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Sets in Excess Demand in Ascending Auctions with Unit-Demand Bidders*

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Abstract

This paper analyzes the problem of selling a number of indivisible items to a set of unit-demand bidders. An ascending auction mechanism called the Excess Demand Ascending Auction (EDAA) is defined. The main results demonstrate that EDAA terminates in a finite number of iterations and that the exact auction mechanism in Demange, Gale and Sotomayor (J. Polit. Economy 94: 863–872, 1986) and its modification based on the Ford-Fulkerson method, proposed by Sankaran (Math. Soc. Sci. 28: 143–150, 1994), reduce to special cases of EDAA.

JEL Classification: C62, D44, D50.
Key Words: Multi-item auction; Unit-demand bidders; Excess demand; Algorithms.

1 Introduction

Economies with indivisible items and money have received considerable attention in the literature since the pioneering work of Shapley and Shubik (1972). Shapley and Shubik (1972) did not only prove the existence of a Walrasian equilibrium but also that the set of Walrasian price vectors forms a complete lattice. Consequently, there exist unique minimum and maximum Walrasian equilibrium price vectors. The existence result has later been refined and generalized by e.g. Demange and Gale (1985), Svensson (1983) and Alkan et al. (1991). Furthermore, the lattice property has been demonstrated to play a key role when designing strategy-proof mechanisms. For example, Andersson and Svensson (2008), Demange and Gale (1985) and Leonard (1983) demonstrate that by regarding the minimum Walrasian price equilibrium as a direct mechanism for allocating the indivisible items no agent can gain by strategic misrepresentation.

An obvious field of application for Walrasian pricing mechanisms is auction design. As emphasized by e.g. Ausubel (2004) and Perry and Reny (2005), dynamic auction mechanisms are overwhelmingly more prevalent than their direct counterparts (i.e. sealed bid auctions)

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because bidders often fear complete revelation of information. Consequently, a variety of different dynamic auction mechanisms has been designed to handle a number of different prerequisites. For example, Demange et al. (1986) defined an ascending multi-item auction mechanism (DGS) under the assumptions that bidders wish to acquire at most one item (unit-demand bidders) and in the absence of income effects (quasi-linear preferences). Their dynamic price adjustment mechanism always terminates at the minimum Walrasian equilibrium prices and it therefore implements the direct mechanism by Leonard (1983). Thus, truthful preference revelation constitutes a Nash equilibrium. In an environment with multi-demand bidders, Gul and Stacchetti (2000) specified a dynamic ascending price adjustment mechanism in the absence of income effects and under the gross substitute assumption of Kelso and Crawford (1982). Their dynamic mechanism also converges to the minimum Walrasian equilibrium prices but the results concerning non-manipulability are somewhat more negative. More explicitly, truthful preference revelation is a perfect Bayesian equilibrium only for a certain sub domain of the (gross substitute) preference domain. The analysis in Gul and Stacchetti (2000) was later simplified by Ausubel (2006) who also developed a dynamic and strategy-proof auction mechanism for multi-demand bidders. Also this mechanism converges to the minimum Walrasian equilibrium prices.

A common ingredient in dynamic auction mechanisms that terminates at the minimum Walrasian equilibrium prices in a unit-demand environment is that prices are updated based on information regarding groups of items that are overdemanded. In this context, a set of items is overdemanded, at a given price vector, if the number of bidders demanding only items in the set is greater than the number of items in the set. This is a natural approach since it is known from a famous theorem by Hall (1935) that a necessary requirement for reaching a Walrasian equilibrium is that all overdemanded sets of items are eliminated. However, it is also well-known that it is in general impossible to reach the minimum Walrasian equilibrium prices by only using information regarding overdemanded sets of items. To see this, suppose that there are three bidders (1, 2 and 3) and two items (A and B). Assume, in addition, that bidders 1 and 2 only demand item A and that bidder 3 only demands item B at the current prices. Clearly, the sets $\{A\}$ and $\{A, B\}$ are overdemanded. If, for example, bidder 1 is indifferent between receiving an item or not if the price of item A is raised by one unit, and the other bidders can tolerate higher price increases, then the minimum Walrasian equilibrium prices can never be reached if the prices of the items in $\{A, B\}$ are raised. In fact, to reach the desired equilibria, it suffices to increase the price of item A by one unit. This demonstrates that if prices are raised in an arbitrary overdemanded set of items, the minimum Walrasian equilibrium prices need not be reached in the process. In for example Demange et al. (1986) and Sun and Yang (2009), this problem is solved by restricting the attention to the family of minimal overdemanded sets, i.e., all overdemanded sets with the property that none of its proper subsets is overdemanded (as e.g. the set $\{A\}$ from the above example). In a modification to the auction mechanism in Demange et al. (1986) proposed by Sankaran (1994), a unique overdemanded, but not necessarily minimal overdemanded, set was identified based on the Ford-Fulkerson method and it was demonstrated that the minimum Walrasian equilibrium price vector will be reached in the process.

This paper considers a dynamic auction mechanism called the Excess Demand Ascending Auction (EDAA, henceforth) designed for economies with unit-demand bidders. As explained above, the price increments cannot be solely based on the overdemand criterion and additional information is needed. EDAA considers a subset of the family of overdemanded sets of items consisting of all "sets in excess demand". Formally, an overdemanded set of items S is in excess demand if the number of items in each subset T of S is strictly smaller than the number of bidders that demand some item in T and in addition only demand items in S. One of the main reasons for the attractiveness of the sets in excess demand is a result from Mishra and Talman (2010, Theorem 2) which states that a price vector equals the minimum Walrasian equilibrium price vector if and only if there are no overdemanded sets of items and no weakly underdemanded sets of items at these prices. Here, weakly underdemand refers to the definition in Mishra and Talman (2010), where a set of items is defined to be weakly underdemanded, at a given price vector, if the price of each item in the set exceeds the reservation price of the seller and the number of bidders demanding items in the set is less than or equal to the number of items in the set. Because there are no weakly underdemanded sets of items at the reservation prices, by definition, and since the price increments for sets in excess demand prescribed by EDAA guarantee that the family of weakly underdemanded sets stays empty, the prices will eventually converge to the minimum Walrasian equilibrium prices because as soon as the price of an item becomes sufficiently large the item cannot belong to any overdemanded set.

Sets in excess demand have a number of attractive properties. First, if a set is minimal overdemanded, at a given price vector, then it is also in excess demand. Hence, the family of minimal overdemanded sets is a subset of the family of sets in excess demand and, as a consequence, the selection of a specific set for price increments in Demange et al. (1986) is reduced to a special case of EDAA. Second, if two sets are in excess demand, at a given price vector, so is their union. Thus, by taking the union of all sets in excess demand, at given prices, it is possible to construct a unique set in excess demand with a largest cardinality. We demonstrate that the modification to the auction mechanism in Demange et al. (1986) proposed by Sankaran (1994) in fact always selects this unique largest set. Thus, the auction mechanisms in both Sankaran (1994) and Demange et al. (1986) can be regarded as special cases of EDAA where the price increments are based on sets of items that belong to specific subsets of the family of sets in excess demand. In this sense, this paper establishes a common framework and a link between Sankaran (1994) and Demange et al. (1986).

By giving the auctioneer the opportunity to base price increments on a larger subset of the family of overdemanded sets, different paths from the reservation prices to the minimum Walrasian equilibrium prices can be identified and some of them may be faster than the previously known paths. To investigate this, a simulation study was conducted. The main insight from the simulations is that by using EDAA, a non-negligible number of new paths appear and in a majority of the cases the fastest path from the reservation prices to the minimum equilibrium prices is neither accessible by DGS nor by the modification to DGS suggested by Sankaran (1994).

The paper is organized as follows. Section 2 introduces the economy. Important set definitions and set results are stated in Sections 3 and 4. Sections 5 and 6 describe EDAA and relate it to the auction in Demange et al. (1986) and its modification proposed by Sankaran (1994). Finally, Section 7 provides the simulation study.

2 The Model

The set of bidders and items are denoted by B and I, respectively. Each item $i \in I$ has a price p_i and a reservation price r_i which for simplicity and without loss of generality is set to zero. The prices are gathered in the vector p. A price vector p is said to be feasible if $p_i \geq 0$ for all

$i \in I$. The value of item $i \in I$ to bidder $b \in B$ is given by v_{bi}. These values are supposed to be integers since in reality no bidder can specify a monetary value more closely than to the nearest dollar or cent. There is a null-item, denoted by 0, whose value is zero to all bidders and whose price is always zero, i.e. $p_0 = 0$ and $v_{b0} = 0$ for all $b \in B$. For notational simplicity we let $I^* = I \cup \{0\}$ and $I^+(p) = \{i \in I \mid p_i > 0\}$. The demand correspondence for bidder $b \in B$ at prices p is defined by:

$$D_b(p) = \{i \in I^* \mid v_{bi} - p_i \geq v_{bj} - p_j \text{ for all } j \in I^*\}.$$

A price vector p is said to be a Walrasian equilibrium price vector if there is an assignment $x : B \mapsto I^*$ such that $x_b \in D_b(p)$ for all $b \in B$ and if $b' \neq b$ and $x_b = x_{b'}$ then $x_b = 0$, i.e. each bidder is assigned an item from his demand set and if two bidders are assigned the same item then both bidders are assigned the null-item. The pair (p, x) is a Walrasian equilibrium if p is a Walrasian equilibrium price vector and if $x_b \neq i$ for all $b \in B$ then $p_i = 0$, i.e. if an item is not assigned to some bidder, then its price equals the zero reservation price. As demonstrated by Shapley and Shubik (1972) the set of competitive price vectors is non-empty and forms a complete lattice. Thus, the existence of a unique minimum Walrasian equilibrium price vector is guaranteed.

3 Set Definitions

This section states a number of basic set definitions. Of particular importance is Definition 4 where so-called sets in excess demand are introduced. As will be demonstrated later, this is a weaker notion than the notion of minimal overdemand (Proposition 1), and as long as the family of sets in excess demand is non-empty there always exists a unique set in excess demand with maximum cardinality (Proposition 2). Minimal overdemanded sets play a key role in the auction mechanism of Demange et al. (1986), DGS for short. The unique set in excess demand with maximum cardinality appears in the modification to DGS based on the Ford-Fulkerson method, as proposed by Sankaran (1994).

The bidders that only demand items in the set $S \subseteq I$ at prices p are collected in the set $O(S, p)$, i.e., $O(S, p) = \{b \in B \mid D_b(p) \subseteq S\}$. A set of items $S \subseteq I$ is said to be overdemanded at prices p if the number of bidders demanding only items in S is greater than the number of items in the set.

Definition 1. *A set of items S is overdemanded at prices p if $S \subseteq I$ and $\#O(S, p) > \#S$.*

The family of overdemanded sets of items at prices p is denoted by $OD(p)$ and is formally given by:

$$OD(p) = \{S \subseteq I \mid \#O(S, p) > \#S\}.$$

A minimal overdemanded set of items is an overdemanded set of items if no proper subset of it is overdemanded.

Definition 2. *A set of items S is minimal overdemanded at prices p if S is overdemanded at prices p and:*

$$\#T \geq \#O(T, p) \text{ for all } T \subset S.$$

The family of minimal overdemanded sets of items at prices p is denoted by $MOD(p)$ and is given by:

$$MOD(p) = \{S \subseteq I \mid \#O(S,p) > \#S \text{ and } \#T \geq \#O(T,p) \text{ for all } T \subset S\}.$$

In order to provide a weakening of Definition 2, some additional notation must be introduced. All bidders that demand some item in the set $S \subseteq I$ at prices p are collected in the set $U(S,p)$, and all bidders in the subset $B' \subseteq B$ that demand some item in the set $S \subseteq I$ at prices p are gathered in the set $U(S,p \mid B')$. Formally:

$$U(S,p) = \{b \in B \mid D_b(p) \cap S \neq \varnothing\},$$
$$U(S,p \mid B') = \{b \in B' \mid D_b(p) \cap S \neq \varnothing\} \text{ for } B' \subseteq B.$$

Note that from the definition of $O(S,p)$ and the above conditions it follows directly that:

(i) $O(S,p) \subseteq U(S,p)$,

(ii) $U(S,p \mid O(S,p)) = O(S,p)$.

The following definition of weakly underdemanded sets can be found in Mishra and Talman (2010, Definition 3).

Definition 3. *A set of items S is weakly underdemanded at prices p if $S \subseteq I^+(p)$ and $\#U(S,p) \leq \#S$.*

The above definition differs slightly from the definition of underdemand in Sotomayor (2002) in the sense that Sotomayor (2002) assumes that there is a dummy bidder that can be allocated more that one item and in addition demands each item at the reservation prices. Note also that the definition requires that S is a subset of $I^+(p)$. Thus, no set S where the price of some item equals the reservation price (zero) can be weakly underdemanded. Consequently, at the reservation prices, no set of items is weakly underdemanded. The family of weakly underdemanded sets of items at prices p is denoted by $WUD(p)$ and is formally defined as follows:

$$WUD(p) = \{S \subseteq I^+(p) \mid \#U(S,p) \leq \#S\}.$$

We are now ready to provide a weakening of Definition 2 and introduce the notion of sets in excess demand. A set of items S is in excess demand at prices p if at prices p the number of items in each subset T of S is strictly smaller than the number of bidders that demand some item in T and in addition only demand items in S. This definition can be formalized as follows.

Definition 4. *A set of items S is in excess demand at prices p if $S \subseteq I$ and:*

$$\#U(T,p \mid O(S,p)) > \#T \text{ for each } T \subseteq S. \tag{1}$$

Note that if a set is in excess demand at prices p it must be overdemanded at p. The collection of sets of items in excess demand at prices p is denoted by $ED(p)$ and can be described by:

$$ED(p) = \{S \in I \mid \#U(T,p \mid O(S,p)) > \#T \text{ for each } T \subseteq S\}.$$

4 Set Results

Given the basic set definitions from the previous section, this section provides a number of properties of the family of sets in excess demand at given prices p. These properties will later be useful to prove that DGS and the Sankaran (1994) modification to DGS are in fact special cases of the algorithm proposed in this paper.

The first result establishes that the family of minimal overdemanded sets is a subset of the family of sets in excess demand. In this sense the notion of sets in excess demand is a weaker notion than minimal overdemand.

Proposition 1. *At any prices p it holds that $MOD(p) \subseteq ED(p)$.*

Proof. Suppose that $S \in MOD(p)$ and $S \notin ED(p)$. $S \in MOD(p)$ implies that $\#O(S,p) > \#S$ and $\#O(T,p) \leq \#T$ for all $T \subset S$. $S \notin ED(p)$ implies that there exists $K \subset S$ such that:

$$\#U(K, p \mid O(S,p)) \leq \#K.$$

Moreover, K is a proper subset of S, since $S \in MOD(p)$ and $U(S,p \mid O(S,p)) = O(S,p)$. Let now $S' = S - K$ and note that S' is a non-empty and proper subset of S. It then follows that $O(S',p) = O(S,p) - U(K,p \mid O(S,p))$. Thus:

$$\begin{aligned} \#O(S',p) &= \#(O(S,p) - U(K,p \mid O(S,p))) \\ &\geq \#O(S,p) - \#U(K,p \mid O(S,p)) \\ &> \#S - \#K \\ &= \#S'. \end{aligned}$$

But then S' is a proper overdemanded subset of S at prices p, which contradicts that $S \in MOD(p)$. □

Note that the conversion of the above proposition is not true, i.e., a set in excess demand need not be a minimal overdemanded set.

The next result states that the union of two sets in excess demand is again a set in excess demand. Note however that the union of two minimal overdemanded sets is not minimal overdemanded by definition. Hence, the weakening of minimal overdemand can be adopted in order to analyze ”larger” sets than in DGS. This has previously been noted in an example in Sankaran (1994, p.146).

Lemma 1. *If $S \in ED(p)$ and $T \in ED(p)$, then $S \cup T \in ED(p)$.*

Proof. To prove the result we need to demonstrate that the following condition is satisfied:

$$\#U(K, p \mid O(S \cup T, p)) > \#K \text{ for each } K \subseteq S \cup T. \tag{2}$$

Note first that if $b \in O(S,p)$ or $b \in O(T,p)$ then $b \in O(S \cup T, p)$. This result together with the observation that there may exist a bidder b with $b \notin O(S,p) \cup O(T,p)$ but $b \in O(S \cup T, p)$ gives:

$$U(L, p \mid O(R,p)) \subseteq U(L, p \mid O(S \cup T, p)) \text{ for each } L \subseteq R \text{ and } R \in \{S, T\}.$$

Consequently, $\#U(L,p \mid O(S \cup T, p)) \geq \#U(L, p \mid O(R,p)) > \#L$ for each $L \subseteq R$ and $R \in \{S,T\}$. The last inequality follows from Definition 4 since $S \in ED(p)$ and $T \in ED(p)$ by assumption. This implies that condition (2) holds if $K \subseteq S$ or $K \subseteq T$.

It remains to prove that condition (2) holds when $K \subseteq S \cup T$ and both $K \nsubseteq S$ and $K \nsubseteq T$. Take any non-empty $A \subseteq S$ with $A \nsubseteq T$ and $C \subseteq T - S$ and let $K = A \cup C$. Because $A \cap C = \varnothing$, it holds that if $b \in O(A,p)$ then $b \notin O(C,p)$, and if $b \in O(C,p)$ then $b \notin O(A,p)$. Consequently, $O(A,p) \cap O(C,p) = \varnothing$. Now, because $S \in ED(p)$ and $T \in ED(p)$ by assumption it follows from Definition 4 that $\#U(A,p \mid O(S,p)) > \#A$ and $\#U(C,p \mid O(T,p)) > \#C$. These facts, together with the observation that there may exist a bidder b with $b \notin O(S,p) \cup O(T,p)$ but $b \in O(S \cup T,p)$, give:

$$\begin{aligned}
\#U(K,p \mid O(S \cup T,p)) &= \#U(A \cup C,p \mid O(S \cup T,p)) \\
&\geq \#U(A \cup C,p \mid O(S,p) \cup O(T,p)) \\
&\geq \#U(A,p \mid O(S,p)) + \#U(C,p \mid O(T,p)) \\
&> \#A + \#C \\
&= \#(A \cup C) \\
&= \#K.
\end{aligned}$$

□

An immediate consequence of Lemma 1 is that in case the family of sets in excess demand is non-empty, there always exists a unique set in excess demand with a maximal cardinality.

Proposition 2. *If $ED(p) \neq \varnothing$ then there exists a unique set $S_* \in ED(p)$ where $\#S_* > \#T$ for any $T \in ED(p) - \{S_*\}$.*

5 The Excess Demand Ascending Auction (EDAA)

This section proposes a new type of auction called the Excess Demand Ascending Auction (EDAA, henceforth). The point of departure is a theorem in Mishra and Talman (2010, Theorem 2) which states that a price vector equals the minimum Walrasian price vector if and only if there are no overdemanded sets of items and no weakly underdemanded sets of items at these prices. To see how this result can be used, recall from Section 3 that there are no weakly underdemanded sets of items at the reservation prices. Hence, we need only to specify price increments that guarantee that the family of weakly underdemanded sets stays empty. The reason for this is that as soon as the price of an item becomes sufficiently large the item cannot belong to any overdemanded set, and then since the price increments guarantee that the family of weakly underdemanded sets is empty, the prices converge to the minimum Walrasian equilibrium prices by the Mishra-Talman's Theorem 2. This idea was in fact outlined in Mishra and Talman (2010, pp.11-12) even though no concrete dynamic procedure or any specific price increments were specified.

Our first task is to find a general price increment which guarantees that the family of weakly underdemanded sets stays empty given that it is empty before the prices are increased. To find such increments, let S be a set in excess demand at prices p and for $n \in \mathbb{N}$ let the price vector $q(S,p,n)$ be given by:

$$q_i(S,p,n) = \begin{cases} p_i + n & \text{if } i \in S, \\ p_i & \text{otherwise.} \end{cases} \tag{3}$$

For bidder $b \in O(S,p)$ we define:

$$k_b(S,p) = \min\{n \in \mathbb{N} \mid D_b(q(S,p,n)) - S \neq \varnothing\}. \tag{4}$$

The interpretation of $k_b(S,p)$ is that it identifies the minimum price increase at prices p of the items in S required for bidder b to leave $O(S,p)$, or equivalently, the amount by which the prices of items in S can be raised at prices p until bidder b becomes indifferent between his currently demanded items and an item not in S. Note that under the separability and linearity assumptions, the minimum price increase $k_b(S,p)$ for the items in S is a positive integer and can easily be computed for any bidder $b \in O(S,p)$.

Lemma 2. *Suppose that $S \in ED(p)$ and $WUD(p) = \varnothing$ and let $k = \min\{k_b(S,p) | b \in O(S,p)\}$. Then $k \geq 1$ and $WUD(q(S,p,k)) = \varnothing$.*

Proof. Let $q = q(S,p,k)$. It is clear that $k \geq 1$ because $k_b(S,p)$ is a positive integer for every $b \in O(S,p)$. In order to show that $WUD(q) = \varnothing$, we need to prove that for an arbitrary $T \subseteq I^+(q)$ it holds that:

$$\#U(T,q) > \#T. \tag{5}$$

Three different cases are considered.

Case (i) $T \subseteq S$. We first make three observations. First, $U(T,q \mid O(S,p)) \subseteq U(T,q)$ because $O(S,p) \subseteq B$. Second, $D_b(p) \subseteq D_b(q)$ for all $b \in O(S,p)$ by definition of $k_b(S,p)$ and construction of q. Consequently, if $b \in U(T,p \mid O(S,p))$, then $b \in U(T,q \mid O(S,p))$, implying that $\#U(T,q \mid O(S,p)) \geq \#U(T,p \mid O(S,p))$. Third, $\#U(T,p \mid O(S,p)) > \#T$ since $S \in ED(p)$. From these three observations, we conclude:

$$\#U(T,q) \geq \#U(T,q \mid O(S,p)) \geq \#U(T,p \mid O(S,p)) > \#T,$$

which demonstrates that condition (5) is satisfied when $T \subseteq S$.

Case (ii) $T \subseteq I^+(p) - S$. Since $WUD(p) = \varnothing$ and $T \subseteq I^+(p)$ it follows that $\#U(T,p) > \#T$. Moreover, by construction of q, an item $i \in T$ belongs to $D_b(q)$ if it belongs to $D_b(p)$, and therefore $U(T,p) \subseteq U(T,q)$. Hence:

$$\#U(T,q) \geq \#U(T,p) > \#T,$$

i.e., condition (5) is satisfied when $T \subseteq I^+(p) - S$.

Case (iii) $T = A \cup C$ where $\varnothing \neq A \subseteq S$ and $\varnothing \neq C \subseteq I^+(p) - S$. Clearly, $A \cap C = \varnothing$. By construction of q it holds that if $i \in C$ belongs to $D_b(p)$ then it also belongs to $D_b(q)$. Thus, $U(C,p) \subseteq U(C,q)$, and as a consequence:

$$\#U(T,q) = \#U(A \cup C,q) = \#(U(A,q) \cup U(C,q)) \geq \#(U(A,q) \cup U(C,p)). \tag{6}$$

Because, by construction of q, $q_i > p_i$ for $i \in A$ and $q_i = p_i$ for $i \in C$, it follows that $U(A,q) \cap U(C,p) = \varnothing$, and therefore:

$$\#(U(A,q) \cup U(C,p)) = \#U(A,q) + \#U(C,p). \tag{7}$$

Since $A \subseteq S$ it follows from Case (i) that $\#U(A,q) > \#A$. Moreover, because $WUD(p) = \varnothing$ and $C \subseteq I^+(p)$ it follows that $\#U(C,p) > \#C$. These observations together with (6) and (7) yield:

$$\#U(T,q) > \#A + \#C = \#(A \cup C) = \#T,$$

which concludes the proof. $\square$

Given the price increments in Lemma 2, EDAA can be formalized as follows.

Algorithm 1 (EDAA)**.** *Introduce an iteration counter t and let p^t denote the price vector in iteration t. Set $t := 0$ and initialize the price vector to the reservation prices, $p^0 := 0$.*

1. *Collect the demand sets $D_b(p^t)$ of every bidder $b \in B$.*

2. *If there is no overdemanded set of items at p^t, the algorithm is terminated. Otherwise:*

3. *Choose a set $S^t \in ED(p^t)$.*

4. *Compute $p^{t+1} = q(S^t, p^t, k^t)$ where $k^t = \min\{k_b(S^t, p^t) \mid b \in O(S^t, p^t)\}$.*

5. *Set $t := t + 1$ and start a new iteration from Step 1.*

Our next theorem states that EDAA converges to the minimum Walrasian equilibrium prices in a finite number of iterations. The proof of the result has already been stated in the beginning of this section. That is, because there are no weakly underdemanded items at the reservation prices, the price increments defined in the algorithm guarantee that the family of weakly underdemanded items stays empty. The proof then follows directly from Mishra and Talman (2010, Theorem 2), and the observation that for sufficiently large price increments no set of items can be overdemanded.

Theorem 1. EDAA *converges to the minimum Walrasian equilibrium prices in a finite number of iterations.*

The next observation is that DGS is a special case of EDAA.

Algorithm 2 (DGS)**.** DGS *replaces Step 3 of Algorithm 1 with:*

3. *Choose a set $S^t \in MOD(p^t)$.*

Proposition 3. DGS *is a special case of* EDAA*.*

Proof. This result is a direct consequence of the fact that the price increments in DGS are based on minimal overdemanded sets of items and by that Proposition 1 a minimal overdemanded set is a set in excess demand. □

Note that in the original description of DGS, the price update for an item $i \in S^t$ is given by $p_i^t + 1$ and not by $p_i^t + k^t$ as in Step 4. However, both DGS and EDAA converge to the minimum Walrasian equilibrium prices regardless of whether the price increments are taken to be equal to 1 or equal to k^t.

We end this section with a numerical example that demonstrates EDAA and highlights the differences between EDAA and DGS.

Example 1. Suppose that $B = \{1, 2, 3, 4, 5\}$ and $I = \{1, 2, 3\}$. The values v_{bi}, $b \in B$, $i \in I$, of the items to the bidders are given by the matrix:

$$V = \begin{bmatrix} 24 & 8 & 32 \\ 0 & 12 & 66 \\ 99 & 66 & 53 \\ 85 & 30 & 18 \\ 45 & 74 & 94 \end{bmatrix}, \tag{8}$$

where the column corresponding to the null-item, $v_{b0} = 0$, has been omitted. At the reservation prices $r = (0, 0, 0)$ the bidders' initial demand sets become $D_1(r) = D_2(r) = D_5(r) = \{3\}$ and $D_3(r) = D_4(r) = \{1\}$. At the reservation prices r we have:

$$ED(r) = \{\{1\}, \{3\}, \{1, 3\}\},$$
$$MOD(r) = \{\{1\}, \{3\}\}.$$

In Figure 1, all possible paths from the seller's reservation prices $r = (0, 0, 0)$ to the minimum Walrasian equilibrium prices $p^{\min} = (79, 46, 66)$ are represented in the form of a graph where each vertex corresponds to a price vector reachable by EDAA and each arc is associated with a specific set in excess demand used to update prices. The solid arcs and the shaded vertices are not reachable using DGS. As can be seen from Figure 1, updating prices based on EDAA leads to a Walrasian equilibrium in between 5-12 iterations depending on the specific choices of sets in excess demand.[1] In this example, it is also interesting to note that if the selection of the specific set in excess demand is based on a rule where minimal overdemanded sets are not chosen whenever possible (e.g. if the set $\{1, 3\}$ is selected in the first price update), EDAA converges to the minimum Walrasian equilibrium prices in at most 8 iterations. DGS on the other hand requires between 8-12 iterations before termination, because DGS can only follow the dashed paths in Figure 1. □

6 Computing Sets in Excess Demand

A for practical purposes important modification to DGS based on the Ford-Fulkerson method was proposed by Sankaran (1994), and has been used by e.g. Mishra and Parkes (2009) for computer simulations. It is motivated by the fact that Steps 2 and 3 of Algorithm 2 imply searching $2^{\#I}$ sets to find a minimal overdemanded set, which is computationally intractable as the number of items increases. The Ford-Fulkerson method can be used to find a (not necessarily minimal overdemanded) set of items whose prices can be raised in polynomial time. In this section, we show that the algorithm proposed by Sankaran (1994) is in fact a special case of EDAA.

The Ford-Fulkerson method is a classical method for network flow problems, described by e.g. Bertsekas (1998), that can be used to find a feasible assignment of maximum cardinality. By feasible assignment we mean a set $X(p) \subseteq \{(b, i) \mid b \in B, i \in I^*\}$ of bidder-item pairs at prices p such that $i \in D_b(p)$ for all $(b, i) \in X(p)$, and all $b \in B$ and all $i \in I$ are part of at most one pair from $X(p)$. No assumption is made on the cardinality of $X(p)$, but if $\#X(p) = \#B$ each bidder is assigned an item and p is therefore a Walrasian equilibrium price vector. This observation can be used to formulate an alternative termination criterion for Step 2 of Algorithm 1.

Starting from an initial feasible assignment, the Ford-Fulkerson method iteratively updates $X(p)$ based on augmenting paths. An augmenting path with respect to $X(p)$ is a sequence on the form $\mathcal{P} = (b_0, i_0, \dots, b_n, i_n)$ such that each bidder b_j, $j \neq 0$, is assigned an item by $X(p)$, each items i_j, $j \neq n$, is assigned to a bidder by $X(p)$, and $i_j \neq 0$ for all $j \neq n$. This informal characterization corresponds to a bipartite matching problem modified to account for the null-item.

[1] There are two paths through the graph that require 5 iterations to converge. Both of them pass trough the vertex $(33, 0, 20)$.

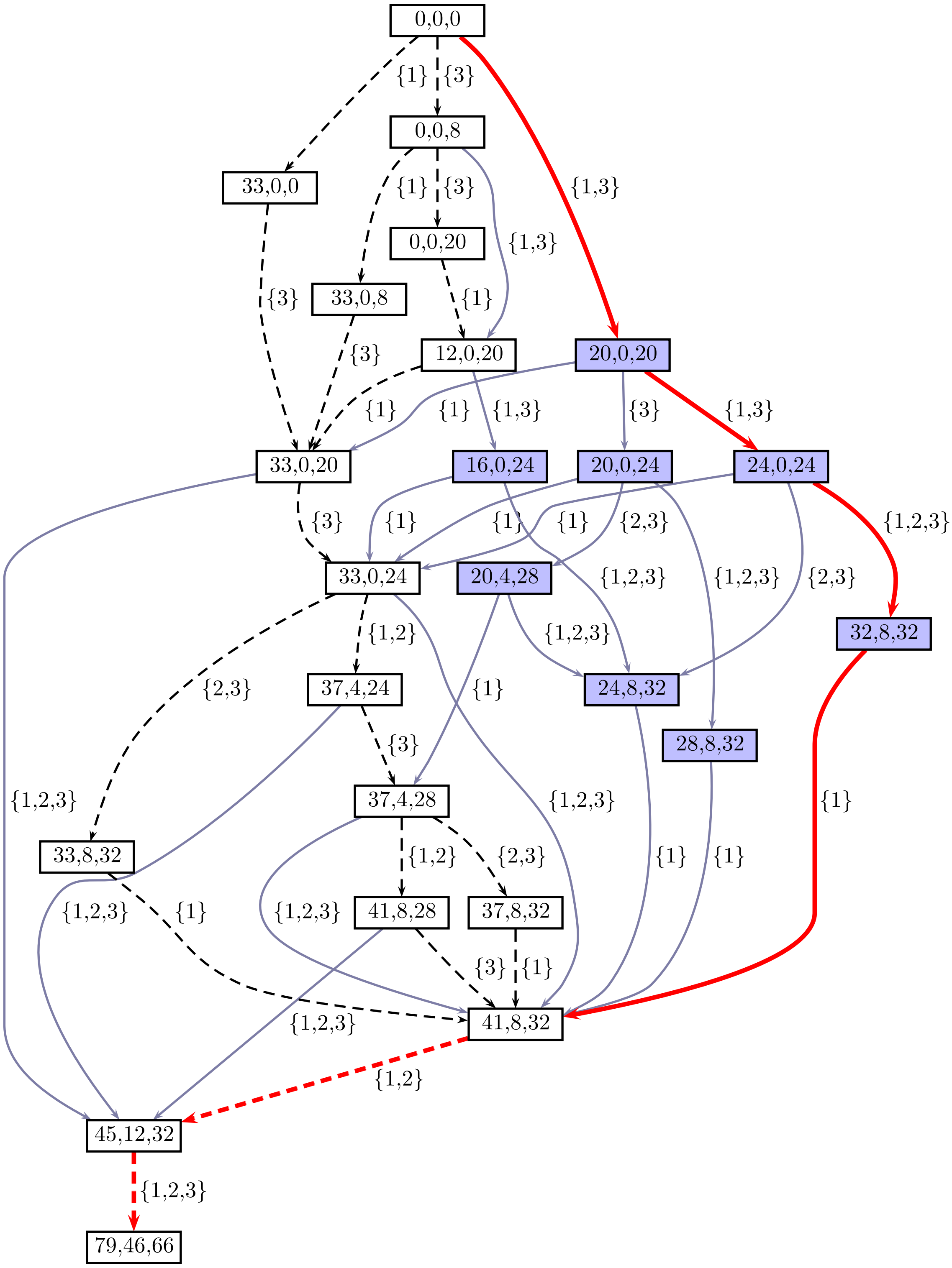

Figure 1: Graph corresponding to an auction with three items and five bidders whose valuations are given by (8). Vertices are labeled with current prices in EDAA. Arcs are labeled with the sets in excess demand used to update prices.

Algorithm 3 (Ford-Fulkerson)**.** *Initialize the feasible assignment to the empty set,* $X(p) := \varnothing$*. For each iteration:*

1. *Find an augmenting path* $\mathcal{P} = (b_0, i_0, \ldots, b_n, i_n)$ *with respect to* $X(p)$*, e.g. using Algorithm* 4*. If no such* $\mathcal{P}$ *exists, then* $\#X(p)$ *is maximized and the algorithm is terminated. Otherwise:*

2. *Augment the assignment along* $\mathcal{P}$*:*

$$X(p) := (X(p) - \{(b_j, i_{j-1}) \mid j \in \{1, 2, \ldots, n\}\}) \cup \{(b_j, i_j) \mid j \in \{0, 1, \ldots, n\}\} .$$

3. *Start a new iteration from Step* 1*.*

For the considered problem the augmentation in Step 2 of Algorithm 3 increases the cardinality of $X(p)$ by one. Thus $\#X(p)$ is strictly increasing as long as there is an augmenting path, and convergence is achieved in less than or equal to $\#B$ iterations. We next consider a breadth-first search for augmenting paths, given in Algorithm 4 for the ascending auction context.

Algorithm 4 (Breadth-first Search for an Augmenting Path)**.** *Given a feasible assignment* $X(p)$*, let the initial set of bidders be defined by:*

$$B_0 = \{b \in B \mid (b, i) \notin X(p) \text{ for all } i \in I^*\}, \tag{9}$$

and label $b \in B_0$ *with* s*. All other bidders and items are initially unlabeled. Introduce the iteration counter* n*. For each iteration* $n = 0, 1, \ldots$*:*

1. *Define the set of items:*

$$I_n = \{i \in I^* - \cup_{j=0}^{n-1} I_j \mid (b, i) \notin X(p) \text{ and } i \in D_b(p) \text{ for some } b \in B_n\}. \tag{10}$$

   Label all $i \in I_n$ *with a respective* b*.*

2. *If* $0 \in I_n$ *or there is an* $i \in I_n$ *such that* $(b, i) \notin X(p)$ *for all* $b \in B$*, a shortest augmenting path has been found. Terminate the algorithm.*

3. *If* $I_n = \varnothing$*, no augmenting path exists. Terminate the algorithm.*

4. *Define the set of bidders:*

$$B_{n+1} = \{b \in B - \cup_{j=0}^{n} B_j \mid (b, i) \in X(p) \text{ for some } i \in I_n\}. \tag{11}$$

   Label all $b \in B_{n+1}$ *with a respective* i*.*

5. *Set* $n := n + 1$ *and start a new iteration from Step 1.*

In case Algorithm 4 terminates with $I_n = \varnothing$, no augmenting path exists and the assignment $X(p)$ has maximum cardinality. Otherwise a shortest augmenting path $(b_0, i_0, \ldots, b_n, i_n)$, with $b_j \in B_j$ and $i_j \in I_j$ for $j = 0, 1, \ldots, n)$, with respect to $X(p)$ can be found by backtracking assigned labels starting from 0 if $0 \in I_n$, or from any $i \in I_n$ such that $(b, i) \notin X(p)$ for all $b \in B$, until a bidder with label s is encountered.

The breadth-first search is guaranteed to find a shortest augmenting path if one exists. Several possible labels may exist in Steps 1 and 4 of Algorithm 4, and there may be more than one shortest augmenting path. The Ford-Fulkerson method converges regardless of the path and labels chosen. Furthermore, if an augmenting path does not exist the set of labeled items at the termination of the search is in excess demand, and can be used in Step 3 of Algorithm 1.

The final result of the paper establishes that DGS with the modification proposed by Sankaran (1994) is a special case of EDAA where the prices are updated for the unique set in excess demand with maximum cardinality (recall from Proposition 2 that such a set exists as long as $MOD(p) \neq \varnothing$). To prove this result, two lemmas are needed.

Lemma 3. *Suppose that Algorithm 4 terminates with $I_n = \varnothing$, and that $S = \cup_{j=0}^{n} I_j$. Then $O(S,p) = \cup_{j=0}^{n} B_j$, and for each $i \in S$ there is some $b \in O(S,p)$ such that $(b,i) \in X(p)$.*

Proof. To show that $O(S,p) = \cup_{j=0}^{n} B_j$ we prove that for any $b \in B$, $b \in \cup_{j=0}^{n} B_j$ if and only if $D_b(p) \subseteq S$. We do this by considering three mutually exclusive cases for bidders $b \in B$.

Case (i) $(b,i) \in X(p)$ for some $i \notin S$. Then $D_b(p) \nsubseteq S$, so it must be shown that b is not an element of some B_j. The condition in (11) that $(b',i') \in X(p)$ for some $i' \in I_j$ is always false for $b' = b$ since $i \notin S$ by assumption. It follows that $b \notin \cup_{j=0}^{n} B_j$.

Case (ii) $(b,i) \in X(p)$ for some $i \in S$. Without loss of generality, assume that $i \in I_m$ for $0 \leq m < n$, where the inequality follows from the assumption $I_n = \varnothing$. Equation (11) gives:

$$\begin{aligned} B_{m+1} &= \{b' \in B - \cup_{j=0}^{m} B_j \mid (b',i') \in X(p) \text{ for some } i' \in I_m\} \\ &\supseteq \{b' \in \{b\} - \cup_{j=0}^{m} B_j\}, \end{aligned}$$

i.e. $b \in B_{m+1}$. To show that $D_b(p) \subseteq S$ we need only compute the set of items I_{m+1} in the next iteration from (10):

$$\begin{aligned} I_{m+1} &= \{i' \in I^* - \cup_{j=0}^{m} I_j \mid (b',i') \notin X(p) \text{ and } i' \in D_{b'}(p) \text{ for some } b' \in B_{m+1}\} \\ &\supseteq \{i' \in I^* - \cup_{j=0}^{m} I_j \mid i' \in D_b(p) - \{i\}\} \\ &= D_b(p) - \cup_{j=0}^{m} I_j, \end{aligned}$$

where $i \in I_m$ has been used in the last equality.

Case (iii) $(b,i) \notin X(p)$ for all $i \in I^*$. From (9) we have $b \in B_0$. Insertion in (10) yields $D_b(p) \subseteq I_0$, which concludes the proof of the first statement.

To prove the second statement we observe that $0 \notin S$, and for each $i \in S$ there is some $b \in B$ such that $(b,i) \in X(p)$. Otherwise the algorithm would terminate in Step 2, which violates the assumption $I_n = \varnothing$. From Case (ii) above it follows that $b \in O(S,p)$. □

Lemma 4. *Suppose that Algorithm 4 terminates with $I_n = \varnothing$, and that $S = \cup_{j=0}^{n} I_j$. If $b \notin O(S,p)$, then $(b,i) \in X(p)$ for some $i \notin S$.*

Proof. The result follows from Lemma 3, and Cases (i) and (iii) from its proof. □

Proposition 4. DGS *with the modification $S^t = \cup_{j=0}^{n} I_j$ in iteration t, as proposed by Sankaran (1994), is a special case of* EDAA, *where S^t is the unique set in excess demand at prices p^t with maximum cardinality.*

Proof. To prove the theorem, we demonstrate that if Algorithm 4 terminates with $I_n = \varnothing$, then $S = \cup_{j=0}^{n} I_j$ is the set in excess demand with maximum cardinality.

By assumption Algorithm 4 is not terminated in Step 2, so it follows that $0 \notin S$, and thus $S \subseteq I$. Lemma 3 states that all items in S are assigned to bidders in $O(S,p)$. It follows directly that $\#U(T,p \mid O(S,p)) \geq \#T$ for all $T \subseteq S$. To prove that $S \in ED(p)$, suppose $\#U(T,p \mid O(S,p)) = \#T$ for some $T \subseteq S$.

Let $K = \{b \in O(S,p) \mid (b,i) \in X(p) \text{ for some } i \in T\}$ denote the set of bidders that are assigned an item from T. If $\#U(T,p \mid O(S,p)) = \#T$ then $D_b(p) \cap T = \varnothing$ holds for all $b \in O(S,p) - K$. Using Lemma 3 we have from (10) that $I_n \cap T \neq \varnothing$ only if $B_n \cap K \neq \varnothing$, and from (11) that $B_{n+1} \cap K \neq \varnothing$ only if $I_n \cap T \neq \varnothing$. Since the initial set of bidders satisfies $B_0 \cap K = \varnothing$ we obtain the contradictions $T \nsubseteq S$ and $K \nsubseteq O(S,p)$. Thus $\#U(T,p \mid O(S,p)) > \#T$ for all $T \subseteq S$, which completes the proof that $S \in ED(p)$.

Because S is a set in excess demand we know from Proposition 2 that there exists a unique $S_* \in ED(p)$ of maximum cardinality, and from Lemma 1 it follows that $S \subseteq S_*$. To prove that $S = S_*$, suppose $S \subset S_*$.

Denote the set of bidders that are assigned an item in $S_* - S$ by $L = \{b \in B - O(S,p) \mid (b,i) \in X(p) \text{ for some } i \in S_* - S\}$, and the set of assigned items in $S_* - S$ by $S_A = \{i \in S_* - S | (b,i) \in X(p) \text{ for some } b \in B - O(S,p)\}$. Without loss of generality we assume that $S_A \neq \varnothing$. Otherwise $O(S_*,p) = O(S,p)$ by Lemma 4, and $U(S_* - S, p \mid O(S_*,p)) = \varnothing$ after applying Lemma 3, thus implying that S_* is not in excess demand.

Note that $D_b(p) \nsubseteq S_*$ may hold for some $b \in L$, and that $U(S_A, p \mid O(S_*,p)) = \varnothing$ from Lemma 3. We obtain:

$$\begin{aligned} \#U(S_A, p \mid O(S_*,p)) &= \#U(S_A, p \mid O(S_*,p) - O(S,p)) \\ &\leq \#U(S_A, p \mid L) = \#S_A. \end{aligned}$$

Hence, if $S \subset S_*$, then S cannot be in excess demand, thereby proving that S is the set in excess demand with maximum cardinality. $\square$

Given Proposition 4 and the description of Algorithms 3 and 4, the Ford-Fulkerson method can be implemented in EDAA to identify the set in excess demand with maximum cardinality.

Algorithm 5 (EDAA with Ford-Fulkerson)**.** *Replace Steps* 2 *and* 3 *of Algorithm* 1 *with:*

2. *Compute an assignment $X(p)$ of maximum cardinality using the Ford-Fulkerson method* (*Algorithm 3*)*. If $|X(p)| = |B|$ the algorithm is terminated. Otherwise:*

3. *Let $S^t = \cup_{j=0}^{n} I_j$ be the set of labeled items upon termination of the augmenting path search* (*Algorithm* 4) *in the last Ford-Fulkerson iteration.*

We end this section by noting that if Algorithm 5 is applied to the problem described in Example 1, the unique path trough the graph in Figure 1 is the path marked by bold (solid or dashed) arcs to the far right in the figure. Thus, the minimum Walrasian equilibrium prices are identified in 7 iterations. This should be compared to the fastest path using EDAA or DGS which is 5 iterations and 8 iterations, respectively.

7 Simulations

This section presents some computational results obtained by numerical simulation. The main objective is to illustrate that EDAA is a significant extension of previously known mechanisms, i.e. that there is a large number of paths from the reservation prices to the minimum Walrasian equilibrium prices that are neither reachable by DGS nor by the modification to DGS suggested by Sankaran (1994). This is achieved by creating 500 sets of randomized auction graphs using the same procedure as in Example 1 (see in particular Figure 1). In the simulations it is assumed that there are five items (not counting the null-item) and eight bidders. The values of items to bidders are given by $v_{bi} = \max\{0, \nu_{bi}\}$ where ν_{bi} are pseudo-random integers uniformly distributed in the range $[-33, 100]$, i.e. each v_{bi} is zero with a probability of approximately 25%. The reservation prices r_i are set to zero for all items $i \in I$.

Given the above setup, Table 1 specifies the mean number of vertices (column 2) and arcs (column 3) for the 500 sample graphs. As can be seen from the table, EDAA significantly increases the average size of the auction graphs and, consequently, also the average number of paths through the graphs. Note also that because the modification to DGS proposed by Sankaran (1994) always gives a unique path, the mean number of vertices and arcs is naturally small.

A for practical purposes important issue is the termination speed. The expected number of iterations based on a random walk assumption for the three auction mechanisms are stated in column 4 of Table 1. As can be seen from the table, DGS is clearly outperformed by the other two mechanisms, and the modification to DGS suggested by Sankaran (1994) performs best in terms of the expected number of iterations. A closer investigation reveals that the Sankaran-modification finds the shortest path in 174 of the 500 investigated cases (34.8%). The DGS subgraph contains it only in 1 of the 500 cases (0.2%). This also means that in the remaining cases, the fastest path is neither accessible by DGS nor by the Sankaran-modification. However, the fastest path is accessible by EDAA in all 500 cases (100%) by construction and Propositions 3 and 4.

Table 1: Mean number of vertices $|V|_{mean}$, mean number of arcs $|A|_{mean}$, and expected number of iterations $E[t]$ for the 500 sample graphs (standard deviation in parenthesis). Values have been rounded to four significant digits.

Auction type	$\lvert V\rvert_{mean}$	$\lvert A\rvert_{mean}$	$E[t]$
EDAA	7925 (8484)	47896 (75291)	13.23 (3.501)
DGS	2600 (2598)	5962 (6418)	22.36 (11.34)
Sankaran	8.750 (1.561)	7.750 (1.561)	8.750 (1.561)

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