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On J.M. Grandmont's Modelling of Behavioral Heterogeneity

by

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Abstract

J.M. Grandmont claims in his paper "Transformations of the Commodity Space, Behavioral Heterogeneity, and the Aggregation Problem" (1992) to model "behavioral heterogeneity". By a specific parametrization he defines a subset of all demand functions and assumes that the distribution of the parameters is getting more dispersed (increasing flatness of the density function). This increasing dispersedness of the parameters is interpreted as "increasing heterogeneity" of the population of households described by the distribution of demand functions. But, due to the specific parametrization, increasing dispersedness of the parameters leads to an increasing concentration of the demand functions. Therefore, roughly speaking, Grandmont rather models increasing "behavioral similarity".

JEL Classification System: D 11, D 30, D41, E 10

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1 Introduction

Behavior of a consumer is described by his demand function, which associates with each price-income situation a demand vector. Given a probability measure τ on the set F of all demand functions, the mean demand of a population of consumers with identical income yet different demand functions is defined by

$$\bar{f}(p, w) = \int_F f(p, w) d\tau.$$

One would tend to speak of "behavioral heterogeneity", if the support of the distribution τ contains "many different" demand functions and, furthermore, the distribution τ is not concentrated on a "small subset". Hence, in this view, a sequence (τ_n) does not display increasing behavioral heterogeneity if with increasing n more and more weight is concentrated on a "small subset" of F. This is not a definition of "behavioral heterogeneity", yet it excludes situations that qualify as "behavioral heterogeneity".

Since the space F of demand functions is an infinite dimensional function space, it is difficult to give a precise meaning of a "small subset" in F. To avoid this conceptional mathematical difficulty many authors therefore consider the following set-up: let $C \subset \mathbb{R}^n$ denote a finite dimensional parameter space and let T be a mapping of C in F. Thus, T defines a parametrization of demand functions. A probability measure μ on C then induces a probability measure τ on F that is defined as the image measure of μ under the mapping T .

Obviously, a parameter distribution μ that is dispersed - a concept which is well defined since μ is a distribution on $\mathbb{R}^n$ - does not necessarily imply that the induced distribution τ on F models "behavioral heterogeneity". Indeed, whether a sequence (μ_n) on C , which is increasingly dispersed, models "increasing heterogeneity" in the above sense, depends entirely on the chosen parametrization T. It might well happen that the sequence (μ_n) is increasingly dispersed yet the induced sequence (τ_n) on F is increasingly concentrated, that is to say, more and more weight is given to a very small subset of F. This is exactly what happens in Grandmont's model.

The intention of this paper is to show, by using Grandmont's well-known model as an example, that every ad hoc parametrization of demand functions contains an inherent danger of misinterpretation. Easily interpretable assumptions on the parameter distribution μ might imply quite unintentional properties of the induced distribution τ of demand functions. This remark, of course, is obvious, yet, it seems that it has been overlooked in the literature on behavioral heterogeneity.

2 Grandmont's Model

Behavior of a consumer is described by his demand function, which is

$$f : \mathbb{P}^l \times \mathbb{P} \rightarrow \mathbb{R}^l_+ \text{ with}$$
$$pf(p,w) = w \text{ and } f(\lambda p, \lambda w) = f(p,w) \text{ for all } p \in \mathbb{P}^l, w \in \mathbb{P}, \lambda \in \mathbb{P}$$

where $\mathbb{P}$ denotes the set of positive numbers and l is the number of commodities.

The set of all continuous demand functions is denoted by F and is endowed with the topology of uniform convergence on compact subsets.

Grandmont gives the following two equivalent parametrizations, where $\otimes$ denotes the componentwise product of two vectors, and $exp(\cdot)$ is applied componentwise. Given a fixed demand function f, called the generating function

1. the parameter set is $\mathbb{P}^l$ and $T(v)(p,w) := v \otimes f(v \otimes p, w), v \in \mathbb{P}^l$
2. the parameter set is $\mathbb{R}^l$ and $T(\alpha)(p,w) := exp(\alpha) \otimes f(exp(\alpha) \otimes p, w), \alpha \in \mathbb{R}^l$.

The demand functions $T(v), T(\alpha)$ are called the transformed functions and, to shorten the notation, are written as f^v, f^α - the latin, greek letter decides which transformation has to be used.

The distribution μ on the α-parameter space $\mathbb{R}^l$ is assumed to be given by a continuously differentiable density function ρ.

Clearly, for each measure μ on $\mathbb{R}^l$ there is by the mapping $exp(\cdot)$ the corresponding image measure ν on $\mathbb{P}^l$. To each such pair of measures (μ, ν) belongs an image measure τ on F by the mappings $T(v), T(\alpha)$. Whether a sequence (μ_n) or (ν_n) of measures on the parameter space $\mathbb{R}^l$ or $\mathbb{P}^l$ models increasing behavioral heterogeneity can only be decided by considering the sequence (τ_n) of measures on F. Grandmont's increasing flatness condition is a condition on the sequence (μ_n). His condition means increasing dispersedness of the parameter α and can be interpreted as increasing heterogeneity of the α-parameters. But this does not necessarily imply that also the sequence (τ_n) of distributions of demand functions models "increasing behavioral heterogeneity". In fact, we will show that Grandmont's increasing flatness condition implies increasing concentration of the measures τ_n, which can be interpreted as "increasing behavioral similarity".

Grandmont considers a set A of "types" of consumers and therefore for each $a \in A$ a generating demand function $f(a,\cdot,\cdot)$ and a density function $\rho(a,\cdot)$ getting flat. Obviously, this more general set-up is not essential for the point to be made in the following two sections.

3 An Example

In this section we choose an example where all calculations can be made explicit. We consider the case of 2 commodities, the distribution of the α-parameters is the 2-dimensional, uncorrelated, symmetric normal distribution, i.e.,

$$\rho(\alpha_1, \alpha_2) = \frac{1}{2\pi s^2} \exp\left(-\frac{\alpha_1^2 + \alpha_2^2}{2s^2}\right), \quad s^2 > 0, \tag{1}$$

and the generating demand function f has the following properties

$$f \text{ is linear in income, i.e.,} \quad f(p, \lambda w) = \lambda f(p, w), \quad \lambda > 0 \tag{2}$$

and

$$\gamma_1 := \lim_{p_1 \to 0} p_1 f_1(p_1, 1, 1) \quad \text{and} \quad \gamma_2 := \lim_{p_2 \to 0} p_2 f(1, p_2, 1) \quad \text{exist.} \tag{3}$$

The normal distribution is chosen only to allow explicit calculations - in the next section we consider the general case. Clearly, Grandmont's increasing flatness condition, i.e.,

$$m(\rho_n) := \max_{h=1,2} \int_{R^2} |\frac{\partial \rho_n}{\partial \alpha_h}(\alpha)| d\alpha \xrightarrow[n \to \infty]{} 0$$

then is equivalent with

$$s^2 \longrightarrow \infty.$$

The assumption of linearity in income is restrictive, but simplifies this example and will be dropped in the next section.

The existence of the limits in (3) means a mild restriction on the boundary behavior of f. The case $\liminf_{p_1 \to 0} p_1 f_1(p_1, 1, 1) < \limsup_{p_1 \to 0} p_1 f_1(p_1, 1, 1)$ would mean that $f_1(p_1, 1, 1)$ would tend to infinity on a "hysteric-like" path.

Proposition 1: *Assume (1), (2) and (3) and define the two demand functions*

$$g^1(p, w) := \left(\frac{\gamma_1}{p_1}, \frac{1-\gamma_1}{p_2}\right) \cdot w \qquad \textit{and} \qquad g^2(p, w) := \left(\frac{1-\gamma_2}{p_1}, \frac{\gamma_2}{p_2}\right) \cdot w\,.$$

Then, for every neighborhood U_1 of g^1 and U_2 of g^2, respectively, with respect to the topology[1] *of uniform convergence on compact subsets,*

$$\lim_{s^2 \to \infty} \mu_s(\{\alpha | f^\alpha \in U_1 \cup U_2\}) = 1,$$

[1] One could also use the topology of uniform convergence on compact subsets of the functionvalues and the derivative.

where μ_s denotes the probability measure belonging to the density ρ_s.

If $U_1 \cap U_2 = \emptyset$, hence $g^1 \neq g^2$, then

$$\lim_{s^2 \to \infty} \mu_s \left(\{\alpha | f^\alpha \in U_1\} \right) = \lim_{s^2 \to \infty} \mu_s \left(\{\alpha | f^\alpha \in U_2\} \right) = \frac{1}{2} \,.$$

Corollary 1: *The image measures τ_s, generated by the mapping $\alpha \mapsto f^\alpha$, on the set F of all continuous demand functions, converge with respect to the weak topology to a probability measure τ_∞ on F, given by*

$$\begin{aligned} \tau_\infty(\{g^1\}) &= \tau_\infty(\{g^2\}) = \frac{1}{2} \quad && \textit{if} \quad g^1 \neq g^2 \\ \tau_\infty(\{g^1\}) &= 1 && \textit{if} \quad g^1 = g^2 \,. \end{aligned}$$

Corollary 2: *The limit mean demand function exists and is given by*

$$\bar{f}_\infty(p, w) := \lim_{s^2 \to \infty} \int_{\mathbb{R}^2} f^\alpha(p, w) \rho_s(\alpha) d\alpha = \left(\frac{\gamma_1 + (1 - \gamma_2)}{2p_1}, \frac{(1 - \gamma_1) + \gamma_2}{2p_2} \right) \cdot w \,.$$

Corollary 3: *If f is a CES-demand function, i.e.,*

$$f(p, w) = \frac{(a^\sigma p_1^{-\sigma}, (1 - a)^\sigma p_2^{-\sigma})}{a^\sigma p_1^{1-\sigma} + (1 - a)^\sigma p_2^{1-\sigma}} \cdot w \qquad 0 < a < 1 \,,\ \sigma > 0,$$

then

$$\bar{f}_\infty(p, w) = \begin{cases} \left(\dfrac{a}{p_1}, \dfrac{(1 - a)}{p_2} \right) \cdot w & \textit{if} \quad \sigma = 1 \\ \left(\dfrac{1}{2p_1}, \dfrac{1}{2p_2} \right) \cdot w & \textit{if} \quad \sigma \neq 1 \,. \end{cases}$$

First we discuss the above results, then we will show in Lemma 1 and 2 the facts which lie behind. Then the proofs of Proposition 1 and its Corollaries will be straightforward.

Since $\rho_s(\alpha) > 0$ for all $\alpha \in \mathbb{R}^2$, $s^2 \in \mathbb{P}$, the support of μ_s is equal to $\mathbb{R}^2$. Hence, the set of demand functions which influence the integral $\int_{\mathbb{R}^2} f^\alpha(p, w) \rho_s(\alpha) d\alpha$ does not depend on s^2. Increasing flatness of ρ_s, i.e. $s^2 \to \infty$, means that the distribution of the α-parameters becomes more dispersed, but, according to Corollary 1, the distribution of the demand functions f^α becomes more concentrated around the two (or one) Cobb-Douglas demand functions g^1 and g^2. Hence

Conclusion: *Increasing flatness of ρ, i.e. increasing dispersedness of the α-parameters, induce increasing concentration of the demand functions f^α, i.e. increasing behavioral similarity.*

The situation becomes clearer, if one changes the parametrization. Note, α enters in the mapping $\alpha \mapsto f^\alpha$ only as $v = \exp(\alpha) \in \mathbb{P}^l$. Hence $v \mapsto f^v$, $v \in \mathbb{P}^l$, is the "natural", or "intrinsic", parametrization. We will see that increasing dispersedness of the α-parameters induces increasing concentration of the v-parameters, and this leads to increasing concentration of the considered demand functions.

Using the transformation $v = \exp(\alpha)$ we replace the α-parameter space $\mathbb{R}^2$ by the strictly positive orthant $\mathbb{P}^2$, the v-parameter space. The probability measure μ on $\mathbb{R}^2$ and the transformation $v = \exp(\alpha)$ generates a probability measure ν on $\mathbb{P}^2$. The connection of the two parameter spaces are illustrated in Figures 1 and 2.

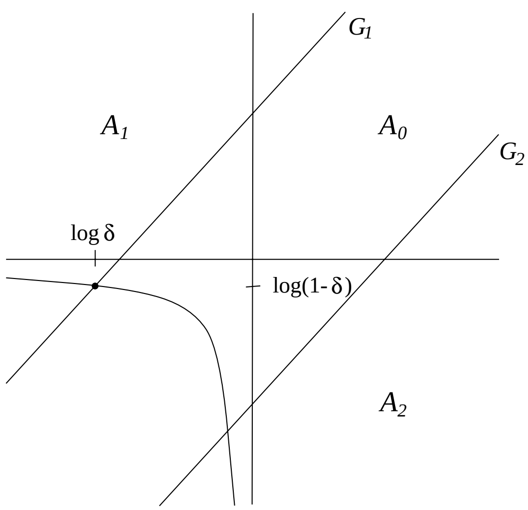

Figure 1: α-space

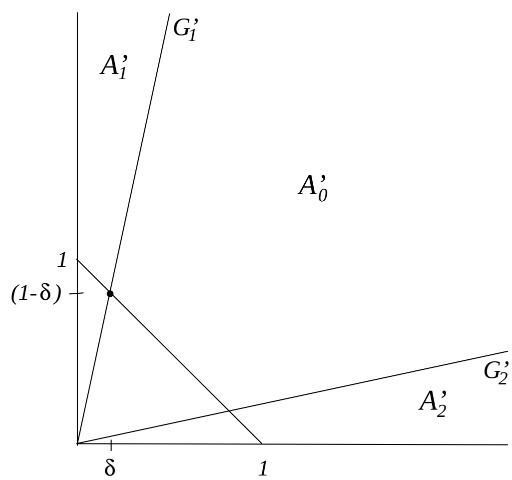

Figure 2: v-space

The lines G_1, G_2 correspond to the (open) rays G'_1, G'_2, the areas A_1 (above G_1), A_0 (between G_1 and G_2), A_2 (below G_2) correspond to A'_1 ("triangle" between the 2-axes and G'_1), A'_0 ("triangle" between G'_1 and G'_2), A'_2 ("triangle" between G'_2 and the 1-axes), and the hyperbel-like graph $\{(\log\delta, \log(1-\delta)) \mid 0 < \delta < 1\}$ corresponds to the open unit simplex. Defining the distance between the two lines G_1 and G_2 as 2β, one obtains

$$\beta = |\log(\delta) - \log(1-\delta)|/\sqrt{2}.$$

One obtains $\mu(A_1)$ by a 45° anticlockwise rotation, i.e. A_1 becomes the strip $\{(\alpha_1, \alpha_2) |\, \alpha_1 \leq -\beta\}$, and then integrating over α_2. Hence

$$\mu(A_1) = \int_{\xi=-\infty}^{\xi=-\beta} \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{\xi^2}{2s^2}\right) d\xi$$

and $\mu(A_0), \mu(A_1)$ when setting the interval of integration to $[-\beta, +\beta], [\beta, +\infty)$, respectively. Since the integrant is the density of the one-dimensional normal distribution it follows the

Lemma 1: *Given* $0 < \delta < \frac{1}{2}$, *then*

$$\left.\begin{array}{ll} \nu_s(A_1') = \mu_s(A_1) & \to \frac{1}{2} \\ \nu_s(A_0') = \mu_s(A_0) & \to 0 \\ \nu_s(A_2') = \mu_s(A_2) & \to \frac{1}{2} \end{array}\right\} \textit{ for } s^2 \to \infty\,.$$

In other words, Lemma 1 states, that increasing flatness of ρ, i.e. increasing dispersedness of the α-parameters, means that the v-parameters are getting more and more concentrated near the axes, in the sense that most v-vectors are contained in two "triangles" which have arbitrarily small chosen angle.

Next we ask, what does the f^v look like for $v \in A_1' \cup A_2'$, when A_1', A_2' are becoming small, i.e. $\delta \to 0$. Because we have assumed that f is linear in income, hence $f^{\lambda v} = f^v, \quad \lambda > 0$, we have only to compute the limits

$$\lim_{\delta\to 0} f^{(\delta, 1-\delta)} \quad \text{and} \quad \lim_{\delta \to 0} f^{(1-\delta,\delta)}\,.$$

We obtain the pointwise convergence

$$\begin{aligned} f_1^{(\delta,1-\delta)}(p,w) &= \delta f_1(\delta p_1, (1-\delta)p_2, w) = \delta f_1\left(\frac{\delta p_1}{(1-\delta)p_2}, 1, \frac{w}{(1-\delta)p_2}\right) \\ &= \frac{w}{p_1} \cdot \frac{\delta p_1}{(1-\delta)p_2} f_1\left(\frac{\delta p_1}{(1-\delta)p_2}, 1, 1\right) \underset{\delta\to 0}{\longrightarrow} \frac{\gamma_1 w}{p_1} \end{aligned}$$

and, analogously,

$$f_2^{(1-\delta,\delta)}(p,w) \underset{\delta\to 0}{\longrightarrow} \frac{\gamma_2 w}{p_2}\,.$$

Obviously, this pointwise convergence is uniform, if p and w are contained in compact sets. Hence, we have shown

Lemma 2:

$$\lim_{\delta \to 0} f^{(\delta, 1-\delta)} = g^1 \quad \textit{and} \quad \lim_{\delta \to 0} f^{1-\delta, \delta} = g^2$$

with

$$g^1(p,w) = \left(\frac{\gamma_1 w}{p_1}, \frac{(1-\gamma_1)w}{p_2} \right) \quad \textit{and} \quad g^2(p,w) = \left(\frac{(1-\gamma_2)w}{p_1}, \frac{\gamma_2 w}{p_2} \right).$$

Now we can give the

Proof of Proposition 1: Using $f^{\lambda v} = f^v, \lambda > 0$, we get from Lemma 2 that for every neighborhood U_1 of g^1 and U_2 of g^2, respectively, there exists a $\delta > 0$ such that

$$f^v \in U_1 \quad \text{if} \quad v_1/v_2 < \delta \quad \text{and} \quad f^v \in U_2 \quad \text{if} \quad v_2/v_1 < \delta.$$

With Lemma 1 follows the claimed convergence

$$\lim_{s^2 \to \infty} \mu \left(\{ \alpha | f^\alpha \in U_1 \cup U_2 \} \right) = 1.$$

The remaining part of Proposition 1 is obvious in the case of a symmetric distribution.

Q.E.D.

Corollaries 1 and 2 follow immediately from the proposition. To prove Corollary 3, one only has to compute the two limit values γ_1 and γ_2.

4 The General Case

In this section we do not assume a functional form of the density function ρ, and the generating demand function f can be any demand function. We shift all proofs to the end of this section.

For $\delta \in I\!R$ with $0 < \delta < 1/l$ we define the following subsets of the open unit simplex

$$\begin{aligned} S &:= \{ v \in I\!R^l \mid v \gg 0, \sum v_h = 1 \} \\ B_\delta &:= \{ v \in S \mid \min v_h < \delta \} \\ I_\delta &:= \{ v \in S \mid \min v_h \geq \delta \}. \end{aligned}$$

Given a vector of direction r, i.e. $\sum r_h^2 = 1$, the probability distribution μ on $I\!R^l$, generated by the density function ρ, defines (according to the Theorem of Fubini) on the one-dimensional subspace $\langle r \rangle$ a marginal measure with density.

$$\varphi_r(\bar{\xi}) = \int_{\langle r \rangle^\perp} \rho_r(\bar{\xi}, x')dx',$$

where ρ_r denotes the function ρ written in the transformed coordinates ξ and x' with respect to the subspaces $\langle r \rangle$ and $\langle r \rangle^\perp$, respectively. But note, without further arguments, we can not exclude the case that $\varphi_r(\bar{\xi}) = \infty$ for a nul-set of $\bar{\xi}$-values.

Denote by $\frac{\partial \rho}{\partial r} := r \operatorname{grad}(\rho)$ the partial derivative of ρ in direction r.

Lemma 3:

$$\text{(i)} \quad \textit{If} \quad m(\rho) := \max_h \int |\frac{\partial \rho}{\partial \alpha_h}(\alpha)| d\alpha < \infty$$

$$\textit{then} \quad m(\rho, r) := \int \left| \frac{\partial \rho}{\partial r}(\alpha) \right| d\alpha \leq \sqrt{l} \cdot m(\rho)$$

$$\text{(ii)} \quad \textit{If} \quad m(\rho, r) := \int \left| \frac{\partial \rho}{\partial r}(\alpha) \right| d\alpha < \infty$$

$$\textit{then} \quad ||\varphi_r|| := \sup\{\varphi_r(\xi) | \xi \in I\!R\} \leq \frac{1}{2} m(\rho, r).$$

Grandmont's increasing flatness condition means that $m(\rho) \to 0$. What is actually used is $||\varphi_r|| \to 0$, a somewhat weaker property.

Proposition 2: *For every number δ with $0 < \delta < 1/l$ and every vector of direction r, which is orthogonal to the diagonal, i.e., $\sum r_h = 0$,*

$$\nu\left(\{\lambda v \mid v \in I_\delta, \lambda > 0\}\right) \leq 2\sqrt{l}\, |\log \delta| \cdot \|\varphi_r\|,$$

where ν denotes the image measure of μ by the transformation $v = \exp(\alpha)$.

If ρ_n is a sequence of densities with $\|\varphi_{r,n}\| \to 0$ then for every $0 < \delta < 1/l$

$$\lim_{n\to\infty} \nu_n\left(\{\lambda v \mid v \in I_\delta, \lambda > 0\}\right) = 0$$

$$\lim_{n\to\infty} \nu_n\left(\{\lambda v \mid v \in B_\delta, \lambda > 0\}\right) = 1.$$

Hence, the integral $\int f^v(p,w)d\nu_n(v) = \int f^\alpha(p,w)d\mu_n(\alpha)$ depends more and more on those f^v, for which $v/\sum v_h$ gets to the boundary. How those f^v are determined shows the following

Proposition 3: *For every compact subset K of $I\!P^l$, there exists a $\delta > 0$, such that*

$$\lambda v \otimes p \notin K \quad \textit{for all} \quad \lambda > 0\,,\ v \in B_\delta\,,\ p \in K.$$

If f and g are two demand functions with $f(p,w) = g(p,w)$ for all $p \notin K$, $w > 0$, then

$$f^{\lambda v}(p,w) = g^{\lambda v}(p,w) \quad \textit{for all} \quad \lambda > 0\,,\ v \in B_\delta\,,\ p \in K\,,\ w > 0.$$

In other words, although the generating demand function is arbitrarily changed on a compact set of prices, for these prices the transformed demand function f^v remains unchanged, if $v/\sum v_h$ is close enough to the boundary. On the other hand, according to Proposition 2, only those f^v contribute substantially to the integral when a sequence of ρ's with increasing flatness is considered. Roughly speaking, it is the boundary behavior of the generating demand function which determines the aggregate demand function. To assume that demand for a commodity tends to infinity when its relative price tends to zero, is a useful technical assumption. But to *base* a theory on the speed, i.e. whether demand runs to infinity slower, faster or with equal speed as the price runs to zero, is not acceptable. In fact, Grandmont "almost" assumes the boundary behavior of a Cobb-Douglas function (p. 18, Assumption (2e)).

If the generating demand function f is linear in income, then the relevant parameter distribution is the measure ν^Δ on the unit simplex, defined by

$$\nu^\Delta(A) := \mu\left(\{v + \lambda \mathbf{1} | v \in A \cap S\}\right), A \subset \Delta := \{v \in I\!R^l_+ | \sum v_n = 1\}.$$

If $m(\rho_n) \to 0$, then the sequence (μ_n) does not converge and the sequence (ν_n) typically does not converge, but for the sequence (ν_n^Δ) we have

Proposition 4: *If $m(\rho_n) \to 0$ then for every neighborhood U of the edge-points of the simplex Δ*

$$\lim \nu_n^\Delta(U) = 1.$$

Every subsequence of (ν_n^Δ) has a convergent subsequence whose limit ν_∞^Δ fulfills

$$\nu_\infty^\Delta(\{\ \textit{edge-points of}\ \Delta\}) = 1.$$

Proof of Lemma 3: Using $\dfrac{\partial\rho}{\partial r} = r \cdot \mathrm{grad}\rho$ and $\max\{\sum |r_i| \mid \sum r_i^2 = 1\} = \sqrt{l}$ we obtain (i) by

$$\begin{aligned}\int |\frac{\partial\rho}{\partial r}| &= \int |\sum r_h \frac{\partial\rho}{\partial\alpha_h}| \le \int \sum |r_h \frac{\partial\rho}{\partial\alpha_h}| = \int \sum |r_h| \cdot |\frac{\partial\rho}{\partial\alpha_h}| \\ &= \sum \left(|r_h| \int |\frac{\partial\rho}{\partial\alpha_h}| \right) \le \left(\sum |r_h| \right) \cdot \max_h \int |\frac{\partial\rho}{\partial\alpha_h}| \le \sqrt{l}\, m(\rho)\,.\end{aligned}$$

Now we prove (ii). If we would know that for every $\bar{\xi}$ there are an $\varepsilon > 0$ and integrable functions $g, g_1 : I\!R^{l-1} \to I\!R_+$ with

$$\begin{aligned}&|\rho_r(\xi, x')| \le g(x') \quad \text{and} \quad \left| \frac{\partial\rho_r}{\partial r}(\xi, x') \right| \le g_1(x') \\ &\text{for all } \xi \in [\bar{\xi} - \varepsilon, \bar{\xi} + \varepsilon] \quad \text{and} \quad x' \in I\!R^{l-1}\end{aligned}$$

then we would obtain (Dieudonné (1970), Th. 13.8.6) that φ_r is continuously differentiable and

$$\varphi_r'(\bar{\xi}) = \int_{\langle r \rangle^\perp} \frac{\partial\rho_r}{\partial\xi}(\bar{\xi}, x')dx'.$$

Clearly, the needed condition is fulfilled if μ is a product measure with respect to the subspaces $\langle r \rangle$ and $\langle r \rangle^\perp$. But in general we have to be more explicit. Define for every natural number k the function

$$\rho_{rk}(\xi, x') := \begin{cases} \rho_r(\xi, x') & \text{if } x' \in [-k, k]^{l-1} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, for ρ_{rk} the above stated condition is fulfilled, while the discontinuity of $\rho_{rk}(\xi, \cdot)$ on the boundary of the cube $[-k, k]^{l-1}$ doesn't matter.

Since $\int \varphi_{rk} \le 1$, there exist sequences $\underline{\xi}_n \to -\infty$ and $\bar{\xi}_n \to +\infty$ with $\varphi_{rk}(\underline{\xi}_n) \to 0$ and $\varphi_{rk}(\bar{\xi}_n) \to 0$ and therefore

$$\varphi_{rk}(\bar{\xi}) = \int_{-\infty}^{\bar{\xi}} \varphi_{rk}'(\xi)d\xi = -\int_{\bar{\xi}}^{\infty} \varphi_{rk}'(\xi)d\xi.$$

Hence,

$$\begin{aligned}0 &= \int_{-\infty}^{\infty} \varphi_{rk}'(\xi)d\xi = \int_{-\infty}^{+\infty} \int_{\langle r \rangle^\perp} \frac{\partial\rho_{rk}}{\partial\xi}(\xi, x')\, dx'd\xi = \int_{I\!R^l} \frac{\partial\rho_k}{\partial r}(\alpha)d\alpha \\ &= \int_{\frac{\partial\rho}{\partial r}(\alpha) \ge 0} \frac{\partial\rho_k}{\partial r}(\alpha)d\alpha + \int_{\frac{\partial\rho}{\partial r}(\alpha) \le 0} \frac{\partial\rho_k}{\partial r}(\alpha)d\alpha.\end{aligned}$$

Therefore we obtain

$$
\begin{aligned}
m(\rho_k, r) &= \int_{I\!R^l} |\frac{\partial \rho_k}{\partial r}(\alpha)| d\alpha \\
&= \int_{\frac{\partial \rho}{\partial r}(\alpha) \geq 0} \frac{\partial \rho_k}{\partial r}(\alpha) d\alpha - \int_{\frac{\partial \rho}{\partial r}(\alpha) \leq 0} \frac{\partial \rho_k}{\partial r}(\alpha) d\alpha = 2 \int_{\frac{\partial \rho}{\partial r}(\alpha) \geq 0} \frac{\partial \rho_k}{\partial r}(\alpha) d\alpha.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\varphi_{rk}(\bar{\xi}) &= \int_{-\infty}^{\bar{\xi}} \varphi'_{rk}(\xi) d\xi \leq \int_{\varphi'_{rk}(\xi) \geq 0} \varphi'_{rk}(\xi) d\xi \\
&\leq \int_{-\infty}^{\infty} \int_{\frac{\partial \rho_r}{\partial \xi}(\xi, x') \geq 0} \frac{\partial \rho_{rk}}{\partial \xi}(\xi, x') dx' d\xi \leq \frac{m(\rho, r)}{2}.
\end{aligned}
$$

Hence, we have shown

$$
\varphi_{rk}(\bar{\xi}) \leq \frac{m(\rho, r)}{2} \text{ for all } k \text{ and } \bar{\xi}.
$$

Since $\rho_{rk}, k = 1, 2, \ldots$, is pointwise increasing, we obtain with the Theorem of Lebesque

$$
\varphi_r(\bar{\xi}) = \lim_{k \to \infty} \varphi_{rk}(\bar{\xi}) \leq \frac{m(\rho, r)}{2}.
$$

Q.E.D.

Proof of Proposition 2: The set $A'_\delta := \{\lambda v | v \in I_\delta, \lambda > 0\}$ is the image of $A_\delta := \{\log v + \lambda \cdot \mathbf{1} \mid v \in I_\delta, \lambda \in I\!R\}$ by the transformation $v = \exp(\alpha)$, and $\mathbf{1}$ denotes the vector with all components equal to one. Therefore we have to show that $\mu(A_\delta) \leq 2\sqrt{l} |\log \delta| \cdot \|\varphi_r\|$.

Clearly, the set A_δ remains unchanged if we replace $\log v$ by the projection of $\log v$ on the hyperplane $\mathbf{1}^\perp$. Setting $C_\delta := \{\log v - \frac{\sum \log v_h}{l} \mathbf{1} \mid v \in I_\delta\}$ we have $A_\delta = \{\alpha + \lambda \mathbf{1} \mid \alpha \in C_\delta, \lambda \in I\!R\}$.

Since $C_\delta \subset \mathrm{proj}_{\langle r \rangle} C_\delta \times \langle r \rangle^\perp$ and $\mathbf{1} \in \langle r \rangle^\perp$, we have

$$
A_\delta \subset \widetilde{A}_\delta := \{\alpha + \alpha' \mid \alpha \in \mathrm{proj}_{\langle r \rangle} C_\delta \, , \ \alpha' \in \langle r \rangle^\perp\}.
$$

Hence

$$
\begin{aligned}
\mu(A_\delta) &\leq \mu(\widetilde{A}_\delta) = \int_{\mathrm{proj}_{<r>} C_\delta} \varphi_r(\xi) d\xi \\
&\leq \|\varphi_r\| \, \mathrm{length}(\mathrm{proj}_{\langle r \rangle} C_\delta).
\end{aligned}
$$

For $\alpha \in C_\delta$ the coordinate of $\text{proj}_{\langle r \rangle}\alpha$ with respect to the space $\langle r \rangle$ is $r \cdot \alpha$. With

$$\begin{aligned} |r\alpha| &= |\sum r_h(\log v_h - \frac{\sum \log v_i}{l})| = |\sum r_h \log v_h - (\sum r_h)\frac{\sum \log v_h}{l}| \\ &= |\sum r_h \log v_h| \leq \sum |r_h||\log v_h| \leq \sum |r_h| \cdot |\log \delta| \\ &\leq \sqrt{l}|\log \delta| \end{aligned}$$

it follows $\text{length}(\text{proj}_{\langle r \rangle} C_\delta) \leq 2\sqrt{l}|\log \delta|$. Hence, the first part of Proposition 2 is proved; the second part is an immediate consequence of the first part.

Q.E.D.

Proof of Proposition 3: Since K is assumed to be a compact subset of $I\!P^l$, we have

$$0 < \beta_1 := \min\{p_h | p \in K\} \quad \text{and} \quad \beta_2 := \max\{p_h | p \in K\} < \infty.$$

We will show that every δ with

$$\frac{\delta}{1-\delta} < \frac{\beta_1^2}{\beta_2^2(l-1)}$$

has the required properties. For this we have to show that

$$\min_h(\lambda v \otimes p)_h < \beta_1 \quad \text{or} \quad \max_h(\lambda v \otimes p)_h > \beta_2 \quad \text{for all} \quad \lambda > 0\,, \; v \in B_\delta\,, \; p \in K.$$

Since $\min_h(\lambda v \otimes p)_h \leq \lambda \cdot \min v_h \cdot \max p_h \leq \lambda\delta \cdot \beta_2$ the assertion is proved, if by luck $\lambda\delta\beta_2 < \beta_1$. Otherwise we have $\lambda \geq \dfrac{\beta_1}{\delta\beta_2}$, and therefore we obtain

$$\max_h\,(\lambda v \otimes p)_h \geq \lambda \max v_h \cdot \min p_h \geq \lambda\frac{1-\delta}{l-1} \cdot \beta_1 \geq \frac{\beta_1^2}{\beta_2} \cdot \frac{1-\delta}{\delta(l-1)} > \beta_2.$$

Q.E.D.

Proof of Proposition 4: Denote

$$V_{ij}^\delta := \{v \in S | v_i, v_j \geq \delta\} \;, \quad i \neq j, \quad \delta > 0.$$

With respect to the direction r defined by $r_i = 1/\sqrt{2}, \quad r_j = -1/\sqrt{2}, \quad r_h = 0$ for $h \neq i, j$, the set V_{ij}^δ has a finite diameter. Hence by Lemma 3

$$\lim_{n\to\infty} \nu_n^\Delta(V_{ij}^\delta) = 0,$$

and therefore

$$\lim_{n\to\infty} \nu_n^{\Delta}\left(\complement \cup_{i\neq j} V_{ij}^{\delta}\right) = 1,$$

where $\complement$ denotes the complement.
Obviously, for the given neighborhood U there exists a $\bar{\delta} > 0$ such that

$$\complement \cup_{i\neq j} V_{ij}^{\bar{\delta}} \subset U,$$

which proves the first part of Proposition 4. The second part is just an application of the weak topology of measures.

5 Discontinuity, Indeterminateness, Types of Consumers

For the example, given in Section 2, we have shown in Corollary 2, that the limit aggregate demand function $\bar{f}_\infty$ exists. But, according to Corollary 3, the limit function $\bar{f}_\infty$ does not depend continuously on the generating demand function: with respect to the topology of uniform convergence on compact sets of the function values and the derivative, the CES-demand function $f(\cdot,\cdot;a,\sigma)$ depends continuously on $0 < a < 1$, $\sigma > 0$, but the limit function $\bar{f}_\infty(\cdot,\cdot)$ does not depend continuously on a and σ when $a \neq \frac{1}{2}$ and $\sigma = 1$. This discontinuity is a consequence of the following two facts:

1. The topology of uniform convergence on compact subsets does not reflect how fast a function runs to infinity at the boundary. Therefore the function $x^{-\delta}, x \in I\!P$, converges to the function $x^{-1}, x \in I\!P$, for $\sigma \to 1$. This means, that a CES-function converges to a Cobb-Douglas function if $\sigma \to 1$.

2. According to Proposition 3, the aggregation process depends only on the boundary behavior of the generating function f.

In general, the limit aggregate demand function $\bar{f}_\infty$ may not exist. First we consider an example.

Let be (ρ_n) a sequence of densities on $I\!R^2$ with $m(\rho_n) \to 0$ and such that for even/odd n the second/fourth quadrant has full measure; such a sequence can be easily constructed. As the generating demand function choose

$$f(p,w) = \lambda\left(\frac{w}{2p_1}, \frac{w}{2p_2}\right) + (1-\lambda)\frac{\left(a^\sigma p_1^{-\sigma}, (1-a)^\sigma p_2^{-\sigma}\right)}{a^\sigma p_1^{1-\sigma} + (1-a)^\sigma p_2^{1-\sigma}} \cdot w$$

with $0 < \lambda < 1$, $a \neq \frac{1}{2}$, $\sigma \neq 1$. This function f fulfills Grandmont's boundary condition (Assumption (2e), p. 18).

Now the sequence $\bar{f}^n(p,w) := \int f(p,w)\rho_n(\alpha)\,d\alpha$ has two accumulation points, which are

$$(p,w) \mapsto \lambda\left(\frac{w}{2p_1}, \frac{w}{2p_2}\right) + (1-\lambda)\begin{cases} (\frac{w}{p_1}, 0) \\ (0, \frac{w}{p_2}) \end{cases} .$$

Although, there are two limit functions, there is no contradiction to Grandmont's result (Theorem 2.3, p. 19), because the derivative of both limit functions is a diagonal matrix with strictly negative diagonal.

Now we consider the general case. For a sequence (ρ_n) of densities with $m(\rho_n) \to 0$, the sequence of measures (μ_n) on $I\!R^l$ does not converge; total mass vanishes to infinity, i.e. $\lim \mu_n(A) = 0$ for every bounded A. Also the image measures (ν_n) on the v-parameter space $I\!P^l$ typically does not converge; some mass moves to the origin and some mass vanishes to infinity. But the measures ν_n^Δ on the unit simplex converge or have several accumulation points. If the generating demand function is linear in income, then ν_n can be replaced by ν_n^Δ, and hence, one expects one or several limit functions. But these functions depend on the boundary behavior of the generating demand function *and* on the manner how total mass vanishes to infinity by the measures μ_n.

Up to now, we have only considered one generating demand function f and density ρ. Grandmont considers a set A of "types" of consumers and therefore for each $a \in A$ a generating demand function $f(a,\cdot,\cdot)$ and a density function $\rho(a,\cdot)$ getting flat. Does this help? No, it makes the story even worse!

Because of Grandmont's independence assumption (p. 18, Assumption (2d)), we can compute the overall aggregate demand function by first integrating over the parameter α with respect to a measure μ which does not depend on a, and then integrating over A, i.e.

$$\int_{a\in A}\left[\int_{R^l} f^\alpha(a,p,w)d\mu\right]da.$$

As we have seen, the inner integral depends, with increasing flatness, more and more on the boundary behavior of $f(a,\cdot,\cdot)$. If the family $\{f(a,\cdot,\cdot) \mid a \in A\}$ would display, "in some sense", behavioral heterogeneity, but with the same boundary behavior, then this heterogeneity would be lost by increasing flatness of $\rho(a,\cdot)$. Hence, increasing flatness of the density ρ can only destroy, but not generate behavioral heterogeneity. Nevertheless, Grandmont's model has been used by many scientific authors.

References

Dieudonné, J. (1970). *Treatise on Analysis, Vol. 2.* New York and London: Academic Press.

Grandmont, J.M. (1992). Transformation of the Commodity Space, Behavioral Heterogeneity, and the Aggregation Problem. *Journal of Economic Theory 57,*1-35.