

A Long-Run Risks Model of Asset Pricing with Fat Tails^{*}

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Abstract

We explore the effects of fat tails on the equilibrium implications of the long run risks model of asset pricing by introducing innovations with dampened power law to consumption and dividends growth processes. We estimate the structural parameters of the proposed model by maximum likelihood. We find that the homoskedastic model with fat tails leads to significant increase in implied risk premia and volatility of price-dividend ratio over the benchmark Gaussian model, but similar volatility of market return, expected risk free rate and its volatility.

Keywords: asset pricing, long run risks, equity risk premium, fat tails, Dampened Power Law, Lévy process

JEL classification: G12, G13, E43

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1 Introduction

The long-run risks model of asset pricing, developed by Bansal and Yaron (2004), provides sound theoretical rationalization for several empirical characteristics of financial markets, such as market risk premium and asset return volatilities. Their model features a long-run risk component, along with stochastic volatility, in consumption and dividend growth processes in a conditionally Gaussian world. Essentially, in this framework, risk-averse agents demand higher equity premium due to persistent effects of the long-run risk component. Bansal (2007) provides a comprehensive review of the long-run risks model.

The presence of fat tails would result in agents with risk aversion demanding higher equity premium than in a Gaussian world, since fat tails imply more frequent occurrence of extreme events. Many financial and macroeconomic time series exhibit fat tails.¹ One could ask how much fat tails would increase the magnitude of implied risk premium in a long-run risks model of Bansal and Yaron (2004) under reasonable assumptions about agents' preferences. We attempt to provide a quantitative assessment of a long-run risks model with fat tails in order to answer this question.

Several papers attempt to document the asset pricing implications of fat tails. Bidarkota and Dupoyet (2007) report that the introduction of fat tails to consumption growth process produces 80% higher risk premium compared to a lognormal process. However, their model does not feature long run risks or recursive utility as in Bansal and Yaron (2004). Shaliastovich and Tauchen (2008) assume that non-normality of consumption and dividend growth comes from a Lévy innovation to an AR(1) economy-wide state variable. This time-varying state variable time-changes both consumption and dividend growth. As in Bansal and Yaron (2004), they assume a utility function of the Epstein and Zin (1989) type. They calibrate the structural parameters of their model and find that their model can generate 4.5% implied

¹Mandelbrot (1963) and Fama (1965) are the early studies documenting fat tails in financial time series. Cont and Tankov (2004) is an excellent exposition on financial modeling under non-Gaussian settings. Blanchard and Watson (1986), Balke and Fomby (1994) and Kiani and Bidarkota (2004) provide empirical evidence on the presence of fat tails in macroeconomic data.

risk premium but only with a very high risk aversion coefficient of 50. By contrast, Bansal and Yaron (2004) are able to generate 6.8% equity risk premium with a risk aversion of 10 assuming stochastic volatility in the consumption and dividend processes. Eraker and Shaliastovich (2007) model volatility of consumption growth as a mean-reverting Gamma-jump process that can accommodate fat tails. They focus on option pricing implications of their model, although they do provide a solution to general asset prices.

Bidarkota, Dupoyet and McCulloch (2007) explore the effects of non-normality on asset pricing through α -stable process under incomplete information. By imposing restrictions on the parameters of the stable distribution, they guarantee finiteness of relevant moments of interest necessary for asset pricing. They generate volatility persistence of implied returns of a magnitude comparable to that in the data. However, their implied risk premium is 4%, well shy of the over-6% value observed in the data. Martin (2008) considers the impact of higher moments of consumption growth process on asset pricing, but without imposing long-run risks. His model captures empirical features more general than fat tails in consumption and dividend growth process by utilizing the cumulant generating function of non-normal processes.

In this paper, we account for possible fat tails in the consumption and dividends growth processes within the framework of long-run risks as in Bansal and Yaron (2004). Fat tails are modeled as a dampened power law (DPL) process, as in Wu (2006b). The representative agent's preferences are assumed to be of Epstein and Zin (1989) recursive type. With this model framework, we first estimate all structural parameters, including persistence of the long run component, via maximum likelihood. We then evaluate the model-implied risk premium and the risk free rate, and their volatilities with the estimated values of the structural parameters.

Using quarterly consumption and dividends data spanning the period from 1947 through 2007, we find that our model with fat tails can generate about 1.92% expected market risk premium and 0.61 % expected risk free rate with the magnitudes of risk aversion and elasticity

of intertemporal substitution being 35 and 1.5, respectively. These values are significantly better than what the benchmark Gaussian model can produce (0.42% equity risk premium and 1.56% risk free rate). We also show that the model with fat tails generates higher volatility of price-dividend ratios. Using an alternative method for estimating the long-run risk component, we report even more impressive empirical results, in which expected market risk premium and risk free rate for the same fat-tailed model are 6.24% and 1.03% (comparable to observed values in the data) compared to 2.95% and 1.42% for the benchmark Gaussian model. In both scenarios, the fat-tailed model exhibits a clear advantage over the benchmark Gaussian model.

The paper is organized as follows. Section 2 introduces the model with long run risks and fat tails, and summarizes the solutions to asset prices in such a setting. Section 3 presents data, discusses estimation methodology, and reports maximum likelihood model estimation results. Section 4 analyzes the asset pricing implications. Section 5 concludes with a brief summary of the main implications of modeling fat tails with long run risks and recursive utility.

2 Model

In this section, we begin with a description of the pricing kernel in a long-run risks model in subsection 2.1 and then propose a consumption growth process with fat tails in subsection 2.2. This is a modification to Bansal and Yaron's (2004) model. We then derive the asset pricing implications under our consumption growth process in the last subsection.

2.1 Pricing Kernel

A representative agent in the economy exhibits recursive preferences as in Epstein and Zin (1989) and Weil (1989). The single period utility separates risk aversion and intertemporal

elasticity of substitution in the following form:

$$U_t = \{(1 - \delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t[U_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}\}^{\frac{\theta}{1-\gamma}} \quad (1)$$

where the parameters δ , γ and ψ are the time discount factor, the risk aversion coefficient and the intertemporal elasticity of substitution (IES), respectively. The parameter θ is defined by $\frac{1-\gamma}{1-\frac{1}{\psi}}$.

The representative agent faces the following first-order condition, or the Euler's equation:

$$E_t[\delta^\theta G_{t+1}^{\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}] = 1 \quad (2)$$

where $R_{i,t+1}$, $R_{a,t+1}$, and G_{t+1} are the gross returns on any asset i , the gross returns on the aggregate consumption portfolio, and the gross growth rate of consumption, respectively. The aggregate consumption portfolio pays aggregate consumption as its dividend every period. $M_{t+1} = \delta^\theta G_{t+1}^{\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)}$ is often called the ‘‘Intertemporal Marginal Rate of Substitution’’ (IMRS) or the pricing kernel, which applies to any asset return $R_{i,t+1}$ in the economy. In order to price any individual asset, we alternatively replace $R_{i,t+1}$ in the above equation with either the aggregate consumption portfolio returns $R_{a,t+1}$, or with the market portfolio returns $R_{m,t+1}$ that pay the aggregate market dividend, or with the risk free asset returns $R_{f,t+1}$ that pay one unit of consumption good as dividends every period.

We use the following notation in the rest of the paper:

$$r_{i,t+1} = \ln R_{i,t+1}$$

$$r_{a,t+1} = \ln R_{a,t+1} = \ln \frac{P_{a,t+1} + C_{a,t+1}}{P_{a,t+1}} \quad (3)$$

$$r_{m,t+1} = \ln R_{m,t+1} = \ln \frac{P_{m,t+1} + D_{t+1}}{P_{m,t+1}} \quad (4)$$

$$r_{f,t+1} = \ln R_{f,t+1}$$

$$m_{t+1} = \ln M_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1)r_{a,t+1} \quad (5)$$

where $P_{a,t+1}$ and $P_{m,t+1}$ are the prices of aggregate consumption and market portfolios, respectively. We drop the subscript “a” in aggregate consumption $C_{a,t+1}$ in the rest of the paper.

The definitions of $r_{a,t+1}$ and $r_{m,t+1}$ in Equations (3) and (4) reflect the fact that the consumption portfolio pays aggregate consumption C_{t+1} as its dividend whereas the market portfolio pays out D_{t+1} . We can relate the prices of consumption and market portfolios to price-dividend ratios of these two assets, namely $z_t = \ln \frac{P_{a,t}}{C_t}$ and $z_{m,t} = \ln \frac{P_t}{D_t}$. Using their definitions, we expand the aggregate and market returns by Taylor’s expansion around the mean of z_t and $z_{m,t}$ respectively as in Campbell and Shiller (1988) to obtain:

$$r_{a,t+1} \simeq k_0 + k_1 z_{t+1} - z_t + g_{c,t+1} \quad (6)$$

$$r_{m,t+1} \simeq k_{0m} + k_{1m} z_{m,t+1} - z_{m,t} + g_{d,t+1} \quad (7)$$

where $g_{c,t+1} = \ln \frac{C_{t+1}}{C_t}$ and $g_{d,t+1} = \ln \frac{D_{t+1}}{D_t}$ are the consumption and dividends growth rates. We complete our model specification by specifying the dynamics of consumption and dividends growth rates in the following section.

2.2 Dynamics of Consumption and Dividends Growth Rates

We first specify the benchmark model - one in which all shocks to consumption and dividend growth rates processes are Gaussian:

$$g_{c,t+1} = \mu_c + x_t + \eta_{c,t+1} \quad (8)$$

$$x_{t+1} = \rho x_t + e_{t+1} \quad (9)$$

$$g_{d,t+1} = \mu_d + \phi x_t + \eta_{d,t+1} \quad (10)$$

where $\eta_{c,t} \sim iidN(0, \sigma_c^2)$, $e_t \sim iidN(0, \sigma_e^2)$ and $\eta_{d,t} \sim iidN(0, \sigma_d^2)$.

This process is the same as the Gaussian no-fluctuating-uncertainty model of Bansal and

Yaron (2004) if we define $\sigma_d = \varphi_d \sigma_c$ and $\sigma_e = \varphi_e \sigma_c$. Consumption growth rates are made up of a non-zero constant mean, a persistent component x_t , and noise. As in Bansal and Yaron (2004), we assume that agents observe the persistent component and set equilibrium asset prices accordingly.

For the more general model, we consider an alternative growth rates process that features non-normality based on the well-documented evidence for the presence of fat tails in macroeconomic (including consumption) data (see footnote 1 for references). As we shall subsequently see in Section 3, the data also show that the deviation of dividend growth rates from normality. However, we choose for brevity the innovations to consumption growth rates $\eta_{c,t+1}$ to follow a fat-tailed distribution while letting shocks to both the dividend growth rates and the persistent component be Gaussian.

As noted in Geweke (2001), we often encounter difficulty in ensuring finiteness of exponential moments of a fat-tailed distribution. This is often essential for ensuring finiteness of asset prices. One approach to overcoming this difficulty is to use “dampened power law” (henceforth DPL) process as in Wu (2006b) to model fat tails. See also Cont and Tankov (2004) and Shaliastovich and Tauchen (2008). An advantage of this approach is tractability when we apply Fourier transform to derive the cumulant generating (and characteristic) function that appears in asset pricing formulae as seen in the following section.

We refer to our model with fat tails in the consumption growth process as “the DPL model”:

$$g_{c,t+1} = \mu_c + x_t + \eta_{c,t+1} \tag{11}$$

$$x_{t+1} = \rho x_t + e_{t+1} \tag{12}$$

$$g_{d,t+1} = \mu_d + \phi x_t + \eta_{d,t+1} \tag{13}$$

where $e_t \sim iidN(0, \sigma_e^2)$, $\eta_{c,t}$ and $\eta_{d,t}$ obey two independent DPL processes. The two DPL

process are defined by their Lévy densities $\nu(\eta)$:

$$\nu(\eta) = \begin{cases} \gamma_+ \eta^{-\beta_+ |\eta|} |\eta|^{-\alpha-1}, & \eta > 0 \\ \gamma_- \eta^{-\beta_- |\eta|} |\eta|^{-\alpha-1}, & \eta < 0. \end{cases}$$

This specification allows for leptokurtosis and skewness in innovations to consumption growth rates. The former is controlled by α , while the latter arises from the asymmetry of the scale parameters γ_+ and γ_- and the dampening parameters β_+ and β_- . A DPL process without dampening, i.e. with $\beta_+ = \beta_- = 0$, becomes an α -stable distribution. Hence, dampened power law is also called a “tempered stable” distribution. DPL process was used in consumption-based asset pricing by Bidarkota and Dupoyet (2007). DPL distribution, without dampening and with $\alpha = 2$, results in the Gaussian distribution.

2.3 Equilibrium

With the specification of exogenous consumption and dividend growth rates, we can proceed to deriving the pricing kernel m_t , returns on the aggregate consumption $r_{a,t}$, the risk-free rate $r_{f,t}$, the market return $r_{m,t}$, and volatilities of asset returns. The key to deriving all these quantities are the log price-dividend ratios z_t and $z_{m,t}$ on the consumption and market portfolios. The linear specification of the growth dynamics guarantees concise solutions to both ratios. Equilibrium solutions to the price-dividend ratios and all other equilibrium quantities of interest in the benchmark model are presented in Bansal and Yaron (2004). We summarize their results using our notation in Appendix A.

In the rest of this subsection, we discuss the solution to the DPL model in some detail. We conjecture that log price-consumption ratio z_t and log price-dividend ratio $z_{m,t}$ in the DPL model take the same form as in the benchmark model, namely that $z_t = b_0 + b_x x_t$ and $z_{m,t} = b_{0m} + b_{xm} x_t$. We derive the asset pricing implications with DPL shocks using an approach similar to that in the benchmark model. The derivations of individual returns, namely aggregate return on the consumption portfolio $r_{a,t+1}$, risk free return $r_{f,t+1}$, and

the market return $r_{m,t+1}$ involve the cumulant exponent of Lévy process. Risk premia and variance of respective returns can then be easily obtained. Detailed derivation is available in Appendix B. Here, we only summarize the main results and briefly discuss the dependence of these results on the persistence of the long run component ρ , the variances of innovation to the long run component σ_e^2 and dividend growth $Var(\eta_d)$.

The price-consumption and price-dividend ratios z_t and $z_{m,t}$ are derived in Appendix B.1 and B.2. The unconditional variance of the market price-dividend ratio is $Var(z_{m,t}) = b_{xm}^2 Var(x_t) = \frac{b_{xm}^2}{1-\rho^2} \sigma_e^2$. Examining the formula reveals that $Var(z_{m,t})$ is positively dependent on the persistence (ρ) and the variance (σ_e^2) of innovation to the long run component.

Returns on the aggregate consumption portfolio are derived as Equation (B13) in Appendix B.3.

The pricing kernel (IMRS) m_{t+1} is derived in Appendix B.4. The unconditional variance of the pricing kernel $Var(m_{t+1})$ is given by Equation (B16). $Var(m_{t+1})$ is determined by the variance of the innovation to the state variable $Var(x_t)$ and the second moment of the innovation to the DPL consumption growth rates.

The expected risk free rate $E(r_{f,t+1})$ is derived as Equation (B19) in Appendix B.5. $E(r_{f,t+1})$ is determined by non-time-varying mean component of consumption growth μ_c and the variance of innovation to the long run component σ_e^2 positively, and cumulant exponent of the DPL component of consumption growth.

The market return and the market risk premium are given by Equations (B21) in Appendix B.6 and (B23) respectively. The market risk premium $E[r_{m,t+1} - r_{f,t}]$ is mainly determined by σ_e^2 , $Var(\eta_d)$ and two cumulant exponents of the DPL innovation positively.

The conditional and unconditional variances of market return are given by Equations (B24) and (B25). The unconditional variance is determined by the variances of the innovations to the state variable σ_e^2 negatively and dividend growth $Var(\eta_d)$ positively.

3 Data and Estimation

This section presents details on the data used, discusses estimation of the consumption and dividends growth processes, and reports their maximum likelihood estimates. Hypotheses tests are also conducted to narrow down a best-fitting model incorporating fat tails.

3.1 Data Description

We employ quarterly US real consumption data on non-durables and services and US real dividends data from the first quarter of 1947 through the fourth quarter of 2007. Consumption data are obtained from the National Income and Product Accounts (NIPA) tables published by the Bureau of Economic Analysis (BEA). Consumer Price Indices (CPI) used to construct real values are obtained from the Bureau of Labor Statistics (BLS) publications. We aggregate monthly dividends data obtained from Robert Shiller's website to quarterly frequency.² Dividends are paid toward the S&P 500 index. Table 1 presents summary statistics for the data and Figure 1 plots the consumption and dividends growth rates.

Annualized standard deviation of consumption growth is 0.0132 during the period 1947-2007, compared to 0.0293 in Bansal and Yaron (2004) (for the period 1929-1998), 0.0357 in Mehra and Prescott (1985) (for the period 1889-1978), and 0.03226 in Bidarkota and Dupoyet (2007) (for the period 1889-1997). Since we use essentially the same source of consumption data as these other studies, the difference arises solely from differing sample periods used. Clearly, post-war consumption is much less volatile than that dating back to 1929 or 1889.

Dividends growth rates are more variable than consumption growth rates. Annualized standard deviation of dividends growth rates is 0.0353 in our sample, compared to 0.115 in Bansal and Yaron (2004), and 0.112 in Campbell (1999) (for the period 1947-1995). The latter two studies use dividends to the CRSP value-weighted NYSE stock index. Differences in summary statistics of consumption and dividend growth rates between our data sample

²<http://www.econ.yale.edu/~shiller/data.htm>

and these other studies have significant implications for asset pricing that we will examine in the next section.

Jarque-Bera tests reported in Table 1 show that both consumption and dividend growth rates exhibit significant non-normality. Based on this observation, we consider model specification in Equations (11-13), namely that non-Gaussian (fat-tailed) shocks drive both consumption and dividend growth rates.

3.2 Model Estimation

Agents are assumed to observe x_t in Equations (8-10) and (11-13). Since we (econometricians) do not have data on x_t , we estimate Equations (8-9) and (11-12) as unobserved components models, and use the resulting filtered mean of x_t as “the data” on x_t that investors are assumed to observe in setting equilibrium asset prices. Estimation of the unobserved components models involves either Kalman filtering in the fully Gaussian model of Equations (8-9), or the more general Sorenson and Alspach (1971) filter in the DPL model of Equations (11-12). In order to avoid complications resulting from bivariate observation equations (8, 10) and (11, 13), especially for the non-Gaussian model, we simplify by ignoring dividends data while estimating the long run risks component x_t . Thus, we estimate Equations (8-9) and (11-12), obtain filtered mean of x_t , and use these values to run regressions in Equations (10) and (13). To check robustness of our results, however, we also reverse the roles of consumption and dividends data in model estimation. We report results for this latter case in subsection 4.4.

In estimating the DPL model, we employ a Bayesian filtering technique proposed by Sorenson and Alspach (1971), which boils down to the Kalman filter under Gaussian innovations, but unlike the latter, is efficient under non-Gaussian innovations as well. The following describes the filtering procedure using consumption process as the observation equation. Denote $G_{c,t}$ as the history of consumption growth up to time t , comprising of $g_{c,1}, g_{c,2}, \dots, g_{c,t}$. The predictive and filtering densities of x_t are obtained by the following rules derived from

Bayes' theorem:

$$p(x_t | G_{c,t-1}) = \int_{-\infty}^{\infty} p(x_t | x_{t-1})p(x_{t-1} | G_{c,t-1})dx_{t-1} \quad (14)$$

$$p(x_t | G_{c,t}) = p(g_{c,t} | x_t)p(x_t | G_{c,t-1})/p(g_{c,t} | G_{c,t-1}) \quad (15)$$

$$p(g_{c,t} | G_{c,t-1}) = \int_{-\infty}^{\infty} p(g_{c,t} | x_t)p(x_t | G_{c,t-1})dx_t \quad (16)$$

The log likelihood function is $\ln[p(g_{c,1}, \dots, g_{c,T})] = \sum_{t=1}^T \ln[p(g_{c,t} | G_{c,t-1})]$. Maximizing the log likelihood function yields the parameter estimates.

3.3 Estimation Results

In this section, we first report maximum likelihood parameter estimates of consumption growth process. We compare the fit of the benchmark model (Equations 8-9) with that of the unrestricted DPL model (Equations 11-12). We also consider the fit of three important restricted versions of the DPL model. We then report estimates of the dividend regression (Equations 10 and 13).

Table 2 reports maximum likelihood estimates for the consumption growth process. The benchmark fully Gaussian model estimates are reported in the first row. The second row reports results for the unrestricted DPL model. Rows 3-5 report results for three restricted versions of the DPL model as follows. The third row reports estimates for the “symmetric dampening” model, obtained by setting $\beta_+^c = \beta_-^c$. The fourth row is the “symmetric scale” model, obtained by restricting $\gamma_+^c = \gamma_-^c$. The fifth row reports estimates for the “symmetric dampening and scale” model, with $\beta_+^c = \beta_-^c$ and $\gamma_+^c = \gamma_-^c$.

Briefly, the main findings from the table can be summarized as follows for all the models reported there. All statistical inferences are reported at the 0.05 significance level, with some exceptions noted below. The time-invariant mean μ_c is significantly positive for both the benchmark and DPL models. The coefficient α^c is significantly less than 2 for all the DPL models, which ensures fat-tails for the DPL process. Dampening coefficients β_+^c and β_-^c are

found to be significantly positive. This guarantees finiteness of moments of all orders for the DPL process, thus ensuring finiteness of equilibrium asset prices. The persistence of the long run component ρ is significantly less than 1 for all the models. This is in contrast to the close-to-one value of 0.979 for ρ calibrated by Bansal and Yaron (2004).

An LR test for the benchmark fully Gaussian model versus the unrestricted DPL model rejects at the 0.05 significance level using the χ^2 distribution with three degrees of freedom. The benchmark model is also rejected versus any of the three restricted versions at the 0.05 significance level.

We also performed likelihood ratio (LR) tests, with each of the three restricted DPL models in turn as the null model versus the most general unrestricted DPL model given in Equations (11-13) as the alternative model. In addition, we performed in turn an LR test with the “symmetric dampening and scale” model as the null model versus the “symmetric dampening” and “symmetric scale” models. In every case, we used critical values from the χ^2 distribution with degrees of freedom equal to the number of restrictions needed on the alternative DPL model to obtain the null model under consideration. A 0.05 significance level is used for each of the tests to draw statistical inference. We next discuss each of these hypotheses tests.

(1) symmetric dampening

The null hypothesis of “symmetric dampening” tests the restriction $\beta_+^c = \beta_-^c$. With symmetric dampening coefficient β^c , a larger negative jump scale estimate γ_-^c versus a smaller positive jump scale estimate γ_+^c results in negative skewness in the innovations. The estimates are consistent with negative skewness (-0.7389) in the consumption growth data. The LR test statistic for this case is 0.0178 which fails to be rejected.

(2) symmetric scale

The null hypothesis of “symmetric scale” tests the restriction $\gamma_+^c = \gamma_-^c$. With symmetric jump scales, a larger positive dampening coefficient β_+^c versus a smaller negative dampening coefficient β_-^c leads to negative skewness of innovations to the consumption growth process.

Again, the estimates are consistent with the statistical properties of the consumption growth data. The LR test statistic for this hypothesis is 1.6512 which fails to be rejected.

(3) symmetric dampening and scale

The “symmetric dampening and scale” model is obtained by setting $\beta_+^c = \beta_-^c$ and $\gamma_+^c = \gamma_-^c$. The model features symmetric innovations to the consumption growth process. An LR test statistic for this null hypothesis against the unrestricted DPL model is 4.1410 which fails to be rejected. Also, an LR test statistic of 2.4858 against the “symmetric scale” alternative model too cannot be rejected. However, an LR test statistic of 4.1232 against the “symmetric dampening” alternative model is rejected at the 0.05 level.

In summary, we cannot reject any of the three restricted cases when tested against the unrestricted DPL model at the 0.05 significance level. The “symmetric dampening and scale” model is rejected against the “symmetric dampening” model. In what follows, we choose the “symmetric dampening” model to capture fat tails in the consumption and dividends growth rates process and study its asset pricing implications.

The upper panel of Figure 2 plots the observed and the filtered mean of consumption growth rates (Equation (8)) for the benchmark model. The upper panel of Figure 3 plots similar quantities for the selected DPL model. These panels show that both models capture trend consumption growth fairly well.

We report maximum likelihood parameter estimates of dividends growth rate processes given in Equations (10) and (13) in Table 3. As indicated at the beginning of subsection 3.2, we use the filtered mean of x_t from the benchmark and DPL models as a proxy for the unobservable persistent component that appears on the right hand sides of these two equations. Results are presented for both the benchmark model as well as for various versions of the DPL model, as in Table 2 for consumption growth process. For simplicity, we assume a similar DPL structure for innovations to dividends growth as that of innovations to consumption growth. Note that regressions for the alternative models are based on different x_t , filtered from the first step of the estimation procedure. Thus, parameter estimates for

the various models exhibit clear differences. Also, a higher likelihood does not necessarily mean a better fit due to the differing x_t for each alternative model.

The lower panel of Figure 2 plots observed dividends and their fitted values in a regression of the former on the filtered mean of the persistent component for the benchmark model. The figure shows that the benchmark model is unable to capture very well fluctuations in dividends growth. The benchmark model overestimates growth during some years, while underestimating variability for most of the sample. The lower panel of Figure 3 plots similar quantities for the selected DPL model. The DPL model produces a more reasonable fit to the data, with a somewhat poor fit at the beginning and end of the sample period.

4 Asset Pricing Implications

In this section, we first discuss model parameterization. We then proceed to computing numerically the equilibrium asset prices and returns implied by our model. We compare our benchmark model implications to the no-fluctuating-uncertainty case in Bansal and Yaron (2004). We then examine whether the DPL model exhibits significant improvement over the benchmark model. We also report our results under an alternative method for estimating the long run component by filtering dividends data.

4.1 Model Parameterization

Asset pricing formulae summarized in subsection 2.3 show that equilibrium returns and other quantities of interest involve three type of parameters: preference parameters that appear in Equation (1), parameters of the stochastic processes for consumption and dividends growth rates that appear in Equations (11-13), and endogenous (implied) parameters that appear in the approximations to the price-dividend ratios on consumption and market portfolios in Equations (6-7). Stochastic process parameter estimates were reported in subsection 3.3. In this subsection, we elaborate on our choice of preference parameters and our methodology

for computing endogenous parameters of price-dividend ratios.

Preference parameters include the risk aversion coefficient γ , the intertemporal elasticity of substitution (IES) ψ , and the time discount factor δ . Our choice of values for these parameters is largely dictated by those used by Bansal and Yaron (2004). The time discount factor δ is set at 0.998 for decisions made at quarterly intervals. In the next two subsections, we discuss asset pricing implications for various alternative values of γ and ψ .

Sections A1, A2, B1 and B2 in the Appendix discuss how to compute endogenous values for the parameters that appear in the approximations to the gross rates of return to the aggregate consumption and market portfolios appearing in Equations (6-7). These are the average values for the price-dividend ratios \bar{z} and \bar{z}_m , and the constants k_0 , k_1 , k_{0m} , and k_{1m} . Table 4 reports these computations for various alternative values of the preference parameters γ and ψ . The upper panel reports values for the benchmark model and the lower panel for the DPL model. It can be seen that all values for the benchmark model are similar to those for the DPL model.

It is worthwhile to compare our parameter values for the benchmark model to those for the no-fluctuating-uncertainty case in Bansal and Yaron (2004). We choose the case of $\gamma = 10$ and $\psi = 1.5$ for ease of comparison. In what follows, we briefly report our parameter values followed by those employed by Bansal and Yaron (2004) in parentheses.³ (1) Consumption dynamics: $\mu_c = 0.0192(0.018)$, $\sigma_c = 0.0102(0.027)$; (2) Long-run risks: $\rho = 0.6857(0.979)$, $\sigma_e = 0.006(0.0012)$; (3) Dividend dynamics: $\mu_d = 0.0228(0.018)$, $\phi = 1.095(3)$, $\sigma_d = 0.0346(0.122)$; (4) Price-dividend ratios: $\bar{z} = 7.644(6.9068)$, $k_0 = 0.004(0.0079)$, $k_1 = 0.9995(0.999)$, $\bar{z}_m = 6.889(5.7105)$, $k_{0m} = 0.008(0.0222)$, and $k_{1m} = 0.999(0.9967)$. Parameter values for our benchmark model are clearly different from those of the equivalent model in Bansal and Yaron (2004).⁴ The latter study uses twice the value of σ_c and thrice the value of σ_d and ϕ than in our model. Differences in all these values have significant

³We thank Dana Kiku for kindly providing these values to us.

⁴Main reasons are twofold: we use different data and we use the estimated (low) value for the persistence of the long run risks component instead of calibrating it to a value of 0.979.

impact on asset pricing implications which we will detail in the following section.

4.2 Benchmark Model

Moments of the model-implied rates of return and price-dividend ratio from the benchmark model are reported in Table 5. These are the unconditional means and volatilities of the market risk premium and the risk free rate, and the volatility of the price-dividend ratio. These statistics are reported for various values of the risk-aversion coefficient γ and the intertemporal elasticity of substitution (IES) parameter ψ .

The expected market risk premium in the benchmark model is no greater than 0.417 percent per annum for any combination of γ and ψ values considered in the table. The reported implied moments are quite low, compared to an annualized expected risk premium of 4.2 percent reported in the no-fluctuating-uncertainty case by Bansal and Yaron (2004). There are several factors that account for this. The market risk premium is primarily determined by the variances of the innovations to the persistent component σ_ϵ^2 and dividend growth σ_d^2 , the persistence of the long run component ρ and the loading factor on the long run component in dividend growth ϕ . This is evident from the formula for the market risk premium given in Equation (A14) in the Appendix, and from examining the numerical values of all the other terms that appear in that formula. As noted in previous section, our model has lower σ_d and lower ρ , which contribute to lower risk premium in our model. The results are intuitive, since (1) lower variance of innovations to dividend growth are in alignment with lower risk premium; (2) less persistence ρ lowers long run risks in the economy, thereby lowering the premium needed to hold risky assets; (3) lower factor loading of long-run risks component on dividend growth ϕ leads to lower market risk premium.

The expected risk free rate in the benchmark model is no lower than 1.562 percent per annum for any combination of γ and ψ considered in the table. All values for $\psi = 1.5$ are higher than those reported in the no-fluctuating-uncertainty case by Bansal and Yaron (2004). The expected risk free rate in the benchmark is negatively dependent on variances

of innovations to consumption growth σ_c^2 and long run risks σ_e^2 as evident from Equation (A19) when θ is negative, or equivalently $\gamma > 1$ and $\psi > 1$. Lower variances of the two components in our sample data contribute to a relatively higher expected risk free rate.

The unconditional volatility of the market return is reported in the third column of Table 5. It is no greater than 4.197 percent per annum for any combination of γ and ψ considered in the table. This is quite low, compared to 16.21 percent reported in the no-fluctuating-uncertainty case by Bansal and Yaron (2004). The market return volatility formula is given in Equation (A16) in the Appendix. It is primarily determined by the variances of the short run innovations to the long run component σ_e^2 and the dividends growth rate σ_d^2 , with the magnitude of the latter being dominantly larger. σ_d^2 is considerably lower in our quarterly dataset (0.013 annualized) than its value estimated from annual datasets (0.12 annualized in Bansal and Yaron (2004)).

The unconditional volatility of the risk free rate is reported in the fourth column of Table 5. The values for different combination of preferences are slightly higher than those reported by Bansal and Yaron (2004), which is due to lower ρ and higher σ_e in our model.

The unconditional volatility of the market price-dividend ratios is reported in the last column of Table 5. These are lower than the values reported by Bansal and Yaron (2004) because of our lower ρ and higher σ_e .

In summary, compared to the no-fluctuating-uncertainty model in Bansal and Yaron (2004), our benchmark model produces lower expected risk premia, higher but comparable risk free rate, lower volatilities of risk premium and price-dividend ratios. Most of these differences can be directly traced to the lower variances of consumption and dividend growth rates in our data, and to our use of a lower estimated value of ρ .

4.3 DPL Model

Moments of the model-implied rates of return and price-dividend ratio from the DPL model are reported in Table 6. These are the unconditional means and volatilities of the market risk

premium and the risk free rate, and the volatility of the price-dividend ratio. These statistics are reported for various values of the risk-aversion coefficient γ and the intertemporal elasticity of substitution (IES) parameter ψ .

The expected market risk premium in the DPL model is as high as 1.917 percent per annum for a combination of $\gamma = 35$ and $\psi = 1.5$ considered in the table. This is significantly higher, compared to a maximum annualized expected risk premium of 0.417 percent reported in the benchmark model. Thus, the DPL model can significantly improve the magnitude of implied risk premia over the benchmark model for the same preference combination.

There are several factors that account for this. The market risk premium is primarily determined by the variances of the innovations to the long run component σ_e^2 and to the dividend growth σ_d^2 , and the coefficients on these variances. Among their coefficients, ϕ and ρ positively affect the market risk premium. This is evident from the formula for the market risk premium given in Equation B23 in the Appendix. However, estimated values of these two variances are seen to be similar in magnitude for both models from Table 2-3. The estimated values of ρ for the two models are only marginally different. The higher loading factor on long run risks ϕ in dividends growth contributes to the consistently higher risk premia for the DPL model.

The expected risk free rate in the DPL model is reported in the fourth column of Table 6. This is lower than in the benchmark model for $\psi = 1.5$. With the same time discount factor δ and the same estimated value of μ_c in both models, the lower estimate of σ_e leads to a lower value in the DPL model as seen in Equation (B19).

The unconditional volatility of the market return is reported in the fifth column of Table 6. The values for the DPL model are comparable to those for the benchmark model. Equation (B25) shows that lower ρ and σ_e estimated in the DPL model contribute to lower volatility of the market return while higher variance of innovations to dividend growth in the DPL model increases the volatility of the market return.

The unconditional volatility of the risk free rate is reported in the sixth column of Table

6. They are marginally lower in the DPL model compared to the benchmark model. As seen in Equation (B20), lower ρ and σ_e contribute to the lower volatility for the DPL model.

The unconditional volatility of the market price-dividend ratios is reported in the last column of Table 6. These are higher in the DPL model for $\psi = 1.5$. Equation (B12) reveals that the significantly larger loading factor ϕ on long-run risks in dividend growth in the DPL model combined with the slightly lower ρ and σ_e leads to higher volatility of price-dividend ratio for the DPL model.

In summary, compared to the benchmark model, our DPL model produces significantly higher expected equity risk premium and higher volatilities of the price-dividend ratios, and comparable magnitudes of the risk free rate and its volatility and that of the market return,

4.4 Filtering x_t Using Dividends Data

The discussion so far on the benchmark and DPL models is based on estimating the long run component x_t through Bayesian filtering using the consumption growth process as the observation equation and the process for x_t as the state transition equation. We now study the robustness of our results to an alternative way of estimating x_t using the dividends growth process, instead of the consumption growth process, as the observation equation.

Maximum likelihood estimation results of the model using dividends process as the observation equation and the x_t process as the state transition equation are reported in Table 7. The table reports results for the benchmark Gaussian model and several versions of the DPL model, as in Table 2. Extensive hypotheses testing along the lines reported for that table in subsection 3.3 pin down the “symmetric dampening and scale” DPL model as giving the best fit. We therefore pursue study of asset pricing implications with this version of the DPL model as the candidate model capturing fat tails.

Maximum likelihood estimation results of the consumption regression equation using x_t obtained by filtering dividends data are reported in Table 8. The table once again reports results for the benchmark Gaussian model and several versions of the DPL model.

To illustrate the asset pricing implications with this alternative approach for estimating x_t , we mainly report results for the parameter combination $\gamma = 35$ and $\psi = 1.5$ for the sake of brevity. The benchmark model under the alternative approach (see Table 9) can generate 2.95 percent equity risk premium (significantly higher than 0.42 percent reported in the earlier benchmark case by filtering consumption growth data for x_t), 1.42 percent risk free rate (compared to 1.56), 5.42 percent volatility of market return (compared to 3.58), 0.51 percent volatility of risk free rate (compared to 0.55), and 0.036 percent volatility of price-dividend ratio (significantly higher than 0.005).

The DPL model under the alternative approach also shows similar improvement over the earlier DPL model based on a comparison of analogous quantities between Table 6 and Table 10. Most significantly, the DPL model under this alternative approach is now able to generate 6.24 percent equity risk premium and 1.03 percent risk free rate, which are close to market data. The main reason for this improvement is that the alternative DPL model now exhibits significantly higher persistence of long-run risks ρ .

We now compare our results to those in Shaliastovich and Tauchen (2008) and Bidarkota and Dupoyet (2007). The former study reports 4.51 percent per annum implied risk premium for their Lévy-process-based model with risk aversion $\gamma = 50$. The latter documents 2.72 percent per annum risk premium with risk aversion $\gamma = 7$ assuming the market portfolio pays aggregate consumption as its dividend. Our DPL model with filtering from consumption data cannot generate high enough equity risk premium as reported earlier. However, the premium for the alternative DPL model with filtering from dividends data is computed to be 6.24 percent per annum with $\gamma = 35$ and $\psi = 1.5$.

Thus, as we have seen above, this alternative approach to estimating x_t produces significantly better empirical results on asset pricing for both the benchmark and DPL models. The results also reaffirm the earlier conclusion that the DPL model represents a clear improvement over the benchmark model. However, our mixed results based on univariate filtering (using either consumption or dividends data alone) highlight the need for entertaining bi-

variate filtering with DPL innovations to consumption and dividends growth rates, which we leave for future research.

5 Conclusions

In this paper, we explore the effects of fat tails on an asset pricing model with long-run risks and recursive utility. Following Bansal and Yaron (2004), we model consumption and dividend growth processes with persistent long run components. Given the evidence of leptokurtosis in consumption and dividends data, we introduce non-normality in shocks to their growth rates via a Lévy process, namely the dampened power law (DPL). We derive the asset pricing implications of the resulting model and study the quantitative importance of modeling fat tails empirically.

When we extract the long run risks component by filtering consumption data, fat tails generate 1.92% expected market risk premium and 0.61% expected risk free rate with the magnitudes of risk aversion and intertemporal elasticity of substitution being 35 and 1.5, respectively. By contrast, when we extract the long run risks component by filtering dividends data, the risk premium and risk free rate become 6.24% and 1.03%, both of which are comparable to those observed in the market. Modeling fat tails leads to clear improvement in implied risk premia and volatility of price-dividend ratios, without deterioration in the magnitudes of other moments of interest. Although the asset pricing model with DPL fat tails can generate higher volatility of market returns, its magnitude (3.77% with consumption filtering and 7.04% with dividend filtering) is well shy of the observed value. This is partly due to the relative smoothness of post-war consumption and dividends growth data compared to pre-war data. Inclusion of pre-war data would undoubtedly generate higher volatility.

Extracting the long-run risks component using both consumption and dividends data is more efficient but involves complications arising from consideration of a bivariate DPL

and/or filtering process. Also, our asset pricing model assumes that agents not only observe the growth rates of consumption and dividends but also their long run persistent component x_t (although it is assumed in estimation that econometricians do not actually observe the true value of x_t but have to learn about it through a Bayesian filtering process). This may not be entirely realistic (Croce, Lettau, and Ludvigson (2006)). It is worth exploring the effects of fat tails on the long run risks model that treats x_t as unobservable even by agents in the model. Solving the asset pricing model in such an incomplete information setting with fat tails poses a challenge. Bidarkota, Dupoyet, and McCulloch ((2007)) study such a model but without long run risks or recursive utility.

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APPENDIX

A Benchmark Model Solution

The benchmark model is represented by the following set of equations:

$$g_{c,t+1} = \mu_c + x_t + \eta_{c,t+1} \tag{A1}$$

$$x_{t+1} = \rho x_t + e_{t+1} \tag{A2}$$

$$g_{d,t+1} = \mu_d + \phi x_t + \eta_{d,t+1} \tag{A3}$$

where $\eta_{c,t+1} \sim iidN(0, \sigma_c^2)$, $e_{t+1} \sim iidN(0, \sigma_e^2)$, and $\eta_{d,t+1} \sim iidN(0, \sigma_d^2)$.

A.1 Price-Consumption Ratio

The price-consumption and price-dividend ratios z_t and $z_{m,t}$ are the only endogenous variables in the model. Once we solve for these, all other equilibrium quantities of interest can be readily derived. We briefly summarize the procedure for deriving z_t here and $z_{m,t}$ in the next section of the Appendix.

The first-order condition for the representative agent given as Equation (2) in the text can be rewritten for returns on the aggregate consumption portfolio as:

$$E_t \left[\exp(\theta \ln \delta - \frac{\theta}{\psi} g_{c,t+1} + \theta r_{a,t+1}) \right] = 1 \tag{A4}$$

We substitute for $r_{a,t+1}$ from Equation 6 and $g_{c,t+1}$ from Equation A1 into the above first-order condition.

Bansal and Yaron (2004) conjecture the following linear solution for the price-consumption ratio as a function of the single state variable x_t in the model: $z_t = b_0 + b_x x_t$ where b_0 and b_x are constants to be determined. We now substitute this conjectured solution for z_t into the resulting first-order condition, and then solve for the constants b_0 and b_x through the

method of undetermined coefficients. The solutions for b_0 and b_x are as follows:

$$b_0 = \frac{\ln\delta + (1 - \frac{1}{\psi}\mu_c + k_0 + 0.5\theta(1 - \frac{1}{\psi})^2\sigma_c^2 + 0.5(\theta k_1 b_x \sigma_e)^2)}{1 - k_1} \quad (\text{A5})$$

$$b_x = \frac{1 - \frac{1}{\psi}}{1 - k_1 \rho} \quad (\text{A6})$$

The approximating constants appearing in Equation (3) in the text k_0 and k_1 are functions of the average level of the price-consumption ratio \bar{z} . Evaluating $z_t = b_0 + b_x x_t$ at \bar{z} and recognizing from Equation A2 that the average value of x_t is zero yields $\bar{z} = b_0$.

Thus, replacing the lhs of Equation A5 with \bar{z} and substituting for b_x from Equation A6 into the rhs gives us a (highly) nonlinear equation in \bar{z} . We can easily solve this equation numerically for \bar{z} . Given \bar{z} , k_0 and k_1 , and hence b_0 and b_x are readily obtained.

A.2 Price-Dividend Ratio

We briefly summarize the procedure for deriving the price-dividend ratio $z_{m,t}$ on the market portfolio here.

The first-order condition for the representative agent given as Equation (2) in the text can be rewritten for returns on the market portfolio as:

$$E_t[\exp(\theta \ln\delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1)r_{a,t+1} + r_{m,t+1})] = 1 \quad (\text{A7})$$

We substitute for $r_{a,t+1}$ from Equation 6 and $r_{m,t+1}$ from Equation 7, $g_{c,t+1}$ from Equation A1 and $g_{d,t+1}$ from Equation A3 into the above first-order condition.

Bansal and Yaron (2004) once again conjecture the following linear solution for the price-dividend ratio as a function of the state variable x_t in the model: $z_{m,t} = b_{0m} + b_{xm}x_t$ where b_{0m} and b_{xm} are constants to be determined. We substitute this conjectured solution for $z_{m,t}$ into the resulting first-order condition, and then solve for the constants b_{0m} and b_{xm} through

the method of undetermined coefficients. The solutions for b_{0m} and b_{xm} are as follows:

$$b_{0m} = \frac{1}{1 - k_{1m}} \left\{ \theta \ln \delta - \frac{\theta}{\psi} \mu_c + (\theta - 1)[k_0 + (k_1 - 1)b_0 + \mu_c] + k_{0m} + \mu_d \right. \\ \left. + 0.5(\theta - \frac{\theta}{\psi} - 1)^2 \sigma_c^2 + 0.5[(\theta - 1)k_1 b_x + k_{1m} b_{xm}]^2 \sigma_e^2 + 0.5\sigma_d^2 \right\} \quad (\text{A8})$$

$$b_{xm} = \frac{(\theta - 1)(k_1 b_x \rho - b_x + 1) - \frac{\theta}{\psi} + \phi}{1 - k_{1m} \rho} \quad (\text{A9})$$

The approximating constants appearing in Equation (4) in the text k_{0m} and k_{1m} are functions of the average level of the price-dividend ratio \bar{z}_m . Evaluating $z_{m,t} = b_{0m} + b_{xm}x_t$ at \bar{z}_m and recognizing from Equation A2 that the average value of x_t is zero yields $\bar{z}_m = b_{0m}$.

Thus, replacing the lhs of Equation A8 with \bar{z}_m and substituting for b_{xm} from Equation A9 into the rhs gives us a nonlinear equation in \bar{z}_m . We can solve this equation numerically for \bar{z}_m . Given \bar{z}_m , k_{0m} and k_{1m} , and hence b_{0m} and b_{xm} are readily obtained.

A.3 Equilibrium Quantities of Interest

The following results are specializations (to the homoskedastic case) of the more general fluctuating-uncertainty model solution derived in the appendix to Bansal and Yaron (2004). These formulae are reproduced here using the notation adopted in our paper for easy reference and for comparison with the solution to the DPL model derived in the next section of the Appendix.

Price-consumption and price-dividend ratios:

$$r_{a,t+1} \simeq k_0 + k_1 z_{t+1} - z_t + g_{c,t+1} \quad (\text{A10})$$

$$z_t = b_0 + b_x x_t \quad (\text{A11})$$

$$b_x = \frac{1 - 1/\psi}{1 - k_1 \rho}$$

$$r_{m,t+1} \simeq k_{0m} + k_{1m} z_{m,t+1} - z_{m,t} + g_{d,t+1} \quad (\text{A12})$$

$$z_{m,t} = b_{0m} + b_{xm} x_t \quad (\text{A13})$$

$$b_{xm} = \frac{\phi - 1/\psi}{1 - k_{1m} \rho}$$

Risk premia, risk free rate and volatilities:

$$E_t[r_{m,t+1}] - r_{f,t} = \beta_{m,e} \lambda_{m,e} \sigma_c^2 - 0.5 \text{Var}_t(r_{m,t+1}) \quad (\text{A14})$$

$$\text{Var}_t(r_{m,t+1}) = \beta_{m,e}^2 \sigma_c^2 + \sigma_d^2 \quad (\text{A15})$$

$$\text{Var}(r_{m,t+1}) = \beta_{m,e}^2 \sigma_c^2 + \sigma_d^2 + \frac{\sigma_e^2}{(1 - \rho^2)\psi^2} \quad (\text{A16})$$

$$E_t[r_{a,t+1}] - r_{f,t} = -\lambda_{m,\eta} \sigma_c^2 + \frac{\lambda_{m,e}^2}{1 - \theta} \sigma_c^2 - 0.5 \text{Var}_t(r_{a,t+1}) \quad (\text{A17})$$

$$\text{Var}_t(r_{a,t+1}) = \sigma_c^2 + (k_1 b_x \sigma_e)^2 \quad (\text{A18})$$

$$\begin{aligned} E[r_{f,t}] &= -\ln \delta + \mu_c / \psi + \frac{1 - \theta}{\theta} [E_t[r_{a,t+1}] - r_{f,t}] - \frac{\lambda_{m,\eta}^2 + \lambda_{m,e}^2}{2\theta} \sigma_c^2 \\ &= -\ln \delta + \mu_c \left(\frac{1}{\psi} + x_t \right) + 0.5(\theta - 1 - \frac{\theta}{\psi^2}) \sigma_c^2 + 0.5(\theta - 1)(k_1 b_x)^2 \sigma_e^2 \end{aligned} \quad (\text{A19})$$

$$\text{Var}(r_{f,t+1}) = \frac{\sigma_e^2}{\psi^2(1 - \rho^2)} \quad (\text{A20})$$

$$\text{Var}(z_{m,t}) = \frac{(b_{xm} \sigma_e)^2}{1 - \rho^2} \quad (\text{A21})$$

where

$$\begin{aligned}\beta_{m,e} &= k_{1m}b_{xm}\sigma_e/\sigma_c = \frac{k_{1m}(\phi - 1/\psi)\sigma_e}{(1 - k_{1m}\rho)\sigma_c} \\ \lambda_{m,e} &= (1 - \theta)k_1b_x\sigma_e/\sigma_c = \frac{(1 - \theta)k_1(1 - 1/\psi)\sigma_e}{(1 - k_1\rho)\sigma_c} \\ \lambda_{m,\eta} &= -\frac{\theta}{\psi} + \theta - 1 = -\gamma\end{aligned}$$

B DPL Model Solution

The DPL model is represented by the following set of equations:

$$\begin{aligned}g_{c,t+1} &= \mu_c + x_t + \eta_{c,t+1} \\ x_{t+1} &= \rho x_t + e_{t+1} \\ g_{d,t+1} &= \mu_d + \phi x_t + \eta_{d,t+1}\end{aligned}$$

where $e_t \sim iidN(0, \sigma_e^2)$, $\eta_{c,t} \sim iidDPL(\gamma_+^c, \gamma_-^c, \beta_+^c, \beta_-^c, \alpha^c)$, and $\eta_{d,t} \sim iidDPL(\gamma_+^d, \gamma_-^d, \beta_+^d, \beta_-^d, \alpha^d)$.

The DPL processes are defined immediately following Equations (11-13) in the main text.

B.1 Price-Consumption Ratio

The price-consumption and price-dividend ratios z_t and $z_{m,t}$ are the only endogenous variables in the model. Once we solve for these, all other equilibrium quantities of interest can be readily derived. We briefly summarize the procedure for deriving z_t here and $z_{m,t}$ in the next section of the Appendix.

The first-order condition for the representative agent given as Equation (2) in the text can be rewritten for returns on the aggregate consumption portfolio as: $E_t\{\exp[\theta \ln \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1)r_{a,t+1}]\} = 1$

We substitute for $r_{a,t+1}$ from Equation (6) and $g_{c,t+1}$ from Equation (11) into the above first-order condition to obtain: $E_t\{\exp[\theta \ln \delta + (\theta - \frac{\theta}{\psi})(\mu_c + x_t + \eta_{c,t+1}) + \theta(k_0 + k_1 z_{t+1} - z_t)]\} = 1$.

As in the benchmark model, we conjecture the following linear solution for the price-consumption ratio as a function of the single state variable x_t in the model: $z_t = b_0 + b_x x_t$, where b_0 and b_x are constants to be determined.

We substitute this conjectured solution for z_t and the process for the long run component x_t from Equation (12) into the resulting first-order condition to obtain:

$$\begin{aligned} 1 = & E_t \{ \exp[\theta(\ln\delta + (1 - \frac{1}{\psi})\mu_c + k_0 + (k_1 - 1)b_0) \\ & + (\theta - \frac{\theta}{\psi} + \theta k_1 b_x \rho - \theta b_x)x_t \\ & + (\theta - \frac{\theta}{\psi})\eta_{c,t+1} + \theta k_1 b_x \sigma_e e_{t+1}] \} \end{aligned}$$

Denote

$$\begin{aligned} A_0 &= \theta(\ln\delta + (1 - \frac{1}{\psi})\mu_c + k_0 + (k_1 - 1)b_0) \\ A_x &= \theta - \frac{\theta}{\psi} + \theta k_1 b_x \rho - \theta b_x \\ A_\eta &= \theta - \frac{\theta}{\psi} \\ A_e &= \theta k_1 b_x \sigma_e \end{aligned}$$

We can now rewrite the first-order condition in a simpler way using the above notation as:

$$E_t \{ \exp[A_0 + A_x x_t + A_\eta \eta_{c,t+1} + A_e e_{t+1}] \} = 1 \quad (\text{B1})$$

The conditional expectation term on the lhs of the above equation can be evaluated using the moment generating function (mgf) of innovations following the normal distribution and the more general DPL process considered here. These are given as:

$$E_t[\exp(A_e e_{t+1})] = \exp[0.5\theta^2 k_1^2 b_x^2 \sigma_e^2] \quad (\text{B2})$$

$$E_t[\exp(A_\eta \eta_{t+1})] = \exp\{\Delta t[\kappa(A_\eta)]\} \quad (\text{B3})$$

$$\kappa(A_\eta) = \Gamma(-\alpha^c)\gamma_+^c[(\beta_+^c - A_\eta)^{\alpha^c} - (\beta_+^c)^{\alpha^c}] + \Gamma(-\alpha^c)\gamma_-^c[(\beta_-^c + A_e)^{\alpha^c} - (\beta_-^c)^{\alpha^c}] + A_e Q \quad (\text{B4})$$

$$Q = \gamma_+^c(\beta_+^c)^{\alpha^c-1}[\Gamma(-\alpha^c)\alpha^c + \Gamma(1 - \alpha^c, \beta_+^c)] - \gamma_-^c(\beta_-^c)^{\alpha^c-1}[\Gamma(-\alpha^c)\alpha^c + \Gamma(1 - \alpha^c, \beta_-^c)] \quad (\text{B5})$$

The cumulant exponent of the DPL innovation $\kappa(A_\eta)$ is derived in Wu (2006a). In our numerical calculations, we set Δt to 0.25 since we use data sampled at quarterly frequency.

We can substitute the above formulae for the mgf's into the simplified first-order condition in Equation B1, and then solve for the constants b_0 and b_x through the method of undetermined coefficients. The solutions for b_0 and b_x are as follows:

$$b_0 = \frac{\theta(\ln\delta + k_0 + \mu_c(1 - 1/\psi)) + 0.5(\theta k_1 b_x \sigma_e)^2 + \Delta t \kappa(A_\eta)}{\theta(1 - k_1)} \quad (\text{B6})$$

$$b_x = \frac{1 - 1/\psi}{1 - k_1 \rho} \quad (\text{B7})$$

As can be readily seen, the solution for b_x in the DPL model is identical to that in the benchmark model derived in Appendix A.

The approximating constants appearing in Equation (6) in the text k_0 and k_1 are functions of the average level of the price-consumption ratio \bar{z} . Evaluating $z_t = b_0 + b_x x_t$ at \bar{z} and recognizing from Equation (12) that the average value of x_t is zero yields $\bar{z} = b_0$.

Thus, replacing the lhs of Equation B6 with \bar{z} and substituting for b_x from Equation B7 into the rhs gives us a (highly) nonlinear equation in \bar{z} . We can easily solve this equation numerically for \bar{z} . Given \bar{z} , k_0 and k_1 , and hence b_0 and b_x are readily obtained.

B.2 Price-Dividend Ratio

We briefly summarize the procedure for deriving the price-dividend ratio $z_{m,t}$ on the market portfolio here.

The first-order condition for the representative agent given as Equation (2) in the text can be rewritten for returns on the market portfolio as:

$$E_t[\exp(\theta \ln \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1)r_{a,t+1} + r_{m,t+1})] = 1 \quad (\text{B8})$$

We first substitute the solution to price-consumption ratio $z_t = b_0 + b_x x_t$ from previous subsection and $g_{c,t+1}$ from Equation 11 into Equation 6 to obtain returns on the aggregate consumption portfolio $r_{a,t+1}$ as follows:

$$r_{a,t+1} = B_{0a} + B_{xa}x_t + B_{ea}e_{t+1} + \eta_{c,t+1}$$

where

$$B_{0a} = k_0 + (k_1 - 1)b_0 + \mu_c$$

$$B_{xa} = k_1 b_x \rho - b_x + 1 = \frac{1}{\psi}$$

$$B_{ea} = k_1 b_x \sigma_e$$

We substitute for $r_{a,t+1}$ from the resulting equation and $r_{m,t+1}$ from Equation 7, $g_{c,t+1}$ from Equation 11 and $g_{d,t+1}$ from Equation 13 into the above first-order condition.

As in the benchmark model, we conjecture the following linear solution for the price-dividend ratio as a function of the state variable x_t in the model: $z_{m,t} = b_{0m} + b_{xm}x_t$ where b_{0m} and b_{xm} are constants to be determined. Substituting this conjectured solution for $z_{m,t}$

into the resulting first-order condition yields:

$$\begin{aligned}
E_t[\exp[\theta \ln \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1)r_{a,t+1} + r_{m,t+1}]] &= 1 \\
E_t[\exp[\theta \ln \delta - \frac{\theta}{\psi} \mu_c + (\theta - 1)B_{0a} + k_{0m} + \mu_d + (k_{1m} - 1)b_{0m} \\
&+ [(\theta - 1)B_{xa} - \frac{\theta}{\psi} + \phi + k_{1m}b_{xm}\rho - b_{xm}]x_t \\
&+ [(\theta - 1)B_{ea} + k_{1m}b_{xm}\sigma_e]e_{t+1} + (\theta - 1 - \frac{\theta}{\psi})\eta_{c,t+1} + \eta_{d,t+1}]] &= 1 \tag{B9}
\end{aligned}$$

This can be abbreviated as:

$$E_t[\exp[A_{0m} + A_{xm}x_t + A_{em}e_{t+1} + A_{\eta m}\eta_{c,t+1} + \eta_{d,t+1}]] = 1$$

where

$$\begin{aligned}
A_{0m} &= \theta \ln \delta - \frac{\theta}{\psi} \mu_c + (\theta - 1)B_{0a} + k_{0m} + \mu_d + (k_{1m} - 1)b_{0m} \\
A_{xm} &= (\theta - 1)B_{xa} - \frac{\theta}{\psi} + \phi + k_{1m}b_{xm}\rho - b_{xm} \\
A_{em} &= (\theta - 1)B_{ea} + k_{1m}b_{xm}\sigma_e \\
A_{\eta m} &= \theta - 1 - \frac{\theta}{\psi}
\end{aligned}$$

The constants b_{0m} and b_{xm} can then be solved by the method of undetermined coefficients:

$$b_{xm} = \frac{(\theta - 1)B_{xa} \frac{\theta}{\psi} + \phi}{1 - k_{1m}\rho} = \frac{(\theta - 1) \frac{\theta}{\psi^2} + \phi}{1 - k_{1m}\rho} \tag{B10}$$

$$b_{0m} = \frac{\theta \ln \delta - \frac{\theta}{\psi} \mu_c + (\theta - 1)B_{0a} + k_{0m} + \mu_d + 0.5A_{em}^2 + \Delta t \kappa_c(A_{\eta m}) + \Delta t \kappa_d(1)}{1 - k_{1m}} \tag{B11}$$

where $\kappa_c(A_{\eta m})$ is the cumulant exponent of the DPL innovation to consumption. The approximating constants appearing in Equation 7 in the text k_{0m} and k_{1m} are functions of the average level of the price-dividend ratio \bar{z}_m . Evaluating $z_{m,t} = b_{0m} + b_{xm}x_t$ at \bar{z}_m and

recognizing from Equation 12 that the average value of $x_{m,t}$ is zero yields $\bar{z}_m = b_{0m}$.

Thus, replacing the lhs of Equation B11 with \bar{z}_m and substituting for b_{xm} from Equation B10 into the rhs gives us a nonlinear equation in \bar{z}_m . We can solve this equation numerically for \bar{z}_m . Given \bar{z}_m , k_{0m} and k_{1m} , and hence b_{0m} and b_{xm} are readily obtained.

Given $z_{m,t} = b_{0m} + b_{xm}x_t$, variance of price-dividend ratio $z_{m,t}$ can be easily obtained as

$$\text{Var}(z_{m,t}) = b_{xm}^2 \text{Var}(x_t) \quad (\text{B12})$$

B.3 Returns on Aggregate Consumption Portfolio

Returns on the aggregate consumption portfolio $r_{a,t+1}$ are given in Equation (6). Using $z_t = b_0 + b_x x_t$ and the DPL process for $g_{c,t+1}$ from Equation (11) yields:

$$\begin{aligned} r_{a,t+1} &= k_0 + (k_1 - 1)b_0 + \mu_c + (k_1 b_x \rho - b_x + 1)x_t + k_1 b_x \sigma_e e_{t+1} + \eta_{c,t+1} \\ &= B_{0a} + B_{xa}x_t + B_{ea}e_{t+1} + \eta_{c,t+1} \end{aligned} \quad (\text{B13})$$

where

$$\begin{aligned} B_{0a} &= k_0 + (k_1 - 1)b_0 + \mu_c \\ B_{xa} &= k_1 b_x \rho - b_x + 1 = \frac{1}{\psi} \\ B_{ea} &= k_1 b_x \sigma_e \end{aligned}$$

Innovations to returns on the aggregate consumption portfolio $r_{a,t+1} - E[r_{a,t+1}]$ can be

expressed as:

$$\begin{aligned}
r_{a,t+1} &= k_0 + k_1 z_{t+1} - z_t + g_{c,t+1} \\
E[r_{a,t+1}] &= k_0 + k_1 E[z_{t+1}] - E[z_t] + E[g_{c,t+1}] \\
r_{a,t+1} - E[r_{a,t+1}] &= k_1(z_{t+1} - E[z_{t+1}]) - (z_t - E[z_t]) + (g_{c,t+1}^c - E[g_{c,t+1}])
\end{aligned}$$

Substituting the conjectured solution for z_t into the above equation and recognizing that $x_{t+1} - E[x_{t+1}] = e_{t+1} - E[e_{t+1}] = e_{t+1}$ yields:

$$\begin{aligned}
r_{a,t+1} - E[r_{a,t+1}] &= k_1 b_x (x_{t+1} - E[x_{t+1}]) + \eta_{c,t+1} \\
&= k_1 b_x \sigma_e e_{t+1} + \eta_{c,t+1}
\end{aligned} \tag{B14}$$

B.4 Pricing Kernel (IMRS)

The (logarithm of the) pricing kernel m_{t+1} is given in Equation (5) in the main text. Substituting for the DPL consumption process from Equation (11) and $r_{a,t+1}$ from Equation (B13) derived in the previous section of this Appendix into the formula for the pricing kernel yields:

$$\begin{aligned}
m_{t+1} &= \theta \ln \delta - \frac{\theta}{\psi} g_{c,t+1} + (\theta - 1) r_{a,t+1} \\
&= \theta \ln \delta - \frac{\theta}{\psi} (\mu_c + x_t + \eta_{c,t+1}) + (\theta - 1) (B_{0a} + B_{xa} x_t + B_{ea} e_{t+1} + \eta_{c,t+1}) \\
&= \theta \ln \delta - \frac{\theta}{\psi} \mu_c + (\theta - 1) B_{0a} + ((\theta - 1) B_{xa} - \frac{\theta}{\psi}) x_t + (\theta - 1) B_{ea} e_{t+1} + (\theta - 1 - \frac{\theta}{\psi}) \eta_{c,t+1}
\end{aligned} \tag{B15}$$

Innovations to the pricing kernel are given as:

$$m_{t+1} - E_t(m_{t+1}) = [(\theta - 1) k_1 b_x - \frac{\theta}{\psi}] \sigma_e e_{t+1} + (\theta - 1 - \frac{\theta}{\psi}) \eta_{c,t+1} \tag{B16}$$

The conditional variance of the pricing kernel can then be obtained as:

$$\begin{aligned} \text{Var}_t(m_{t+1}) &= E_t[(m_{t+1} - E_t(m_{t+1}))^2] \\ &= [(\theta - 1)k_1b_x - \frac{\theta}{\psi}]^2\sigma_e^2 + (\theta - 1 - \frac{\theta}{\psi})^2\sigma_\eta^2 \end{aligned} \quad (\text{B17})$$

where $\sigma_\eta^2 = \Delta t\Gamma(2 - \alpha)(\gamma_+\beta_+^{\alpha-2} + \gamma_-\beta_-^{\alpha-2})$ is the second moment of the innovation following the DPL process.

B.5 Risk Free Rate

The risk free asset pays one unit of consumption good as dividends every period. The first-order condition for the representative agent given as Equation (2) in the text can be rewritten for risk free returns as: $E_t[\exp(m_{t+1}r_{f,t+1})] = 1$. Recognizing that the risk free rate $r_{f,t+1}$ is known as of time t , and using Equation (B15) for m_{t+1} and Equation (B13) for $r_{a,t+1}$, we can derive the risk free rate as:

$$\begin{aligned} \exp(r_{f,t+1}) &= 1/E_t\{\exp[\theta \ln \delta + (\theta - 1)(k_0 + k_1b_0 - b_0) + \mu_c(\theta - 1 - \frac{\theta}{\psi}) \\ &\quad + [(\theta - 1)(k_1b_x\rho - b_x + 1) - \frac{\theta}{\psi}]x_t \\ &\quad + (\theta - 1)k_1b_x\sigma_e e_{t+1} + (\theta - 1 - \frac{\theta}{\psi})\eta_{c,t+1}]\} \\ r_{f,t+1} &= -B_{0f} - 0.5B_{ef}^2 - \Delta t\kappa(B_{\eta f}) - B_{xf}x_t \end{aligned} \quad (\text{B18})$$

where

$$\begin{aligned}
B_{0f} &= \theta \ln \delta + (\theta - 1)(k_0 + k_1 b_0 - b_0) + \mu_c \left(\theta - 1 - \frac{\theta}{\psi} \right) \\
&= \ln \delta - \frac{\mu_c}{\psi} + 0.5(1 - \theta)\theta(k_1 b_x \sigma_e)^2 + \frac{1 - \theta}{\theta} \Delta t \kappa(A_\eta) \\
B_{xf} &= (\theta - 1)(k_1 b_x \rho - b_x + 1) - \frac{\theta}{\psi} \\
B_{ef} &= (\theta - 1)k_1 b_x \sigma_e \\
B_{\eta f} &= \theta - 1 - \frac{\theta}{\psi} = \gamma
\end{aligned}$$

The second equality for B_{0f} is obtained by substituting Equation (B6) for b_0 into the rhs of the first equation.

Using $E[x_t] = 0$ and $A_\eta = \theta - \frac{\theta}{\psi} = 1 - \gamma$, the unconditional expectation of the risk free rate $E[r_{f,t}]$ is given by:

$$\begin{aligned}
E[r_{f,t+1}] &= -B_{0f} - 0.5B_{ef}^2 - \Delta t \kappa(B_{\eta f}) \\
&= -\ln \delta + \frac{\mu_c}{\psi} + 0.5(\theta - 1)\theta(k_1 b_x)^2 \sigma_e^2 + \Delta t \left[\left(1 - \frac{1}{\theta}\right) \kappa(1 - \gamma) - \kappa(\gamma) \right] \quad (\text{B19})
\end{aligned}$$

Unconditional variance of the risk free rate $Var(r_{f,t+1})$ can be easily obtained from Equation (B18) as:

$$Var(r_{f,t+1}) = B_{xf}^2 Var(x_t) = \frac{B_{xf}^2}{1 - \rho^2} \sigma_e^2 \quad (\text{B20})$$

where $Var(x_t) = \frac{1}{1 - \rho^2} \sigma_e^2$ as a straightforward result of $x_{t+1} = \rho x_t + e_{t+1}$.

B.6 Market Risk Premium

Returns on the market portfolio $r_{m,t+1}$ are given in Equation (7). Using $z_{m,t} = b_{0m} + b_{xm}x_t$ and substituting Equations (11) and (13) for $g_{c,t+1}$ and $g_{d,t+1}$ respectively yields:

$$\begin{aligned} r_{m,t+1} &= k_{0m} + (k_{1m} - 1)b_{0m} + \mu_d + (k_{1m}b_{xm}\rho - b_{xm} + \phi)x_t + k_{1m}b_{xm}\sigma_e e_{t+1} + \eta_{d,t+1} \\ &= B_{0m} + B_{xm}x_t + B_{em}e_{t+1} + \eta_{c,t+1} \end{aligned} \quad (\text{B21})$$

where

$$\begin{aligned} B_{0m} &= k_{0m} + (k_{1m} - 1)b_{0m} + \mu_d \\ B_{xm} &= k_{1m}b_{xm}\rho - b_{xm} + \phi \\ B_{em} &= k_{1m}b_{xm}\sigma_e \end{aligned}$$

Subtracting the expected risk free rate in Equation (B19) from returns on the market portfolio in Equation B21 yields conditional market risk premium:

$$\begin{aligned} E_t[r_{m,t+1} - r_{ft}] &= k_{0m} + k_{1m}b_{0m} - b_{0m} + \mu_d + B_{0f} \\ &\quad + 0.5((\theta - 1)k_1b_{xf}\sigma_e)^2 + \kappa_c(B_{\eta f}) \\ &\quad + (k_{1m}b_{xm}\rho - b_{xm} + \phi + B_{xf})x_t \end{aligned} \quad (\text{B22})$$

and unconditional market risk premium:

$$\begin{aligned} E[r_{m,t+1} - r_{ft}] &= k_{0m} + k_{1m}b_{0m} - b_{0m} + \mu_d + B_{0f} \\ &\quad + 0.5((\theta - 1)k_1b_{xf}\sigma_e)^2 + \kappa_c(B_{\eta f}) \\ &= [(1 - \theta)k_1b_x - 0.5k_{1m}b_{xm}]k_{1m}b_{xm}\sigma_e^2 - \Delta t\kappa_d(1) \end{aligned} \quad (\text{B23})$$

where the cumulant exponent $\kappa_d(1)$ is computed through Equation B4, but with the DPL

parameters to the innovation to consumption growth $\gamma_+^c, \gamma_-^c, \beta_+^c, \beta_-^c, \alpha^c$ being replaced by the parameters to dividend growth $\gamma_+^d, \gamma_-^d, \beta_+^d, \beta_-^d, \alpha^d$.

B.7 Variance of Market Returns

We can derive the conditional innovations to excess market returns as: $r_{m,t+1} - E_t[r_{m,t+1}] = k_{1m}b_{xm}\sigma_e e_{t+1} + \eta_{d,t+1}$.

The conditional and unconditional variances of market returns can then be obtained as follows:

$$\begin{aligned} Var_t(r_{m,t+1}) &= E_t[r_{m,t+1} - E_t[r_{m,t+1}]]^2 \\ &= k_{1m}^2 b_{xm}^2 \sigma_e^2 + Var_d \end{aligned} \tag{B24}$$

$$\begin{aligned} Var(r_{m,t+1}) &= (k_{1m}b_{xm}\rho - b_{xm} + \phi)Var(x_t) + k_{1m}^2 b_{xm}^2 \sigma_e^2 + Var(\eta_d) \\ &= \frac{1}{\psi^2}Var(x_t) + k_{1m}^2 b_{xm}^2 \sigma_e^2 + Var(\eta_d) \end{aligned} \tag{B25}$$

where $Var(x_t) = \sigma_e^2/(1 - \rho^2)$ and $Var(\eta_d) = \Delta t \Gamma(2 - \alpha^d)[\gamma_+^d(\beta_+^d)^{\alpha^d-2} + \gamma_-^d(\beta_-^d)^{\alpha^d-2}]$.

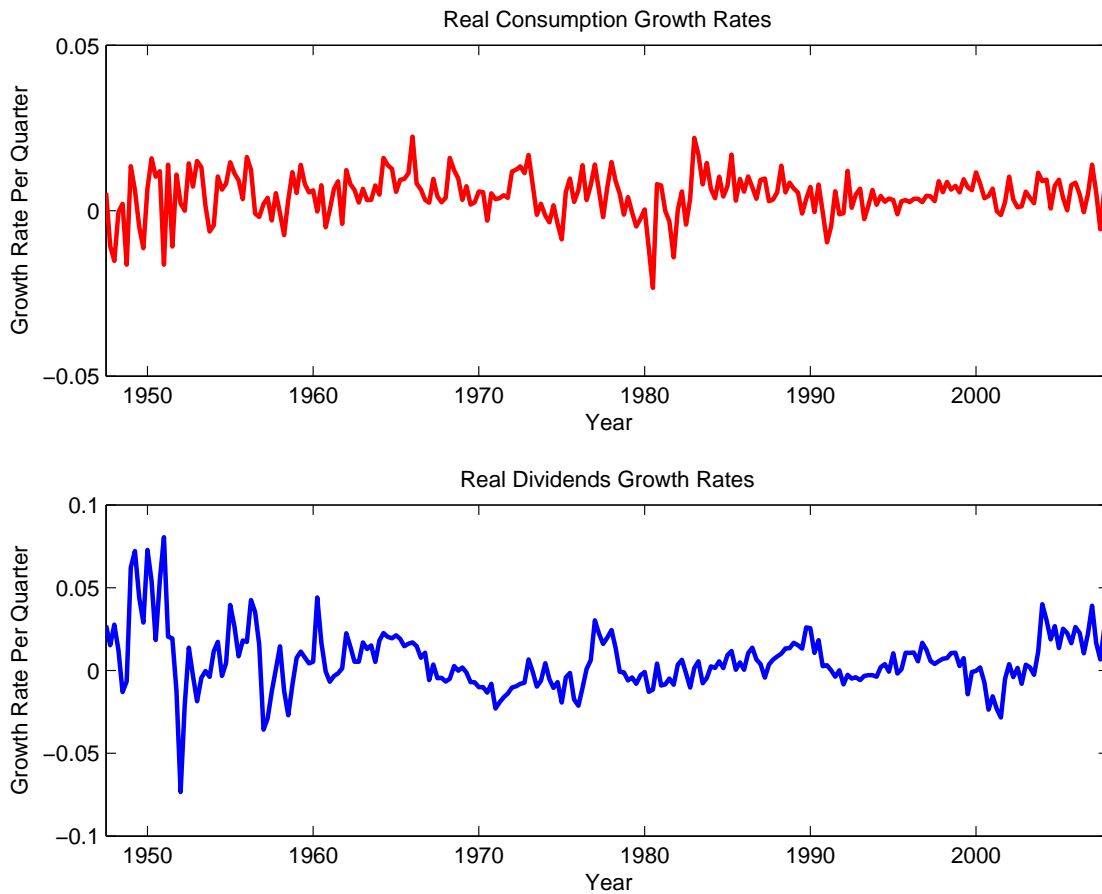


Figure 1: Quarterly consumption and dividends growth rates

Quarterly consumption and dividends growth rates span the period 1947:I through 2007:IV. Consumption includes non-durable goods and services from the NIPA tables. Dividends, paid toward the S&P 500 index, are obtained from Robert Shiller's website. Nominal consumption and dividends are deflated by the CPI series to obtain real quantities.

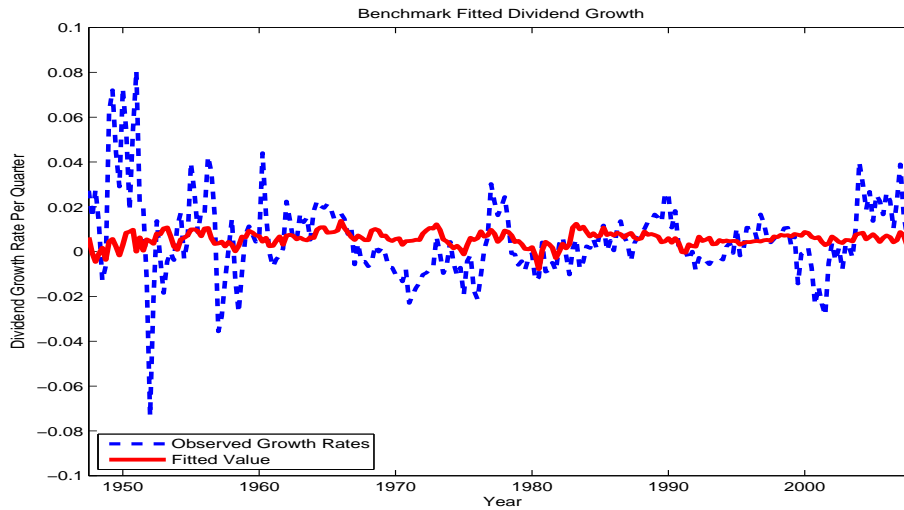
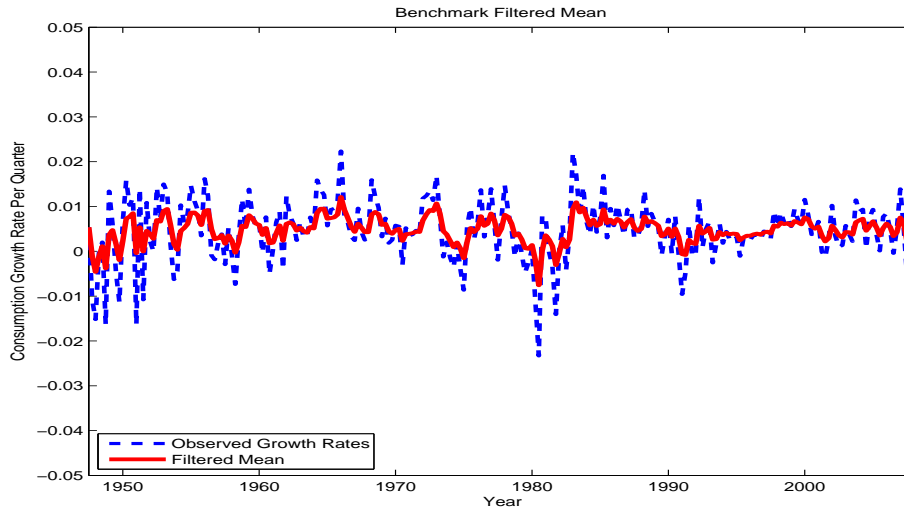


Figure 2: Filtered consumption and fitted dividend growth: the benchmark model
 The upper panel plots observed consumption and its filtered mean for the benchmark model, where all innovations are assumed to be Gaussian. The lower panel plots observed dividends and their fitted values in a regression of the former on the filtered mean of the persistent component (this is the filtered mean series plotted in the upper panel, adjusted for a non-zero time-invariant mean as in Equation (8)).

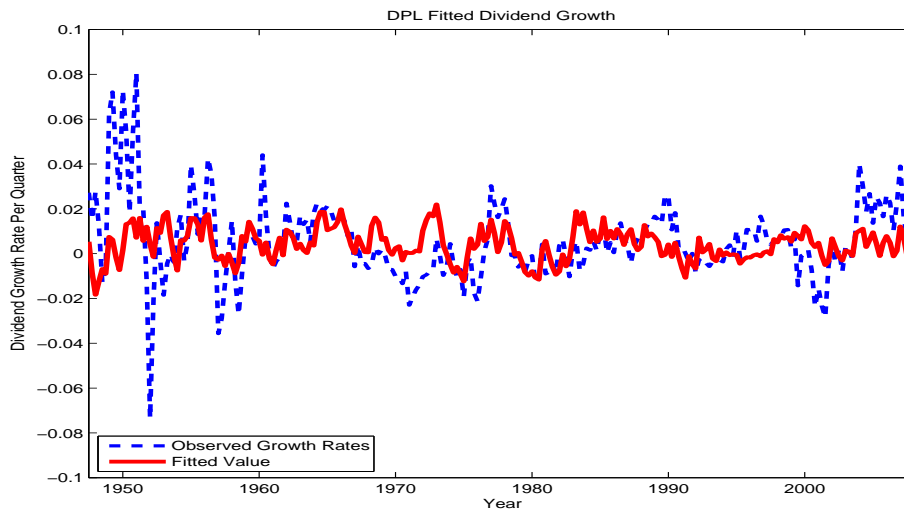
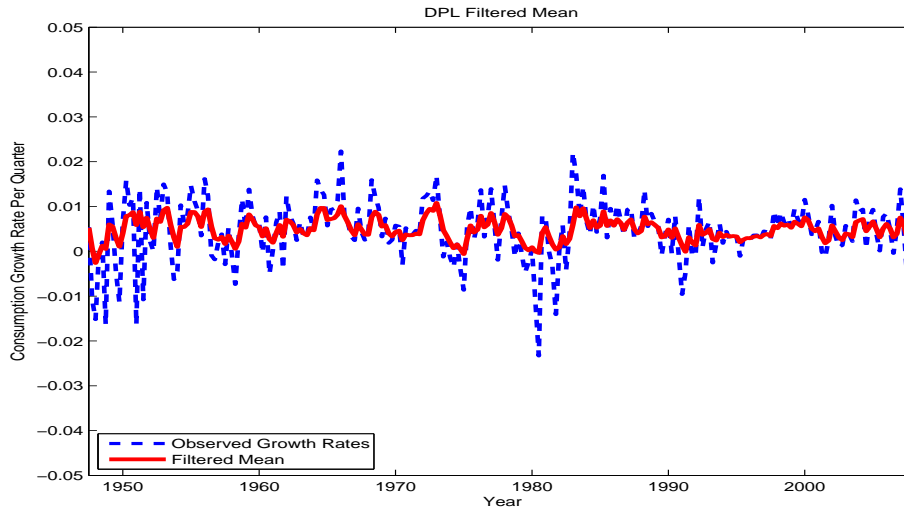


Figure 3: Filtered consumption and fitted dividend growth: the DPL model
 The upper panel plots observed consumption and its filtered mean for the DPL model, where iid innovations to consumption growth rates (but not to the persistent component) are assumed to follow the dampened power law. The lower panel plots observed dividends and their fitted values in a regression of the former on the filtered mean of the persistent component (this is the filtered mean series plotted in the upper panel, adjusted for a non-zero time-invariant mean as in Equation (11)).

Table 1: Summary Statistics for Consumption and Dividends

	Mean	Std. Dev.	Skewness	Kurtosis	J-B Test
Consumption Growth Rates	0.0047 (0.0004)	0.00662 (0.0008)	-0.7389 (0.0000)	4.8688 (0.0000)	57.4736 (0.0000)
Dividends Growth Rates	0.0057 (0.0011)	0.01764 (0.0008)	0.6914 (0.0000)	6.9019 (0.0000)	173.5148 (0.0000)

The table presents summary statistics for quarterly real per capita consumption and dividends growth rates over the period 1947:I-2007:IV. Numbers in parentheses are standard errors for columns 1 and 2, and p-values for columns 3-5. Consumption data, for non-durable goods and services, are obtained from NIPA tables. Dividends, paid toward the S&P 500 index, are obtained from Robert Shiller's website.

Table 2: Maximum Likelihood Parameter Estimates of Consumption Growth Process

Model	μ_c	σ_c	ρ	σ_e							LogL
Benchmark	0.0048 0.0007	0.0052 0.0006	0.6857 0.1669	0.0030 0.0011							881.0295
DPL	μ_c	σ_e	ρ	γ_+^c	γ_-^c	$\gamma_+^c = \gamma_-^c$	β_+^c	β_-^c	$\beta_+^c = \beta_-^c$	α^c	LogL
Unrestricted	0.0049	0.0026	0.6701	0.0006	0.0017		68.3914	67.5682		1.2617	891.8692
DPL	0.0006	0.0000	0.0088	0.0004	0.0005		0.0013	0.0071		0.0000	
Sym. Damp.	0.0049 0.0006	0.0026 0.0000	0.6708 0.0688	0.0006 0.0011	0.0017 0.0021				68.2415 1.8168	1.2569 0.2232	891.8603
Sym. Scale	0.0049 0.0006	0.0016 0.0000	0.7821 0.0557			0.0009 0.0001	121.6673 0.0470	38.5114 0.0553		1.3240 0.0038	891.0436
Sym. Damp. and Scale	0.0053 0.0000	0.0016 0.0000	0.7781 0.0000			0.0041 0.0000			97.0592 0.0001	1.1000 0.0000	889.7987

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- 1 Consumption Growth Process: $g_{c,t+1} = \mu_c + x_t + \eta_{c,t+1}$
DPL model: $\eta_{c,t+1} \sim iidDPL(\gamma_+^c, \gamma_-^c, \beta_+^c, \beta_-^c, \alpha^c)$
Benchmark model: $\eta_{c,t+1} \sim iidN(0, \sigma_c^2)$
- 2 State transition: $x_{t+1} = \rho x_t + e_{t+1}$. $e_{t+1} \sim iidN(0, \sigma_e^2)$.
- 3 For each model, parameter estimates are reported in the first row, and standard errors in the second.
- 4 “Sym. Damp.” refers to the DPL model with “Symmetric Dampening”.

Table 3: Maximum Likelihood Parameter Estimates of Dividend Growth Process

Model	μ_d	ϕ	σ_d							LogL
Benchmark	0.0057 0.0011	1.0950 0.3926	0.0173 0.0008							640.6437
DPL	μ_d	ϕ	γ_+^d	γ_-^d	$\gamma_+^d = \gamma_-^d$	β_+^d	β_-^d	$\beta_+^d = \beta_-^d$	α^d	LogL
Unrestricted	0.0047	3.0956	0.0018	0.0013		10.2087	9.6859		1.4444	667.7964
DPL	0.0000	0.0001	0.0003	0.0003		0.0211	0.0351		0.0013	
Sym. Damp.	0.0043 0.0000	3.0120 0.0000	0.0942 0.0018	0.0769 0.0000				43.3669 0.0106	0.7052 0.0143	668.6691
Sym. Scale	0.0045 0.0000	3.1008 0.0000			0.0086 0.0003	20.1681 3.8624	35.8592 7.4751		1.1405 0.0023	670.7876
Sym. Damp. and Scale	0.0043 0.0000	3.0074 0.0003			0.0061 0.0004			18.9258 4.5241	1.1895 0.0338	667.9971

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- 1 Dividend Growth Process: $g_{d,t+1} = \mu_d + \phi x_t + \eta_{d,t+1}$
DPL model: $\eta_{d,t+1} \sim iidDPL(\gamma_+^d, \gamma_-^d, \beta_+^d, \beta_-^d, \alpha^d)$
Benchmark model: $\eta_{d,t+1} \sim iidN(0, \sigma_d^2)$

2 For each model, parameter estimates are reported in the first row, and standard errors in the second.

3 “Sym. Damp.” refers to the DPL model with “Symmetric Dampening”.

Table 4: Parameters Determining Price-Consumption and Price-Dividend Ratios

γ	ψ	\bar{z}	k_0	k_1	\bar{z}_m	k_{0m}	k_{1m}
Benchmark Consumption							
gamma	psi	z	k0	k1	zm	k0m	k1m
7.5	0.5	5.044	0.039	0.9936	5.248	0.033	0.995
7.5	1.5	7.543	0.005	0.9995	3.307	0.153	0.965
10	0.5	5.066	0.038	0.9937	5.286	0.032	0.995
10	1.5	7.454	0.005	0.9994	3.309	0.153	0.965
25	0.5	5.215	0.034	0.9946	5.545	0.025	0.996
25	1.5	7.044	0.007	0.9991	3.321	0.151	0.965
35	0.5	5.328	0.031	0.9952	5.766	0.021	0.997
35	1.5	6.841	0.008	0.9989	3.328	0.150	0.965
DPL Consumption							
7.5	0.5	5.056	0.038	0.994	4.983	0.041	0.993
7.5	1.5	7.496	0.005	0.999	6.848	0.008	0.999
10	0.5	5.090	0.037	0.994	5.011	0.040	0.993
10	1.5	7.375	0.005	0.999	6.753	0.009	0.999
25	0.5	5.329	0.031	0.995	5.208	0.034	0.995
25	1.5	6.847	0.008	0.999	6.351	0.013	0.998
35	0.5	5.533	0.026	0.996	5.391	0.029	0.995
35	1.5	6.600	0.010	0.999	6.205	0.015	0.998

¹ \bar{z} and \bar{z}_m are the average values of the price-consumption and price-dividend ratios for the aggregate consumption and market portfolios, respectively.

² k_0 , k_1 , k_{0m} , and k_{1m} are the constants appearing in the approximate equations for the gross returns to the consumption and market portfolios in Equations (6-7).

Table 5: Asset Pricing Implications - Benchmark Model

γ	ψ	$E(r_m - r_f)$	$E(r_f)$	$\sigma(r_m)$	$\sigma(r_f)$	$\sigma(p - d)$
7.5	0.5	-0.249	4.626	4.192	1.649	0.012
7.5	1.5	0.033	1.975	3.579	0.550	0.005
10	0.5	-0.329	4.629	4.193	1.649	0.012
10	1.5	0.068	1.938	3.579	0.550	0.005
25	0.5	-0.811	4.651	4.195	1.649	0.012
25	1.5	0.277	1.712	3.579	0.550	0.005
35	0.5	-1.136	4.667	4.197	1.649	0.012
35	1.5	0.417	1.562	3.579	0.550	0.005

The table reports implied expected market risk premium and the risk free rate along with their volatilities, and the volatility of the implied price-dividend ratio for the benchmark fully Gaussian model for various values of the risk aversion coefficient γ and the intertemporal elasticity of substitution ψ .

Table 6: Asset Pricing Implications - DPL Model

γ	ψ	$E(r_m - r_f)$	$E(r_f)$	$\sigma(r_m)$	$\sigma(r_f)$	$\sigma(p - d)$
7.5	0.5	0.117	4.336	2.199	1.402	0.011
7.5	1.5	0.326	1.819	3.778	0.467	0.025
10	0.5	0.178	4.200	2.199	1.402	0.011
10	1.5	0.471	1.716	3.777	0.467	0.025
25	0.5	0.546	3.358	2.203	1.402	0.011
25	1.5	1.340	1.077	3.770	0.467	0.025
35	0.5	0.796	2.743	2.206	1.402	0.011
35	1.5	1.917	0.610	3.767	0.467	0.025

The table reports implied expected market risk premium and the risk free rate along with their volatilities, and the volatility of the implied price-dividend ratio for the DPL model for various values of the risk aversion coefficient γ and the intertemporal elasticity of substitution ψ .

Table 7: Maximum Likelihood Parameter Estimates of Dividend Growth Process with Dividend Filtering

Model	μ_d	ϕ	σ_d	σ_e	ρ							LogL
Benchmark	0.0069 0.0021	3.7387 0.7071	0.0065 0.0033	0.0028 0.0008	0.6754 0.0881							699.2127
DPL	μ_d	ϕ	σ_e	ρ	γ_+^d	γ_-^d	$\gamma_+^d = \gamma_-^d$	β_+^d	β_-^d	$\beta_+^d = \beta_-^d$	α^d	LogL
Unrestricted DPL	0.0051 0.0000	3.1003 0.0000	0.0019 0.0000	0.8709 0.0355	0.0297 0.0008	0.0172 0.0033		24.8812 5.6452	19.8334 0.0018		0.6613 0.0006	733.9235
Sym. Damp.	0.0051 0.0000	3.1017 0.0000	0.0021 0.0000	0.8394 0.0329	0.0088 0.0000	0.0047 0.0009				9.0270 0.0000	0.8465 0.0285	732.8834
Sym. Scale	0.0052 0.0000	3.0196 0.0000	0.0025 0.0000	0.8076 0.0433			0.0065 0.0009	3.6454 0.0012	23.327 0.0023		0.8380 0.0000	731.3902
Sym. Damp. and Scale	0.0052 0.0000	3.0212 0.0000	0.0021 0.0000	0.8595 0.0000			0.0023 0.0001			3.4880 0.0000	1.0833 0.0000	731.1811

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1 Dividend Growth Process: $g_{d,t+1} = \mu_d + \phi x_t + \eta_{d,t+1}$

DPL model: $\eta_{d,t+1} \sim iidDPL(\gamma_+^d, \gamma_-^d, \beta_+^d, \beta_-^d, \alpha^d)$

Benchmark model: $\eta_{d,t+1} \sim iidN(0, \sigma_d^2)$

2 State: $x_{t+1} = \rho x_t + e_{t+1}$. $e_{t+1} \sim iidN(0, \sigma_e^2)$.

3 For each model, parameter estimates are reported in the first row, and standard errors in the second.

4 “Sym. Damp.” refers to the DPL model with “Symmetric Dampening”.

Table 8: Parameter Estimates of Consumption Growth Process with Dividend Filtering

Model	μ_c	σ_c							LogL
Benchmark	0.0050 0.0004	0.0069 0.0003							864.0457
DPL	μ_c	γ_+^c	γ_-^c	$\gamma_+^c = \gamma_-^c$	β_+^c	β_-^c	$\beta_+^c = \beta_-^c$	α^c	LogL
Unrestricted	0.0040	0.0156	0.0068			194.8290	36.4532	0.9932	877.5458
DPL	0.0000	0.0000	0.0000			0.0001	0.0000	0.0000	
Sym. Damp.	0.0035 0.0000	0.3498 0.0000	1.0279 0.0000			181.05 0.0000		0.3329 0.0000	874.9804
Sym. Scale	0.0047 0.0000			0.0217 0.0017	151.66 0.0000	97.099 0.0018		0.8763 0.0000	878.0937
Sym. Damp. and Scale	0.0050 0.0000			0.0195 0.0005			107.45 9.7561	0.8858 0.0009	873.7761

¹ Consumption Growth Process:

$$g_{c,t+1} = \mu_c + x_t + \eta_{c,t+1}; \text{ DPL model: } \eta_{c,t+1} \sim iidDPL(\gamma_+^c, \gamma_-^c, \beta_+^c, \beta_-^c, \alpha^c)$$

$$\text{Benchmark model: } \eta_{c,t+1} \sim iidN(0, \sigma_c^2)$$

² Assuming consumption growth process share the same DPL structure with dividends growth process.

³ For each model, parameter estimates are reported in the first row, and standard errors in the second.

⁴ “Sym. Damp.” refers to the DPL model with “Symmetric Dampening”.

Table 9: Asset Pricing Implications - Benchmark Model with Dividend Filtering

γ	ψ	$E(r_m - r_f)$	$E(r_f)$	$\sigma(r_m)$	$\sigma(r_f)$	$\sigma(p - d)$
7.5	0.5	0.222	4.684	3.564	1.519	0.020
7.5	1.5	0.436	1.988	4.931	0.506	0.033
10	0.5	0.347	4.649	3.562	1.519	0.020
10	1.5	0.662	1.936	5.115	0.506	0.034
25	0.5	1.094	4.436	3.553	1.519	0.020
25	1.5	2.057	1.624	5.444	0.506	0.036
35	0.5	1.591	4.296	3.546	1.519	0.020
35	1.5	2.946	1.416	5.418	0.506	0.036

The table reports implied expected market risk premium and the risk free rate along with their volatilities, and the volatility of the implied price-dividend ratio for the benchmark fully Gaussian model for various values of the risk aversion coefficient γ and the intertemporal elasticity of substitution ψ .

Table 10: Asset Pricing Implications - DPL Model with Dividend Filtering

γ	ψ	$E(r_m - r_f)$	$E(r_f)$	$\sigma(r_m)$	$\sigma(r_f)$	$\sigma(p - d)$
7.5	0.5	0.336	4.820	4.347	1.643	0.029
7.5	1.5	1.090	1.914	7.488	0.548	0.068
10	0.5	0.542	4.848	4.339	1.643	0.028
10	1.5	1.593	1.835	7.441	0.548	0.067
25	0.5	1.766	5.027	4.295	1.643	0.028
25	1.5	4.456	1.357	7.187	0.548	0.065
35	0.5	2.571	5.151	4.267	1.643	0.028
35	1.5	6.238	1.028	7.043	0.548	0.064

The table reports implied expected market risk premium and the risk free rate along with their volatilities, and the volatility of the implied price-dividend ratio for the DPL model for various values of the risk aversion coefficient γ and the intertemporal elasticity of substitution ψ .