MARKET VOLATILITY AND FEEDBACK EFFECTS FROM DYNAMIC HEDGING[1]

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Abstract

In this paper we analyze the manner in which the demand generated by dynamic hedging strategies affects the equilibrium price of the underlying asset. We derive an explicit expression for the transformation of market volatility under the impact of such strategies. It turns out that volatility increases and becomes time and price dependent. The strength of these effects however depends not only on the share of total demand that is due to hedging, but also significantly on the heterogeneity of the distribution of hedged payoffs. We finally discuss in what sense hedging strategies derived from the assumption of constant volatility may still be appropriate even though their implementation obviously violates this assumption.

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1 Introduction

Standard derivative pricing theory is based on arbitrage arguments, which in turn rest on three key hypotheses about the markets for the underlying asset. Markets are assumed to be complete, frictionless and perfectly elastic. Clearly, this is a very stylized view of real financial markets, in which these assumptions are satisfied only up to a certain extent. This is why a rapidly growing literature has concentrated on the implications of relaxing one or more of them. In this paper we drop the elasticity assumption and study the manner in which the demand generated by dynamic hedging strategies affects the underlying asset's equilibrium price, in particular its volatility structure.

These hedging strategies are derived from specific assumptions on the stochastic law that governs the underlying's price dynamics. In practice they are seen both as a theoretical valuation concept and, more importantly for our analysis, as a device to manage risk as incurred for instance by selling OTC derivative contracts.

We believe an analysis of the feedback effects caused by dynamic hedging in imperfectly elastic markets to be important for a number of reasons. To begin with, when carried out

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on a large scale, dynamic hedging is most likely to perturb the very stochastic law it is based upon. We ask how hedging strategies perform when we allow the underlying's price process to be affected by their implementation, even if this effect is not fully taken into account in designing them. Moreover, hedging is mostly used to replicate payoffs that are convex functions of the underlying asset's price, requiring the investor to sell shares of the underlying asset when its price declines and to buy when its price goes up.[1] Therefore one should expect an increase of market volatility under the impact of such trading behaviour. Thus dynamic hedging is likely to have a destabilizing effect on prices.

There have been a number of studies on the impact of dynamic hedging on the price of the underlying asset. Grossman (1988) focuses on informational differences between buying an option and running the corresponding replicating strategy. Gennotte and Leland (1990) study the effects of portfolio insurance in a model with asymmetric information similar for instance to the one considered by Grossman and Stiglitz (1980). They find that the better this activity is understood by the other market participants, the weaker is the effect of hedging.

Jarrow (1994) analyzes, in a discrete-time model, how standard option pricing theory can be extended to a situation where there is a feedback effect from the demand of a "large trader" on the underlying price process. Platen and Schweizer (1994) use the feedback effect of portfolio insurance to explain the "smile pattern" of implied volatilities that is observed in practice. Their model relies to a large extent on our previous paper (Frey and Stremme 1994).

Brennan and Schwartz (1989) address an issue very similar to the one discussed here. They analyze the transformation of market volatility under the impact of portfolio insurance. They consider a finite-horizon economy in which securities are traded continuously but consumption takes place only at the terminal date. Agents are hence only concerned about the long-term prospects of the asset. Since the risky asset's terminal value is entirely determined by an exogeneously given random variable, which is interpreted as the fundamental value of the asset, agents' expectations are solely driven by the successively revealed information about the value of this state variable. In particular, agents do not alter their expectations in reaction to changes in current price. Markets are thus very liquid, causing the feedback effect of hedging on volatility to be relatively small.

Empirical evidence however suggests that in many situations there is "an enormous amount of short-term position taking", whereas the funds dedicated to long-term investment are limited by uncertainty and agents' risk aversion; see for instance Goodhart (1988). A theoretical justification for this kind of trading behaviour is given by De Long, Shleifer, Summers, and Waldmann (1990a). Moreover, when making trading decisions for very short periods, like in intra-day dealing, investors seem to rely more on the information conveyed by current price movements than on the long-term fundamental prospects of the assets. This "Keynesian" view of investment is supported by evidence reported by the Group of Ten (1993) or Allen and Taylor (1992). In fact there even seems to be a positive feedback effect of current price changes on expectations, see De Long, Shleifer, Summers,

[1] This kind of trading behaviour is also referred to as "Portfolio Insurance".

and Waldmann (1990b).

In the present paper we develop a framework in which the effect of dynamic hedging on the underlying asset's price process can be studied. We start by constructing a general discrete-time temporary equilibrium model in which short-term investment can be modelled. To get a clearer picture of the equilibrium price process, and in particular of its volatility, we then pass to a limiting continuous-time diffusion. This approach to the construction of diffusion models for asset prices was first proposed by Föllmer and Schweizer (1993).

As a special case we consider an economy populated by traders whose preferences over future wealth exhibit constant relative risk-aversion as in the Brennan-Schwartz study. Agents take changes in current prices as signals for future price movements. If they were the only traders in the market the equilibrium price process would be a geometric Brownian motion as in the classical Black-Scholes option pricing model. However, if they interact with program traders who are running dynamic hedging strategies, the structure of the equilibrium price process becomes more complex: while it still can be represented as an Itô process, its volatility increases and becomes time and price dependent. A comparison reveals that the increase in volatility is much more pronounced in our study than that observed by Brennan and Schwartz (1989). This finding underlines the importance of agents' expectations in determining the liquidity of the market. Moreover it illustrates that there exist realistic scenaros in which the effect of hedging is far larger than predicted by Brennan and Schwartz.

We derive an explicit expression for the transformation of market volatility under the impact of hedging. We use this transformation rule to study in particular the feedback effects generated by the strategies derived from the classical Black-Scholes formula. It also allows us to study the importance of different payoff structures being hedged. We show that increasing heterogeneity of the distribution of hedged contracts reduces both level and price sensitivity of volatility.

As reported by the Group of Ten (1993) the effects predicted by our analysis are indeed observed in practice:

> "[T]he existence of options and related dynamic hedging could increase volatility, especially in the smaller and less liquid currency segments, as the spot exchange rate approaches the strike price. When strike prices and/or option maturities are highly concentrated, a large volume of one-way hedging could occur in a short period. Market participants reported that sharp [...] movements in spot prices were frequently observed as a result of such concentrations."

Price dependent volatility, as results from hedging in our model, causes problems in practical application. While hedging strategies can still be shown to exist they can no longer be calculated explicitly. This is why most practioners rely on the classical Black-Scholes formula. Using an idea of El Karoui, Jeanblanc-Picqué, and Shreve (1995) we are able to show that simple strategies derived from a constant-volatility Black-Scholes model are still sufficient to completely hedge the risk incurred by selling OTC derivatives. This remains true even if their implementation causes the actual volatility to be price dependent, as is the case in our model. However, the misperception of the feedback effect on volatility generates a "tracking error": the terminal value of the hedge portfolio might exceed the payoff it was supposed to replicate, thus requiring an initial "over-investment" in the strategy. But again, heterogeneity proves to be beneficial: the tracking error and hence the over-investment diminishes with increasing heterogeneity of the distribution of hedged payoffs.

The remainder of the paper is organized as follows. In Section 2 we develop the general discrete-time temporary equilibrium framework. We then specify a concrete sample economy in which agents whose preferences exhibit constant relative risk-aversion interact with program traders who run dynamic hedging strategies. Section 3 is devoted to the passage to the continuous-time limit. We characterize the limiting price process as the solution of an Itô type Stochastic Differential Equation, thus obtaining an explicit expression for its volatility. In Section 4 we conduct a detailed study of the feedback effect caused by the implementation of Black-Scholes hedging strategies and directly compare our findings to those of Brennan and Schwartz (1989). Section 5 concludes.

2 The Discrete-Time Model

We consider a sequence of discrete-time infinite-horizon economies. More precisely, for each $n = 1, 2, \ldots$ there is a sequence of times $0 = t_0^n < t_1^n < \ldots < t_k^n < \ldots$ at which trading takes place on a Walrasian market. Since we are mainly interested in the continuous-time limit we assume that

$$\Delta^n := \sup_k (t_{k+1}^n - t_k^n) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Traded Assets: There are two assets in the economy, a riskless one (typically a bond or money market account), and a risky one (typically a stock or foreign exchange rate). We take the riskless asset as numéraire, thereby making interest rates implicit in our model. Moreover, we assume the market for the riskless security to be perfectly elastic. This is an idealization of the fact that money markets are far more liquid than those for the typical risky asset considered here. The equilibrium price at time t_k^n of the risky asset, accounted in units of the numéraire, is denoted by X_k^n.

In the present paper we are only interested in the feedback effect of hedging on the underlying's volatility, and not in developing a *pricing* theory. In finitely liquid markets it is no longer obvious how to derive option prices from the prices of the underlying. To avoid the price inconsistencies that can arise from an inadequate modelling of the relationship between stock and options markets we assume that there is no liquid market for options on the risky asset.

Aggregate Demand Schedule: At any time t_k^n the aggregate demand function for the risky asset is assumed to be given by a smooth function $G : [0, \infty) \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ in the form

$$x \mapsto G^n(t_k^n, F_k^n, x). \tag{2.1}$$

Here, x is the (proposed) Walrasian price. $(F_k^n)_{k=0,1,\dots}$ is a stochastic process describing the current state of the economy, to be specified in more detail later. Note that the above form of the demand function implies in particular that all the information necessary for the investors to form their demand can be summarized in F_k^n and x.

EQUILIBRIUM: We normalize total supply of the risky asset to one, hence the equilibrium price X_k^n at time t_k^n is determined by the *market clearing equation*

$$G^n(t_k^n, F_k^n, X_k^n) \equiv 1. \tag{2.2}$$

The following assumptions are of technical nature. They ensure existence and uniqueness of equilibria and guarantee convergence of the equilibrium price processes (see Section 3). We will see later how these assumption can be achieved in a more concrete specification of the economy (see Section 2.1 and Corollary 3.3).

Assumption (A.1) *The demand functions G^n are smooth, and the sequence $\{G^n\}_{n=1,2,\dots}$ converges uniformly on compacts to a smooth function $G : [0,\infty) \times \mathbb{R}^2_{++} \to \mathbb{R}$. Moreover,*

- *(i) For every pair $(t, f) \in [0,\infty) \times \mathbb{R}_{++}$ the equations $G^n(t, f, x) = 1$ and $G(t, f, x) = 1$ have exactly one solution in x, denoted by $\psi^n(t, f)$ and $\psi(t, f)$, respectively.*
- *(ii) For every compact set $K \subset\subset \mathbb{R}^2_{++}$ the sequence $\{\psi^n\}_{n=1,2,\dots}$ is relatively compact in the space of all bounded functions on K endowed with the supremum norm.*
- *(iii) For any fixed t and f, the derivatives of G^n satisfy "in equilibrium":*

$$\frac{\partial G^n}{\partial x}(t,f,x)\bigg|_{x=\psi^n(t,f)} < 0 \quad \text{and} \quad \frac{\partial G^n}{\partial f}(t,f,x)\bigg|_{x=\psi^n(t,f)} > 0,$$

and the analogous statements hold for the limit function G.

REMARKS: Note that (i) guarantees that there is a unique solution to the market clearing equation in each discrete-time economy characterized by G^n as well as in the continuous-time limit economy characterized by G. The first inequality in (iii) together with the Implicit Function Theorem implies in particular that the solution in the limit economy depends smoothly on f, i.e. the function $\psi : [0,\infty) \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ is smooth. The second inequality in (iii) implies in addition that for fixed t the mapping $f \mapsto \psi(t, f)$ is invertible, i.e. there exists a smooth function $\psi^{-1} : [0,\infty) \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ such that $\psi^{-1}(t, \psi(t, f)) = f$ and $\psi(t, \psi^{-1}(t, x)) = x$ for all t, f and x. Finally note that if the G^n are differentiable, then so are the ψ^n by (iii). In this case the Arzéla-Ascoli Theorem implies that (ii) holds if the ψ^n and their first derivatives are uniformly bounded on compacts.

AGENTS: There are two groups of agents in the market, called "reference traders" and "program traders", respectively. The economy in which there are only reference traders active constitutes the benchmark case for our analysis. It will be compared with the case in which reference traders interact with program traders, who are running dynamic hedging strategies. We neglect the aggregation problem and specify a representative reference trader, whose demand function for the risky asset at time t_k^n is assumed to take the form

$$x \mapsto D^n(F_k^n, x). \tag{2.3}$$

A more detailed specification in which the reference trader's preferences exhibit constant relative risk-aversion will be given in Section 2.1.

A typical program trader might be a bank hedging a portfolio of written OTC contracts by running a dynamic trading strategy in the underlying asset. Since the majority of the demand for such contracts is motivated by considerations beyond the scope of our model,[2] we take the extreme view that the hedging objectives of our program traders are exogenously given. Moreover, the hedging strategy for a portfolio of payoffs is just the portfolio of the hedging strategies for the individual payoffs. Thus we can concentrate on a representative program trader, whose demand function for the risky asset at time t_k^n takes the form

$$x \mapsto \rho \cdot \phi^n(t_k^n, x). \tag{2.4}$$

Here, ϕ^n is a normalized strategy function and $0 \leq \rho < 1$ is the fraction of the market portfolio that is being managed by program traders. We make the following assumptions on the strategy functions ϕ^n:

Assumption (A.2) *The functions ϕ^n are smooth, and the sequence $\{\phi^n\}_{n=1,2,\ldots}$ converges uniformly on compacts to a smooth function $\phi : [0,\infty) \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$. Moreover,*

(i) for every compact set $K \subset [0,\infty) \times \mathbb{R}_{++}$ we have

$$\sup_n \sup_{(t,x)\in K} \left| \frac{\partial \phi^n}{\partial t}(t,x) \right| < \infty,$$

(ii) ϕ^n is increasing in the underlying price, i.e.

$$\frac{\partial \phi^n}{\partial x}(t,x) > 0 \quad \text{for all } t \geq 0 \text{ and all } x \in \mathbb{R}_{++},$$

(iii) and ϕ^n is normalized in such a way that

$$\sup_{t,x} |\phi^n(t,x)| = 1.$$

Finally, the limit ϕ satisfies the analogous conditions to (ii) and (iii).

The total aggregate demand function will hence be of the form

$$G^n(t,f,x) = D^n(f,x) + \rho\phi^n(t,x). \tag{2.5}$$

Note that by (iii) ρ can indeed be interpreted as the fraction of the total supply of the risky asset that is subject to portfolio insurance; in a slight abuse of language we simply refer to ρ as *market weight* of program traders. Also note that (ii) reflects the fact that a typical hedging strategy requires that shares of the underlying be sold when its price has declined and vice versa, as was mentioned in the introduction.

[2] The Group of Ten (1993) for example reports that "derivative instruments were primarily used for risk hedging purposes".

Assumption (A.2) is satisfied in particular if ϕ^n and ϕ are mixtures of hedging strategies as given by the Black-Scholes formula. It is also possible to consider strategies ϕ^n derived from the discrete state-space model of Cox, Ross, and Rubinstein (1979). Convergence of such strategies to their continuous-time counterpart ϕ follows, for instance, from results by He (1990).

2.1 A Case with Constant Relative Risk Aversion

In this section we provide a concrete specification of the preferences and beliefs of the representative reference trader. The assumptions introduced in this section ensure that the aggregate demand functions of this particular economy satisfy Assumption (A.1) (see Corollary 3.3 below). The model considered here is closely related to the kind of temporary equilibrium models discussed by De Long, Shleifer, Summers, and Waldmann (1990a). Consider an overlapping generations model without bequests, in which agents live for two periods. When young, the representative reference trader receives an exogenous stochastic income F_k^n which she invests in the available assets. When old, she just consumes all her wealth and then disappears from the market. Thus, at any time t_k^n the young agent chooses the number d of shares of the risky asset she wants to hold in order to maximize expected utility of next period's wealth. Given her income is f, her demand function will be

$$D^n(f,x) = \arg\max_{d \geq 0} E\left[\ u\left(f + d \cdot (\tilde{X}_{k+1}^n(x) - x)\right)\ \right],$$

where u is her von Neumann-Morgenstern utility function and $\tilde{X}_{k+1}^n(x)$ the agent's belief about next period's price. Note here that we allow the expected future price to depend explicitly on the current price x, i.e. agents may update their expectations in reaction to changes in current prices.

Of course this overlapping generations scenario must not be taken literally. It is a stylized model of a market where agents' investment decisions are made sequentially over time and where each decision is determined mainly by myopic optimization. For example, market participants might be managers of investment funds who are managing a stochastically fluctuating amount of funds. Typically fund managers are (at least partly) compensated according to the performance of their portfolio, evaluated at certain predetermined dates. Therefore their investment decisions are often predominantly aimed at the next evaluation date.

Assumption (A.3) *Reference traders' beliefs and preferences and the evolution of their income over time are assumed to be given as follows:*

(i) The representative reference trader's preferences exhibit constant relative risk aversion, i.e. her von Neumann-Morgenstern utility function u satisfies $u'(z) = z^{-\gamma}$ for some $\gamma > 0$.

(ii) Given current price x, the agent believes next period's price $\tilde{X}_{k+1}^n(x)$ to be of the form $\tilde{X}_{k+1}^n(x) = x \cdot \xi_k^n$ for some random variable ξ_k^n. We assume $(\xi_k^n)_{k=0,1,\ldots}$ to be serially independent and independent of x.

(iii) Given current income F_k^n, next period's income F_{k+1}^n is given by $F_{k+1}^n = F_k^n \cdot \zeta_k^n$ for some random shock $\zeta_k^n > 0$. We assume $(\zeta_k^n)_{k=0,1,\ldots}$ to be serially independent.

Note that by (ii) there is a *positive feedback* from current price x into agents' expectations: after a rise of x they anticipate a rise in future prices and in the case of a price decline they expect future prices to fall as well. A list of striking observations which emphasize the importance of such extrapolative expectations on financial markets has been compiled by De Long, Shleifer, Summers, and Waldmann (1990b). We will see in Section 4.3 that this way of expectation formation leads to destabilizing effects of dynamic hedging which are much larger than those observed in the framework of Brennan and Schwartz (1989).

By Assumption (A.3) the solution to the agent's utility maximization problem, given income f and proposed price x, is uniquely determined by the first order condition

$$
\begin{aligned}
(2.6) \qquad 0 &= E\left[\, u'\left(f + d \cdot (\tilde{X}_{k+1}^n(x) - x)\right) \cdot (\tilde{X}_{k+1}^n(x) - x)\,\right] \\
&= E\left[\, (f + d \cdot x \cdot (\xi_k^n - 1)\,)^{-\gamma} \cdot x \cdot (\xi_k^n - 1)\,\right].
\end{aligned}
$$

As an immediate consequence of this characterization we get

Lemma 2.1 *Under Assumptions (A.3) the representative reference trader's demand function D^n has the following properties:*

(i) D^n is homogenous of degree one w.r.t. f, formally: $D^n(\alpha f, x) = \alpha D^n(f,x)\,\forall \alpha$.

(ii) D^n is homogenous of degree zero w.r.t. (f,x), formally: $D^n(\alpha f, \alpha x) = D^n(f,x)\,\forall\alpha$.

In particular, using these homogeneities we find

$$
(2.7) \qquad D^n(f,x) = D^n(x \cdot \frac{f}{x}, x) = \frac{f}{x} \cdot D^n(1,1) =: \frac{f}{x} \cdot D_*^n.
$$

Equilibrium without Program Traders: In the absence of program traders the market clearing equation takes the form

$$
(2.8) \qquad 1 = D_*^n \cdot \frac{F_k^n}{X_k^n} \qquad \Longrightarrow \qquad X_k^n = D_*^n \cdot F_k^n.
$$

Using Assumption (A.3) (iii) this implies $X_{k+1}^n = D_*^n \cdot F_{k+1}^n = D_*^n \cdot F_k^n \cdot \zeta_k^n = X_k^n \cdot \zeta_k^n$. We summarize this in the following

Lemma 2.2 *In the economy specified in Assumption (A.3) without program traders the equilibrium price process $(X_k^n)_{k=0,1,\ldots}$ is given by*

$$
(2.9) \qquad X_{k+1}^n = X_k^n \cdot \zeta_k^n.
$$

In particular, expectations are rational if and only if $\xi_k^n \equiv \zeta_k^n$ for all k.

As an example for income dynamics consider the sequence

$$
\zeta_k^n = \exp\left((\mu - \frac{1}{2}\eta^2)(t_{k+1}^n - t_k^n) + \eta\sqrt{t_{k+1}^n - t_k^n} \cdot \varepsilon_{k+1}^n\right),
$$

where $(\varepsilon_k^n)_{k=0,1,\ldots}$ is an i.i.d. sequence of random variables. If, for instance, the ε_k^n are standard-normally distributed, then prices follow a discretized geometric Brownian Motion. If, on the other hand, the ε_k^n are just the increments of a random walk, then prices are given by a geometric random walk as in Cox, Ross, and Rubinstein (1979).

Equilibrium with Program Traders: In the presence of program traders the market clearing equation becomes

$$1 = D_*^n \cdot \frac{F_k^n}{X_k^n} + \rho \cdot \phi^n(t_k^n, X_k^n). \tag{2.10}$$

In order to ensure existence of a unique equlibrium we have to make an additional technical assumption. It will turn out later that this is mainly a restriction on the market weight ρ of program traders (see Section 4 below).

Assumption (A.4) *There exists a constant $K > 0$ such that for each n*

$$1 - \rho \cdot \phi^n(t,x) - \rho \cdot x \frac{\partial \phi^n}{\partial x}(t,x) \geq K \qquad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}_{++}. \tag{2.11}$$

Moreover, we require (2.11) also to hold for the limit function ϕ.

Proposition 2.3 *Given Assumption (A.4), there exists for every market weight $\rho \in (0,1)$ a unique equilibrium in the economy specified in Assumptions (A.3) and (A.2).*

Remarks: As the proof of Proposition 2.3 below shows, condition (2.11) guarantees that in equilibrium the aggregate demand function is strictly *decreasing* in x, such that equilibrium prices will depend smoothly on the reference trader's income F_k^n. This rules out price jumps and "crashes" of the kind discussed by Gennotte and Leland (1990) and Schönbucher (1993). Moreover, (2.11) ensures that the equilibrium is stable under the usual Walrasian tatônnement process. We believe that this is an important feature of our model, which contrasts with the analysis of Platen and Schweizer (1994). In order for them to explain volatility smiles by feedback effects from dynamic hedging they have to assume an (excess) demand function for their reference traders that is *increasing* in price. Not only does this result in the equilibrium being unstable, it even gives rise to arbitrage opportunities for the program traders in the sense of Jarrow (1994).

Proof: Existence of an equilibrium follows from continuity of the demand functions and

$$\limsup_{x \to \infty} \left(D^n(f,x) + \rho \phi^n(t,x) \right) \leq \rho < 1 \quad \text{and} \quad \lim_{x \to 0} \left(D^n(f,x) + \rho \phi^n(t,x) \right) = +\infty$$

for all $t, f > 0$. To prove uniqueness it is sufficient to show that $\frac{\partial}{\partial x}\left(D^n(f,x) + \rho \phi^n(t,x) \right) < 0$ whenever t, f and x solve the market clearing equation (2.10). A direct computation using Lemma 2.1 gives

$$\frac{\partial}{\partial x}\left(D^n(f,x) + \rho \phi^n(t,x) \right) = -\frac{1}{x} \cdot \left(D^n(f,x) - \rho \cdot x \frac{\partial \phi^n}{\partial x}(t,x) \right).$$

The term in the brackets on the right-hand side equals $1 - \rho\phi^n(t,x) - \rho \cdot x \frac{\partial \phi^n}{\partial x}(t,x)$ in equilibrium, which is positive by Assumption (A.4). □

3 The Continuous-Time Model

In order to get a clearer picture of the equilibrium price process and in particular of its volatility structure, we will now pass to the limiting continuous-time model. This also brings us closer to the original Black-Scholes model. To maintain a maximal level of generality we turn back to the situation described in the beginning of Section 2. That is, we assume a general aggregate demand function of the form (2.1) satisfying Assumption (A.1). For each $n = 1, 2, \ldots$ let $(X_k^n)_{k=0,1,\ldots}$ be the unique equilibrium price process, i.e. solution to the market clearing equation (2.2), which we know exists by Assumption (A.1) (i).

In order to formulate our results we need a common base space on which the distributions of all these processes can be compared. Let $\mathcal{D}^d[0,\infty)$ denote the d-dimensional *Skorohod-Space*; cf. Jacod and Shiryaev (1987). We identify any sequence $\xi_0^n, \xi_1^n, \ldots$ defined for times $t_0^n, t_1^n, \ldots$ with the RCLL function

$$\xi_t^n := \sum_{k=0}^{\infty} \xi_k^n \cdot 1_{\{t_k^n \le t < t_{k+1}^n\}}.$$

Let $(X_t^n)_{t\geq 0}$ and $(F_t^n)_{t\geq 0}$ denote the RCLL versions of $(X_k^n)_{k=0,1,\ldots}$ and $(F_k^n)_{k=0,1,\ldots}$, respectively. For the passage to the limit we require the state variable processes $(F_k^n)_{k=0,1,\ldots}$ to converge to a continuous-time limit. Remember that for the CRRA case without program traders the equilibrium price process $(X_k^n)_{k=0,1,\ldots}$ is proportional to $(F_k^n)_{k=0,1,\ldots}$; cf. Lemma 2.2. Since our objective is to study the effect of hedging in a Black-Scholes type environment we assume that the limiting state variable process is a geometric Brownian Motion.[3]

Assumption (A.5) *Suppose that the sequence $\{F^n\}_{n=1,2,\ldots}$ of state variable processes converges in distribution to a geometric Brownian Motion with constant drift and volatility parameters μ and η, respectively.*

We are now ready to state the main result of this section:

Theorem 3.1 *Suppose the sequences $\{G^n\}_{n=1,2,\ldots}$ and $\{F^n\}_{n=1,2,\ldots}$ of aggregate demand functions and state variable processes satisfy Assumptions (A.1) and (A.5), respectively.*

Then the sequence $\{X^n\}_{n=1,2,\ldots}$ of equilibrium price processes converges in distribution, and the limit distribution is uniquely characterized as the law of the solution $(X_t)_{t\geq 0}$ of the SDE

$$(3.12)\quad X_t = X_0 - \int_0^t \left(\frac{\frac{\partial G}{\partial f}(s, F_s, X_s)}{\frac{\partial G}{\partial x}(s, F_s, X_s)} \eta F_s \right) dW_s - \int_0^t \left(\frac{\frac{\partial G}{\partial t}(s, F_s, X_s)}{\frac{\partial G}{\partial x}(s, F_s, X_s)} + \frac{\frac{\partial G}{\partial f}(s, F_s, X_s)}{\frac{\partial G}{\partial x}(s, F_s, X_s)} \mu F_s - \frac{1}{2} H(s, F_s, X_s) \eta^2 F_s^2 \right) ds,$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process, $(F_t)_{t\geq 0}$ is just short for $F_t = \psi^{-1}(t, X_t)$, and H is a smooth function that depends only on first and second order derivatives of G. In

[3] The main result of this section, Theorem 3.1, easily carries over to more general diffusion processes.

particular, the instantaneous volatility of the process $(X_t)_{t\geq 0}$ at time t is given by

$$(3.13)\qquad v(t,X_t)\cdot\eta := -\frac{\frac{\partial G}{\partial f}(t,\psi^{-1}(t,X_t),X_t)}{\frac{\partial G}{\partial x}(t,\psi^{-1}(t,X_t),X_t)}\cdot\frac{\psi^{-1}(t,X_t)}{X_t}\cdot\eta.$$

Proof: By Assumption (A.1) (i) the equilibrium price process in economy n is given by $X_t^n = \psi^n(\tau_t^n, F_t^n)$, where $\tau_k^n = t_k^n$ if $t_k^n \leq t < t_{k+1}^n$. We first show that the ψ^n converge:

Lemma 3.2 *The sequence $\{\psi^n\}_{n=1,2,\ldots}$ converges uniformly on compacts to the smooth function ψ defined by the relation $G(t,f,\psi(t,f))\equiv 1$.*

Smoothness of ψ follows from the Implicit Function Theorem and Assumption (A.1) (iii). Convergence is shown in Appendix A.1. By Assumption (A.5) the sequence $\{F^n\}_{n=1,2,\ldots}$ converges in distribution by to a process $(F_t)_{t\geq 0}$ satisfying

$$(3.14)\qquad F_t = F_0 + \int_0^t \eta F_s\, dW_s + \int_0^t \mu F_s\, ds$$

for some standard Wiener process $(W_t)_{t\geq 0}$. A version of the Continuous Mapping Theorem then implies convergence in distribution of the triplets of processes $(\tau_t^n, F_t^n, \psi^n(\tau_t^n, F_t^n))$ to $(t, F_t, \psi(t,F_t))$. To characterize the limiting distribution on $\mathcal{D}^3[0,\infty)$ we apply Itô's Lemma to $\psi(t,F_t)$ and use (3.14) to obtain

$$\begin{aligned}\psi(t,F_t) &= \psi(0,F_0) + \int_0^t \left(\frac{\partial\psi}{\partial f}(s,F_s)\eta F_s\right) dW_s \\ &\quad + \int_0^t \left(\frac{\partial\psi}{\partial t}(s,F_s) + \frac{\partial\psi}{\partial f}(s,F_s)\mu F_s + \frac{1}{2}\frac{\partial^2\psi}{\partial f^2}(s,F_s)\eta^2 F_s^2\right) ds.\end{aligned}$$

By differentiating the defining equation $G(t,f,\psi(t,f))\equiv 1$, the derivatives of ψ can be expressed in terms of those of G, proving that $X_t = \psi(t,F_t)$ indeed solves the SDE (3.12). Expression (3.13) for the volatility is obtained by simply plugging $F_t = \psi^{-1}(t,X_t)$ back into (3.12). To complete the proof note that the drift and dispersion functions in equation (3.12) are smooth by assumption and thus locally Lipschitz. This implies pathwise uniqueness of (3.12) and hence uniqueness in distribution. □

We now relate the concrete specification with CRRA utility as outlined in Assumptions (A.3) and (A.2) to the general situation characterized by Assumption (A.1), and deduce the shape of the volatility in this special case.

Corollary 3.3 *Suppose that the sequence of discrete-time economies specified in Assumptions (A.3) and (A.2) satisfies in addition Assumption (A.4). If the D_*^n from (2.7) converge to some D_* as $n\longrightarrow\infty$, then the corresponding sequence $\{G^n\}_{n=1,2,\ldots}$ of demand functions satisfies (A.1) with a limiting demand function G of the form*

$$G(t,f,x) = D_*\cdot\frac{f}{x} + \rho\cdot\phi(t,x).$$

Hence, under (A.5) the corresponding sequence of equilibrium price processes $\{X^n\}_{n=1,2,\ldots}$ *by Theorem 3.1 converges in distribution to a continuous-time diffusion process* $(X_t)_{t\geq 0}$ *whose instantaneous volatility at any time* t *is given by*

$$v(t, X_t) \cdot \eta := \frac{1 - \rho\phi(t, X_t)}{1 - \rho\phi(t, X_t) - \rho X_t \frac{\partial \phi}{\partial x}(t, X_t)} \cdot \eta. \tag{3.15}$$

In particular, volatility is increasing *in the market weight* ρ *of program traders, bounded below by the "reference volatility"* η *and bounded above by* η/K*. Note also that in the absence of program traders, i.e. when* $\rho = 0$*,* $v(t, X_t) \equiv 1$*. Market volatility then equals the volatility* η *of the exogenous state variable process* $(F_t)_{t\geq 0}$*.*

Proof: The convergence of G^n to G is obvious by the assumed convergence of D_*^n to D_* and (A.2). It is easily seen that (A.4) implies (A.1) (i) and (iii). Finally (A.1) (ii) follows since the ψ^n and their first derivatives are uniformly bounded on compacts by (A.4) and (A.2) (i). The form of the volatility function is derived in Appendix A.1. □

Remarks: From (3.13) it can be seen that under some technical assumptions on the aggregate demand function G the resulting continuous-time model will still be complete (see for instance Duffie (1992, Section 2)). However, had we allowed the aggregate hedging function ϕ or the weight ρ to depend on some exogenous uncertainty, we would have typically ended up with an incomplete model in which volatility is stochastic. Such models recently have become a focus of attention, for example Föllmer and Schweizer (1991) and Hull and White (1987).

4 Feedback-Effects from Black-Scholes Trading

Price dependent volatility—as generated by dynamic hedging in our model—causes major problems in practical applications of option pricing theory. Although hedging strategies may still be shown to exist they can in most cases no longer be calculated explicitly. This is why in practice most investors base their trading on the classical Black-Scholes Formula, which postulates constant volatility. In this section we therefore study in more detail the feedback effect generated by the corresponding strategies and analyze the extent to which they are still appropriate, when the effect of their implementation on prices is taken into account.

We work directly in the limiting diffusion model, because the explicit expression for the volatility faciliates the analysis. Therefore we contend ourselves with specifying properties only of the limiting demand function G which we assume to be of the form $G = D + \rho\phi$ with ϕ being a mixture of Black-Scholes trading strategies. We remark, however, that under Assumption (A.2) the weak convergence of the equilibrium price processes implies the convergence of the corresponding gains from trade; see Duffie and Protter (1992). Hence our results on the performance of hedge strategies are meaningful for the discrete-time models of Section 2, too.

4.1 Hedge Demand Generated by Black-Scholes Strategies

First we want to specify the strategy used by the representative program trader in more detail. As shown by Leland (1980) every convex payoff can be represented as the terminal value of a portfolio consisting of a mixture of European call options and a static position in the underlying. Therefore we concentrate on such portfolios. Consider first the problem of replicating the payoff of one single call option with strike price K and maturity date T. As was shown in the seminal paper by Black and Scholes (1973), if the trading decisions are based on the assumption of the underlying asset price following a geometric Brownian Motion with constant volatility σ, the corresponding price at any time t is given by the solution $c(t, X_t)$ of the terminal value problem

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) c(t,x) \equiv 0, \qquad c(T,x) = [x-K]^+, \tag{4.16}$$

and the corresponding strategy is $\frac{\partial c}{\partial x}(t, X_t)$. We denote the price and strategy functionals for a fixed contract (K,T) by $C(\sigma, K, T-t, x)$ and $\varphi(\sigma, K, T-t, x)$, respectively.[4] The terminal value problem (4.16) is explicitely solvable and the strategy function is given by

$$\varphi(\sigma, K, \tau, x) := \mathcal{N}\left(\frac{\log x - \log K}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}\right), \tag{4.17}$$

where $\mathcal{N}$ is the standard normal distribution function and τ denotes the time to maturity $T-t$. In the sequel we will refer to σ also as the *input volatility*, since it is the volatility used for the computation of hedging strategies.

We assume that the aggregate demand of our representative program trader is independent of time. This models the scenario of many program traders entering and leaving the market at random times, so that the average composition of payoffs being replicated is constant over time. Formally, the representative program trader's demand is given by $\rho\phi(\sigma, x)$ where ρ is the market weight and

$$\phi(\sigma, x) = a + \int_{\mathbb{R}^2_+} \varphi(\sigma, K, \tau, x)\, \nu(dK \otimes d\tau). \tag{4.18}$$

Here, a represents the static position in the underlying and ν is a measure on $\mathbb{R}^2_+$ that describes the distribution of strike prices K and times-to-maturity τ in the portfolio. For convenience we define $\Gamma(\sigma, x) := x\frac{\partial \phi}{\partial x}(\sigma, x)$.[5] We want to show that the feedback effects of portfolio insurance are mitigated if the distribution of strike prices and times to maturity is relatively heterogeneous. To this end we concentrate on the following extreme case:

Assumption (A.6) *ν has a smooth density with respect to the Lebesgue-measure, i.e. ν is of the form $\nu(dK \otimes d\tau) = g(K,\tau)\, dK \otimes d\tau$, where $g : \mathbb{R}_+ \times [0,\infty) \longrightarrow \mathbb{R}_+$ is a smooth density function having compact support in $\mathbb{R}_+ \times [0,\infty)$.*[6]

[4] Note that both value and strategy function depend on current time t and maturity time T only via their difference, the time-to-maturity $\tau = T - t$.

[5] In the standard option pricing theory the price derivative of a strategy function is known as the strategy's "gamma", which motivates this notation.

[6] Note that we explicitly allow $g(K, 0) > 0$ for some K, i.e. arbitrarily small times to maturity.

Next we want to verify that Assumption (A.6) ensures that for ρ sufficiently small there is a unique equilibrium in the economy with CRRA agents (i.e. that Assumption (A.4) holds). Observe first that on the single contract level the function $x\frac{\partial\varphi}{\partial x}(\sigma, K, \tau, x)$ explodes when $x \to K$ and $\tau \to 0$. This corresponds to the well-known fact that option hedging strategies require extremely large changes of the hedge portfolio when the option is at the money and close to maturity. Surprisingly, this problem disappears in the aggregate, if the distribution ν is non-singular. The following proposition shows that bounds on $\Gamma(x, \sigma)$ can be found that depend only on the degree of heterogeneity of the distribution ν:

Proposition 4.1 *Suppose that $\sigma > \underline{\eta}$ for some $\underline{\eta} > 0$. Under Assumption (A.6) we have the following estimates for all $x \in \mathbb{R}_{++}$:*

(*i*)
$$|\Gamma(\sigma, x)| \leq \int_0^\infty \int_0^\infty \left| \frac{\partial}{\partial K} \left(Kg(K, \tau) \right) \right| dK\, d\tau,$$

(*ii*)
$$\left| \frac{\partial}{\partial \sigma} \Gamma(\sigma, x) \right| \leq \frac{2}{\underline{\eta}} \cdot \int_0^\infty \int_0^\infty \left| \frac{\partial^2}{\partial \tau \partial K} \left(\tau K g(K, \tau) \right) \right| dK\, d\tau.$$

The proof is given in Appendix A.2. Because of (i) we can achieve Assumption (A.4) without further restrictions on the distribution ν simply by requiring the porfolio insurance weight ρ not to be too large. Calculations with some sample density functions have shown that any reasonable value for ρ can be permitted.

Remark: By $dK = Kd(\log K)$ the expression on the right-hand side of (i) above can be interpreted as a measure of the heterogeneity of the distribution of logarithmic strike prices, averaged over the time to maturity τ. An inspection of equation (4.19) below reveals that the feedback effect of dynamic hedging on market volatility mainly manifests itself through the appearence of $\Gamma(\sigma, x)$ in the denominator of $v(\sigma, x)$. Hence by (i) we see that this "disturbance" is controlled by the degree of heterogeneity of ν. This is most apparent in the economy with CRRA agents. Here we get from Corrollary 3.3

$$v(\sigma, x) = \frac{1 - \rho\phi(\sigma, x)}{1 - \rho\phi(\sigma, x) - \rho\Gamma(\sigma, x)} \leq \frac{1 - \rho}{1 - \rho - \rho \sup \Gamma(\sigma, x)},$$

such that even the maximal increase in volatility is controlled by the degree of heterogeneity of ν.

4.2 Rational Black-Scholes Trading

We now investigate the extent to which the Black-Scholes formula is still appropriate for the design of hedge strategies in our setting. We work with a limiting demand function of the form[7]

$$G(\sigma, f, x) = D(f, x) + \rho \cdot \phi(\sigma, x),$$

[7] In the subsequent analysis σ is a parameter which does not vary with f or x. Its appearence does not alter the validity of Theorem 3.1.

where $\phi(\sigma, x)$ is as in (4.18). Throughout the rest of this section we require only Assumptions (A.1) and (A.6) to hold. In particular we do not require the specific CRRA utility. By Theorem 3.1 the volatility of the limiting diffusion is

$$v(\sigma, x) \cdot \eta = \frac{\frac{\partial D}{\partial f}\left(\psi^{-1}(x), x\right)}{x \cdot \frac{\partial D}{\partial x}\left(\psi^{-1}(x), x\right) - \rho\Gamma(\sigma, x)} \cdot \psi^{-1}(x) \cdot \eta. \tag{4.19}$$

As a first step, we use an idea of El Karoui, Jeanblanc-Picqué, and Shreve (1995) to derive a formula for the "tracking error". This number measures the difference between the *actual* and the *theoretical* value of a self-financing hedge portfolio for a European call calculated from the Black-Scholes formula with constant volatility σ. Recall that the theoretical value is given by $C_t := C(\sigma, K, T-t, X_t)$. The actual value V_t of the self-financing portfolio defined by initially investing $V_0 = C_0$ and holding $\varphi(\sigma, K, T-t, X_t)$ shares of the underlying at any time $t \leq T$ is given by the cumulative gains from trade, i.e.

$$V_t = V_0 + \int_0^t \varphi(\sigma, K, T-s, X_s)\, dX_s.$$

The *tracking error* e_t is then defined as the difference between actual and theoretical value:

$$e_t := V_t - C_t.$$

Since $C_T = [X_T - K]^+$, e_T measures the deviation of the hedge portfolio's terminal value from the payoff it is supposed to replicate. In particular, if the tracking error is always positive, the terminal value of the hedge portfolio of an investor following the strategy $\varphi(\sigma, K, T-t, X_t)$ always completely covers the option's payoff. The following is a simplified version of Theorem 2 in El Karoui, Jeanblanc-Picqué, and Shreve (1995).

Proposition 4.2 *Suppose that the underlying asset's price follows a diffusion with volatility (4.19). Then the tracking error for a single option is given by*

$$e_t = \frac{1}{2}\int_0^t \left(\sigma^2 - \eta^2 v^2(\sigma, X_s)\right) \cdot X_s^2 \frac{\partial^2 C}{\partial x^2}(\sigma, K, T-s, X_s)\, ds. \tag{4.20}$$

In particular, if $\sigma \geq \eta v(\sigma, x)$, the tracking error is always positive.

Proof: By Itô's Lemma,

$$C_t = \underbrace{C_0 + \int_0^t \varphi(\sigma, K, T-s, X_s) dX_s}_{= V_t}$$
$$+ \int_0^t \left(\frac{\partial C}{\partial t}(\sigma, K, T-s, X_s) + \frac{1}{2}\eta^2 v^2(\sigma, X_s) \cdot X_s^2 \frac{\partial^2 C}{\partial x}(\sigma, K, T-s, X_s)\right) ds.$$

Substituting (4.16) into the above equation yields the desired expression for the tracking error. Moreover, $C(\sigma, K, T-t, x)$ being convex in x, its second derivative is always positive. Hence by (4.20) the sign of the tracking error is entirely determined by the sign of the volatility difference $\sigma^2 - \eta^2 v^2(\sigma, x)$. □

As we have just seen, if the terminal value of the hedge portfolio is to be no smaller than the payoff it is supposed to cover, the input volatility σ used for the computation of the hedging strategy must be no smaller than the actual market volatility. On the other hand, following the Black-Scholes strategy corresponding to a certain input volatility σ requires an initial investment of $C(\sigma, K, T, X_0)$. Since C is increasing in σ, to keep the initial "over-investment" as low as possible, investors should seek to find the smallest such σ. This motivates the following

Definition 4.3 *The constant $\overline{\sigma}$ is called a* super-volatility *if $\overline{\sigma}$ is the smallest positive solution of the equation*

$$\sigma = \sup \{ \eta v(\sigma, x) \, : \, x \in \mathbb{R}_{++} \} . \tag{4.21}$$

Note the recursive structure: Since the choice of the input volatility and hence of the trading strategy affects the actual volatility, σ appears on both sides of (4.21). It will turn out that sufficient for the existence of a super-volatility is the following

Assumption (A.7) *The volatility function (4.19) has the following properties:*

(i) There are constants $0 < \underline{\eta} < \overline{\eta} < \infty$ so that $\underline{\eta} \le v(\sigma, x) \le \overline{\eta}$, $\forall \, \sigma \in [\underline{\eta}, \overline{\eta}]$, $\forall \, x \in \mathbb{R}_{++}$.

(ii) There is a positive constant β so that $\eta \cdot \frac{\partial v}{\partial \sigma}(\sigma, x) \le 1 - \beta$, $\forall \, \sigma \in [\underline{\eta}, \overline{\eta}]$, $\forall \, x \in \mathbb{R}_{++}$.

Essentially (ii) means that variations in the input volatility do not affect the actual market volatility too much. Since $\frac{\partial}{\partial \sigma}\Gamma(\sigma, x)$ is bounded according to Proposition 4.1, this assumption holds as long as ρ is not too large. Of course to check if (i) is fulfilled one has to know the function D. In case of the economy with CRRA agents (i) is implied by Assumption (A.4).

Proposition 4.4 *Suppose that Assumption (A.7) holds. Then the super-volatility in the sense of Definition 4.3 exists and is given by*

$$\overline{\sigma} := \sup\{\sigma^*(x) \, : x \, \in \mathbb{R}_{++}\},$$

where $\sigma^*(x)$ is the unique solution of the fixed point problem $\eta v(\sigma, x) = \sigma$.

Proof: Assumption (A.7) implies that the equation $\eta v(\sigma, x) - \sigma = 0$ has a unique solution $\sigma^*(x)$ for each $x > 0$. Now by definition $\overline{\sigma} \ge \sigma^*(x)$ for any $x \in \mathbb{R}_{++}$. Since by Assumption (A.7) (ii) the mapping $\sigma \mapsto \eta v(\sigma, x) - \sigma$ is strictly decreasing, we get

$$\eta v(\overline{\sigma}, x) - \overline{\sigma} \le \eta v(\sigma^*(x), x) - \sigma^*(x) = 0$$

for all $x \in \mathbb{R}_{++}$, hence $\overline{\sigma}$ is indeed an upper bound for $\eta v(\sigma, x)$. It is also the smallest such bound. This is obvious in the case when the supremum in (4.21) is attained for some x, and it follows from Assumption (A.7) (ii) in the general case. To prove that $\overline{\sigma}$ is also the smallest solution to (4.21), simply note that by definition $\sigma^*(x)$ gets arbitrarily close to $\overline{\sigma}$. Hence for $\tilde{\sigma} < \overline{\sigma}$ there is always an x with $\tilde{\sigma} < \sigma^*(x)$ such that by monotonicity

$$\eta v(\tilde{\sigma}, x) - \tilde{\sigma} > \eta v(\sigma^*(x), x) - \sigma^*(x) = 0.$$

□

REMARK: From a practioner's viewpoint it is reasonable to use a Black-Scholes strategy based on the supervolatility $\overline{\sigma}$ for hedging purposes as long as the difference between $\overline{\sigma}$ and $\inf\{\eta v(\overline{\sigma}, x),\ x \in \mathbb{R}_{++}\}$ and hence the possible initial overinvestment is relatively small. As we will see in the simulations reported below, this issue, and hence the robustness of the Black-Scholes Formula with respect to the feedback from dynamic hedging, depends largely on the heterogeneity of the insured payoffs.

4.3 Comparison with the Brennan and Schwartz study

Using explicit numerical computations (see below for details) we can compare our results to those obtained by Brennan and Schwartz (1989). Table 1 lists the ratios of the volatility in the presence of program traders to that in the benchmark economy without program traders. While the utility functions, and hence notably the risk aversion, of the reference traders are identical in both models, an inspection of Table 1 shows that the effects of program trading on volatility are much stronger in our model. As was explained in the introduction this is due to the different expectation formation of the reference traders in the two models.

4.4 Numerical Computations

First we computed the resulting volatility function $\eta h(\overline{\sigma}, x)$ as a function of x, using as hedging input the super-volatility $\overline{\sigma}$ for a variety of different weights ρ, different reference volatilities η, and different levels of heterogeneity of ν. Figures 1 and 2 show the dramatic effect of heterogeneity. Here, we graphed the reference volatility "$\circ$" and the resulting volatility "$\bullet$" against the current price using a value of $\rho = 10\%$.[8] All numerical results, including those not featured in this paper, support our findings from the qualitative analysis: Volatility increases with the market share ρ of portfolio insurance as well as with reference volatility η. Both the level of increase and the price dependency are reduced by heterogeneity.

We then ran Monte Carlo simulations to generate sample price paths and used the tracking error formula (4.20) to compute the terminal value of a hedging portfolio based on the super-volatility $\overline{\sigma}$. We compared the results to the payoff of the option it was supposed to duplicate, again for a variety of different parameter constellations. Figures 3 and 4 again capture the striking effect of heterogeneity. For every sample path we graphed the terminal value "$\bullet$" of the portfolio against the terminal price of the underlying. The straight line depicts the exact option payoff. Again all results we obtained strongly support our qualitative findings: The tracking error is largest around the option's striking price and

[8] The fraction of the aggregate equity value subject to formal portfolio insurance prior to the events of October 1987 was approximately 5%. However, one should bear in mind that the amount of "informal" portfolio insurance may have amounted to considerabely more than this. Moreover, part of the aggregate equity supply is held because of the associated control rights and not for speculative reasons, such that the actual "supply" should be considered smaller than aggregate equities. Hence the "actual" market weight of program trading might be larger than just these 5%.

almost vanishes as the option gets deeper in the money or out of the money. We see that even a comparatively low level of heterogeneity is sufficient for the super-hedging portfolio to duplicate the option's payoff almost perfectly.

5 Conclusions

In this paper we have analyzed the feedback effect of dynamic hedging strategies on the equilibrium price process of the underlying asset in an economy where the market for the latter is only finitely elastic. We gave an explicit expression for the transformation of market volatility which allowed us to carry out a detailed study of the feedback effects caused by dynamic hedging. A comparison with the analysis of Berennan and Schwartz revealed the importance of agents' expectations in determining market liquidity and hence the amplitude of the feedback effect on volatility. Adding to the existing literature, we identified heterogeneity of the distribution of hedged contracts as one of the key determinants for the transformation of volatility. Moreover, we showed that simple hedging strategies derived from the assumption of constant volatility may still be appropriate even though their implementation obviously violates this assumption. However, investors might have to "over-invest" in their hedging strategies. To sum up, we find that classical Black-Scholes theory is quite robust with respect to the feedback effects discussed, as long as the distribution of different payoff claims being hedged does not become too homogeneous. Nonetheless future research is needed to extend the work of Jarrow (1994) on option pricing theory in an economy where agents' hedging strategies affect the underlying asset's price process.

A Mathematical Appendix

A.1 Complements to Section 3

Proof of Lemma 3.2: First we prove the pointwise convergence of the ψ^n to ψ. For t and f fixed, every subsequence of the sequence $x_n := \psi^n(t,f)$ contains a further subsequence $\{x_{n_j}\}_{j=1,2,\ldots}$ which converges to some $x \in \mathbb{R}_+$ by Assumption (A.1) (ii). Now we have the following estimate:

$$\left|G^{n_j}(t,f,x_{n_j}) - G(t,f,x)\right| \le \left|G^{n_j}(t,f,x_{n_j}) - G(t,f,x_{n_j})\right| + \left|G(t,f,x_{n_j}) - G(t,f,x)\right|.$$

The first term on the right-hand side tends to zero for $j \longrightarrow \infty$ because G^n converges to G uniformly on compacts. The second term tends to zero because of the uniform continuity of G on compacts. It follows that $G(t,f,x) = \lim_j G^{n_j}(t,f,x_{n_j}) = 1$ and hence $x = \psi(t,f)$. Now by Assumption (A.1) (ii) the sequence $\{\psi^n\}_{n=1,2,\ldots}$ also converges uniformly on compacts to ψ. □

Proof of Corollary 3.3: Using (3.13) and the particular form of the limiting demand function together with $F_t = \psi^{-1}(t, X_t)$ we get

$$v(t,X_t) = \frac{-\frac{\partial G}{\partial f}(t,\psi^{-1}(t,X_t),X_t)}{\frac{\partial G}{\partial x}(t,\psi^{-1}(t,X_t),X_t)} \cdot \frac{\psi^{-1}(t,X_t)}{X_t} = \frac{-D_* \cdot \frac{F_t}{X_t}}{-D_* \cdot \frac{F_t}{X_t} + \rho X_t \frac{\partial \phi}{\partial x}(t,X_t)}.$$

Using the market clearing equation $1 \equiv D_* \frac{F_t}{X_t} + \rho\phi(t, X_t)$ to substitute for $D_* \frac{F_t}{X_t}$ in the above expression gives the desired shape of the volatility of the limiting diffusion. □

A.2 Complements to Section 4:

Estimates for $\Gamma(\sigma, x)$: Observe first that by (4.17) we have

$$x \frac{\partial}{\partial x} \varphi(\sigma, K, \tau, x) = -K \frac{\partial}{\partial K} \varphi(\sigma, K, \tau, x),$$

which implies:

$$\begin{aligned} \Gamma(\sigma, x) &= \int_0^\infty \int_0^\infty x \frac{\partial}{\partial x} \varphi(\sigma, K, \tau, x) g(K, \tau)\, dK\, d\tau \\ &= -\int_0^\infty \int_0^\infty K \frac{\partial}{\partial K} \varphi(\sigma, K, \tau, x) g(K, \tau)\, dK\, d\tau \\ &= \int_0^\infty \int_0^\infty \varphi(\sigma, K, \tau, x) \frac{\partial}{\partial K} (K g(K, \tau))\, dK\, d\tau \end{aligned}$$

by partial integration and the assumption of g having compact support. But since $0 \leq \varphi \leq 1$ by (4.17), this implies

$$|\Gamma(\sigma, x)| \leq \int_0^\infty \int_0^\infty \left| \frac{\partial}{\partial K} (K g(K, \tau)) \right| dK\, d\tau.$$

Estimates for $\frac{\partial}{\partial \sigma} \Gamma(\sigma, x)$: Observe first that by (4.17) we have

$$\sigma \frac{\partial}{\partial \sigma} \varphi(\sigma, K, \tau, x) = 2\tau \frac{\partial}{\partial \tau} \varphi(\sigma, K, \tau, x),$$

which together with the results from the previous paragraph implies

$$\begin{aligned} \frac{\partial}{\partial \sigma} \Gamma(\sigma, x) &= \int_0^\infty \int_0^\infty \frac{\partial}{\partial \sigma} \varphi(\sigma, K, \tau, x) \frac{\partial}{\partial K} (K g(K, \tau))\, dK\, d\tau \\ &= \int_0^\infty \int_0^\infty 2 \frac{\tau}{\sigma} \frac{\partial}{\partial \tau} \varphi(\sigma, K, \tau, x) \frac{\partial}{\partial K} (K g(K, \tau))\, dK\, d\tau \\ &= -\frac{2}{\sigma} \int_0^\infty \int_0^\infty \varphi(\sigma, K, \tau, x) \frac{\partial^2}{\partial \tau \partial K} (\tau K g(K, \tau))\, dK\, d\tau \end{aligned}$$

again by partial integration and the assumption of g having compact support. But since $0 \leq \varphi \leq 1$ by (4.17) and furthermore $\sigma \geq \underline{\eta}$, this implies

$$\left| \frac{\partial}{\partial \sigma} \Gamma(\sigma, x) \right| \leq \frac{2}{\underline{\eta}} \cdot \int_0^\infty \int_0^\infty \left| \frac{\partial^2}{\partial \tau \partial K} (\tau K g(K, \tau)) \right| dK\, d\tau.$$

References

Allen, H., and M. Taylor (1992): "The Use of Technical Analysis in the Foreign Exchange Market," *Journal of International Money and Finance*, 11(3), 304–314.

Black, F., and M. Scholes (1973): "The Pricing of Options and other Corporate Liabilities," *Journal of Political Economy*, 81(3), 637–654.

Brennan, M., and E. Schwartz (1989): "Portfolio Insurance and Financial Market Equilibrium," *Journal of Business*, 62(4), 455–472.

Cox, J., S. Ross, and M. Rubinstein (1979): "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, 229–263.

De Long, J., A. Shleifer, L. Summers, and R. Waldmann (1990a): "Noise Trader Risk in Financial Markets," *Journal of Political Economy*, 98(4), 703–738.

De Long, J., A. Shleifer, L. Summers, and R. Waldmann (1990b): "Positive Feedback Investment Strategies and Destabilizing Rational Speculation," *Journal of Finance*, 45(2), 381–395.

Duffie, D. (1992): *Dynamic Asset Pricing Theory*. Princeton University Press, Princeton, New Jersey.

Duffie, D., and P. Protter (1992): "From Discrete to Continuous Time Finance: Weak Convergence of the Financial Gain Process," *Mathematical Finance*, 2(1), 1–15.

El Karoui, N., M. Jeanblanc-Picqué, and S. Shreve (1995): "Robustness of the Black and Scholes Formula," preprint, forthcoming in Mathematical Finance.

Föllmer, H., and M. Schweizer (1991): "Hedging of Contingent-Claims under Incomplete Information," in *Applied Stochastic Analysis*, pp. 205–223. Gordon & Breach, London.

Föllmer, H., and M. Schweizer (1993): "A Microeconomic Approach to Diffusion Models for Stock Prices," *Mathematical Finance*, 3(1), 1–23.

Frey, R., and A. Stremme (1994): "Portfolio Insurance and Volatility," Discussion Paper 188, Financial Markets Group, London School of Economics.

Gennotte, G., and H. Leland (1990): "Market Liquidity, Hedging and Crashes," *American Economic Review*, 80, 999–1021.

Goodhart, C. (1988): "The Foreign Exchange Market: A Random Walk with a Dragging Anchor," *Economica*, 55, 437–460.

Grossman, S. (1988): "An Analysis of the Implications for Stock and Futures Price Volatility of Program Trading and Dynamic Hedging Strategies," *Journal of Business*, 61, 275–298.

Grossman, S., and J. Stiglitz (1980): "On the Impossibility of Informationally Efficient Markets," *American Economic Review*, 70, 393–408.

Group of Ten (1993): "International Capital Movement and Foreign Exchange Markets: A Report to the Ministers and Governors by the Group of Deputies," Rome.

He, H. (1990): "Convergence from Discrete to Continuous Time Contingent Claim Prices," *Review of Financial Studies*, 3, 523–546.

HULL, J., and A. WHITE (1987): "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42(2), 281–300.

JACOD, J., and A. SHIRYAEV (1987): *Limit Theorems for Stochastic Processes*. Springer Verlag, Berlin.

JARROW, R. (1994): "Derivative Securities Markets, Market Manipulation and Option Pricing Theory," *Journal of Financial and Quantitative Analysis*, 29, 241–261.

LELAND, H. (1980): "Who Should Buy Portfolio Insurance?," *Journal of Finance*, 35(2), 581–594.

PLATEN, E., and M. SCHWEIZER (1994): "On Smile and Skewness," Statistics Research Report 027-94, Australian National University, Canberra.

SCHÖNBUCHER, P. (1993): "Option Pricing and Hedging in Finitely Liquid Markets," Master's thesis, Exeter College, Oxford.

	Program Trader's Market Weight					
	1 %		5 %		10 %	
Price	F-S	B-S	F-S	B-S	F-S	B-S
80	1.01	1.00	1.06	1.02	1.13	1.04
90	1.02	1.00	1.08	1.02	1.18	1.04
100	1.02	1.00	1.09	1.02	1.19	1.05
110	1.02	1.00	1.08	1.02	1.18	1.05
120	1.01	1.00	1.06	1.02	1.14	1.05

Table 1: *This table shows the ratios of actual to reference volatility in our model (F-S) and the Brennan-Schwartz model (B-S) for different market weights of the program trader.*

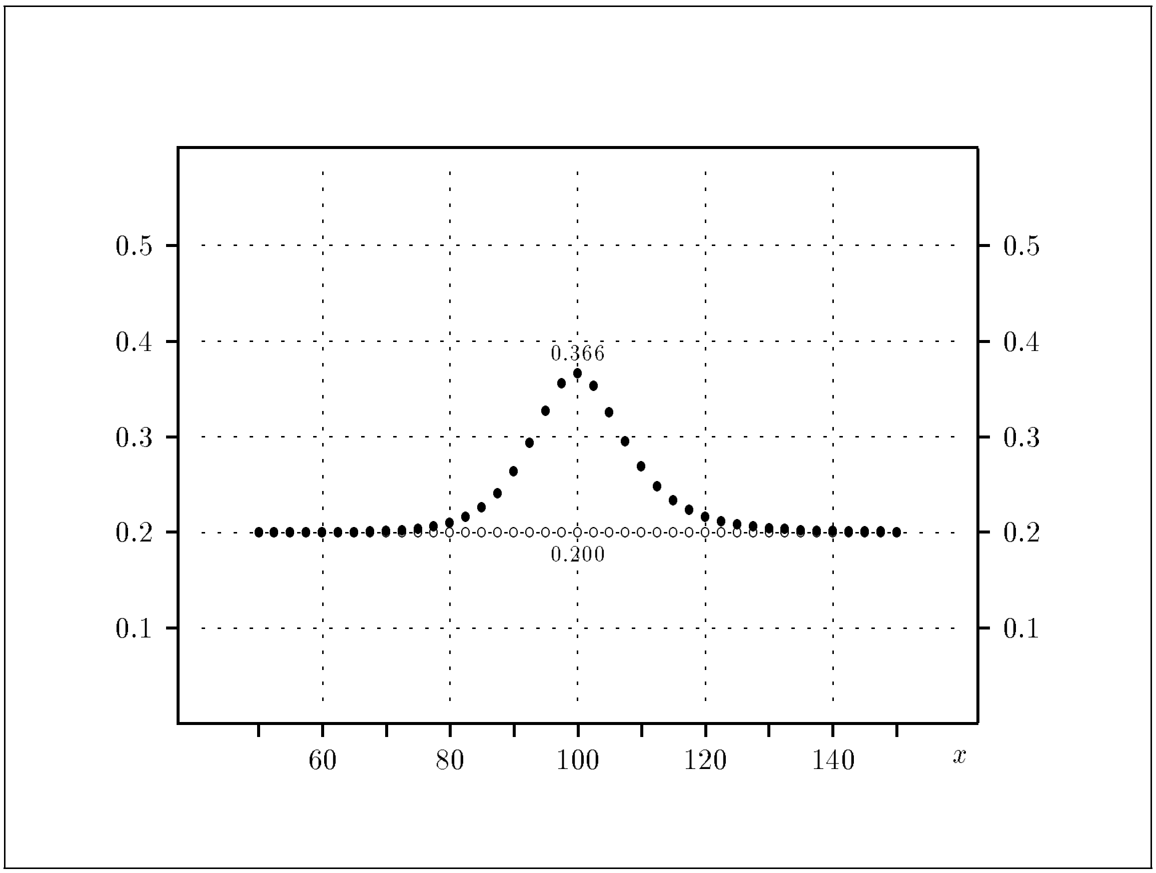

Figure 1: *This graph shows the actual market volatility (•) compared to reference volatility (∘) when hedging is based on the super-volatility* $\overline{\sigma}$. *The variances of strike prices and times-to-maturity are* 0.0625, *i.e. the distribution is highly concentrated. The market weight of portfolio insurance is* $\rho = 10\%$.

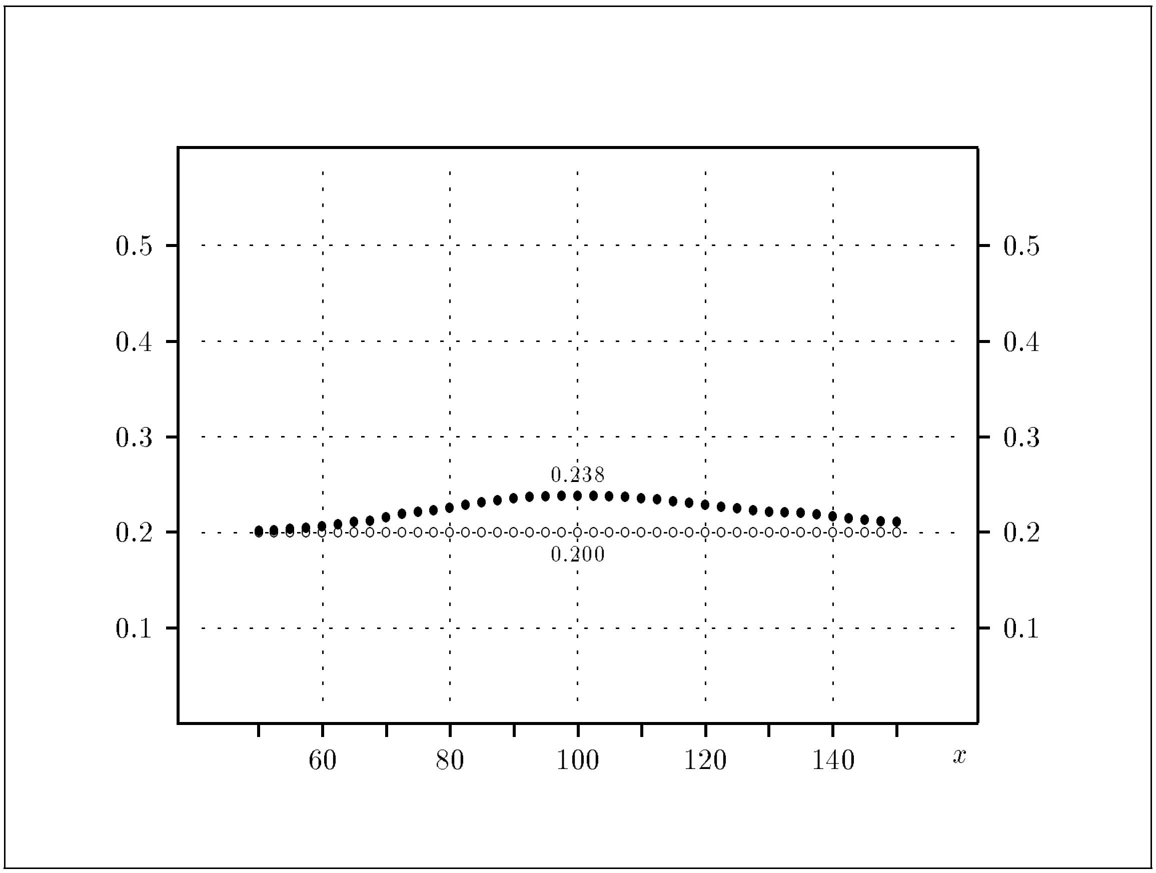

Figure 2: *This graph shows the actual market volatility (•) compared to reference volatility (∘) when hedging is based on the super-volatility $\overline{\sigma}$. The variances of strike prices and times-to-maturity are 0.5, i.e. the distribution is relatively heterogeneous. The market weight of portfolio insurance is $\rho = 10\%$.*

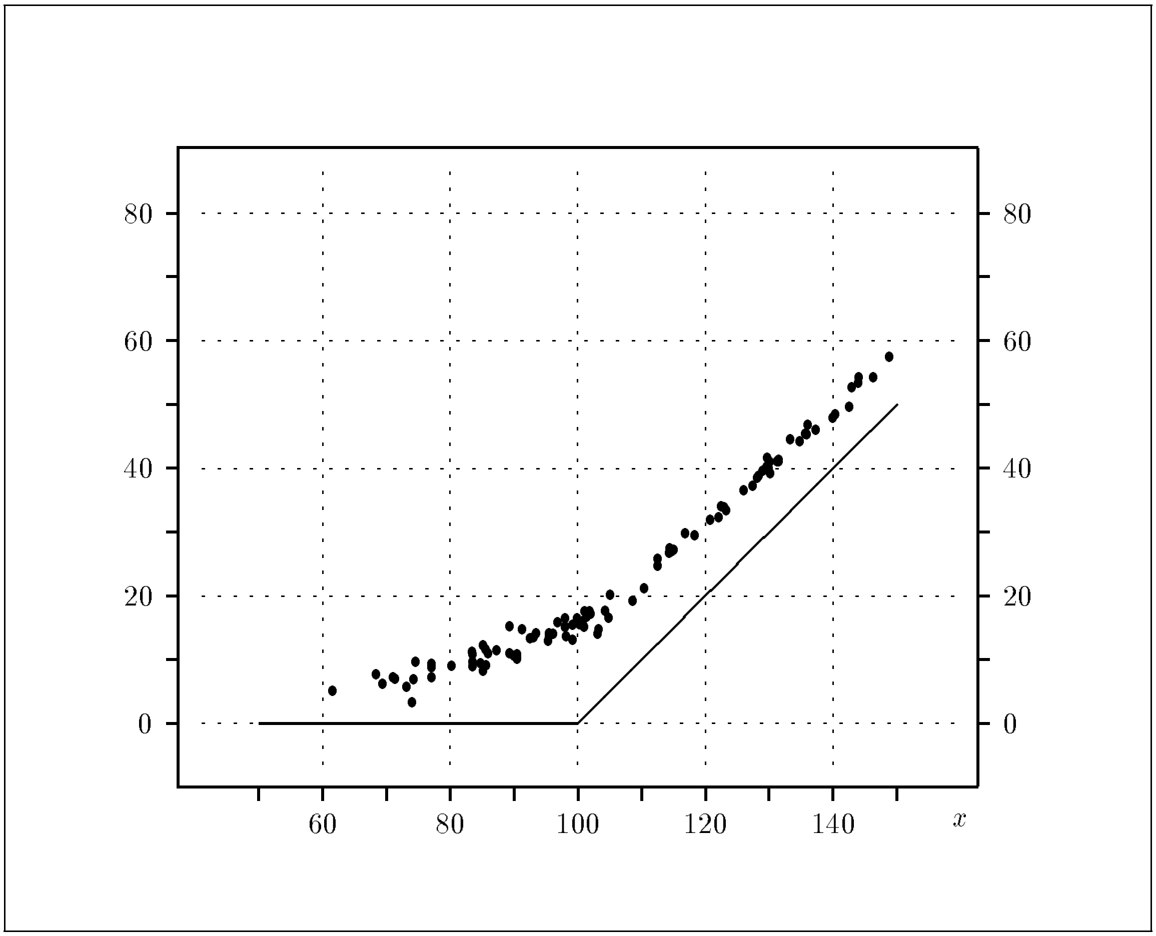

Figure 3: *This graph shows for* 100 *sample paths the terminal value* (•) *of a portfolio designed to hedge a European call with strike price* 100.0*, based on the super-volatility* $\overline{\sigma}$*, compared to the option's pay-off itself. The variances of strike prices and times-to-maturity are* 0.0625*, i.e. the distribution is highly concentrated. The market weight of portfolio insurance is* $\rho = 10\%$.

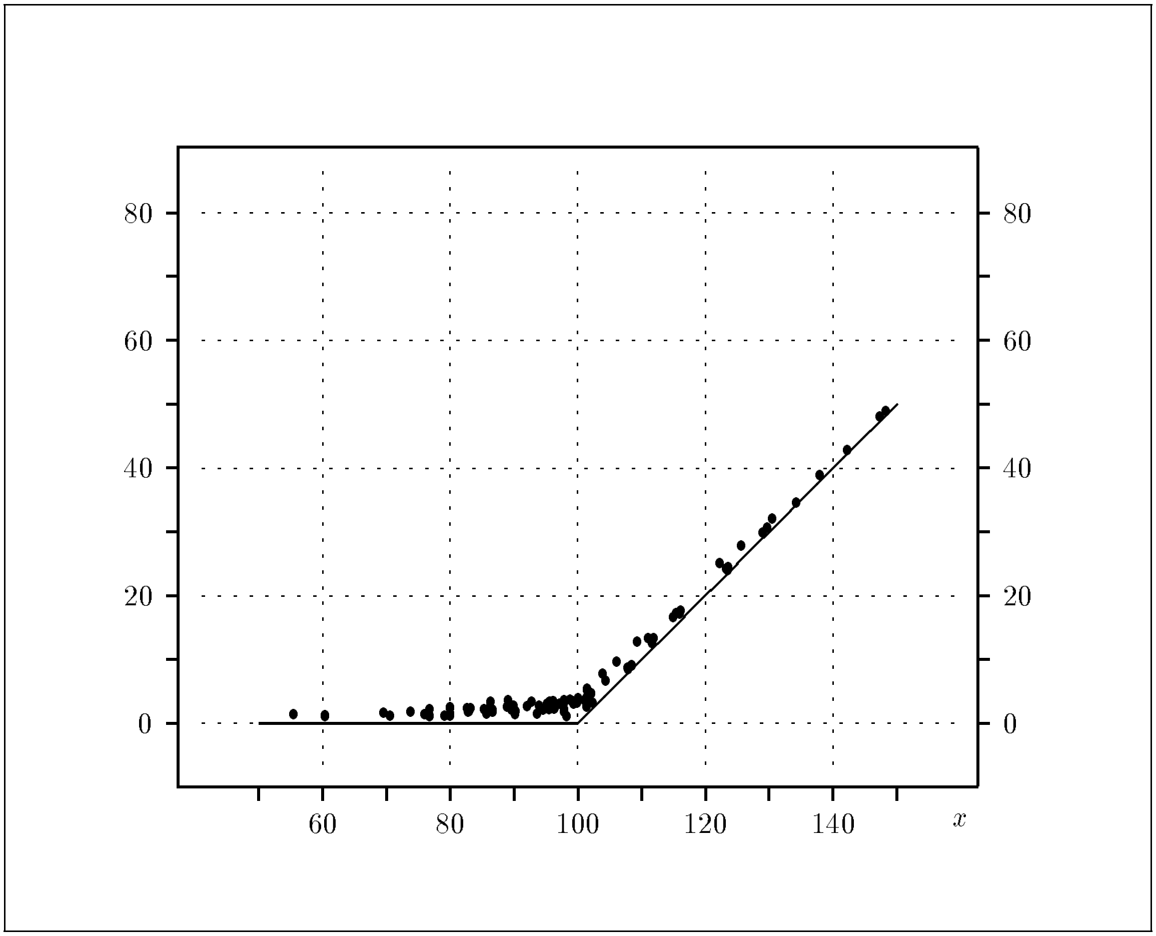

Figure 4: *This graph shows for* 100 *sample paths the terminal value* (•) *of a portfolio designed to hedge a European call with strike price* 100.0*, based on the super-volatility* $\overline{\sigma}$*, compared to the option's pay-off itself. The variances of strike prices and times-to-maturity are* 0.5*, i.e. the distribution is relatively heterogeneous. The market weight of portfolio insurance is* $\rho = 10\%$*.*