{i-1}}]^+$$

and the present value at date t_{i-1} is determined by

$$P(t_{i-1}, t_i)V\alpha\big[L - f(t_{i-1}, t_{i-1}, \alpha)\big]^+ = V(1 + \alpha L)\big[F(t_{i-1}, t_{i-1}, \alpha) - \frac{1}{1 + \alpha L}\big]^+. \tag{7}$$

Therefore, the payoff of a cap or a floor at each date t_{i-1} is equivalent to $V(1 + \alpha L)$ times the payoff of a European put option or a European call option, respectively, with exercise date t_{i-1}, exercise price $K = \frac{1}{1+\alpha L}$, and a zero coupon bond with maturity $t_i = t_{i-1} + \alpha$ as the underlying security. Thus the arbitrage price of a cap or a floor is equal to the arbitrage price of a portfolio of European put options or European call options, respectively.

As already noted by Müller (1985) a (trivially self-financing) trading strategy can be constructed using forward contracts on the underlying security if and only if there exists a self-financing trading strategy that duplicates this contingent claim on the spot market. That is, we are free to choose whether we consider the spot market or the forward market to hedge a given contingent claim.

Since we, in this paper, have derived the price process of the forward price, $F(t, T, \alpha)$, of the underlying bond, it is natural to hedge in the forward market as also noted by Rady and Sandmann (1994).

Theorem 3.1. *Suppose*

$$\mathrm{vol}\big(dF(\cdot, T, \alpha)_t\big) = \nu\big(t, F(t, T, \alpha)\big)dW_t,$$

and that $u(t, x)$ is a continuous function on $[0, T] \times [0, 1]$ which solves the PDE

$$u_t(t, x) + \frac{1}{2}\nu^2(t, x)u_{xx}(t, x) = 0, \tag{8}$$

default-free volatility with the volatility of the defaultable rate and (ii) using the defaultable rate as the short rate in the option pricing model. We are indebted to Darrell Duffie for pointing this out to us.

on $[0,T]\times(0,1)$ *with boundary conditions*

$$\begin{aligned} &u(T,x) = g(x), \qquad x\in[0,1],\\ &h_l(x)\le u(t,x)\le h_u(x), \qquad t\in[0,T], \quad x\in[0,1]. \end{aligned}$$

Then the (self-financing) trading strategy $\{\phi_s\}_{s\in[t,T]}$ *has the (date* T*) forward value*

$$FV(\{\phi_s\}_{s\in[t,T]}) = u\big(t, F(t,T,\alpha)\big),$$

at date t*, or the present value*

$$PV(\{\phi_s\}_{s\in[t,T]}) = P(t,T)u\big(t, F(t,T,\alpha)\big),$$

at date t*.* ϕ_s *denotes the number of forward contracts to hold, at each date* $s\in[t,T]$*. The forward contract to hold, matures at date* T *and is written on a bond with maturity date* $T+\alpha$*. Moreover,* ϕ_t *is defined by*

$$\phi_t = u_x\big(t, F(t,T,\alpha)\big).$$

Proof. Define the Itô process, $\{V_t\}$, as

$$V_t = u\big(t, F(t,T,\alpha)\big).$$

By Itô's lemma and the PDE (8), this implies that

$$\begin{aligned} dV_t &= \Big(u_t\big(t,F(t,T,\alpha)\big) + \frac{1}{2}\nu^2\big(t,F(t,T,\alpha)\big)u_{xx}\big(t,F(t,T,\alpha)\big)\Big)dt\\ &\quad + u_x\big(t,F(t,T,\alpha)\big)dF(\cdot,T,\alpha)_t\\ &= u_x\big(t,F(t,T,\alpha)\big)dF(\cdot,T,\alpha)_t\\ &= \phi_t dF(\cdot,T,\alpha)_t. \end{aligned}$$

Thus, $\{V_t\}$ is the forward value process of the trading strategy $\{\phi_s\}_{s\in[t,T]}$, that is, $V_t = FV(\{\phi_s\}_{s\in[t,T]})$. By the usual no-arbitrage argument, the forward value of a contingent claim with terminal payoff $V_T = u\big(T,F(T,T,\alpha)\big)$ at date T is, therefore, equal to $V_t = u\big(t,F(t,T,\alpha)\big)$. Similarly, the spot arbitrage price of this contingent claim is given by $P(t,T)u\big(t,F(t,T,\alpha)\big)$. □

From Equation (5) we have

$$\nu\big(t,F(t,T,\alpha)\big) = -\gamma(t,T)F(t,T,\alpha)\big(1-F(t,T,\alpha)\big).$$

Therefore, our PDE (8) is

$$u_t(t,x) + \frac{1}{2}\gamma^2(t,T)x^2(1-x)^2u_{xx}(t,x) = 0. \tag{9}$$

Furthermore, we have the following boundary conditions for a European call option with exercise price K[8] on a zero-coupon bond

$$\begin{aligned} &u(T,x) = [x-K]^+, \qquad x\in[0,1],\\ &[x-K]^+ \le u(t,x)\le \min\{x, 1-K\}, \qquad t\in[0,T], \quad x\in[0,1], \end{aligned}$$

and for the put option

$$\begin{aligned} &u(T,x) = [K-x]^+, \qquad x\in[0,1],\\ &0\le u(t,x)\le K, \qquad t\in[0,T], \quad x\in[0,1]. \end{aligned}$$

We can now derive closed form solutions for interest rate contingent claims.

[8] To avoid trivial cases we assume that $K\in[0,1]$.

Corollary 3.2. *The arbitrage price of a European call option with exercise price K and exercise date T written on a zero coupon bond with maturity date $T+\alpha$ is equal to*

$$\begin{aligned} \text{Call} &= P(t,T+\alpha)N(e_1) - KP(t,T)N(e_2) - KP(t,T+\alpha)\big(N(e_1)-N(e_2)\big) \\ &= (1-K)P(t,T+\alpha)N(e_1) - K\big(P(t,T)-P(t,T+\alpha)\big)N(e_2), \end{aligned} \tag{10}$$

with

$$\begin{aligned} e_1 &= \frac{1}{\sigma(t,T)}\Big(\ln\frac{P(t,T+\alpha)(1-K)}{\big(P(t,T)-P(t,T+\alpha)\big)K} + \frac{\sigma^2(t,T)}{2}\Big), \\ e_2 &= e_1 - \sigma(t,T), \\ \sigma^2(t,T) &= \int_t^T \gamma^2(s,T)ds, \end{aligned}$$

where $N(\cdot)$ denotes the cumulative distribution function of a standard normally distributed variable.

Corollary 3.3. *The arbitrage price of a European put option with exercise price K and exercise date T written on a zero coupon bond with maturity date $T+\alpha$ is equal to*

$$\begin{aligned} \text{Put} &= KP(t,T)N(-e_2) - P(t,T+\alpha)N(-e_1) - KP(t,T+\alpha)\big(N(e_1)-N(e_2)\big) \\ &= K\big(P(t,T)-P(t,T+\alpha)\big)N(-e_2) - (1-K)P(t,T+\alpha)N(-e_1), \end{aligned} \tag{11}$$

with

$$\begin{aligned} e_1 &= \frac{1}{\sigma(t,T)}\Big(\ln\frac{P(t,T+\alpha)(1-K)}{\big(P(t,T)-P(t,T+\alpha)\big)K} + \frac{\sigma^2(t,T)}{2}\Big), \\ e_2 &= e_1 - \sigma(t,T), \\ \sigma^2(t,T) &= \int_t^T \gamma^2(s,T)ds, \end{aligned}$$

where $N(\cdot)$ denotes the cumulative distribution function of a standard normally distributed variable.

We have written two versions of the closed form solutions. The former has three terms where the first two terms look similar to the Black-Scholes formula, i.e. price of underlying security times N(of something) minus present value of the exercise price times N(of something minus volatility), and then there is a third correction term. The latter is in structure a Black-Scholes formula where $(1-K)P(t,T+\alpha)$ should be interpreted as the price of the underlying security and $K\big(P(t,T)-P(t,T+\alpha)\big)$ as the present value of the exercise price.

Within the context of a bond price based model the closed form solution, (10), was first derived by Käsler (1991). A discussion of this model relative to other bond price based models can be found in Rady and Sandmann (1994). The proof of Corollary 3.2–3.3 follows the presentation in Rady and Sandmann (1994).[9]

Note that a crucial assumption to derive these closed form solutions is that the underlying bond is maturing exactly α time units after maturity of the option, where α is the compounding period of our log-normally distributed forward rates. That is, to price a specific contingent claim we must choose a specific forward rate to model as log-normal. If we again look at Figure 1 then α must be chosen such that the option matures at date $s+(n-1)\alpha$ and that the underlying bond matures at date $s+n\alpha$.

We can now apply Corollary 3.2–3.3 to the pricing of interest rate caps and floors.

Corollary 3.4. *Consider a cap with interest rate level L, face value V, underlying interest rate process $\{f(t,t,\alpha)\}$, and payment dates $t_1, \ldots, t_N$, with $\alpha = t_i - t_{i-1}$, $i = 2, \ldots, N$. The arbitrage price of this*

[9] For completeness we give the outline of the proof in Appendix A.

cap at date $t \leq t_0 = t_1 - \alpha$ is

$$\text{Cap} = \alpha V \sum_{i=0}^{N-1} P(t, t_{i+1})\Big(f(t, t_i, \alpha) N\big(d_1(t, t_i, \alpha)\big) - L N\big(d_2(t, t_i, \alpha)\big)\Big), \tag{12}$$

with

$$\begin{aligned}
d_1(t, s, \alpha) &= \frac{1}{\sigma(t, s)}\big(\ln \frac{f(t, s, \alpha)}{L} + \frac{\sigma^2(t, s)}{2}\big), \\
d_2(t, s, \alpha) &= d_1(t, s, \alpha) - \sigma(t, s), \\
\sigma^2(t, s) &= \int_t^s \gamma^2(r, s) dr,
\end{aligned}$$

where $N(\cdot)$ denotes the cumulative distribution function of a standard normally distributed variable.

Proof. By simple arbitrage we can simplify the problem to the pricing of a caplet with payment date t_i, without loss of generality. From Equation (6), the payoff of the caplet is equivalent to $V(1 + \alpha L)$ times the payoff of a European put option with exercise price $\frac{1}{1+\alpha L}$, where the underlying security is a zero coupon bond with maturity date t_i. That is, according to Corollary 3.3, the arbitrage price of the caplet is

$$\begin{aligned}
\text{Caplet} &= V(1 + \alpha L)\Big(\frac{1}{1 + \alpha L}\big(P(t, t_{i-1}) - P(t, t_i)\big) N(-e_2) - \big(1 - \frac{1}{1 + \alpha L}\big) P(t, t_i) N(-e_1)\Big) \\
&= V P(t, t_i)\Big(\big(\frac{P(t, t_{i-1})}{P(t, t_i)} - 1\big) N(-e_2) - \alpha L N(-e_1)\Big) \\
&= V P(t, t_i)\big(\alpha f(t, t_{i-1}, \alpha) N(-e_2) - \alpha L N(-e_1)\big),
\end{aligned}$$

with

$$\begin{aligned}
-e_2 &= \frac{-1}{\sigma(t, t_{i-1})}\Big(\ln \frac{P(t, t_i)\big(1 - \frac{1}{1+\alpha L}\big)}{\big(P(t, t_{i-1}) - P(t, t_i)\big)\frac{1}{1+\alpha L}} - \frac{\sigma^2(t, t_{i-1})}{2}\Big) \\
&= \frac{-1}{\sigma(t, t_{i-1})}\Big(\ln \frac{1 - \frac{1}{1+\alpha L}}{\big(\frac{P(t, t_{i-1})}{P(t, t_i)} - 1\big)\frac{1}{1+\alpha L}} - \frac{\sigma^2(t, t_{i-1})}{2}\Big) \\
&= \frac{-1}{\sigma(t, t_{i-1})}\big(\ln \frac{\alpha L}{\alpha f(t, t_{i-1}, \alpha)} - \frac{\sigma^2(t, t_{i-1})}{2}\big) \\
&= \frac{1}{\sigma(t, t_{i-1})}\big(\ln \frac{\alpha f(t, t_{i-1}, \alpha)}{\alpha L} + \frac{\sigma^2(t, t_{i-1})}{2}\big) \\
&= d_1(t, t_{i-1}, \alpha).
\end{aligned}$$

By summing the respective caplets, this yields the pricing formula for a cap. □

The pricing formula (12) for a cap is a modification of the Black-Scholes formula for a call option. The reason for this is the assumption of log-normally distributed α-compounding interest rates.

A similar proof gives

Corollary 3.5. *Consider a floor with interest rate level L, face value V, underlying interest rate process $\{f(t, t, \alpha)\}$, and payment dates $t_1, \ldots, t_N$, with $\alpha = t_i - t_{i-1}$, $i = 2, \ldots, N$. The arbitrage price of this floor at date $t \leq t_0 = t_1 - \alpha$ is*

$$\text{Floor} = \alpha V \sum_{i=0}^{N-1} P(t, t_{i+1})\Big(L N\big(-d_2(t, t_i, \alpha)\big) - f(t, t_i, \alpha) N\big(-d_1(t, t_i, \alpha)\big)\Big), \tag{13}$$

with

$$d_1(t,s,\alpha) = \frac{1}{\sigma(t,s)}\big(\ln\frac{f(t,s,\alpha)}{L} + \frac{\sigma^2(t,s)}{2}\big),$$
$$d_2(t,s,\alpha) = d_1(t,s,\alpha) - \sigma(t,s),$$
$$\sigma^2(t,s) = \int_t^s \gamma^2(r,s)dr,$$

where $N(\cdot)$ *denotes the cumulative distribution function of a standard normally distributed variable.*

4. The Supporting Continuously Compounded Term Structure of Interest Rates Model

We want to show how to specify the volatility of the Heath-Jarrow-Morton model such that this model will give the crucially needed log-normal α-compounding forward rates.[10] In the Heath-Jarrow-Morton model the continuously compounded forward rate, $\{f(t,T,0)\}_{t\in[0,T]}$, for $T \in [0,\tau]$, is the basic modelling element. This process is modelled as an Itô process in the following way

$$df(\cdot,T,0)_t = \mu\big(t,T,f(t,T,0)\big)dt + \sigma\big(t,T,f(t,T,0)\big)dW_t.$$

The relation between the continuously compounding forward rates and the α-compounding rates is given by

$$\frac{1}{1+\alpha f(t,T,\alpha)} = F(t,T,\alpha) = e^{-\int_T^{T+\alpha} f(t,s,0)ds}, \qquad t \le T.$$

Defining

$$Y(t,T,\alpha) = -\ln F(t,T,\alpha)$$

then

$$\frac{\partial}{\partial T}Y(t,T,\alpha) = f(t,T+\alpha,0) - f(t,T,0). \tag{14}$$

On the other hand

$$Y(t,T,\alpha) = \ln\big(1+\alpha f(t,T,\alpha)\big),$$

therefore,

$$\frac{\partial}{\partial T}Y(t,T,\alpha) = \frac{1}{1+\alpha f(t,T,\alpha)}\alpha f_T(t,T,\alpha) = F(t,T,\alpha)\alpha f_T(t,T,\alpha), \tag{15}$$

where $f_T(t,T,\alpha)$ denotes $\frac{\partial}{\partial T}f(t,T,\alpha)$. Combining Equation (14) and (15) yields

$$f(t,T+\alpha,0) - f(t,T,0) = \alpha F(t,T,\alpha)f_T(t,T,\alpha). \tag{16}$$

Solving the simple difference equation (16) gives

$$f(t,s+n\alpha,0) = f(t,s,0) + \sum_{i=0}^{n-1}\alpha F(t,s+i\alpha,\alpha)f_T(t,s+i\alpha,\alpha), \qquad s\in[t,t+\alpha), \tag{17}$$

with initial condition

$$\int_t^{t+\alpha} f(t,s,0)ds = 1 + \alpha f(t,t,\alpha). \tag{18}$$

This is compatible with our earlier findings in Section 2, that is, when specifying the Itô process of the α-compounding forward rates, we do not specify the continuously compounded interest rates in the time interval $[t,t+\alpha]$. So any (non-negative) value of the continuously compounded forward rate in that interval, fulfilling the initial condition (18), is valid, because the continuously compounded rates specified

[10] This is also the purpose of Musiela (1994). However, Musiela (1994) is working with instantaneous forward rates. That is, Musiela's $q(t,x)$ is the nominal annual rate prevailing at time t over the time interval $[t+x,t+x+dx]$ compounded m times during a year, whereas our $f(t,T,\alpha)$ is the interest rate with compounding period of length α prevailing at date t for the future time interval $[T,T+\alpha]$.

in the time interval $[t+\alpha, \tau]$ by Equation (17) takes care off integrating up to the right bond prices. Again, the reader is referred to Figure 1 to get the intuition.

To find the volatility of the corresponding continuously compounded interest rates we just have to find $\operatorname{vol}\big(dX(\cdot, T, \alpha)_t\big)$, where

$$X(t,T,\alpha) = \alpha f_T(t,T,\alpha) F(t,T,\alpha)$$

and then use Equation (17).

We already know $\operatorname{vol}\big(dF(\cdot, T, \alpha)_t\big)$ from Equation (5). Moreover, using the Itô process description from Equation (4) and a result from Fernique et al. (1983, Chapter 2), $\operatorname{vol}\big(df_T(\cdot, T, \alpha)_t\big)$ can be calculated as

$$\operatorname{vol}\big(df_T(\cdot,T,\alpha)_t\big) = \frac{\partial}{\partial T}\big(f(t,T,\alpha)\gamma(t,T)\big)\,dW_t = \big(f_T(t,T,\alpha)\gamma(t,T) + f(t,T,\alpha)\gamma_T(t,T)\big)dW_t.$$

Finally, using Itô's lemma

$$\begin{aligned}
\operatorname{vol}\big(dX(\cdot,T,\alpha)_t\big) &= \alpha\Big(-f_T(t,T,\alpha)F(t,T,\alpha)\big(1-F(t,T,\alpha)\big)\gamma(t,T) \\
&\quad + F(t,T,\alpha)\big(f_T(t,T,\alpha)\gamma(t,T) + f(t,T,\alpha)\gamma_T(t,T)\big)\Big)dW_t \\
&= \Big(\alpha f_T(t,T,\alpha)F(t,T,\alpha)\gamma(t,T)\big(-1+F(t,T,\alpha)+1\big) \\
&\quad + \alpha F(t,T,\alpha)f(t,T,\alpha)\gamma_T(t,T)\Big)dW_t \\
&= \Big(\alpha f_T(t,T,\alpha)F^2(t,T,\alpha)\gamma(t,T) + \big(1-F(t,T,\alpha)\big)\gamma_T(t,T)\Big)dW_t.
\end{aligned} \tag{19}$$

Using Itô's lemma on Equation (17) and the result of Equation (19) yields the volatility of the corresponding continuously compounded forward rate model.

5. Conclusion

We will use this conclusion to discuss the limitations of the closed form solutions presented in the light of the underlying α-compounding forward rate model. First, for the caps and floor formulas it is a crucial assumption in the derivation of the closed form solutions that the underlying interest rate process is the α-compounding process, where α is also the difference between payoff dates on the contract. If this is not the case, there are two ways to proceed either (i) you can solve the corresponding PDE by numerical procedures or (ii) you can assume that the true underlying interest rate process is distributed in such a way that the α-compounding interest rate is log-normal. Surely, at first glance (ii) seems like cheating. But why? After all, the whole idea of this exercise is to come up with a set of plausible assumptions on the interest rate process such that we get the closed form solutions. From an economic point of view, any non-negative interest rate process with reasonable steady state properties is plausible.

Second, recall that a crucial assumption to derive closed form solutions for puts and calls is that the underlying bond is maturing exactly α time units after maturity of the option, where α is the compounding period of our log-normally distributed forward rates. This is fatal if we are trying to price two different options with different maturity on the same underlying bond. Only one of them can be priced consistently using our closed form solution, for the other option we are bound to numerical procedures. The same problem arises if we are pricing caps and floors with different payout intervals or if we are pricing caps or floors and puts or calls within the same model. Further research is needed to measure the size of this problem. That is, how big is the mispricing if, in spite of the inconsistency, one, after all, uses the closed form solutions to price two different options with different maturity on the same underlying option instead of consistently pricing one of the options using numerical procedures. This mispricing should be counterbalanced with the extra calculations needed to do numerical procedures.

This second problem is analogous to the problem of using the Black-Scholes formula on individual assets simultaneously with using the Black-Scholes formula on an arithmetic index of the same assets. An inconsistency problem which practitioners do not care much about, because the magnitude of this problem is smaller than many other theoretically inconsistency problems of using the Black-Scholes formula.

Appendix A. Proofs

A.1. **Proof of Corollary 3.2.** The proof follows exactly the arguments given in Rady and Sandmann (1994).

Proof. Given the assumptions of Corollary 3.2, we have to solve the PDE (9) on $[0,T]\times(0,1)$, i.e.,

$$u_t(t,x)+\frac{1}{2}\gamma^2(t,T)x^2(1-x)^2u_{xx}(t,x)=0,$$

with boundary conditions

$$\begin{aligned}&u(T,x)=[x-K]^+, \qquad x\in[0,1],\\ &[x-K]^+\le u(t,x)\le\min\{x,1-K\}, \qquad t\in[0,T], \quad x\in[0,1],\end{aligned}$$

where $u(t,x)$ is the date T forward value of the option contract. This problem is transformed by introducing the new time variable

$$s=s(t,T)=\int_t^T\gamma^2(r,T)dr$$

and the new space variable

$$z=\ln\frac{x}{1-x}$$

which is equivalent to

$$x=\frac{1}{1+\exp(-z)}$$

and finally setting

$$u(t,x)=a(z)b(s)h(s,z).$$

The idea is now to choose differentiable functions $a(\cdot)$ and $b(\cdot)$ in such a way that any solution $h(\cdot,\cdot)$ of the heat conduction equation yields a solution $u(\cdot,\cdot)$ of the original partial differential equation. As shown in Rady and Sandmann (1994) this can be done by setting

$$\begin{aligned}a(z)&=\frac{1}{\exp\left(\frac{z}{2}\right)+\exp\left(-\frac{z}{2}\right)},\\ b(s)&=e^{-\frac{s}{8}}.\end{aligned}$$

That is,

$$u(t,x)=\frac{1}{\exp\left(\frac{z}{2}\right)+\exp\left(-\frac{z}{2}\right)}e^{-\frac{s}{8}}h(s,z).$$

The transformed problem on $[0,T]\times\mathbb{R}$ is

$$\frac{1}{2}h_{zz}-h_s=0,$$

with boundary condition

$$h(0,z)=\left(e^{\frac{z}{2}}+e^{-\frac{z}{2}}\right)\Big[\frac{1}{1+\exp(-z)}-K\Big]^+.$$

The well-known solution to this problem is

$$\begin{aligned}h(s,z)&=\frac{1}{\sqrt{2\pi}}\int_{\frac{1}{\sqrt{s}}(\ln\frac{K}{1-K}-z)}^{\infty}\left(e^{\frac{1}{2}(z+\rho\sqrt{s})}+e^{-\frac{1}{2}(z+\rho\sqrt{s})}\right)\left(\frac{1}{1+\exp(-(z+\rho\sqrt{s}))}-K\right)e^{-\frac{\rho^2}{2}}d\rho\\ &=(1-K)I_1-KI_2,\end{aligned}$$

with

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{s}}(\ln \frac{K}{1-K} - z)}^{\infty} e^{\frac{1}{2}(z+\rho\sqrt{s})} e^{-\frac{\rho^2}{2}} d\rho = e^{\frac{z}{2}} e^{\frac{s}{8}} N\Big(\frac{1}{\sqrt{s}}(z + \ln \frac{1-K}{K} + \frac{s}{2})\Big),$$

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{s}}(\ln \frac{K}{1-K} - z)}^{\infty} e^{-\frac{1}{2}(z+\rho\sqrt{s})} e^{-\frac{\rho^2}{2}} d\rho = e^{-\frac{z}{2}} e^{\frac{s}{8}} N\Big(\frac{1}{\sqrt{s}}(z + \ln \frac{1-K}{K} - \frac{s}{2})\Big).$$

Therefore,

$$u(t,x) = \frac{\exp\left(-\frac{s}{8}\right)}{\exp\left(\frac{z}{2}\right) + \exp\left(-\frac{z}{2}\right)} h(s,z) = (1-K) \underbrace{\frac{\exp\left(\frac{z}{2}\right)}{\exp\left(\frac{z}{2}\right) + \exp\left(-\frac{z}{2}\right)}}_{=x} N(e_1) - K \underbrace{\frac{\exp\left(-\frac{z}{2}\right)}{\exp\left(\frac{z}{2}\right) + \exp\left(-\frac{z}{2}\right)}}_{=1-x} N(e_2)$$

and since

$$P(t, T+\alpha) = P(t,T) F(t,T,\alpha) = P(t,T) \frac{1}{1 + \alpha f(t,T,\alpha)}$$

the spot arbitrage price of the European call option is

$$\text{Call} = P(t,T) u\big(t, F(t,T,\alpha)\big).$$

□

A.2. **Proof of Corollary 3.3.** For the European put option the put-call parity leads to the corresponding result.

Proof. By put-call parity

$$\text{Put} = \text{Call} + KP(t,T) - P(t, T+\alpha)$$

Hence, under the assumptions of Corollary 3.3, the arbitrage price of a European put option with exercise price K, exercise date T, and underlying zero coupon bond with maturity $T+\alpha$ is

$$\text{Put} = KP(t,T)N(-e_2) - P(t,T+\alpha)N(-e_1) - KP(t,T+\alpha)\big(N(e_1) - N(e_2)\big)$$

□

References

Brace, A. and M. Musiela (1993): "Swap Derivatives in a Gaussian HJM Framework," Working Paper, University of New South Wales, Sydney, Australia.

Duffie, J. D. (1994): "Forward Rate Curves with Default Risk," Working Paper, Graduate School of Business, Stanford University, Stanford, California 94305–5015, USA.

Duffie, J. D., M. Schroder, and C. Skiadas (1994): "Recursive Valuation of Defaultable Securities and the Timing of Resolution of Uncertainty," Working Paper, Graduate School of Business, Stanford University, Stanford, California 94305–5015, USA.

Duffie, J. D. and K. J. Singleton (1994): "Econometric Modeling of Term Structures of Defaultable Bonds," Working Paper, Graduate School of Business, Stanford University, Stanford, California 94305–5015, USA.

El Karoui, N. and J.-C. Rochet (1989): "A Pricing Formula for Options on Coupon Bonds," Working Paper, University of Paris 6, Paris, France.

Fernique, X., P. W. Millar, D. W. Stroock, and M. Weber (1983): *Ecole d'Eté de Probabilités de Saint-Flour XI – 1981,* volume 976 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, Germany.

Goldys, B., M. Musiela, and D. Sondermann (1994): "Log-Normality of Rates and Term Structure Models," Working Paper, University of New South Wales, Sydney, Australia.

Harrison, M. J. and D. M. Kreps (1979): "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20:381–408.

Harrison, M. J. and S. R. Pliska (1981): "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and their Applications*, 11:215–260. Addendum: Harrison and Pliska (1983).

——— (1983): "A Stochastic Calculus Model of Continuous Trading: Complete Markets," *Stochastic Processes and their Applications*, 15:313–316.

Heath, D., R. Jarrow, and A. J. Morton (1992): "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 60(1):77–105.

Ho, T. S., R. C. Stapleton, M. G. Subrahmanyam, and C. Thanassoulas (1994): "A Two-Factor Model for the Risk Management of Interest Rate Derivatives," Working Paper, Dept. of Accounting and Finance, The Management School, Lancaster University, Lancaster La1 4YX, UK.

Hogan, M. and K. Weintraub (1993): "The Log-Normal Interest Rate Model and Eurodollar Futures," Working Paper, Citibank, New York, New York, USA.

Jamshidian, F. (1990): "Bond and Option Evaluation in the Gausian Interest Rate Model," Working Paper, Financial Strategies Group, Merrill Lynch Capital Markets, World Financial Center, North Tower, New York, New York 10281–1315, USA.

Käsler, J. (1991): *Optionen auf Anleihen,* Ph.D. dissertation, Universität Dortmund, Germany.

Miltersen, K. R. (1993): "An Empirical Study of the Term Structure of Interest Rates," *Scandinavian Journal of Management*, 9:S29–S46. Supplementary Issue: Proceedings of the Nordic Symposium on Contingent Claims Analysis in Finance, 1992.

——— (1994): "An Arbitrage Theory of the Term Structure of Interest Rates," *The Annals of Applied Probability*, 4(4):953–967.

Morton, A. J. (1988): "Arbitrage and Martingales," Technical Report 821, School of Operations Research and Industrial Engineering, College of Engineering, Cornell University, Ithaca, New York 14853, USA.

Müller, S. (1985): *Arbitrage Pricing of Contingent Claims,* Springer-Verlag, New York, New York, USA.

Musiela, M. (1994): "Nominal Annual Rates and Log-Normal Volatility Structure," Working Paper, School of Mathematics, University of New South Wales, Sydney, Australia.

Rady, S. and K. Sandmann (1994): "The Direct Approach to Debt Option Pricing," *The Review of Futures Markets*, 13(2):461–514.

Sandmann, K. and D. Sondermann (1993): "A Term Structure Model and the Pricing of Interest Rate Derivatives," *The Review of Futures Markets*, 12(2):391–423.

——— (1994): "On the Stability of Log-Normal Interest Rate Models," Discussion Paper No. B–263, Department of Statistics, University of Bonn, Adenauerallee 24–42, D-53113 Bonn, Federal Republic of Germany.

Sandmann, K., D. Sondermann, and K. R. Miltersen (1994): "Closed Form Term Structure Derivatives in a Heath-Jarrow-Morton Model with Log-Normal Annually Compounded Interest Rates," Discussion Paper No. B–285, Department of Statistics, University of Bonn, Adenauerallee 24–42, D-53113 Bonn, Federal Republic of Germany.

Department of Management, Odense Universitet, Campusvej 55, DK-5230 Odense M, Denmark

E-mail address: krm@busieco.ou.dk

Department of Statistics, Rheinische Friedrich-Wilhelms-Universität Bonn, Adenauerallee 24–42, D-53113 Bonn, Germany

E-mail address: sandmann@addi.or.uni-bonn.de

Department of Statistics, Rheinische Friedrich-Wilhelms-Universität Bonn, Adenauerallee 24–42, D-53113 Bonn, Germany

E-mail address: sondermann@addi.or.uni-bonn.de