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Testing for bivariate spherical symmetry*

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Abstract

An omnibus test for spherical symmetry in $\mathbb{R}^2$ is proposed, employing localized empirical likelihood. The thus obtained test statistic is distribution-free under the null hypothesis. The asymptotic null distribution is established and critical values for typical sample sizes, as well as the asymptotic ones, are presented. In a simulation study, the good performance of the test is demonstrated. Furthermore, a real data example is presented.

JEL codes: C12, C14.
Key words: Asymptotic distribution, distribution-free, empirical likelihood, hypothesis test, spherical symmetry.

1 Introduction

Spherically symmetric distributions are an important class of distributions: They are a generalization of the multivariate standard normal distribution and include, amongst others, also multivariate Laplace and t distributions. Furthermore, spherical symmetry is a distributional assumption which is associated

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with many statistical models, see [6]. For instance, only recently a relationship between L_1 spherical symmetry and Archimedean copulas was discovered in [11]. Another example is [9], where univariate general linear models are considered with an error term that is spherically symmetric distributed. More applications of spherically symmetric distributions in statistics, such as in minimax estimation or stochastic processes, are discussed in [3]. For a general introduction to symmetry see [13]. Our focus is on spherical symmetry in $\mathbb{R}^2$.

There exist several approaches to test for spherical symmetry, cf. the survey papers [7] or [10] for a good overview. An often used basis, that is also underlying this paper, is the stochastic representation: Let $X = (X_1, X_2)$ be a bivariate random vector. Define the radius $S := \sqrt{X_1^2 + X_2^2}$ and the direction $Z := X/S$. Then X is bivariate spherically symmetric (in the L_2-norm) if and only if S is independent of Z, and Z is uniformly distributed on the unit circle. Other nonparametric tests based on this stochastic representation include [15] and [1], whereas the test proposed in [8] uses multivariate distribution functions (df's) and a multivariate extension of quantile functions.

We will moreover use that uniform random variables on a circle which are projected on a tangent to that circle are Cauchy distributed on that tangent (see, amongst others, [16]). More precisely, set $Y := X_2/X_1$; $(1, Y)$ is the projection of Z on the tangent at $(1, 0)$. If Z is uniformly distributed on the unit circle it follows that Y is standard Cauchy distributed. Since such a projection cannot distinguish between (X_1, X_2) and $(-X_1, -X_2)$, we project those (X_1, X_2) with $X_1 > 0$ on the line $x_1 = 1$, whereas the (X_1, X_2) with $X_1 < 0$ are projected on $x_1 = -1$. Denoting $\delta := \mathrm{sign}(X_1)$, both $Y \mid \delta = -1$ and $Y \mid \delta = 1$ are then also standard Cauchy distributed.

We wish to test

$$H_0: \ X \text{ is spherically symmetric around the origin}$$

on the basis of (S, Y, δ), but, for the first time, the test is developed in an empirical likelihood framework. The empirical likelihood method has the nice features which are known from parametric likelihood theory, but the data are used directly, i.e. in a nonparametric manner (see the monograph [12]). By localizing a functional equation, see [5], we create an omnibus test for spherical symmetry. More precisely, a functional equation is 'split up' in infinitely many pointwise equations and then standard empirical likelihood theory is used to deal with these pointwise constraints. Finally the infinitely many likelihood ratios are considered simultaneously as a stochastic process and an integral of this stochastic process is taken.

In Section 2, we derive the test statistic and present its limiting behavior under H_0. The test is consistent against all alternatives. In Section 3, critical values are computed, and in a simulation study we examine the performance of the test by power calculations for normal distributions and by a comparison to the test proposed in [8]. Furthermore, an application to a financial data set is presented. The proof of the main result is deferred to Section 4.

2 Main result

Let (S, Y, δ), as introduced in Section 1, have df F with marginals F_S, F_Y, and F_δ. Define the subdistribution functions by $F^-(s,y) := F(s,y,-1)$ and $F^+(s,y) := F(s,y,1) - F^-(s,y)$ and denote their marginals with $F_S^\pm$ and $F_Y^\pm$. Then the null hypothesis of spherical symmetry can be written as

$$H_0: \; F^-(s,y) = F^+(s,y) = \tfrac{1}{2}F_S(s)G(y), \quad \text{for all } s \in \mathbb{R}^+,\, y \in \mathbb{R},$$

with G denoting the standard Cauchy df.

Consider n independent random variables $(X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})$ distributed as (X_1, X_2). Write (S_i, Y_i, δ_i), $i = 1, \ldots, n$, for the transformed random vectors and denote with F_n their empirical df. Define the nonparametric likelihood $L(\tilde{F}) = \prod_{i=1}^n \tilde{P}(\{(S_i, Y_i, \delta_i)\})$, where $\tilde{P}$ is the probability measure corresponding to $\tilde{F}$. Furthermore, define for *fixed* $(s,y) \in \mathbb{R}^+ \times \mathbb{R}$ the *localized* empirical likelihood ratio

$$R(s,y) = \frac{\sup^\star\{L(\tilde{F})\}}{\sup\{L(\tilde{F})\}}\,, \tag{1}$$

where $\sup^\star$ is the supremum taken under the constraints given by H_0 and the corresponding marginal constraints:

$$\begin{array}{ll} \tilde{F}^-(s,y) \;=\; \tilde{F}_S^-(s)G(y), & \tilde{F}^+(s,y) \;=\; \tilde{F}_S^+(s)G(y), \\ \tilde{F}_Y^-(y) \;=\; \tfrac{1}{2}G(y), & \tilde{F}_Y^+(y) \;=\; \tfrac{1}{2}G(y), \\ \tilde{F}_S^-(s) \;=\; \tilde{F}_S^+(s) \;=\; \tfrac{1}{2}\tilde{F}_S(s), & \tilde{F}^-(\infty,\infty) = \tilde{F}^+(\infty,\infty) \;=\; \tfrac{1}{2}, \end{array}$$

and sup is the maximum over the unrestricted likelihood obtained at $\tilde{F} = F_n$, i.e., giving each observation mass $\frac{1}{n}$.

Define, for either choice of sign, the bivariate empirical subdistribution functions

$$F_n^{\pm}(s,y) = \tfrac{1}{n}\sum_{i=1}^{n} 1_{[0,s]\times(-\infty,y]\times\{\pm 1\}}(S_i, Y_i, \delta_i),$$

and write $N := nF_n^{-}(\infty,\infty)$. Observe that N is the number of data points with $X_{1i} \le 0$.

Consider for (S_i, Y_i, δ_i), $i = 1,\dots,n$, and either choice of sign, the regions

$$\begin{aligned} A_3^{\pm} &= [0,s]\times(y,\infty)\times\{\pm 1\}, & A_4^{\pm} &= (s,\infty)\times(y,\infty)\times\{\pm 1\},\\ A_1^{\pm} &= [0,s]\times(-\infty,y]\times\{\pm 1\}, & A_2^{\pm} &= (s,\infty)\times(-\infty,y]\times\{\pm 1\}. \end{aligned}$$

Denote with P_n the empirical measure corresponding to F_n. Let $F_{Sn}^{\pm}$ and $F_{Tn}^{\pm}$ denote the respective marginal df's of $F_n^{\pm}$. Observe that

$$\begin{aligned} P_n(A_3^{\pm}) &= F_{Sn}^{\pm}(s) - F_n^{\pm}(s,y), & P_n(A_4^{\pm}) &= F_n^{\pm}(\infty,\infty) - F_{Sn}^{\pm}(s) - F_{Sn}^{\pm}(y)\\ & & &\quad + F_n^{\pm}(s,y),\\ P_n(A_1^{\pm}) &= F_n^{\pm}(s,y), & P_n(A_2^{\pm}) &= F_{Sn}^{\pm}(y) - F_n^{\pm}(s,y). \end{aligned}$$

To maximize the numerator of (1), $\tilde{F}$ should put equal mass p_j^-, say, on each observation in A_j^- and mass p_j^+ on each observation in A_j^+, $j = 1,\dots,4$. Hence we need to maximize

$$\prod_{j=1}^{4} \left(p_j^-\right)^{nP_n(A_j^-)} \left(p_j^+\right)^{nP_n(A_j^+)}$$

under the constraints

$$\begin{aligned} nP_n(A_1^-)p_1^- &= \left(nP_n(A_1^-)p_1^- + nP_n(A_3^-)p_3^-\right) G(y),\\ nP_n(A_1^+)p_1^+ &= \left(nP_n(A_1^+)p_1^+ + nP_n(A_3^+)p_3^+\right) G(y),\\ nP_n(A_1^-)p_1^- + nP_n(A_2^-)p_2^- &= \tfrac{1}{2}G(y),\\ nP_n(A_1^+)p_1^+ + nP_n(A_2^+)p_2^+ &= \tfrac{1}{2}G(y),\\ nP_n(A_1^-)p_1^- + nP_n(A_3^-)p_3^- &= nP_n(A_1^+)p_1^+ + nP_n(A_3^+)p_3^+,\\ \sum_{j=1}^{4} p_j^- nP_n(A_j^-) = \tfrac{1}{2},&\\ \sum_{j=1}^{4} p_j^+ nP_n(A_j^+) = \tfrac{1}{2}.& \end{aligned}$$

This yields, for either choice of sign, the maximum empirical likelihood estimators

$$\hat{p}_3^{\pm} = \frac{F_{Sn}(s)(1-G(y))}{2nP_n(A_3^{\pm})}, \qquad \hat{p}_4^{\pm} = \frac{(1-F_{Sn}(s))(1-G(y))}{2nP_n(A_4^{\pm})},$$
$$\hat{p}_1^{\pm} = \frac{F_{Sn}(s)G(y)}{2nP_n(A_1^{\pm})}, \qquad \hat{p}_2^{\pm} = \frac{(1-F_{Sn}(s))G(y)}{2nP_n(A_2^{\pm})}.$$

Define, for either choice of sign,

$$\log R^{\pm}(s,y) = nP_n(A_1^{\pm}) \log \frac{F_{Sn}(s)G(y)}{2P_n(A_1^{\pm})} \; + \; nP_n(A_2^{\pm}) \log \frac{(1-F_{Sn}(s))\,G(y)}{2P_n(A_2^{\pm})}$$
$$+ \, nP_n(A_3^{\pm}) \log \frac{F_{Sn}(s)(1-G(y))}{2P_n(A_3^{\pm})} \; + \; nP_n(A_4^{\pm}) \log \frac{(1-F_{Sn}(s))\,(1-G(y))}{2P_n(A_4^{\pm})}, \tag{2}$$

where $0 \log(a/0) = 0$, then we have

$$\log R(s,y) = \log R^{-}(s,y) + \log R^{+}(s,y).$$

Consider the test statistic

$$T_n = -2 \int_{-\infty}^{\infty} \int_{0}^{\infty} \log R(s,y) dF_{Sn}(s) dG(y).$$

Clearly, T_n is distribution-free; selected critical values are provided in Table 1.

We now consider the limiting distribution of T_n. In order to define the limiting random variable, we denote with W a standard Wiener process on $[0,1]^3$, i.e. a centered Gaussian process with $Cov\,(W(u,v,w), W(\tilde{u},\tilde{v},\tilde{w})) = (u \wedge \tilde{u})(v \wedge \tilde{v})(w \wedge \tilde{w})$, and with $B(u,v,w) = W(u,v,w) - uvwW(1,1,1)$ the standard trivariate Brownian bridge. We also define $B^{-}(u,v) := B(u,v,\frac{1}{2})$ and $B^{+}(u,v) := B(u,v,1) - B^{-}(u,v)$. Observe $B^{-}(1,1) = -B^{+}(1,1)$. Furthermore, let, for either choice of sign, $W_0^{\pm}$ be a four-sided tied-down "half" Wiener process on $[0,1]^2$ defined by $W_0^{\pm}(u,v) := B^{\pm}(u,v) - vB^{\pm}(u,1) - uB^{\pm}(1,v) + uvB^{\pm}(1,1)$. Finally write

$$\begin{aligned} K(u,v) \;\; = \;\; & \frac{W_0^{-}(u,v)^2 + W_0^{+}(u,v)^2}{\frac{1}{2}u(1-u)v(1-v)} \; + \; 4B^{-}(1,1)^2 \\ & + \frac{\left[B^{-}(u,1) - uB^{-}(1,1) - B^{+}(u,1) + uB^{+}(1,1)\right]^2}{u(1-u)} \\ & + \frac{\left[B^{-}(1,v) - vB^{-}(1,1)\right]^2 \; + \; \left[B^{+}(1,v) - vB^{+}(1,1)\right]^2}{\frac{1}{2}v(1-v)}. \end{aligned}$$

Theorem 2.1 *Let F_S be continuous. Then, under H_0,*

$$T_n \xrightarrow{d} \int_0^1 \int_0^1 K(u,v)\, du\, dv.$$

The proof of the theorem is given in Section 4.

Note that for fixed s and y, under H_0,

$$-2 \log R(s,y) \xrightarrow{d} K(F_S(s), G(y)) \overset{d}{=} \chi^2_6.$$

This is a special case of Owen's [12] nonparametric version of the classical Wilks theorem.

Also note that within the localized empirical likelihood framework a test based directly on (S, Z) can be constructed as well, but such a test has typically less power.

3 Simulation study and real data example

Table 1 provides selected critical values for the proposed test statistic T_n. The values for $n = 50$, 100 and 200 are based on 100 000 samples in each case. For $n = \infty$, the quantiles of the limiting distribution are given, also based on 100 000 repetitions.

	Percentage points			
n	90%	95%	97.5%	99%
50	8.83	10.01	11.23	12.81
100	8.83	9.99	11.20	12.80
200	8.77	9.96	11.17	12.74
∞	8.61	9.83	11.02	12.66

Table 1: Critical values for the test for bivariate spherical symmetry.

To evaluate the power of the test (based on the critical values from Table 1), we regard data from a bivariate normal distribution with means 0, variances 1, and correlation ρ. The calculations, which are presented in Table 2, are based on 1000 replications. At the 5% significance level we see a high power for $\rho = 0.6$ ($n = 100$), and for $n = 200$, $\rho = 0.4$ is already well detected.

	$n = 100$				$n = 200$			
	Significance level							
ρ	10%	5%	2.5%	1%	10%	5%	2.5%	1%
0.1	0.10	0.05	0.03	0.01	0.17	0.08	0.04	0.02
0.2	0.20	0.11	0.07	0.03	0.37	0.22	0.12	0.06
0.3	0.38	0.24	0.14	0.06	0.68	0.54	0.41	0.24
0.4	0.61	0.46	0.31	0.17	0.94	0.86	0.78	0.60
0.5	0.86	0.75	0.62	0.44	1.00	0.99	0.98	0.93
0.6	0.98	0.95	0.90	0.75	1.00	1.00	1.00	1.00
0.7	1.00	1.00	0.99	0.97	1.00	1.00	1.00	1.00
0.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 2: Power of the test for bivariate normal distributions with different correlations for sample sizes $n = 100$ and $n = 200$.

Next, we compare the performance of our localized empirical likelihood (LEL-) test with the test proposed in [8] (KL-test), see Table 3. It needs to be pointed out that the null hypothesis in [8] is broader: There the center is unknown. Therefore the powers cannot be likened directly: A positive comparison for the LEL-test can be seen as an advise to use that test in case a center is given. We consider all the alternatives introduced in [8]:

$H_1^{(1)}$: $X_1 \sim Exp(1)$ and $X_2 \sim Exp(2)$, X_1 and X_2 independent, with $Exp(\lambda)$ the exponential distribution with mean $1/\lambda$;

$H_1^{(2)}$: $X_1 \sim N(0,1)$ and $X_2 \sim Exp(1)$, X_1 and X_2 independent;

$H_1^{(3)}$: Mixture (with parameter 1/2) of two normal distributions with identity covariance matrices and with means $(-1.5, 0)$ and $(1.5, 0)$;

$H_1^{(4)}$: Uniform distribution on an equilateral triangle, centered at the origin.

Especially $H_1^{(1)}$ and $H_1^{(2)}$ are clearly visible as non-symmetric by the naked eye and should therefore lead to a high power. To center the data around the origin, we transform the data of $H_1^{(1)}$ and $H_1^{(2)}$ by subtracting the medians, hence we consider $(X_1 - \text{med}(X_1), X_2 - \text{med}(X_2))$. This is in line with [8], where the empirical spatial median is chosen to estimate the center. The results are

Distribution	n	Significance level					
		10%		5%		1%	
		LEL	KL	**LEL**	KL	**LEL**	KL
$H_1^{(1)}$	100	**1.00**	0.23	**1.00**	0.16	**1.00**	0.04
	200	**1.00**	0.94	**1.00**	0.86	**1.00**	0.55
$H_1^{(2)}$	100	**0.97**	0.92	**0.89**	0.90	**0.46**	0.52
	200	**1.00**	1.00	**1.00**	0.92	**1.00**	0.63
$H_1^{(3)}$	100	**0.93**	0.14	**0.83**	0.11	**0.39**	0.02
	200	**1.00**	0.92	**1.00**	0.83	**0.99**	0.44
$H_1^{(4)}$	100	**0.73**	0.47	**0.53**	0.21	**0.21**	0.07
	200	**0.99**	0.81	**0.97**	0.57	**0.78**	0.19

Table 3: Powers of the localized empirical likelihood (*LEL*) test and the test in [8] (*KL*).

again based on 1000 repetitions of the LEL-test, whereas the results for the KL-test are taken from [8] (100 repetitions). The LEL-test outperforms the KL-test in nearly every setting and typically performs even considerably better. For the alternative hypotheses $H_1^{(1)}$, $H_1^{(3)}$, and $H_1^{(4)}$, the LEL-test has for $n = 100$ already about the same power as the KL-test for $n = 200$. Only for $H_1^{(2)}$, $n = 100$, both tests have comparable power.

Finally we present a real data application. The bivariate data are the daily exchange rate log-returns of the Yen to the Dollar and the Pound to the Euro from January 2nd, 2009, to December 31st, 2009. The data set has size $n = 251$ and is available from `http://wrds-web.wharton.upenn.edu`, see Figure 1. The returns are known to be centered at the origin; this is affirmed by an estimated spatial median of $(-3.3 \cdot 10^{-5}, -3.0 \cdot 10^{-4})$. We want to test whether these data are spherically symmetric and find $T_n = 6.84$, which is clearly below the asymptotic critical value at the 10% significance level. Therefore the null hypothesis is not rejected. As a consequence, a further statistical analysis of these data leads to more accurate inference, since it can be performed assuming spherical symmetry.

As an example, consider the estimation of the probability p that X_1 and X_2 are both positive (gains for Dollar and Euro) *and* that the radius S (the size of the gains) is above a certain threshold s_0. In general, we can estimate

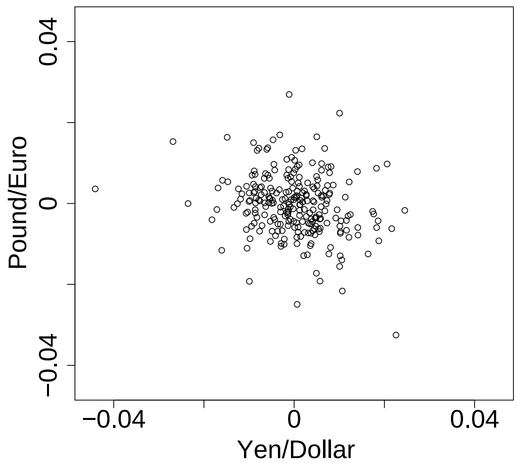

Figure 1: Daily exchange rate log-returns of Yen-Dollar and Pound-Euro, from January 2, 2009 to December 31, 2009.

this with the empirical probability $\hat{p}$, but under the null hypothesis we can estimate it with $\frac{1}{4}\hat{p}_{s_0}$, with $\hat{p}_{s_0}$ the empirical probability of $\{(x_1, x_2) : x_1^2 + x_2^2 > s_0^2\}$. For $s_0 = 0.015$ this leads to an asymptotic 95% confidence interval of $(0.0303, 0.0534)$, whereas the confidence interval based on $\hat{p}$ is more than double as wide: $(0.0214, 0.0742)$.

4 Proof of Theorem 2.1

Write Q_S, Q for the quantile functions corresponding to F_S, G, set $U_i = F_S(S_i)$ and $V_i = G(Y_i)$, and let Γ_n be the empirical df of the $(U_i, V_i, F_\delta(\delta_i))$ and Γ_{Sn}, Γ_{Yn}, and $\Gamma_{\delta n}$ the corresponding marginals. Furthermore, write $\Gamma_n^-(u,v) := \Gamma_n(u, v, \frac{1}{2})$, hence Γ_n^- is the empirical subdistribution function of the (U_i, V_i), for which $\delta_i = -1$, with marginals Γ_{Sn}^- and Γ_{Yn}^-, and note that $\Gamma_n^-(1,1) = \frac{N}{n}$. Define Γ_n^+ similarly.

Let $0 < \varepsilon \leq \frac{1}{2}$. It suffices to show that, as $n \to \infty$,

$$
\begin{aligned}
T_{1n} &= -2 \int_{Q(\varepsilon)}^{Q(1-\varepsilon)} \int_{Q_S(\varepsilon)}^{Q_S(1-\varepsilon)} \log R(s,y)\, dF_{Sn}(u)\, dG(y) \\
&\xrightarrow{d} \int_{\varepsilon}^{1-\varepsilon} \int_{\varepsilon}^{1-\varepsilon} K(u,v)\, du\, dv, \qquad (3)
\end{aligned}
$$

and

$$T_{2n} = T_n - T_{1n} = O_P(\sqrt{\varepsilon}) \tag{4}$$

uniformly in ε; see [2] (Theorem 4.2).

First, consider T_{1n} and decompose it further to

$$\begin{aligned} T_{1n} &= -2\int_{\varepsilon}^{1-\varepsilon}\int_{\varepsilon}^{1-\varepsilon} \log R^-(Q_S(u),Q(v))\,d\Gamma_{Sn}(u)dv \\ &\quad -2\int_{\varepsilon}^{1-\varepsilon}\int_{\varepsilon}^{1-\varepsilon} \log R^+(Q_S(u),Q(v))\; d\Gamma_{Sn}(u)\,dv \;=:\; T_{1n}^- + T_{1n}^+. \end{aligned}$$

Because of symmetry, we will first only consider T_{1n}^-. From (2), applying a Taylor expansion of $\log(1{+}x)$, it follows that, uniformly in $s \in [Q_S(\varepsilon), Q_S(1-\varepsilon)]$ and $y \in [Q(\varepsilon), Q(1-\varepsilon)]$,

$$\begin{aligned} \log R^-(s,y) \;&=\; \frac{n}{2} - N \;-\; \frac{n}{8}\Bigg[\frac{\big(F_{Sn}(s)G(y) - 2P_n(A_1^-)\big)^2}{P_n(A_1^-)} \\ &+ \frac{\big((1-F_{Sn}(s))\,G(y) - 2P_n(A_2^-)\big)^2}{P_n(A_2^-)} + \frac{\big(F_{Sn}(s)\,(1-G(y)) - 2P_n(A_3^-)\big)^2}{P_n(A_3^-)} \\ &+ \frac{\big((1-F_{Sn}(s))\,(1-G(y)) - 2P_n(A_4^-)\big)^2}{P_n(A_4^-)}\Bigg] \;+\; o_P(1) \\ =\; &\frac{n}{2} - N \;-\; \frac{F_{Sn}^-(s)}{8P_n(A_1^-)P_n(A_3^-)}\left[\sqrt{n}\,\big(F_{Sn}(s)G(y) - 2P_n(A_1^-)\big)\right]^2 \\ &- \frac{\frac{N}{n} - F_{Sn}^-(s)}{8P_n(A_2^-)P_n(A_4^-)}\left[\sqrt{n}\,\big((1-F_{Sn}(s))\,G(y) - 2P_n(A_2^-)\big)\right]^2 \\ &- \frac{\frac{N}{n} - F_{Tn}^-(y)}{8P_n(A_3^-)P_n(A_4^-)}\left[\sqrt{n}\,\big(F_{Sn}^+(s) - F_{Sn}^-(s)\big)\right]^2 \;-\; \frac{\left[\sqrt{n}\,\big(1-2\frac{N}{n}\big)\right]^2}{8P_n(A_4^-)} \\ &+ 2\frac{\sqrt{n}\,\big(1-2\frac{N}{n}\big)}{8P_n(A_4^-)}\Big[\sqrt{n}\,\big(F_{Sn}^+(s) - F_{Sn}^-(s)\big) + \sqrt{n}\,((1-F_{Sn}(s))\,G(y) \\ &- 2P_n(A_2^-))\Big] \;+\; 2\frac{\sqrt{n}\,\big(F_{Sn}^+(s) - F_{Sn}^-(s)\big)}{8P_n(A_3^-)}\sqrt{n}\,\big(F_{Sn}(s)G(y) - 2P_n(A_1^-)\big) \\ &- 2\frac{\sqrt{n}\,\big(F_{Sn}^+(s) - F_{Sn}^-(s)\big)}{8P_n(A_4^-)}\sqrt{n}\,\big((1-F_{Sn}(s))\,G(y) - 2P_n(A_2^-)\big) \;+\; o_P(1). \end{aligned}$$

Observe that

$$\begin{aligned}
&\sqrt{n}\left(F_{Sn}(s)G(y) - 2P_n(A_1^-)\right) \\
&\quad = \sqrt{n}\left(F_{Sn}(s) - F_S(s)\right)G(y) \; - \; 2\sqrt{n}\left(P_n(A_1^-) - \tfrac{1}{2}F_S(s)G(y)\right),
\end{aligned}$$

$$\begin{aligned}
&\sqrt{n}\left(\left(1 - F_{Sn}(s)\right)G(y) - 2P_n(A_2^-)\right) \\
&\quad = -2\sqrt{n}\left(F_{Tn}^-(y) - \tfrac{1}{2}G(y)\right) - \sqrt{n}\left(F_{Sn}(s)G(y) - 2P_n(A_1^-)\right),
\end{aligned}$$

and

$$\sqrt{n}\left(F_{Sn}^+(s) - F_{Sn}^-(s)\right) = \sqrt{n}\left(F_{Sn}^+(s) - F_S^+(s)\right) - \sqrt{n}\left(F_{Sn}^-(s) - F_S^-(s)\right).$$

It follows from the Glivenko-Cantelli theorem that

$$\tfrac{N}{n} \xrightarrow{P} \tfrac{1}{2},$$

$$\begin{aligned}
&\sup_{\substack{Q_S(\varepsilon)\le s\le Q_S(1-\varepsilon)\\ Q(\varepsilon)\le y\le Q(1-\varepsilon)}} \left| \frac{F_{Sn}^-(s)}{8P_n(A_1^-)P_n(A_3^-)} - \frac{1}{4F_S(s)G(y)\left(1-G(y)\right)} \right| \; = \; o_P(1), \\
&\sup_{\substack{Q_S(\varepsilon)\le s\le Q_S(1-\varepsilon)\\ Q(\varepsilon)\le y\le Q(1-\varepsilon)}} \left| \frac{\frac{N}{n} - F_{Sn}^-(s)}{8P_n(A_2^-)P_n(A_4^-)} - \frac{1}{4\left(1-F_S(s)\right)G(y)\left(1-G(y)\right)} \right| \; = \; o_P(1), \\
&\sup_{\substack{Q_S(\varepsilon)\le s\le Q_S(1-\varepsilon)\\ Q(\varepsilon)\le y\le Q(1-\varepsilon)}} \left| \frac{\frac{N}{n} - F_{Tn}^-(y)}{8P_n(A_3^-)P_n(A_4^-)} - \frac{1}{4F_S(s)\left(1-F_S(s)\right)\left(1-G(y)\right)} \right| \; = \; o_P(1), \\
&\sup_{\substack{Q_S(\varepsilon)\le s\le Q_S(1-\varepsilon)\\ Q(\varepsilon)\le y\le Q(1-\varepsilon)}} \left| \frac{1}{8P_n(A_3^-)} - \frac{1}{4F_S(s)\left(1-G(y)\right)} \right| \; = \; o_P(1), \\
&\sup_{\substack{Q_S(\varepsilon)\le s\le Q_S(1-\varepsilon)\\ Q(\varepsilon)\le y\le Q(1-\varepsilon)}} \left| \frac{1}{8P_n(A_4^-)} - \frac{1}{4\left(1-F_S(s)\right)\left(1-G(y)\right)} \right| \; = \; o_P(1). \qquad (5)
\end{aligned}$$

Writing $\alpha_n^-(u,v) := \sqrt{n}\left(\Gamma_n^-(u,v) - \frac{1}{2}uv\right)$, $\alpha_n^+(u,v) := \sqrt{n}\left(\Gamma_n^+(u,v) - \frac{1}{2}uv\right)$, and $\alpha_n(u,v) := \alpha_n^-(u,v) + \alpha_n^+(u,v)$, we have, using (5) uniformly for $\varepsilon \le u, v \le$

$1-\varepsilon$,

$$
\begin{aligned}
&\log R^-(Q_S(u),Q(v))\\
&= \frac{n}{2} - N - \frac{\left[\sqrt{n}\left(\Gamma_{Sn}(u)-u\right)v - 2\sqrt{n}\left(\Gamma_n^-(u,v)-\frac{1}{2}uv\right)\right]^2}{4uv(1-v)}\\
&\quad - \frac{\left[2\sqrt{n}\left(\Gamma_{Yn}^-(v)-\frac{1}{2}v\right)+\sqrt{n}\left(\Gamma_{Sn}(u)-u\right)v - 2\sqrt{n}\left(\Gamma_n^-(u,v)-\frac{1}{2}uv\right)\right]^2}{4(1-u)v(1-v)}\\
&\quad - \frac{\left[\sqrt{n}\left(\Gamma_{Sn}^+(u)-\frac{1}{2}u\right)-\sqrt{n}\left(\Gamma_{Sn}^-(u)-\frac{1}{2}u\right)\right]^2}{4u(1-u)(1-v)} - \frac{\left[\sqrt{n}\left(\Gamma_n^-(1,1)-\frac{1}{2}\right)\right]^2}{(1-u)(1-v)}\\
&\quad - \frac{\sqrt{n}\left(\Gamma_n^-(1,1)-\frac{1}{2}\right)}{(1-u)(1-v)}\left[\sqrt{n}\left(\Gamma_{Sn}^+(u)-\frac{1}{2}u\right)-\sqrt{n}\left(\Gamma_{Sn}^-(u)-\frac{1}{2}u\right)\right.\\
&\qquad \left. -2\sqrt{n}\left(\Gamma_{Yn}^-(v)-\frac{1}{2}v\right)-\sqrt{n}\left(\Gamma_{Sn}(u)-u\right)v+2\sqrt{n}\left(\Gamma_n^-(u,v)-\frac{1}{2}uv\right)\right]\\
&\quad + \frac{\sqrt{n}\left(\Gamma_{Sn}^+(u)-\frac{1}{2}u\right)-\sqrt{n}\left(\Gamma_{Sn}^-(u)-\frac{1}{2}u\right)}{2u(1-u)(1-v)}\left[\sqrt{n}\left(\Gamma_{Sn}(u)-u\right)v\right.\\
&\qquad \left. -2\sqrt{n}\left(\Gamma_n^-(u,v)-\frac{1}{2}uv\right)+2u\sqrt{n}\left(\Gamma_{Yn}^-(v)-\frac{1}{2}v\right)\right] + o_P(1)\\
&= \frac{n}{2} - N - \frac{\left[v\alpha_n(u,1)-2\alpha_n^-(u,v)\right]^2}{4uv(1-v)} - \frac{\left[2\alpha_n^-(1,v)+v\alpha_n(u,1)-2\alpha_n^-(u,v)\right]^2}{4(1-u)v(1-v)}\\
&\quad - \frac{\left[\alpha_n^+(u,1)-\alpha_n^-(u,1)\right]^2}{4u(1-u)(1-v)} - \frac{\alpha_n^-(1,1)^2}{(1-u)(1-v)}\\
&\quad - \frac{\alpha_n^-(1,1)}{(1-u)(1-v)}\left[\alpha_n^+(u,1)-\alpha_n^-(u,1)-2\alpha_n^-(1,v)-v\alpha_n(u,1)+2\alpha_n^-(u,v)\right]\\
&\quad + \frac{\alpha_n^+(u,1)-\alpha_n^-(u,1)}{2u(1-u)(1-v)}\left[v\alpha_n(u,1)+2u\alpha_n^-(1,v)-2\alpha_n^-(u,v)\right] + o_P(1). \qquad (6)
\end{aligned}
$$

Applying

$$
\begin{aligned}
&-\frac{\left[v\alpha_n(u,1)-2\alpha_n^-(u,v)\right]^2}{4uv(1-v)} - \frac{\left[2\alpha_n^-(1,v)+v\alpha_n(u,1)-2\alpha_n^-(u,v)\right]^2}{4(1-u)v(1-v)}\\
&\quad = -\frac{\left[v\alpha_n(u,1)-2\alpha_n^-(u,v)\right]^2}{4u(1-u)v(1-v)} - \frac{\alpha_n^-(1,v)}{(1-u)v(1-v)}\left[v\alpha_n(u,1)-2\alpha_n^-(u,v)\right]\\
&\qquad -\frac{\alpha_n^-(1,v)^2}{(1-u)v(1-v)},
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\left[v\alpha_n(u,1)-2\alpha_n^-(u,v)\right]^2}{4uv(1-v)(1-u)} - \frac{\left[\alpha_n^+(u,1)-\alpha_n^-(u,1)\right]^2}{4u(1-u)(1-v)} = -\frac{\left[\alpha_n^+(u,1)-\alpha_n^-(u,1)\right]^2}{4u(1-u)}\\
&\quad -\frac{\left[v\alpha_n^-(u,1)-\alpha_n^-(u,v)\right]^2}{u(1-u)v(1-v)} - \frac{\alpha_n^+(u,1)-\alpha_n^-(u,1)}{2u(1-u)(1-v)}\left[v\alpha_n(u,1)-2\alpha_n^-(u,v)\right],
\end{aligned}
$$

$$-\frac{\left[v\alpha_n^-(u,1)-\alpha_n^-(u,v)\right]^2}{uv(1-v)(1-u)}-\frac{\alpha_n^-(1,v)^2}{(1-u)v(1-v)} \;=\; 2\frac{\alpha_n^-(1,v)}{(1-u)v(1-v)}\left[v\alpha_n^-(u,1)\right.$$
$$\left.-\alpha_n^-(u,v)\right]-\frac{\left[v\alpha_n^-(u,1)-\alpha_n^-(u,v)+u\alpha_n^-(1,v)\right]^2}{u(1-u)v(1-v)}-\frac{\alpha_n^-(1,v)^2}{v(1-v)},$$

$$-\frac{\left[v\alpha_n^-(u,1)-\alpha_n^-(u,v)+u\alpha_n^-(1,v)\right]^2}{uv(1-v)(1-u)}-\frac{\alpha_n^-(1,1)^2}{(1-u)(1-v)}$$
$$= \;-\frac{\left[v\alpha_n^-(u,1)+u\alpha_n^-(1,v)-\alpha_n^-(u,v)-uv\alpha_n^-(1,1)\right]^2}{uv(1-v)(1-u)}-(1-uv)$$
$$\cdot\frac{\alpha_n^-(1,1)^2}{(1-u)(1-v)}-2\frac{\alpha_n^-(1,1)}{(1-u)(1-v)}\left[v\alpha_n^-(u,1)+u\alpha_n^-(1,v)-\alpha_n^-(u,v)\right],$$

$$-\frac{\alpha_n^-(1,v)^2}{v(1-v)}+\frac{uv\alpha_n^-(1,1)^2}{(1-u)(1-v)} \;=\; -\frac{\left[\alpha_n^-(1,v)-v\alpha_n^-(1,1)\right]^2}{v(1-v)}$$
$$+2\frac{\alpha_n^-(1,v)}{1-v}\alpha_n^-(1,1)+\frac{v\alpha_n^-(1,1)^2}{(1-u)(1-v)},$$

$$-\frac{\left[\alpha_n^+(u,1)-\alpha_n^-(u,1)\right]^2}{4u(1-u)}-\frac{\alpha_n^-(1,1)}{1-u}\left[\alpha_n^+(u,1)-\alpha_n^-(u,1)\right]$$
$$= \;-\frac{\left[\alpha_n^+(u,1)-\alpha_n^-(u,1)+2u\alpha_n^-(1,1)\right]^2}{4u(1-u)}+\frac{u\alpha_n^-(1,1)^2}{1-u},$$

to the right-hand side of (6) yields

$$\log R^-(Q_S(u),Q(v))$$
$$= \;\frac{n}{2}-N-\frac{\left[v\alpha_n^-(u,1)+u\alpha_n^-(1,v)-\alpha_n^-(u,v)-uv\alpha_n^-(1,1)\right]^2}{uv(1-v)(1-u)}-\alpha_n^-(1,1)^2$$
$$-\frac{\left[\alpha_n^-(1,v)-v\alpha_n^-(1,1)\right]^2}{v(1-v)}-\frac{\left[\alpha_n^+(u,1)-\alpha_n^-(u,1)+2u\alpha^-(1,1)\right]^2}{4u(1-u)}+\;o_P(1).$$

Because of symmetry we obtain a similar expression for $\log R^+(Q_S(u),Q(v))$.

Hence we find

$$
\begin{aligned}
&-2\log R(Q_S(u),Q(v))\\
&= \frac{[\alpha_n^-(u,v)-v\alpha_n^-(u,1)-u\alpha_n^-(1,v)+uv\alpha_n^-(1,1)]^2}{\frac{1}{2}u(1-u)v(1-v)}\\
&\quad+\frac{[\alpha_n^+(u,v)-v\alpha_n^+(u,1)-u\alpha_n^+(1,v)+uv\alpha_n^+(1,1)]^2}{\frac{1}{2}u(1-u)v(1-v)}\\
&\quad+\frac{[\alpha_n^-(1,v)-v\alpha_n^-(1,1)]^2}{\frac{1}{2}v(1-v)}+\frac{[\alpha_n^+(1,v)-v\alpha_n^+(1,1)]^2}{\frac{1}{2}v(1-v)}+\frac{\alpha_n^-(1,1)^2}{\frac{1}{4}}\\
&\quad+\frac{[\alpha_n^+(u,1)-u\alpha_n^+(1,1)-\alpha_n^-(u,1)+u\alpha_n^-(1,1)]^2}{u(1-u)}+\ o_P(1).
\end{aligned}
$$

Standard empirical process theory and the Skorohod construction (but keeping the same notation), yield, for either choice of sign,

$$\sup_{0\le u,v\le 1}\left|\alpha_n^{\pm}(u,v)-B^{\pm}(u,v)\right|\to 0\quad\text{a.s.}$$

Hence T_{1n} can be replaced by

$$\int_{\varepsilon}^{1-\varepsilon}\int_{\varepsilon}^{1-\varepsilon}K(u,v)\,dv\,d\Gamma_{Sn}(u).$$

Because the integrand is uniformly continuous, this implies (3) by the Helly-Bray theorem.

To show (4), we only consider integration over the L-shaped region

$$C_\varepsilon=\left\{(u,v)\in(0,1)^2:\ 0<u\le\varepsilon,\ 0<v\le\tfrac{1}{2}\ \text{ or }\ 0<u\le\tfrac{1}{2},\ 0<v\le\varepsilon\right\},$$

because of symmetry arguments. Consider the following five regions

$$
\begin{aligned}
C_{\varepsilon,1,1}&=\left\{(u,v)\in(0,1)^2:\ 0<u\le n^{-3/5},\ n^{-3/8}\le v\le\tfrac{1}{2}\right\},\\
C_{\varepsilon,1,2}&=\left\{(u,v)\in(0,1)^2: n^{-3/8}\le u\le\tfrac{1}{2},\ 0<v\le n^{-3/5}\right\},\\
C_{\varepsilon,2}&=\left\{(u,v)\in(0,n^{-3/8}]^2\right\},\\
C_{\varepsilon,3,1}&=\left\{(u,v)\in(0,1)^2:\ n^{-3/5}<u\le\varepsilon,\ n^{-3/8}\le v\le\tfrac{1}{2}\right\}\\
C_{\varepsilon,3,2}&=\left\{(u,v)\in(0,1)^2:\ n^{-3/8}\le u\le\tfrac{1}{2},\ n^{-3/5}<v\le\varepsilon\right\},
\end{aligned}
$$

which cover C_ε. We will use the following bound: For any $\eta>0$ there exists a positive constant M_η, such that

$$\mathbb{P}\left(\Gamma_{Sn}(u)\ \le\ uM_\eta,\ \Gamma_{Yn}(u)\ \le\ uM_\eta,\ \text{ for all } 0\le u\le 1\right)\ >\ 1-\eta\,,\qquad(7)$$

see [14], p. 419.

For $C_{\varepsilon,1,1}$, $C_{\varepsilon,1,2}$, and $C_{\varepsilon,2}$ we only consider $\log R^-$, $\log R^+$ is treated similarly. We regard the four terms of (2) separately. For $C_{\varepsilon,1,1}$ and $C_{\varepsilon,2}$ we get, with (7) and if $P_n(A_j^\pm) \geq \frac{1}{n}$, $j = 1, \ldots, 4$, with probability $1 - \eta$,

$$\begin{aligned}
&\left| n\Gamma_n^-(u,v) \log \frac{\Gamma_{Sn}(u)v}{2\Gamma_n^-(u,v)} \right| \;\leq\; n\Gamma_{Sn}(u) \log \left(\frac{v}{\frac{2}{n}} \vee \frac{2\Gamma_{Yn}(v)}{\Gamma_{Sn}(u)v} \right) \\
&\leq\; M_\eta u n \log \left(n \vee \frac{2M_\eta}{\frac{1}{n}} \right) \;\leq\; M_\eta u n \log(2M_\eta n),
\end{aligned}$$

and

$$\begin{aligned}
&\left| n\left(\Gamma_{Sn}^-(u) - \Gamma_n^-(u,v)\right) \log \frac{\Gamma_{Sn}(u)(1-v)}{2\left(\Gamma_{Sn}^-(u) - \Gamma_n^-(u,v)\right)} \right| \\
&\leq\; n\Gamma_{Sn}(u) \log \left(n \vee \frac{2(1-\Gamma_{Yn}(v))}{\Gamma_{Sn}(u)(1-v)} \right) \;\leq\; M_\eta u\, n \log(4n),
\end{aligned}$$

and, with $|\log(1+x)| \leq 2|x|$ for $x \geq -0.5$, with probability $1 - \eta$,

$$\begin{aligned}
&\left| n\left(\tfrac{N}{n} - \Gamma_{Sn}^-(u) - \Gamma_{Yn}^-(v) + \Gamma_n^-(u,v)\right) \log \frac{\left(1-\Gamma_{Sn}(u)\right)(1-v)}{2\left(\frac{N}{n} - \Gamma_{Sn}^-(u) - \Gamma_{Yn}^-(v) + \Gamma_n^-(u,v)\right)} \right| \\
&\leq n \left| \left(1-\Gamma_{Sn}(u)\right)(1-v) - 2\tfrac{N}{n} + 2\Gamma_{Sn}^-(u) + 2\Gamma_{Yn}^-(v) - 2\Gamma_n^-(u,v) \right| \\
&\leq n \left| \Gamma_{Sn}(u)v - 2\Gamma_n^-(u,v) \right| \;+\; n\left|2\Gamma_{Yn}^-(v) - v\right| + n\left|\Gamma_{Sn}(u) - 2\Gamma_{Sn}^-(u)\right| \\
&\quad + n\left|1 - 2\tfrac{N}{n}\right| \\
&\leq n\left(\Gamma_{Sn}(u) + 2\Gamma_n^-(u,v)\right) \;+\; n\left|2\Gamma_{Yn}^-(v) - v\right| + n\left(\Gamma_{Sn}(u) + 2\Gamma_{Sn}^-(u)\right) \\
&\quad + n\left|1 - 2\tfrac{N}{n}\right| \\
&\leq 6nM_\eta u \;+\; 2n\left|\Gamma_{Yn}^-(v) - \tfrac{1}{2}v\right| +\; 2n\left|\Gamma_n^-(1,1) - \tfrac{1}{2}\right| \\
&=\; 6nM_\eta u \;+\; 2n^{1/2}\left|\alpha_n^-(1,v)\right| + 2n^{1/2}\left|\alpha_n^-(1,1)\right|. \qquad (8)
\end{aligned}$$

Furthermore, for $C_{\varepsilon,2}$, we have, with probability $1 - \eta$,

$$\left| n\left(\Gamma_{Yn}^-(v) - \Gamma_n^-(u,v)\right) \log \frac{\left(1-\Gamma_{Sn}(u)\right)v}{2\left(\Gamma_{Yn}^-(v) - \Gamma_n^-(u,v)\right)} \right| \;\leq\; M_\eta v\, n \log(2M_\eta n),$$

and for $C_{\varepsilon,1,1}$, employing the Taylor expansion as in (8), with probability $1-\eta$,

$$\begin{aligned}
&\left| n\left(\Gamma_{Yn}^-(v) - \Gamma_n^-(u,v)\right) \log \frac{\left(1-\Gamma_{Sn}(u)\right)v}{2\left(\Gamma_{Yn}^-(v) - \Gamma_n^-(u,v)\right)} \right| \\
&\leq n\left| \left(1-\Gamma_{Sn}(u)\right)v - 2\Gamma_{Yn}^-(v) + 2\Gamma_n^-(u,v) \right| \;\leq 3nM_\eta u \;+\; 2n\left|\Gamma_{Yn}^-(v) - \tfrac{1}{2}v\right| \\
&=\; 3nM_\eta u \;+\; 2n^{1/2}\left|\alpha_n^-(1,v)\right|.
\end{aligned}$$

Combining the above, we have that with probability $1 - 2\eta$

$$\iint_{C_{\varepsilon,1,1}} \left|\log R^-(Q_S(u), Q(v))\right| \, d\Gamma_{Sn}(u)\, dv \;\le\; \iint_{C_{\varepsilon,1,1}} M_\eta un \left(\log(2M_\eta n)\right.$$

$$\left. + \log(4n)\right) + 9nM_\eta u + 4n^{1/2} \left|\alpha_n^-(1,v)\right| + 2n^{1/2} \left|\alpha_n^-(1,1)\right| \, d\Gamma_{Sn}(u)\, dv$$

$$\le \left(2M_\eta n^{2/5} \log(4M_\eta n) + 9M_\eta n^{2/5} + 4n^{1/2} \sup_{0\le v\le 1} \left|\alpha_n^-(1,v)\right| + 2n^{1/2} \left|\alpha_n^-(1,1)\right|\right)$$

$$\cdot \int_{n^{-3/8}}^{1/2} \int_0^{n^{-3/5}} d\Gamma_{Sn}(u)\, dv \;\to\; 0,$$

and

$$\iint_{C_{\varepsilon,2}} \left|\log R^-(Q_S(u), Q(v))\right| \, d\Gamma_{Sn}(u)\, dv \;\le\; \iint_{C_{\varepsilon,2}} (M_\eta u + M_\eta v)\, n \log(2M_\eta n)$$

$$+ M_\eta un \log(4n) + 6nM_\eta u + 2n^{1/2} \left|\alpha_n^-(1,v)\right| + 2n^{1/2} \left|\alpha_n^-(1,1)\right| \, d\Gamma_{Sn}(u)\, dv$$

$$\le \left(3M_\eta n^{5/8} \log(4M_\eta n) + 6M_\eta n^{5/8} + 2n^{1/2} \sup_{0\le v\le 1} \left|\alpha_n^-(1,v)\right| + 2n^{1/2} \left|\alpha_n^-(1,1)\right|\right)$$

$$\cdot \int_0^{n^{-3/8}} \int_0^{n^{-3/8}} d\Gamma_{Sn}(u)\, dv \;\to\; 0.$$

The region $C_{\varepsilon,1,2}$ can be treated in a similar way as $C_{\varepsilon,1,1}$.

For $C_{\varepsilon,3,1}$ and $C_{\varepsilon,3,2}$ we use $|\log(1+x) - x| \le x^2$, for $x \ge -0.5$, and the convergence in probability of P_n/P uniform over certain rectangles (the $A_j^\pm$) to 1. This follows from, e.g., [4], Inequality 2.9 or Theorem 3.3. Then, with probability tending to 1,

$$|\log R(Q_S(u), Q(v))| \;\le\; \frac{\left[v\alpha_n^-(u,1) - uv\alpha_n^-(1,1) - \alpha_n^-(u,v) + u\alpha_n^-(1,v)\right]^2}{\frac{1}{2}u(1-u)v(1-v)}$$

$$+ \frac{\left[v\alpha_n^+(u,1) - uv\alpha_n^+(1,1) - \alpha_n^+(u,v) + u\alpha_n^+(1,v)\right]^2}{\frac{1}{2}u(1-u)v(1-v)}$$

$$+ \frac{\left[\alpha_n^-(1,v) - v\alpha_n^-(1,1)\right]^2}{\frac{1}{2}v(1-v)} + \frac{\left[\alpha_n^+(1,v) - v\alpha_n^+(1,1)\right]^2}{\frac{1}{2}v(1-v)}$$

$$+ \frac{\left[\alpha_n^+(u,1) - u\alpha_n^+(1,1) - \alpha_n^-(u,1) + u\alpha_n^-(1,1)\right]^2}{u(1-u)} + 4\alpha_n^-(1,1)^2$$

$$\le 4\frac{\alpha_n^-(u,v)^2 + v^2\alpha_n^-(u,1)^2 + u^2\alpha_n^-(1,v)^2 + u^2v^2\alpha_n^-(1,1)^2}{\frac{1}{2}u(1-u)v(1-v)}$$

$$+ 4\frac{\alpha_n^+(u,v)^2 + v^2\alpha_n^+(u,1)^2 + u^2\alpha_n^+(1,v)^2 + u^2v^2\alpha_n^+(1,1)^2}{\frac{1}{2}u(1-u)v(1-v)}$$

$$
\begin{aligned}
&+ 2\frac{\alpha_n^-(1,v)^2 + v^2\alpha_n^-(1,1)^2}{\frac{1}{2}v(1-v)} + 2\frac{\alpha_n^+(1,v)^2 + v^2\alpha_n^+(1,1)^2}{\frac{1}{2}v(1-v)} + 2\alpha_n^+(1,1)^2 \\
&+ 4\frac{\alpha_n^+(u,1)^2 + u^2\alpha_n^+(1,1)^2 + \alpha_n^-(u,1)^2 + u^2\alpha_n^-(1,1)^2}{u(1-u)} + 2\alpha_n^-(1,1)^2 \\
\leq\; & 32\left[\frac{\alpha_n^-(u,v)^2 + \alpha_n^+(u,v)^2}{uv} + \frac{\alpha_n^-(u,1)^2 + \alpha_n^+(u,1)^2}{u}\right. \\
&\left. + \frac{\alpha_n^-(1,v)^2 + \alpha_n^+(1,v)^2}{v} + 2\alpha_n^-(1,1)^2\right].
\end{aligned}
$$

Theorem 3.1 in [4] yields, for either choice of sign,

$$
\sup_{0<u,v\leq 1} \frac{|\alpha_n^{\pm}(u,v)|}{(uv)^{1/4}} = O_P(1).
$$

Hence we find

$$
\begin{aligned}
&\iint_{C_{\varepsilon,3,1}\cup C_{\varepsilon,3,2}} |\log R(Q_S(u), Q(v))| \; d\Gamma_{Sn}(u)\, dv \\
&= O_P(1)\cdot \iint_{C_{\varepsilon,3,1}\cup C_{\varepsilon,3,2}} \left(\tfrac{1}{\sqrt{uv}} + \tfrac{1}{\sqrt{u}} + \tfrac{1}{\sqrt{v}} + 1\right) d\Gamma_{Sn}(u)\, dv \\
&= O_P(1)\cdot \iint_{C_{\varepsilon,3,1}\cup C_{\varepsilon,3,2}} \tfrac{1}{\sqrt{uv}} d\Gamma_{Sn}(u)\, dv \;=\; O_P(\sqrt{\varepsilon}),
\end{aligned}
$$

uniformly in ε, because of (7). This completes the proof of (4). □

References

[1] Baringhaus, L. (1991), "Testing for spherical symmetry of a multivariate distribution," *The Annals of Statistics*, 19, 899–917.

[2] Billingsley, P. (1968), *Convergence of Probability Measures*, Wiley, New York.

[3] Chmielewski, M.A. (1981), "Elliptically symmetric distributions: A review and bibliography," *International Statistical Review*, 49, 67–74.

[4] Einmahl, J.H.J. (1987), *Multivariate Empirical Processes*, CWI Tract 32. Centre for Mathematics and Computer Science, Amsterdam.

[5] Einmahl, J.H.J., and McKeague, I.W. (2003), "Empirical likelihood based hypothesis testing," *Bernoulli*, 9, 267–290.

[6] Fang, K.-T., Kotz, S., and Ng, K.W. (1990), *Symmetric Multivariate and Related Distributions*, Chapman and Hall, London.

[7] Fang, K.-T., and Liang, J.J. (1999), "Testing spherical and elliptical symmetry". In S. Kotz, C.B. Read, D.L. Banks (Eds.), *Encyclopedia of Statistical Sciences* (Update), 686–691 (Vol 3), Wiley, New York.

[8] Koltchinskii, V.I, and Li, L. (1998), "Testing for spherical symmetry of a multivariate distribution," *Journal of Multivariate Analysis*, 65, 228–244.

[9] Laurent , A.G. (1974), "The intersection of random spheres and the non-central radial error distribution for spherical models," *The Annals of Statistics*, 2, 182–189.

[10] Liang, J., Fang, K.-T., Hickernell, F.J. (2008), "Some necessary uniform tests for spherical symmetry," *Annals of the Institute of Statistical Mathematics,* 60, 679–696.

[11] McNeil, A.J., and Nešlehovà, J. (2009), "Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions", *The Annals of Statistics*, 37, 3059–3097.

[12] Owen, A.B. (2001), *Empirical Likelihood*, Chapman and Hall/CRC, Boca Raton.

[13] Serfling, R.J. (2006) "Multivariate symmetry and asymmetry". In S. Kotz, N. Balakrishnan, C.B. Read, B. Vidakovic (Eds.), *Encyclopedia of Statistical Sciences*, 5338–5345 (2nd Ed.), Wiley, New York.

[14] Shorack, G.R., and Wellner, J.A. (1986), *Empirical Processes with Applications to Statistics*, Wiley, New York.

[15] Smith, P.J. (1977), "A nonparametric test for bivariate circular symmetry based on the empirical cdf," *Communications in Statistics - Theory and Methods*, 6, 209–220.

[16] Szabłowski, P.J. (1998), "Uniform distributions on spheres in finite dimensional L_α and their generalizations," *Journal of Multivariate Analysis*, 64, 103–117.