UCDAVIS

DEPARTMENT OF ECONOMICS

Working Paper Series

Awareness-Dependent Subjective Expected Utility

Burkhard Schipper
University of California Davis

December 21, 2011

Paper # 10-22

We develop awareness-dependent subjective expected utility by taking unawareness structures introduced in Heifetz, Meier, and Schipper (2006, 2008, 2011a) as primitives in the Anscombe-Aumann approach to subjective expected utility. We observe that a decision maker is unaware of an event if and only if her choices reveal that the event is ``null'' and the negation of the event is ``null''. Moreover, we characterize ``impersonal'' expected utility that is behaviorally indistinguishable from awareness-dependent subject expected utility and assigns probability zero to some subsets of states that are not necessarily events. We discuss in what sense probability zero can model unawareness.

Department of Economics
One Shields Avenue
Davis, CA 95616
(530)752-0741

http://www.econ.ucdavis.edu/working_search.cfm

Awareness-Dependent Subjective Expected Utility*

Burkhard C. Schipper †

This Version: December 21, 2011
First Version: February 2, 2009

Abstract

We develop awareness-dependent subjective expected utility by taking unawareness structures introduced in Heifetz, Meier, and Schipper (2006, 2008, 2011a) as primitives in the Anscombe-Aumann approach to subjective expected utility. We observe that a decision maker is unaware of an event if and only if her choices reveal that the event is "null" and the negation of the event is "null". Moreover, we characterize "impersonal" expected utility that is behaviorally indistinguishable from awareness-dependent subject expected utility and assigns probability zero to some subsets of states that are not necessarily events. We discuss in what sense probability zero can model unawareness.

Keywords. Unawareness, awareness, unforeseen contingencies, null, probability zero, subjective expected utility, Anscombe-Aumann, small worlds.

JEL-Classifications. C70, C72, D03, D80, D81.

---

*I thank two anonymous referees for helpful suggestions that improved the exposition of the paper. Moreover, I thank Christopher Chambers, Adam Dominiak, Konrad Grabiszewski, Aviad Heifetz, Bart Lipman, Martin Meier, Klaus Nehring as well as two anonymous referees for the LOFT 2010 conference for helpful comments. This paper is closely related to prior and ongoing work on unawareness with my friends, Aviad Heifetz and Martin Meier. All mistakes are mine. Financial support through NSF SES-0647811 is gratefully acknowledged. An earlier version was circulated under the title "Revealed Unawareness".

†Department of Economics, University of California, Davis, Email: bcschipper@ucdavis.edu

1 Introduction

Unawareness refers to the lack of conception rather than the lack of information. There is a fundamental difference between uncertainty about which event obtains and the inability to conceive of some events. In the literature, unawareness has been defined epistemically using syntactic and semantic approaches.[1] While epistemic characterizations are conceptually insightful, the behavioral content of unawareness remains unclear. For instance, a referee of a recent report on Heifetz, Meier, and Schipper (2011b) wrote "It has become a folk wisdom among readers of this literature that unawareness is often nothing but another name for 0-probability belief. ... Is unawareness really nothing but another name for 0-probability belief? I don't know."

Heifetz, Meier, and Schipper (2006, 2008, 2011a) introduced a syntax-free semantics of unawareness using state spaces familiar to economists, decision theorists, and game theorists.[2] Instead of one state space, it consists of a lattice of disjoint spaces, where every space in the lattice captures one particular horizon of meanings or propositions. Higher spaces capture wider horizons, in which states correspond to situations described by a richer vocabulary. In the present paper, we replace the standard state space in the Anscombe and Aumann (1963) approach to subjective utility theory by a lattice of spaces. This is done because Dekel, Lipman, and Rustichini (1998) showed that standard state spaces preclude unawareness while Heifetz, Meier, and Schipper (2006) showed that nontrivial unawareness obtains in a lattice of spaces. In this richer framework, we are able to characterize awareness-dependent subjective expected utility. We choose the Anscombe-Aumann approach not because we think it is the most natural one in the context of unawareness but because it is perhaps the most "standard" approach and starting point. Apart from the lattice of spaces, the setting should be entirely familiar and thus easily accessible to the reader. The message we like to convey is that unawareness structures lend themselves in a straightforward way as primitives in subjective expected utility theory.

Acts are now defined on the union of all spaces and are interpreted as labels for actions. The interpretation of those label depends on the awareness of the decision maker and the decision maker may evaluate acts differently depending on her awareness. For instance, consider a potential investor who considers the act "invest in firm X". Firm X is a bundle of potential opportunities and liabilities that depend on the states of nature. Which of

[1] For a bibliography see http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm

[2] Apart from having a syntax-free semantics, Heifetz, Meier, and Schipper (2006, 2008) generalize Modica and Rustichini (1999) and a version of Fagin and Halpern (1988) to the multi-agent case. The precise connection between Fagin and Halpern (1988), Modica and Rustichini (1999), Halpern (2001) and Heifetz, Meier, and Schipper (2006) is understood from Halpern and Rêgo (2008) and Heifetz, Meier, and Schipper (2008). The connection between Heifetz, Meier, and Schipper (2006, 2008) and Galanis (2011a) is explored in Galanis (2011b). The relationship between Board and Chung (2009) and Heifetz, Meier, and Schipper (2006) is studied in Board, Chung, and Schipper (2011). Heinsalu (2011) studies the relationship between Li (2009) and Fagin and Halpern (1988). The connection to Feinberg (2009) is yet to be explored.

these opportunities and liabilities the investor has in mind is determined by her awareness of these events. An investor being aware of a potential lawsuit that involves the firm but unaware of a potential innovation that may enhance the value of the firm may evaluate the act differently than an investor who is unaware of the former but aware of the latter.[3]

Preferences of the decision maker are defined on those modified acts, one preference relation for each awareness level, so that the same decision maker at different awareness levels can be compared. Standard properties on preferences are imposed for each awareness level, and an additional property is imposed that confines the perception of an act to one awareness level. An awareness-dependent subjective expected utility representation is then characterized in an embarrassingly straightforward way. Indeed, the first positive main message of this paper for the applied economist may be that it is straightforward to characterize subjective expected utility in unawareness structures. This closes an important gap in the literature, as we do not know of any other choice-theoretic model that allows for nontrivial unawareness satisfying epistemic properties as introduced in Fagin and Halpern (1988), Modica and Rustichini (1999), and Dekel, Lipman, and Rustichini (1998). On one hand, standard choice-theory uses a state-space as a primitive such that nontrivial unawareness is precluded by Dekel, Lipman, and Rustichini (1998)'s impossibility result. On the other hand, the literature on unawareness defines unawareness epistemically but provides no choice-theoretic characterization of it. This critique applies also to our own prior work. Originally, we studied just epistemic properties of unawareness structures in Heifetz, Meier, and Schipper (2006). Logical foundations have been provided by Halpern and Rego (2008) and Heifetz, Meier, and Schipper (2008). Unawareness structures have been applied to speculative trade in Heifetz, Meier, and Schipper (2011a) and Meier and Schipper (2010), to Bayesian games in Meier and Schipper (2011), and to dynamic games Heifetz, Meier, and Schipper (2011b). Yet, until now, notions of utility and beliefs have been taken as primitives in those structures. The current paper shows that they can be derived from choices within unawareness structures.[4]

The second goal is to apply the representation theorem to analyze the behavioral implications of unawareness. Consider an outside observer who wishes to know from the choices of a decision maker conforming to the Anscombe-Aumann approach whether she is unaware of an event or not. It is shown that a decision maker is unaware of the event if and only if her choices reveal that the event is "null" *and* the negation of the event is "null". This distinguishes unawareness from subjective probability zero belief, for which the event is null but its negation cannot be null. Thus unawareness does have behavioral implications different from subjective probability zero belief. The following example illustrates the point: Consider a potential buyer of a firm. Agreements on the change of ownerships of private firms may be very complex, involving many pages of legal

[3]See Heifetz, Meier, and Schipper (2011a) and Meier and Schipper (2010) for the analysis of speculative trade in such a setting.

[4]This claim must come with a caveat. Like universal type-spaces in standard game theory, the universal unawareness type–space in Heifetz, Meier, and Schipper (2011a) contains all hierarchies of beliefs. It is hard to conceive of choice experiments that would "reveal" all those higher order beliefs.

documents. It is not inconceivable that the buyer may miss certain important clauses and may not think about them when contemplating the transaction. In particular, the buyer may be unaware of a specific costly future lawsuit that the firm may or may not be involved in. Assume that the buyer can choose among two contracts. Under contract 1 the potential lawsuit is the buyer's responsibility. Under contract 2, the potential lawsuit is the seller's responsibility. Otherwise both contracts are the same in content. Being indifferent between both contracts is consistent with assigning probability zero to the event of the lawsuit. Assume now that a third contract is available. Under contract 3 the potential lawsuit is the seller's responsibility but the seller receives an additional compensation from the buyer in the event that the lawsuit does not obtain. Apart from this clause, the content of contract 3 is the same as the other contracts. Being indifferent between contract 3 and 2 is consistent with assigning probability zero to the event of "no lawsuit". Indifference between *all* three contracts is consistent with being unaware of "lawsuit", but not with assigning probability zero to either the events "lawsuit" or "no lawsuit" because probability zero cannot be assigned to an event and its negation.[5]

The third goal of this note is to explore to what extent behavior of unaware decision makers can be captured with probability zero nevertheless. We characterize "impersonal" expected utility that is behaviorally indistinguishable from awareness-dependent expected utility. The representation delivers a probability measure on the "flattened state space", the union of all state spaces in the lattice, that assigns zero probability not only to null events but also to any subsets of states (that may not necessarily be events) that the decision maker does "not reason" about. We discuss in Subsection 6.1 that probability zero per se cannot model unawareness. It requires additionally the "rich language" of unawareness structures. But in unawareness structures, awareness-dependent subjective expected utility should be preferable.

Awareness-dependent expected utility may be seen as a step towards analyzing Savage's (1954) "small worlds" assumption. Savage (1954, p. 82-83) used the term for the space of states of nature to indicate the "...practical necessity to confining attention, or isolating, relatively simple situations...". Savage (1954, p. 16) felt that he "was unable to formulate criteria for selecting these small worlds...". While we cannot deliver such a criterion either, our approach allows the modeler to analyze the decision maker in various sets of "small worlds", partially ordered by their richness. The representation theorem should be interpreted either from the modeler's point of view as contemplating a decision maker's (admittedly counterfactual) choices at various awareness levels, or from the decision maker's point of view conditional on her awareness level.[6]

[5]See Section 4 for a more detailed discussion of the example and Figure 2.

[6]Intuitively, the awareness level corresponds to the richest description of events that can be detected from the decision maker's choices. See the discussion of Property 6. In an extended model with states of the world (as in Heifetz, Meier, and Schipper, 2011a) rather than states of nature, that is, in which states also encode the preference and thus beliefs of the decision maker, the decision maker at a given awareness level could also reason about her own decisions at lower awareness levels. See Schipper (2011) for such an extended model.

The paper is organized as follows: In Section 2, we present primitives of unawareness structures. In Section 3, we develop awareness-dependent subjective expected utility. This is applied to the problem of revealing unawareness in Section 4. In Section 5, we characterize impersonal expected utility and discuss its relation to awareness-dependent subjective expected utility. In Section 6, we finish with a discussion of probability zero, caveats, and extensions, as well as the related literature. All proofs are collected in an appendix.

2 Primitives of Unawareness Structures

2.1 State spaces

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a finite lattice of disjoint *state spaces*, with a partial order $\preceq$ on $\mathcal{S}$. For simplicity we assume in this paper that each S is finite. If S_α and S_β are such that $S_\alpha \succeq S_\beta$ we say that "S_α is more expressive than S_β – states of S_α describe situations with a richer vocabulary than states of S_β ".[7] Denote by $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the union of these spaces.

Spaces in the lattice can be more or less "rich" in terms of facts that may or may not obtain in them. The partial order relates to the "richness" of spaces.

2.2 Projections

For every S and S' such that $S' \succeq S$, there is a surjective projection $r_S^{S'} : S' \to S$, where r_S^S is the identity. ("$r_S^{S'}(\omega)$ is the restriction of the description ω to the more limited vocabulary of S.") Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: If $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

Projections "translate" states in "more expressive" spaces to states in "less expressive" spaces by "erasing" facts that cannot be expressed in a lower space.

These surjective projections may embody Savage's idea that "(i)t may be well, however, to emphasize that a state of the smaller world corresponds not to a state of the larger, but to a set of states" (Savage, 1954, p. 9).

2.3 Events

Denote $g(S) = \{S' : S' \succeq S\}$. For $D \subseteq S$, denote $D^{\uparrow} = \bigcup_{S' \in g(S)} \left(r_S^{S'}\right)^{-1}(D)$. ("All the extensions of descriptions in D to at least as expressive vocabularies.")

[7] Here and in what follows, phrases within quotation marks hint at intended interpretations, but we emphasize that these interpretations are not part of the definition of the set-theoretic structure.

An *event* is a pair (E, S), where $E = D^{\uparrow}$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$.[8] If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Ω is an event.

Some fact may obtain in a subset of a space. Then this fact should be also "expressible" in "more expressive" spaces. Therefore the event contains not only the particular subset but also its inverse images in "more expressive" spaces.

Let Σ be the set of events of Ω. Note that unless $\mathcal{S}$ is a singleton, Σ is not an algebra because it contains distinct vacuous events $\emptyset^S$ for all $S \in \mathcal{S}$. These vacuous events correspond to contradictions with differing "expressive power".

2.4 Negation

If $(D^{\uparrow}, S)$ is an event where $D \subseteq S$, the negation $\neg(D^{\uparrow}, S)$ of $(D^{\uparrow}, S)$ is defined by $\neg(D^{\uparrow}, S) := ((S \setminus D)^{\uparrow}, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^{\uparrow} := (S \setminus D)^{\uparrow}$. Note that by our notational convention, we have $\neg S^{\uparrow} = \emptyset^S$ and $\neg \emptyset^S = S^{\uparrow}$, for each space $S \in \mathcal{S}$. The event $\emptyset^S$ should be interpreted as a "logical contradiction phrased with the expressive power available in S." $\neg D^{\uparrow}$ is typically a proper subset of the complement $\Omega \setminus D^{\uparrow}$. That is, $(S \setminus D)^{\uparrow} \subsetneqq \Omega \setminus D^{\uparrow}$.

Intuitively, there may be states in which the description of an event $D^{\uparrow}$ is both expressible and true – these are the states in $D^{\uparrow}$; there may be states in which its description is expressible but false – these are the states in $\neg D^{\uparrow}$; and there may be states in which neither its description nor its negation are expressible – these are the states in

$$\Omega \setminus \left(D^{\uparrow} \cup \neg D^{\uparrow}\right) = \Omega \setminus S\left(D^{\uparrow}\right)^{\uparrow}.$$

Thus our structure is not a standard state space model in the sense of Dekel, Lipman, and Rustichini (1998).

2.5 Conjunction and Disjunction

If $\left\{\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)\right\}_{\lambda \in L}$ is a collection of events (with $D_{\lambda} \subseteq S_{\lambda}$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)$ is defined by $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right) := \left(\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right), \sup_{\lambda \in L} S_{\lambda}\right)$. Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda \in L} S_{\lambda}$ exists. If $S = \sup_{\lambda \in L} S_{\lambda}$, then we have $\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right) = \left(\bigcap_{\lambda \in L}\left(\left(r_{S_{\lambda}}^{S}\right)^{-1}\left(D_{\lambda}\right)\right)\right)^{\uparrow}$. Again, abusing notation, we write $\bigwedge_{\lambda \in L} D_{\lambda}^{\uparrow} := \bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$ interchangeably).

[8]For instance, for the event (E, S) the base-space is $S(E) = S$, for the event (F, S') the base-space is $S(F) = S'$.

We define the relation $\subseteq$ between events (E,S) and (F,S'), by $(E,S) \subseteq (F,S')$ if and only if $E \subseteq F$ as sets *and* $S' \preceq S$. If $E \neq \emptyset$, we have that $(E,S) \subseteq (F,S')$ if and only if $E \subseteq F$ as sets. Note however that for $E = \emptyset^S$ we have $(E,S) \subseteq (F,S')$ if and only if $S' \preceq S$. Hence we can write $E \subseteq F$ instead of $(E,S) \subseteq (F,S')$ as long as we keep in mind that in the case of $E = \emptyset^S$ we have $\emptyset^S \subseteq F$ if and only if $S \succeq S(F)$. It follows from these definitions that for events E and F, $E \subseteq F$ is equivalent to $\neg F \subseteq \neg E$ only when E and F have the same base, that is, $S(E) = S(F)$.

The disjunction of $\left\{D_\lambda^\uparrow\right\}_{\lambda \in L}$ is defined by de Morgan's Law: $\bigvee_{\lambda \in L} D_\lambda^\uparrow = \neg\left(\bigwedge_{\lambda \in L} \neg\left(D_\lambda^\uparrow\right)\right)$. Typically, $\bigvee_{\lambda \in L} D_\lambda^\uparrow \subsetneqq \bigcup_{\lambda \in L} D_\lambda^\uparrow$, and, if all D_λ are nonempty, we have that $\bigvee_{\lambda \in L} D_\lambda^\uparrow = \bigcup_{\lambda \in L} D_\lambda^\uparrow$ holds if and only if all the $D_\lambda^\uparrow$ have the same base space. Note that by these definitions, the conjunction and disjunction of events is an event.

2.6 Probability Measures

Let $\Delta(S)$ be the set of probability measures on S.

For a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}(D) := \mu\left(\left(r_S^{S'}\right)^{-1}(D)\right), \quad D \subseteq S.$$

Let S_μ be the space on which μ is a probability measure. If $S_\mu \succeq S(E)$ for an event $E \in \Sigma$, then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \npreceq S_\mu$, then we say that $\mu(E)$ is undefined.

2.7 Unawareness

We follow the literature on unawareness, and define an epistemic notion of unawareness that in the subsequent sections will be characterized behaviorally.

Definition 1 (Unawareness) *A decision maker is* unaware *of the event E if her belief is represented by a probability measure $\mu \in \Delta(S)$ with $S \nsucceq S(E)$.*

This notion follows the definition of unawareness in a more sophisticated model in which states of the world rather than states of nature are considered. That is, states also capture beliefs of agents. In such a richer setting, unawareness of an agent may differ from state to state even within the same space. Unawareness operators on events can be defined, and it can be shown that these operators satisfy all properties of unawareness that have been suggested in the literature. See Heifetz, Meier, and Schipper (2011a) for details.

If a decision maker's belief is represented by a probability measure $\mu \in \Delta(S)$, we sometimes refer to S as the "awareness level" of the decision maker.

Since $S(E) = S(\neg E)$ by definition, we have the following observation.

Remark 1 (Symmetry) *A decision maker is unaware of the event E if and only if she is unaware of the event $\neg E$.*

3 Subjective Expected Utility

3.1 Outcomes

Let X be an arbitrary space of outcomes or prizes. We denote by $\Delta(X)$ the set of simple probability measures on X, that is, the set of finitely additive probability measures with finite support (see Fishburn, 1970, Section 8.2). For $p \in \Delta(X)$, we denote by *supp*(p) the support of p.

3.2 Acts

An *act* is a function $f : \Omega \longrightarrow \Delta(X)$. Note that, unlike Anscombe-Aumann acts, f is not defined on just one state space, but on the union Ω of spaces. This is interpreted as follows: Let's say an individual considers buying a firm (e.g., the act f). A firm can be viewed as a bundle of uncertain opportunities and liabilities. The decision maker may perceive only a subset of opportunities and liabilities depending on her awareness level that may be influenced by her prior experience or her reading of the "fine print" of the share sales and purchase agreement. In our setting, the uncertain opportunities and liabilities are represented by lotteries over outcomes contingent on states (i.e., acts). Some of the opportunities and liabilities (i.e., events) may not be expressible in some of the spaces in $\mathcal{S}$. That is, they are not perceived when having certain awareness levels. For instance, if $\omega \in S$ and the decision maker's awareness level is given by space $S' \prec S$, then the lottery perceived is not $f(\omega)$ but $f(\omega_{S'})$. So an act denotes simultaneously more or less rich descriptions of those uncertain opportunities and liabilities. In the present example, it is essentially a label of the action "buy the firm". Which opportunities and liabilities are perceived by the decision maker, that is, the awareness level of the decision maker, will be captured by the preferences over acts introduced below.

For any event E and acts f and g, define a *composite act* $f_E g$ by

$$f_E g(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in E \\ g(\omega) & \text{otherwise.} \end{cases}$$

Note that unlike composite acts in the Anscombe-Aumann approach, g is not only prescribed on the negation of E but also on all states that are neither in E nor in $\neg E$. Unlike acts defined on a standard state space, we have in general $f_E g \neq g_{\neg E} f$.

For any collection of pairwise disjoint events $E_1, E_2, ..., E_n \in \Sigma$ and acts $f^1, f^2, ..., f^n, g \in \mathcal{A}$, let $f^1_{E_1} f^2_{E_2} ... f^n_{E_n} g$ denote the composite act that yields $f^i(\omega)$ if $\omega \in E_i$ for $i = 1, ..., n$, and $g(\omega)$ otherwise.

If f and g are acts and $\alpha \in [0,1]$ then $\alpha f + (1-\alpha)g$ is an act defined pointwise by $(\alpha f + (1-\alpha)g)(\omega) = \alpha f(\omega) + (1-\alpha)g(\omega)$ for all $\omega \in \Omega$.

We abuse notation slightly and let p also denote the constant act that yields $p \in \Delta(X)$ in every state in Ω.

Let $\mathcal{A}$ denote the set of all acts.

Remark 2 *$\mathcal{A}$ is a* mixture space*. That is, for all $f, g \in \mathcal{A}$ and all $\alpha, \beta \in [0,1]$, (i) $1f + 0g = f$, (ii) $\alpha f + (1-\alpha)g = (1-\alpha)g + \alpha f$, and (iii) $\alpha[\beta f + (1-\beta)g] + (1-\alpha)g = \alpha\beta f + (1-\alpha\beta)g$.*

Note that we do not impose a measurability condition on acts in the sense that for any $f \in \mathcal{A}$ and $p \in \Delta(X)$, the set of states $\{\omega \in \Omega : f(\omega) = p\}$ is an event in Σ as defined previously. While such a measurability assumption may be justified in some applications, it may not be applicable in general. Consider, for instance, the example of a potential investor buying a firm discussed in the introduction. Contracts 1 and 3 are not measurable in the above sense.[9] A practical framework of decision making under unawareness should not rule out such examples.

3.3 Preferences

The decision maker's choices are represented by a collection of preferences, $\{\succsim_S\}_{S \in \mathcal{S}}$, one for each space $S \in \mathcal{S}$, with each $\succsim_S$ defined on $\mathcal{A}$. The preference relation of the decision maker with awareness level S is $\succsim_S$.

For each $S \in \mathcal{S}$, strict preference, $\succ_S$, is defined on $\mathcal{A}$ by $\succsim_S$ and not $\precsim_S$. Indifference, $\sim_S$, is defined on $\mathcal{A}$ by $\succsim_S$ and $\precsim_S$.

Preferences are allowed to vary with state spaces. The idea is that an act f may be preferred over the act g at a certain awareness level but g may be preferred over f at a different awareness level. For example, suppose that you may prefer onions to any other food, but, if you were aware that an eminent expert suspects that onions cause the fatal disease cuppacuppitis then you may rank onions below some other vegetable.

3.4 Assumptions on Preferences

The following five well known properties are standard in the Anscombe-Aumann approach, but adapted here to the lattice of state spaces.

[9] See Figure 2 in Section 4 for a more detailed exposition. In this figure, for the act depicted in the left graph (contract 1), the set of states that yield outcome 100 is not an event in our structure.

Property 1 (Weak Order) *For all $S \in \mathcal{S}$, $\succ_S$ is complete and transitive.*

Property 2 (Archimedean Continuity) *For all $S \in \mathcal{S}$ and $f, g, h \in \mathcal{A}$, if $f \succ_S g \succ_S h$, then there exists $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ_S g \succ_S \beta f + (1-\beta)h$.*

Property 3 (Independence) *For all $S \in \mathcal{S}$, $f, g, h \in \mathcal{A}$ and $\alpha \in (0,1)$, if $f \succ_S g$ then $\alpha f + (1-\alpha)h \succ_S \alpha g + (1-\alpha)h$.*

Definition 2 (Null Event) *An event E is S-*null *if $S(E) \preceq S$ and $f_E g \sim_S h_E g$ for all $f, g, h \in \mathcal{A}$. A state ω is S-null if $\{\omega\}^{\uparrow}$ is S-null. An event E is S-nonnull if $S(E) \preceq S$ and $f_E g \succ_S h_E g$ for some $f, g, h \in \mathcal{A}$.*

This definition generalizes Savage's notion of null event to our structure. We will show that it captures "events conceived but assigned probability zero" rather than "events not conceived of". We think that indeed this is in the spirit of Savage's notion of null event because, in Savage, "events not conceived of" are simply not considered in the decision maker's small world.

Remark 3 *For each $S \in \mathcal{S}$:*

(i) For any event F with $S(F) \not\preceq S$, F is neither S-null nor S-nonnull.

(ii) $\emptyset^{S'}$ is S-null if and only if $S' \preceq S$.

Property 4 (Nondegeneracy) *For all $S \in \mathcal{S}$ there exist $f, g \in \mathcal{A}$ such that $f \succ_S g$.*

Property 5 (State Independence) *If $f \in \mathcal{A}$, $p, q \in \Delta(X)$ are such that $p_{\{\omega\}^{\uparrow}} f \succ_S q_{\{\omega\}^{\uparrow}} f$ for some ω, then for all S-nonnull ω' we have $p_{\{\omega'\}^{\uparrow}} f \succ_S q_{\{\omega'\}^{\uparrow}} f$*

If the decision maker has preference $\succsim_S$, then the following property suggests the interpretation that she has *"awareness level"* S. This property is trivially satisfied in standard state space models; it is key in the current approach.

Property 6 (Confined Extensionality) *For any $S \in \mathcal{S}$, if $f, g \in \mathcal{A}$ are such that $f(\omega) = g(\omega)$ for all $\omega \in S$, then $f \sim_S g$.*

The examples in Figure 1 illustrate Property 6. There are only two spaces, S_1 and S_2 with $S_1 \succ S_2$. Different shades represent different outcomes. For instance, the left structure in Figure 1 represents the choice between two different composite acts. The left composite act yields "grey" in state ω_1 but "white" in states ω_2 and ω_3. The right composite act yields "white" in states ω_1 and ω_3, and "grey" in state ω_2. If the decision maker's awareness level is given by the lower space, S_2, then she does not care what happens in the upper space because she is unaware of those events. She is indifferent

Figure 1: Illustration of Property 6

between the two composite acts. The right structure of Figure 1 illustrates that if the decision maker's awareness level is given by the upper space, S_1, then she cares only about states in S_1. She neglects whatever happens in lower spaces presumably because everything that can be expressed in S_1 can also be expressed in S_2. Consequently, she is indifferent between the two composite acts.

Remark 4 *Property 6 implies that for all events E, if $S(E) \not\preceq S$, then*

(i) $f_E g \sim_S g$ for all $f, g \in \mathcal{A}$.

(ii) $f_E g \sim_S h_E g$ and $f_{\neg E} g \sim_S h_{\neg E} g$ for all $f, g, h \in \mathcal{A}$.

Remark 5 *Properties 1 and 6 imply that if $S' \preceq S$, then $f_{S'\uparrow} g \succsim_S h_{S'\uparrow} g$ if and only if $f \succsim_S h$.*

Remark 6 *Properties 1, 4, and 6 imply that, for each $S \in \mathcal{S}$, there exists a state $\omega \in S$ that is S-nonnull.*

3.5 Awareness-Dependent Subjective Expected Utility

Definition 3 (ASEU) *We say that $\{\succsim_S\}_{S\in\mathcal{S}}$ admits an* awareness-dependent subjective expected utility *(ASEU) representation if there exists a collection of nonconstant von Neumann-Morgenstern utility functions $\{u_S : X \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ and a collection of probability measures $\{\mu_S \in \Delta(S)\}_{S\in\mathcal{S}}$ such that for all $S \in \mathcal{S}$ and $f, g \in \mathcal{A}$, $f \succ_S g$ if and only if*

$$\sum_{\omega\in S}\left(\sum_{x\in supp(f(\omega))} u_S(x) f(\omega)(x)\right)\mu_S(\omega) \;>\; \sum_{\omega\in S}\left(\sum_{x\in supp(g(\omega))} u_S(x) g(\omega)(x)\right)\mu_S(\omega), \quad (1)$$

and $\mu_S(\{\omega\}) = 0$ if and only if ω is S-null.

Moreover, if there exists another collection of von Neumann-Morgenstern utility functions $\{v_S : X \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ *and a collection of probability measures* $\{\nu_S \in \Delta(S)\}_{S\in\mathcal{S}}$*, then, for all* $S \in \mathcal{S}$*, there are constants* $a_S > 0$ *and* b_S *such that* $v_S(x) = a_S u_S(x) + b_S$ *and* $\nu_S = \mu_S$*.*

The specification outlined so far allows me to apply the Anscombe and Aumann (1963) approach to each $S \in \mathcal{S}$ separately to prove in the appendix the following result.

Theorem 1 (Representation) $\{\succsim_S\}_{S\in\mathcal{S}}$ *admits an awareness-dependent subjective expected utility representation if and only if it satisfies Properties 1 to 6.*

Definition 4 *An awareness-dependent subjective expected utility representation has* awareness-independent utilities *if for all* $S, S' \in \mathcal{S}$ *there exist constants* $a_{S'S} > 0$ *and* $b_{S'S}$ *such that* $u_S = a_{S'S} u_{S'} + b_{S'S}$*.*

If an awareness-dependent subjective expected utility representation has awareness-independent utilities, then the utility function u_S at awareness level S is also a utility function for any awareness level $S' \in \mathcal{S}$ because conditional on each awareness level, utilities are unique up to affine transformations. We believe that in reality this may not be satisfied except in rather special cases.

Property 7 (Awareness-Independent Ranking) *For* $p, q \in \Delta(X)$*,* $p \succ_S q$ *if and only if* $p \succ_{S'} q$ *for all* $S', S \in \mathcal{S}$*.*

Proposition 1 $\{\succsim_S\}_{S\in\mathcal{S}}$ *admits an awareness-dependent subjective expected utility representation with awareness-independent utilities if and only if it satisfies Properties 1 to 7.*

4 Revealed Unawareness

In this section, we want to apply the representation to "reveal" a decision maker's unawareness of an event. Suppose an outside observer wishes to infer from the choices of a decision maker whether she is unaware of an event E or not. The outside observer does not know the preferences of the decision maker nor does he know which preference relation is related to which awareness level (the mapping from state spaces to binary relations over acts). All he knows is that no matter what the awareness level of the decision maker is, her choices conform to Properties 1 to 6. Given that the outside observer knows that the decision maker is an awareness-dependent subjective expected utility maximizer but does not know the awareness level of the decision maker, can he learn from the choices of the decision maker whether she is aware of an event E? We denote by $\succsim$ the observed choices and define $\succ$ and $\sim$ as usual.

The following proposition summarizes the behavioral implications of unawareness.

Proposition 2 (Revealed Unawareness) *Assume that* $\{\succsim_S\}_{S\in\mathcal{S}}$ *satisfy Properties 1 to 6, and let* $\succsim \in \{\succsim_S\}_{S\in\mathcal{S}}$. *A decision maker is unaware of the event* E *if and only if for all events* F *with* $S(F) = S(E)$, $f_F g \sim h_F g$ *for all* $f, g, h \in \mathcal{A}$.

We may take the characterization of Proposition 2 as a behavioral definition of unawareness and call it revealed unawareness.

Consider now an outside observer who knows that the decision maker is an awareness-dependent subjective expected utility maximizer, does not know the awareness level of the decision maker, and wishes to infer from choices of a decision maker whether she attaches subjective probability zero belief to the event E or whether she is unaware of the event E. We say that the decision maker ascribes subjective probability zero to the event E if in the representation of Definition 3, $\mu_S(E) = 0$ for some $S \succeq S(E)$. The following proposition states the different behavioral implications of unawareness and subjective probability zero belief. With the structure in place, the proof is straightforward.

Proposition 3 (Null versus Unawareness) *Assume that* $\{\succsim_S\}_{S\in\mathcal{S}}$ *satisfy Properties 1 to 6, and let* $\succsim \in \{\succsim_S\}_{S\in\mathcal{S}}$.

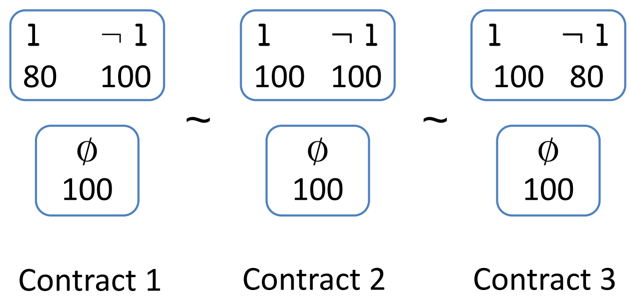
(i) Unawareness: A decision maker is unaware of the event E *if and only if* $f_E g \sim h_E g$ *and* $f_{\neg E} g \sim h_{\neg E} g$ *for all* $f, g, h \in \mathcal{A}$.

(ii) Subjective Probability Zero Belief: A decision maker ascribes subjective probability zero to the event E *if and only if* $f_E g \sim h_E g$ *and not* $f_{\neg E} g \sim h_{\neg E} g$ *for all* $f, g, h \in \mathcal{A}$.

A decision maker is unaware of an event E if and only if she considers both E and the negation of E to be "null". This is different from assigning subjective probability zero to the event E which is characterized by considering E to be null but the negation of E to be nonnull.

Recall the example from the introduction in which a potential investor may buy a firm using one of three contracts. The investor may or may not be aware of a potential lawsuit involving the firm. The contracts differ in the "fine print" with respect to this lawsuit. The three contracts are represented in the three graphics of Figure 2, respectively. Each contract describes in the upper space the monetary payoffs to the buyer in the event of lawsuit, $[\ell]$, not lawsuit, $\neg[\ell]$, and in the lower space the status quo payoff to the buyer when she does not think about the lawsuit, $\emptyset$. For instance, the act/contract depicted in the left graph assigns probability 1 to the monetary payoff of 80 if the state is ℓ. For a buyer conforming to awareness-dependent expected utility, indifference among all contracts is consistent with being unaware of $[\ell]$, that is, having the preference relation $\succsim_{S_\emptyset}$. But it is inconsistent with the event $[\ell]$ or $\neg[\ell]$ being null. To see this, note that *contract 1* $\sim_{S_{\{\ell\}}}$ *contract 2* is consistent with the event lawsuit, $[\ell]$, being null. But then $\neg[\ell]$ cannot be null, that is, *contract 2* $\nsim_{S_{\{\ell\}}}$ *contract 3*.

Figure 2: Example



5 Impersonal Expected Utility

In what sense could a probability zero approach "model" behavior under unawareness nevertheless?

Given a lattice of spaces $\mathcal{S}$, we follow Heifetz, Meier, and Schipper (2011a) in defining the *flattened state space* associated with $\mathcal{S}$ simply by the union of all spaces, $\Omega = \bigcup_{S \in \mathcal{S}} S$. Note that the set of all subsets 2^{Ω} may contain elements that are not events in the unawareness structure (unless the lattice is trivially a singleton). That is, typically $\Sigma \subsetneqq 2^{\Omega}$.

A probability measure μ_S on S is extended to a probability measure φ_S on the flattened state space Ω by[10]

$$\varphi_S(E) \quad := \quad \begin{cases} \mu_S(E \cap S) & \text{if } E \cap S \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Note that Ω is just a standard state space. The probability measure is extended by assigning probability zero to all subsets of Ω that are "not reasoned" about by the decision maker. Such subsets may not be events in the unawareness structure.

Consider a composite act of the form

$$f_{\{\omega\}}g(\omega') = \begin{cases} f(\omega') & \text{if } \omega = \omega' \\ g(\omega') & \text{otherwise.} \end{cases} \tag{2}$$

At first glance, it may appear that $f_{\{\omega\}}g$ is not an act in the set $\mathcal{A}$ since $\{\omega\}$ may not be an event in the unawareness structure. However, for every $f, g \in \mathcal{A}$ we can find $h \in \mathcal{A}$ such that $h(\omega) = f(\omega)$ and $h(\omega') = g(\omega')$ for $\omega' \neq \omega$. Clearly, $f_{\{\omega\}}g = h_{\{\omega\}^{\uparrow}}g$ and $h_{\{\omega\}^{\uparrow}}g \in \mathcal{A}$. Thus, $f_{\{\omega\}}g \in \mathcal{A}$.

The following remark characterizes "null" in the flattened state space by S-null or unawareness.

[10]Recall that if $S \not\succeq S(E)$ then $\mu_S(E)$ is undefined. In this case too, we now require $\varphi_S(E) = 0$.

Remark 7 *Properties 1 and 6 imply that* $f_{\{\omega\}}g \sim_S h_{\{\omega\}}g$ *for all* $f, g, h \in \mathcal{A}$ *if and only if* $\omega \in \Omega$ *is* S*-null or* $\omega \notin S$.

Definition 5 (IEU) *We say that* $\{\succsim_S\}_{S \in \mathcal{S}}$ *admits an* impersonal expected utility *(IEU) representation if there exists a collection of nonconstant von Neumann-Morgenstern utility functions* $\{w_S : X \longrightarrow \mathbb{R}\}_{S \in \mathcal{S}}$ *and a collection of probability measures* $\{\varphi_S \in \Delta(\Omega)\}_{S \in \mathcal{S}}$ *such that, for all* $f, g \in \mathcal{A}$, $f \succ_S g$ *if and only if*

$$\sum_{\omega \in \Omega} \left(\sum_{x \in supp(f(\omega))} w_S(x) f(\omega)(x) \right) \varphi_S(\omega) \;\; > \;\; \sum_{\omega \in \Omega} \left(\sum_{x \in supp(g(\omega))} w_S(x) g(\omega)(x) \right) \varphi_S(\omega), \tag{3}$$

and $\varphi_S(\{\omega\}) = 0$ *if and only if* ω *is* S*-null or* $\omega \notin S$.

Moreover, if there exists another collection of von Neumann-Morgenstern utility functions $\{v_S : X \longrightarrow \mathbb{R}\}_{S \in \mathcal{S}}$ *and a collection of probability measures* $\{\phi_S \in \Delta(\Omega)\}_{S \in \mathcal{S}}$*, then, for all* $S \in \mathcal{S}$*, there are constants* $a_S > 0$ *and* b_S *such that* $v_S(x) = a_S w_S(x) + b_S$ *and* $\phi_S = \varphi_S$.

Compared to awareness-dependent subjective expected utility, there are two key differences. First, we sum over states in the *union of spaces* Ω rather than states in S. This is exactly the deference between Propositions 4 and 5 in the appendix. Second, we use the extended probability measure φ_S defined on Ω instead of μ_S defined on S. Consequently, probability zero is now attached also to states ω that are not "reasoned about" by the decision maker with awareness level S, that is all $\omega \notin S$.

Theorem 2 (Characterization) $\{\succsim_S\}_{S \in \mathcal{S}}$ *admits an impersonal expected utility representation if and only if it satisfies Properties 1 to 6.*

Corollary 1 $\{\succsim_S\}_{S \in \mathcal{S}}$ *admits an impersonal expected utility representation if and only if it admits an awareness-dependent subjective expected utility representation.*

Corollary 2 *Let* $\{\succsim_S\}_{S \in \mathcal{S}}$ *satisfy Properties 1 to 6. Denote by* $\{\mu_S \in \Delta(S)\}_{S \in \mathcal{S}}$ *the collection of subjective probability measures from the awareness-dependent subjective expected utility representation of* $\succsim_S$*, and by* $\{\varphi_S \in \Delta(\Omega)\}_{S \in \mathcal{S}}$ *the collection of probability measures from the impersonal representation of* $\succsim_S$*. Then, for all* $S \in \mathcal{S}$, $\mu_S(E) = \varphi_S(E)$ *for all events* $E \in \Sigma$ *with* $S(E) \preceq S$.

6 Discussion

6.1 Can we model unawareness with probability zero?

Both Theorems 1 and 2 provide characterizations of Properties 1 to 6. In Section 4 we showed that unawareness of an event is behaviorally distinct from the event being null. In the awareness-dependent subjective expected utility representation of Theorem 1, null is captured by probability zero and revealed unawareness is captured by undefined probability. This is in contrast to the impersonal expected utility representation of Theorem 2 that must represent both null and revealed unawareness with probability zero.

Suppose that we want to model a the decision maker who is unaware of an event E. Under impersonal expected utility, if the decision maker is unaware of the event E, then she assigns probability zero to it. But zero probability is just a necessary condition. A necessary and sufficient condition would be that probability zero is assigned to the event E and probability zero is assigned to the negation of the event, $\neg E$. But to model this, we would be required to consider negation as defined in Subsection 2.4 and thus (the special event structure of) unawareness structures. Thus, probability zero alone won't allow us to model unawareness, and we are required to consider unawareness structures. However, using the unawareness structure in the first place makes the impersonal expected utility approach obsolete, since every behavior that can be captured in impersonal expected utility theory can also be captured in awareness-dependent expected utility, but the latter has the advantage of clearly distinguishing between revealed unawareness of an event and the event being null in the representation. This should be of practical interest in applications where we work with representations rather than with (properties of) preferences.

It has been argued in game theory that a "good" model should model the relevant factors as perceived by the decision maker (Rubinstein, 1991). However, note that probability measures in impersonal expected utility cannot be interpreted as a "personal" or "subjective" probabilities of the decision maker. (Hence, the attribute "impersonal".) Statements like "I am assigning probability zero to the event E because I am unaware of it" are nonsensical, since the very statement implies that I think about the event E.[11] Historically, it was precisely the goal of subjective expected utility theory to make sense of statements like "I find the event E more likely than the event F" (see for instance Savage, 1954, p. 27). For me, the attraction of subjective expected utility theory is that choices provide a window into the decision maker's reasoning. This attraction is lost with impersonal expected utility but not with awareness-dependent expected utility. In the latter representation, it makes sense to interpret the probability measures as "personal" or "subjective" beliefs of a decision maker given her awareness level. In contrast, the probability measures in impersonal expected utility can only be interpreted as an artificial construct ascribed to the decision maker by an outside modeler. The issue here is more severe than the usual "as if" assumption in decision theory. In subjective expected utility, the decision maker may not really reason with the subjective probabilities ascribed to her by her choices. But it is not impossible that she could use them for reasoning. In impersonal expected utility, it is impossible that the decision maker uses such impersonal

[11] Indeed, one of the epistemic properties of unawareness (in a model with states of the world rather than with states of nature) is that if a decision maker is aware that she is unaware of the event E then she is aware of the event E. This is called AU-introspection in Dekel, Lipman, and Rustichini (1998), and it holds in unawareness structures; see Heifetz, Meier, and Schipper (2011a, Proposition 3).

probabilities and is at the same time be unaware of some events.[12]

6.2 Caveats

Property 6 implies that events of which the decision maker is unaware of do not affect her ranking of acts. This holds even for composite acts conditioned on events that the decision maker is unaware of. More generally, it rules out that a decision maker becomes aware of an event merely from facing an act. While this is also the implicit assumption in standard decision theory (i.e., different acts do change the subset of "small worlds" that a decision maker pays attention to), it may be unrealistic in some situations. Sometimes, when facing an act, a decision maker may become in very subtle ways a bit more careful with the "fine print" of acts, and this care may lead her to become aware of events. For example, a buyer facing a decision about whether or not to buy a certain insurance contract may become aware of events that she was previously unaware of when reading all the fine print of the contract. If ex ante an outside observer does not know how acts affect the awareness of a decision maker, can he still elicit whether or not a decision maker is unaware of an event?

To answer this question, we could consider a modified framework in which acts may influence the awareness of a decision maker. When an outside observer presents the decision maker with acts, he may change the decision maker's awareness. So, in this sense, the outside observer may destroy the unawareness of the decision maker with the experiment to measure it. While this weakens substantially the usefulness of the approach for revealing unawareness of events, it still allows the outside observer to measure at least sometimes whether or not a decision maker was unaware of some events ex ante (i.e., before the experiment). For instance, in this framework we may observe a sequence of choices $f \succ g$, $g \succ h$ and $h \succ f$ that constitute a cycle. The reason is that when the decision maker faced act h, she became aware of an event that led her to evaluate acts differently from the time she faced the choice between f and g. But without further assumptions, the outside observer cannot draw inferences with regard to which event(s) the decision maker was unaware of. Moreover, after observing $h \succ f$, when the outside observer lets the decision maker choose again between f and g for a second time, the outside observer should observe $g \succ f$.[13] That is, in this setting arbitrary choice cycles are still ruled out and limited to one-time cycles.

A second caveat applies to the approach presented in this paper. In what sense do Theorems 1 and 2 really provide behavioral representations of awareness-dependent subjective expected utility and impersonal expected utility, respectively? How would one go about practically testing these theories? An experiment would require us to test

[12]This is not surprising since (in a model with states of the world rather than with states of nature) the impossibility result by Dekel, Lipman, and Rustichini (1998) applies, because the flattened state space is a standard state space.

[13]Unless in between the decision maker faced some act different from f, g and h that made her aware of yet other events such that again $f \succ g$.

Properties 1 to 6 for *every* space $S \in \mathcal{S}$. Suppose we first run such an experiment for space S. Afterwards we could mention an event E with $S(E) \npreceq S$ to the decision maker and run the experiment again, this time effectively for the join of spaces S and $S(E)$. We could continue step-by-step in this fashion, at each step mentioning a new event to the decision maker at which she was previously unaware of. But this procedure would allow us to test the theory for a finite chain of state spaces only and not for a finite lattice of spaces in general. The problem is that in order to run such an experiment also for a space that is *unordered* to S, we would require a procedure that makes the decision maker unaware of an event. Although it is not inconceivable that, in the moment of taking a decision, the decision maker may not think about a relevant event that he previously thought about, we still believe that it poses an unsurmountable challenge to find an experiment design that would make a decision maker systematically unaware of events. This practical challenge is analogous to challenges shared by other decision theories that require tests of continuity properties of preferences or properties of preferences conditional on events that are physically or logically impossible, unverifiable, or counterfactual.

The caveat extends also to our Propositions 2 and 3 on the behavioral implications of unawareness. These results show the possibility of revealing unawareness of an event *assuming* that the decision maker is an awareness-dependent subjective expected utility maximizer. More desirable would be to reveal jointly that the decision maker is an awareness-dependent subjective expected utility maximizer *and* is unaware of this or that event.

6.3 Related Literature

The present approach is silent on the decision maker's reasoning about his preferences. It due to the fact that we work with states of nature rather than with states of the world, a feature shared with standard decision theoretic models. In a states-of-nature approach, the preferences or beliefs of the decision maker are not part of the description of the state, and beliefs are elicited for non-epistemic events only. This is in contrast to Bayesian games or epistemic game theory, in which states of the world also describe a player's preferences, beliefs, beliefs about beliefs and so on. To bridge the gap, Morris (1996) introduced preference-based definitions of knowledge and beliefs in states-of-the-world models. Schipper (2011) extends this approach to unawareness structures and provides a preference-based characterization of knowledge and unawareness. In such a structure, a decision maker can reason about his own preferences and those of other players at (weakly) lower awareness levels.

Li (2008) analyzes unawareness versus zero probability in a model different from ours. Her study is a bit more ambitious than mine, as she considers a two-period model in which an initially unaware decision maker becomes aware in the second period. The decision maker chooses among bets defined on her first-period "subjective" states. This requires her to specify how those "subjective bets" correspond to "objective" bets in the second period. In contrast, in my model, acts are already defined on all states, although the decision maker may have a limited understanding of them. Li (2008) considers various specifications, including one in which unawareness of an event may be thought of "as if" the decision maker believes that the event does not obtain.

Ahn and Ergin (2010) study framing that may also be due to lack of awareness. They take more or less fine partitions of a state space as the primitive. Since the set of all partitions forms a lattice, we believe that their analysis could be "translated" into unawareness structures. In their approach, acts are defined to be measurable with respect to some of the partitions. When a decision maker faces an act that is measurable with respect to some partition, then she evaluates the act on at least the events of that partition. Intuitively, they assume that a decision maker always reads the "fine print" of an act. This is important for their aim of studying how decisions are affected by framing through acts. It is in contrast to the approach taken in Section 3 because I define acts on all partitions simultaneously. One interesting feature of their representation is a (not necessarily additive) set function from which the partition-dependent probability measure is defined. It allows them to relate beliefs across partitions. They discuss various interpretations of this set function. In particular, their approach is an extension and axiomatization of support theory in psychology.

Ahn and Ergin's (2010) notion of "completely overlooked" differs from the notion of (propositional) unawareness in the epistemic literature. It is consistent with their model that an event is "completely overlooked" while its negation is not. This is in contrast with the symmetry property of unawareness: if a decision maker can reason about the negation of an event, then she can reason about the event (and vice versa).

Blume, Easley, and Halpern (2009) take a syntactic approach to subjective expected utility theory in which primitives in standard subjective expected utility theory such as the state space, outcome space, and acts are replaced by syntactic descriptions. This requires a modified set of properties which are used to characterize subjective expected utility theory including the primitives. It would be intriguing to extend their approach to unawareness structures.

The caveats discussed in Section 6.2 suggest that decision making under unawareness may be more appropriately studied in a dynamic setting in which awareness can grow over time. Karni and Vierø (2011) present a decision theoretic model in which the decision maker can become aware of new acts, consequences, as well as links between acts and consequence. The preference relations over the expanding sets of acts are linked by an underlying preference over satisfactions of basic needs. They develop representation theorems and updating rules for beliefs over expanding state spaces. Grant and Quiggin (2011) consider also dynamic decisions with differential levels of awareness and associate a language to such dynamic decision problems that allows them to study whether it is possible for a decision maker to believe that there exists a proposition of which he is unaware. They then argue that based on the experience of becoming aware of propositions, the proposition that this will continue to be so in future is supported by historical induction. Van Ditmarsch and French (2011) present a purely epistemic framework to study agents that may become aware of propositional variables. They present a complete axiomatization of their logic and show that it is decidable. Their approach can be also interpreted as deriving a complete lattice of spaces of Heifetz, Meier, and Schipper (2006) via their notion of awareness bisimulation.

A Proofs

A.1 Proof of Remark 4

(i) If $S(E) \not\preceq S$, then by the definition of composite acts, $f_E g(\omega) = g(\omega)$ for all $\omega \in S$ for all $f, g \in \mathcal{A}$. Hence by Property 6, $f_E g \sim_S g$ for all $f, g \in \mathcal{A}$.

(ii) If $S(E) \not\preceq S$, then $f_E g(\omega) = h_E g(\omega)$ for all $\omega \in S$. Hence by Property 6, $f_E g \sim_S h_E g$. Since $S(E) = S(\neg E)$, by analogous arguments, $f_{\neg E} g \sim_S h_{\neg E} g$. $\square$

A.2 Proof of Remark 5

If $S' \preceq S$, then $f_{S'^\uparrow} g(\omega) = f(\omega)$ and $h_{S'^\uparrow} g(\omega) = h(\omega)$ for all $\omega \in S'^\uparrow \cap S$. Since $S' \preceq S$, we have $S'^\uparrow \cap S = S$. Hence by Property 6 and transitivity (Property 1), $f_{S'^\uparrow} g \sim_S f \succsim_S h \sim_S h_{S'^\uparrow} g$ imply $f_{S'^\uparrow} g \succsim_S h_{S'^\uparrow} g$ and vice versa. $\square$

A.3 Proof of Remark 6

Assume that Properties 1, 4, and 6 hold, and suppose by way of contradiction that, for some $S \in \mathcal{S}$ all states in S are S-null. Since S is finite, number its states $1, ..., |S|$. Then for all $g, h \in \mathcal{A}$, $g \sim_S h_{\{\omega_1\}^\uparrow} g \sim_S h_{\{\omega_1, \omega_2\}^\uparrow} g \sim_S ... \sim_S h_{\{\omega_1, \omega_2, ..., \omega_{|S|-1}\}^\uparrow} g \sim_S h_{S^\uparrow} g \sim_S h$, where the last $\sim_S$ follows from Property 6. By transitivity (Property 1), we have $g \sim_S h$ for all $g, h \in \mathcal{A}$, a contradiction to Property 4. $\square$

A.4 Proof of Remark 7

"$\Leftarrow$": If $\omega \notin S$, then $f_{\{\omega\}} g(\omega') = g(\omega') = h_{\{\omega\}} g(\omega')$ for all $\omega' \in S$ and all $f, g, h \in \mathcal{A}$. Thus by Property 6, $f_{\{\omega\}} g \sim_S h_{\{\omega\}} g$ for all $f, g, h \in \mathcal{A}$.

State ω being S-null means $S(\{\omega\}^\uparrow) \preceq S$ and $f_{\{\omega\}^\uparrow} g \sim_S h_{\{\omega\}^\uparrow} g$ for all $f, g, h \in \mathcal{A}$. If $S(\{\omega\}^\uparrow) = S$, then $f_{\{\omega\}} g(\omega') = f_{\{\omega\}^\uparrow} g(\omega')$ and $h_{\{\omega\}} g(\omega') = h_{\{\omega\}^\uparrow} g(\omega')$ for all $\omega' \in S$ and all $f, g, h \in \mathcal{A}$. By Property 6, $f_{\{\omega\}} g \sim_S f_{\{\omega\}^\uparrow} g$ and $h_{\{\omega\}} g \sim_S h_{\{\omega\}^\uparrow} g$ for all $f, g, h \in \mathcal{A}$. Thus, by Property 1, $f_{\{\omega\}} g = h_{\{\omega\}} g$ for all $f, g, h \in \mathcal{A}$. If $S(\{\omega\}^\uparrow) \prec S$, then $\omega \notin S$. Thus, in this case the result follows from the arguments above.

"$\Rightarrow$": Suppose to the contrary that $f_{\{\omega\}} g \sim_S h_{\{\omega\}} g$ for all $f, g, h \in \mathcal{A}$, but ω is S-nonnull and $\omega \in S$. ω being S-nonnull and $\omega \in S$ means that $f_{\{\omega\}^\uparrow} g \succ_S h_{\{\omega\}^\uparrow} g$ for some $f, g, h \in \mathcal{A}$. Since $\omega \in S$, $f_{\{\omega\}^\uparrow} g(\omega') = f_{\{\omega\}} g(\omega')$ and $h_{\{\omega\}^\uparrow} g(\omega') = h_{\{\omega\}} g(\omega')$ for all $\omega' \in S$. By Property 6, $f_{\{\omega\}^\uparrow} g \sim_S f_{\{\omega\}} g$ and $h_{\{\omega\}^\uparrow} g \sim_S h_{\{\omega\}} g$. From Property 1, it follows that $f_{\{\omega\}} g \succ_S h_{\{\omega\}} g$, a contradiction. $\square$

A.5 Proofs of Theorems 1 and 2

The proofs follow essentially Fishburn (1970, Chapter 13.1 and 13.2). We point out minor differences along the way. We present the proofs of both results side-by-side so that the interested reader can compare the differences. Moreover, this presentation helps to minimize redundancies.

First we show the following representation results in terms of state-dependent utilities or additively separable utilities.

Proposition 4 *$\{\succsim_S\}_{S\in\mathcal{S}}$ satisfies Properties 1 to 3 and 6 if and only if there exists a collection of functions $\{u_S : X \times S \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ such that for all $S \in \mathcal{S}$ and $f, g \in \mathcal{A}$,*

$f \succ_S g$ if and only if

$$\sum_{\omega\in S}\sum_{x\in supp(f(\omega))} u_S(x,\omega) f(\omega)(x) > \sum_{\omega\in S}\sum_{x\in supp(g(\omega))} u_S(x,\omega) g(\omega)(x). \tag{4}$$

Moreover, if $\{v_S : X \times S \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ is another collection of functions satisfying formula (4), then for each $S \in \mathcal{S}$ there exist constants $a_S \in \mathbb{R}_{++}$ and $b_S \in \mathbb{R}$ such $a_S u_S(\cdot,\omega) + b_S = v_S(\cdot,\omega)$ for each $\omega \in S$.

Proposition 5 *$\{\succsim_S\}_{S\in\mathcal{S}}$ satisfies Properties 1 to 3 and 6 if and only if there exists a collection of functions $\{w_S : X \times \Omega \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ such that for all $S \in \mathcal{S}$ and $f, g \in \mathcal{A}$,*

$f \succ_S g$ if and only if

$$\sum_{\omega\in \Omega}\sum_{x\in supp(f(\omega))} w_S(x,\omega) f(\omega)(x) > \sum_{\omega\in \Omega}\sum_{x\in supp(g(\omega))} w_S(x,\omega) g(\omega)(x). \tag{5}$$

Moreover, if $\{z_S : X \times \Omega \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ is another collection of functions satisfying formula (5), then for each $S \in \mathcal{S}$ there exist constants $a_S \in \mathbb{R}_{++}$ and $b_S \in \mathbb{R}$ such $a_S w_S(\cdot,\omega) + b_S = z_S(\cdot,\omega)$ for each $\omega \in \Omega$.

Proofs of Propositions. Under Properties 1 to 3 and 6, the existence of a collection of functions $\{U_S : \mathcal{A} \longrightarrow \mathbb{R}\}_{S\in\mathcal{S}}$ such that for $f, g \in \mathcal{A}$

$$f \succ_S g \text{ if and only if } U_S(f) > U_S(g) \tag{6}$$

and U_S being affine, that is,

$$U_S(\alpha f + (1-\alpha) g) = \alpha U_S(f) + (1-\alpha) U_S(g), \text{ for all } \alpha \in [0,1], \tag{7}$$

follows from applying the Mixture-Space Theorem (Herstein and Milnor, 1953, see also Fishburn, 1970, Section 8.4) for each $S \in \mathcal{S}$. Moreover, for each $S \in \mathcal{S}$, U_S is unique up to positive affine transformations.

We want to show that for $f \in \mathcal{A}$,

$$U_S(f) = \sum_{\omega\in S}\sum_{x\in supp(f(\omega))} u_S(x,\omega) f(\omega)(x) \tag{8}$$

for some function $u_S : X \times S \longrightarrow \mathbb{R}$ for every $S \in \mathcal{S}$.

We want to show that for $f \in \mathcal{A}$,

$$U_S(f) = \sum_{\omega\in \Omega}\sum_{x\in supp(f(\omega))} w_S(x,\omega) f(\omega)(x) \tag{9}$$

for some function $w_S : X \times \Omega \longrightarrow \mathbb{R}$ for every $S \in \mathcal{S}$.

The next step in the proof of Proposition 4 differs slightly from the Anscombe-Aumann approach.

We claim that Property 6 implies that

$$\frac{1}{|S|}f + \frac{|S|-1}{|S|}g \sim_S \sum_{\omega \in S} \frac{1}{|S|} f_{\{\omega\}^{\uparrow}} g. \tag{10}$$

To see the claim, number states in S by $1, ..., |S|$, and observe that for all $\omega \in S$,

$$\begin{aligned}
&\frac{1}{|S|}f(\omega) + \frac{|S|-1}{|S|}g(\omega) \\
&= \frac{1}{|S|} f_{\{\omega_1\}^{\uparrow}} g(\omega) + \cdots + \frac{1}{|S|} f_{\{\omega_{|S|}\}^{\uparrow}} g(\omega) \\
&= \sum_{\omega' \in S} \frac{1}{|S|} f_{\{\omega'\}^{\uparrow}} g(\omega).
\end{aligned}$$

Hence, Property 6 implies the claim.

By equations (6) and (7), we have

$$\begin{aligned}
&\frac{1}{|S|}U_S(f) + \frac{|S|-1}{|S|}U_S(g) \\
&= \frac{1}{|S|} \sum_{\omega \in S} U_S\left(f_{\{\omega\}^{\uparrow}} g\right).
\end{aligned} \tag{12}$$

Define $u_S : \Delta(X) \times S \longrightarrow \mathbb{R}$ by

$$u_S(p, \omega) := U_S\left(p_{\{\omega\}^{\uparrow}} g\right) - \frac{|S|-1}{|S|}U_S(g). \tag{14}$$

For $f \in \mathcal{A}$,

$$u_S(f(\omega), \omega) = U_S\left(f_{\{\omega\}^{\uparrow}} g\right) - \frac{|S|-1}{|S|}U_S(g). \tag{16}$$

Summing over $\omega \in S$ and dividing by $|S|$, we obtain

$$\begin{aligned}
&\frac{1}{|S|} \sum_{\omega \in S} u_S(f(\omega), \omega) \\
&= \frac{1}{|S|} \sum_{\omega \in S} U_S\left(f_{\{\omega\}^{\uparrow}} g\right) - \frac{|S|-1}{|S|}U_S(g).
\end{aligned} \tag{18}$$

Comparing equation (18) with (12), we have

$$U_S(f) = \sum_{\omega \in S} u_S(f(\omega), \omega). \tag{20}$$

Combining equations (14) and (7) yields, for $p, q \in \Delta(X)$, $\omega \in S$, and $\alpha \in [0, 1]$,

$$\begin{aligned}
&u_S(\alpha p + (1-\alpha)q, \omega) \\
&= \alpha u_S(p, \omega) + (1-\alpha)u_S(q, \omega).
\end{aligned} \tag{22}$$

We claim

$$\frac{1}{|\Omega|}f + \frac{|\Omega|-1}{|\Omega|}g = \sum_{\omega \in \Omega} \frac{1}{|\Omega|} f_{\{\omega\}} g. \tag{11}$$

To see the claim, number states in Ω by $1, ..., |\Omega|$, and observe that for all $\omega \in S$,

$$\begin{aligned}
&\frac{1}{|\Omega|}f(\omega) + \frac{|\Omega|-1}{|\Omega|}g(\omega) \\
&= \frac{1}{|\Omega|} f_{\{\omega_1\}} g(\omega) + \cdots + \frac{1}{|\Omega|} f_{\{\omega_{|\Omega|}\}} g(\omega) \\
&= \sum_{\omega' \in \Omega} \frac{1}{|\Omega|} f_{\{\omega'\}} g(\omega).
\end{aligned}$$

Note that Property 6 is not required for the claim since equation (11) holds with equality.

By equations (6) and (7), we have

$$\begin{aligned}
&\frac{1}{|\Omega|}U_S(f) + \frac{|\Omega|-1}{|\Omega|}U_S(g) \\
&= \frac{1}{|\Omega|} \sum_{\omega \in \Omega} U_S\left(f_{\{\omega\}} g\right).
\end{aligned} \tag{13}$$

Define $w_S : \Delta(X) \times \Omega \longrightarrow \mathbb{R}$ by

$$w_S(p, \omega) := U_S\left(p_{\{\omega\}} g\right) - \frac{|\Omega|-1}{|\Omega|}U_S(g). \tag{15}$$

For $f \in \mathcal{A}$,

$$w_S(f(\omega), \omega) = U_S\left(f_{\{\omega\}} g\right) - \frac{|\Omega|-1}{|\Omega|}U_S(g). \tag{17}$$

Summing over $\omega \in \Omega$ and dividing by $|\Omega|$, we obtain

$$\begin{aligned}
&\frac{1}{|\Omega|} \sum_{\omega \in \Omega} w_S(f(\omega), \omega) \\
&= \frac{1}{|\Omega|} \sum_{\omega \in \Omega} U_S\left(f_{\{\omega\}} g\right) - \frac{|\Omega|-1}{|\Omega|}U_S(g).
\end{aligned} \tag{19}$$

Comparing equation (19) with (13), we have

$$U_S(f) = \sum_{\omega \in \Omega} w_S(f(\omega), \omega). \tag{21}$$

Combining equations (15) and (7) yields, for $p, q \in \Delta(X)$, $\omega \in \Omega$, and $\alpha \in [0, 1]$,

$$\begin{aligned}
&w_S(\alpha p + (1-\alpha)q, \omega) \\
&= \alpha w_S(p, \omega) + (1-\alpha)w_S(q, \omega).
\end{aligned} \tag{23}$$

For $x \in X$, let $u_S(x, \omega) = u_S(\delta_x, \omega)$, with δ_x being the Dirac measure with unit mass on x. Since the support of a simple probability measure is finite,

$$u_S(p, \omega) = \sum_{x \in supp(p)} u_S(x, \omega). \tag{24}$$

Combining the representation in formula (6) with equation (20) yields inequality (4) for $f, g \in \mathcal{A}$. Repeat this construction for each $S \in \mathcal{S}$.

Uniqueness up to positive linear transformations follows from the uniqueness of U_S. If $v_S(\cdot, \omega)$ satisfies formula (4) in place of $u_S(\cdot, \omega)$, then

$$V_S(f) = \sum_{\omega \in S} \sum_{x \in supp(f(\omega))} v_S(x, \omega) f(\omega)(x),$$

$V_S = a_S U_S + b_S$, and $a_S > 0$. Holding $f(\omega')(x)$ fixed for all $\omega' \in S$, $\omega' \neq \omega$, it then follows that $v_S(\cdot, \omega) = a_S(\omega) u_S(\cdot, \omega) + b_S(\omega)$. This holds for each $\omega \in S$.

Note that $u_S(\cdot, \omega)$ is constant on X if and only if $\omega \in S$ is S-null. To see this, note that $\omega \in S$ being S-null means (with some slight abuse of notation) $x_{\{\omega\}^\uparrow} g \sim_S g$ for all $x \in X$ and $g \in \mathcal{A}$, which is equivalent by formula (6) to $U_S(x_{\{\omega\}^\uparrow} g) = U_S(g)$ for all $x \in X$ and $g \in \mathcal{A}$. But $u_S(x, \omega) = U_S(g) - \frac{|S|-1}{|S|} U_S(g) = \frac{1}{|S|} U_S(g)$, which is independent of x and thus constant in x.

For the converse, we prove only the nonstandard Property 6. Suppose, by way of contradiction, that we have the representation in formula (4) but Property 6 is violated. Then there exist a space $S \in \mathcal{S}$ and acts $f, g \in \mathcal{A}$ with $f(\omega) = g(\omega)$ for all $\omega \in S$ but $f \succ_S g$. But this contradicts formula (4). $\square$

.

.

For $x \in X$, let $w_S(x, \omega) = w_S(\delta_x, \omega)$, with δ_x being the Dirac measure with unit mass on x. Since the support of a simple probability measure is finite,

$$w_S(p, \omega) = \sum_{x \in supp(p)} w_S(x, \omega). \tag{25}$$

Combining the representation in formula (6) with equation (21) yields inequality (5) for $f, g \in \mathcal{A}$. Repeat this construction for each $S \in \mathcal{S}$.

Uniqueness up to positive linear transformations follows from the uniqueness of U_S. If $z_S(\cdot, \omega)$ satisfies formula (5) in place of $w_S(\cdot, \omega)$, then

$$Z_S(f) = \sum_{\omega \in \Omega} \sum_{x \in supp(f(\omega))} z_S(x, \omega) f(\omega)(x),$$

$Z_S = a_S U_S + b_S$, and $a_S > 0$. Holding $f(\omega')(x)$ fixed for all $\omega' \in \Omega$, $\omega' \neq \omega$, it then follows that $z_S(\cdot, \omega) = a_S(\omega) w_S(\cdot, \omega) + b_S(\omega)$. This holds for each $\omega \in \Omega$.

Note that $w_S(\cdot, \omega)$ is constant on X if and only if $\omega \in \Omega$ is S-null or $\omega \in \Omega \setminus S$. To see this, note that by Remark 7 (Property 1 and 6), $\omega \in \Omega$ is S-null or $\omega \notin S$ if and only if (with some slight abuse of notation) $x_{\{\omega\}} g \sim_S g$ for all $x \in X$ and $g \in \mathcal{A}$, which is equivalent by formula (6) to $U_S(x_{\{\omega\}^\uparrow} g) = U_S(g)$ for all $x \in X$ and $g \in \mathcal{A}$. But $u_S(x, \omega) = U_S(g) - \frac{|\Omega|-1}{|\Omega|} U_S(g) = \frac{1}{|\Omega|} U_S(g)$, which is independent of x and thus constant in x.

For the converse, we prove only the nonstandard Property 6. Suppose by way of contradiction, that we have the representation in formula (5) but Property 6 is violated. Then there exist a space $S \in \mathcal{S}$ and acts $f, g \in \mathcal{A}$ with $f(\omega) = g(\omega)$ for all $\omega \in S$ but $f \succ_S g$.

It follows from the fact that $f(\omega) = g(\omega)$ for all $\omega \in S$ that

$$\sum_{\omega \in S} \sum_{x \in supp(f(\omega))} w_S(x, \omega) f(\omega)(x) \tag{26}$$
$$= \sum_{\omega \in S} \sum_{x \in supp(g(\omega))} w_S(x, \omega) g(\omega)(x).$$

Note that

$$\sum_{\omega \in \Omega \setminus S} \sum_{x \in supp(f(\omega))} w_S(x, \omega) f(\omega)(x) \tag{27}$$
$$= \sum_{\omega \in \Omega \setminus S} \sum_{x \in supp(g(\omega))} w_S(x, \omega) g(\omega)(x),$$

.

since, as we noted earlier, $w_S(\cdot,\omega)$ is constant on X for all $\omega \in \Omega \setminus S$. But this contradicts formula (5). $\square$

We continue with the proof of Theorems 1 and 2, respectively. Fix an space $S \in \mathcal{S}$. By Remark 6, there exists an S-nonnull state $\omega^\circ \in S$. Let $p, q \in \Delta(X)$, and let $\omega \in S$ be an S-nonnull state.

For $f \in \mathcal{A}$,

$$\sum_{x\in supp(p)} u_S(x,\omega)p(x) > \sum_{x\in supp(q)} u_S(x,\omega)q(x) \quad (28)$$

if and only if

$$U_S(p_{\{\omega\}^\uparrow} f) > U_S(q_{\{\omega\}^\uparrow} f), \quad (29)$$

if and only if by Proposition 4

$$p_{\{\omega\}^\uparrow} f \succ_S q_{\{\omega\}^\uparrow} f, \quad (30)$$

if and only if, by Property 5,

$$p_{\{\omega^\circ\}^\uparrow} f \succ_S q_{\{\omega^\circ\}^\uparrow} f, \quad (31)$$

if and only if, by Proposition 4,

$$U_S(p_{\{\omega^\circ\}^\uparrow} f) > U_S(q_{\{\omega^\circ\}^\uparrow} f), \quad (32)$$

if and only if

$$\sum_{x\in supp(p)} u_S(x,\omega^\circ)p(x) > \sum_{x\in supp(q)} u_S(x,\omega^\circ)q(x). \quad (33)$$

.

.

For $f \in \mathcal{A}$,

$$\sum_{x\in supp(p)} w_S(x,\omega)p(x) > \sum_{x\in supp(q)} w_S(x,\omega)q(x) \quad (34)$$

if and only if

$$U_S(p_{\{\omega\}^\uparrow} f) > U_S(q_{\{\omega\}^\uparrow} f). \quad (35)$$

To see this, note that

$$\begin{aligned} U_S(p_{\{\omega\}^\uparrow} f) &= \sum_{x\in supp(p)} w_S(x,\omega)p(x) \\ &+ \sum_{\omega'\in\{\omega\}^\uparrow\setminus\{\omega\}} \sum_{x\in supp(p)} w_S(x,\omega')p(x) \\ &+ \sum_{\omega'\in\Omega\setminus\{\omega\}^\uparrow} \sum_{x\in supp(f(\omega'))} w_S(x,\omega')f(\omega')(x). \end{aligned}$$

It is sufficient to show that

$$\begin{aligned} &\sum_{\omega'\in\{\omega\}^\uparrow\setminus\{\omega\}} \sum_{x\in supp(p)} w_S(x,\omega')p(x) \quad (36) \\ &= \sum_{\omega'\in\{\omega\}^\uparrow\setminus\{\omega\}} \sum_{x\in supp(q)} w_S(x,\omega')q(x). \end{aligned}$$

Since $\omega \in S$, $\omega' \in \{\omega\}^\uparrow \setminus \{\omega\}$ implies that $\omega' \notin S$. By arguments in the proof of Proposition 5, $w_S(\cdot,\omega')$ is constant in X. This yields inequality (36). Inequality (35) holds if and only if, by Proposition 5,

$$p_{\{\omega\}^\uparrow} f \succ_S q_{\{\omega\}^\uparrow} f, \quad (37)$$

if and only if, by Property 5,

$$p_{\{\omega^\circ\}^\uparrow} f \succ_S q_{\{\omega^\circ\}^\uparrow} f, \quad (38)$$

if and only if, by Proposition 5,

$$U_S(p_{\{\omega^\circ\}^\uparrow} f) > U_S(q_{\{\omega^\circ\}^\uparrow} f), \quad (39)$$

if and only if (by analogous arguments as for inequality (35))

$$\sum_{x\in supp(p)} w_S(x,\omega^\circ)p(x) > \sum_{x\in supp(q)} w_S(x,\omega^\circ)q(x). \quad (40)$$

By the uniqueness of von Neumann-Morgenstern utilities, there exist constants $a_S(\omega) > 0$ and $b_S(\omega)$ such that

$$a_S(\omega)u_S(\cdot,\omega^\circ)+b_S(\omega)=u_S(\cdot,\omega). \quad (41)$$

For S-null states, let $a_S(\omega)=0$, since we observed in the proof of Proposition 4 that ω is S-null if and only if $u_S(\cdot,\omega)$ is constant on X.

Define $u_S(x):=u_S(x,\omega^\circ)$, that is, $a_S(\omega^\circ)=1$ and $b_S(\omega^\circ)=0$. Then the representation in formula (4) becomes

$f \succ_S g$ if and only if

$$\sum_{\omega\in S}\sum_{x\in supp(f(\omega))}(a_S(\omega)u_S(x)+b_S(\omega))f(\omega)(x) \quad (43)$$
$$> \sum_{\omega\in S}\sum_{x\in supp(g(\omega))}(a_S(\omega)u_S(x)+b_S(\omega))g(\omega)(x),$$

which simplifies to

$$\sum_{\omega\in S}\left(b_S(\omega)+a_S(\omega)[\sum_{x\in supp(f(\omega))}u_S(x)f(\omega)(x)]\right)$$
$$>\sum_{\omega\in S}\left(b_S(\omega)+a_S(\omega)[\sum_{x\in supp(g(\omega))}u_S(x)g(\omega)(x)]\right).$$

We cancel $b_S(\omega)$, divide by $\sum_{\omega\in S}a_S(\omega)$, and define

$$\mu_S(\omega):=\frac{a_S(\omega)}{\sum_{\omega'\in S}a_S(\omega')} \quad (45)$$

to obtain inequality (1).

$$a_S(\omega)w_S(\cdot,\omega^\circ)+b_S(\omega)=w_S(\cdot,\omega). \quad (42)$$

If ω is S-null or $\omega\in\Omega\setminus S$, let $a_S(\omega)=0$, since we observed in the proof of Proposition 5 that ω is S-null or $\omega\in\Omega\setminus S$ if and only if $w_S(\cdot,\omega)$ is constant on X.

Define $w_S(x):=w_S(x,\omega^\circ)$, that is, $a_S(\omega^\circ)=1$ and $b_S(\omega^\circ)=0$. Then the representation in formula (5) becomes

$f \succ_S g$ if and only if

$$\sum_{\omega\in\Omega}\sum_{x\in supp(f(\omega))}(a_S(\omega)w_S(x)+b_S(\omega))f(\omega)(x) \quad (44)$$
$$> \sum_{\omega\in\Omega}\sum_{x\in supp(g(\omega))}(a_S(\omega)w_S(x)+b_S(\omega))g(\omega)(x),$$

which simplifies to

$$\sum_{\omega\in\Omega}\left(b_S(\omega)+a_S(\omega)[\sum_{x\in supp(f(\omega))}w_S(x)f(\omega)(x)]\right)$$
$$>\sum_{\omega\in\Omega}\left(b_S(\omega)+a_S(\omega)[\sum_{x\in supp(g(\omega))}w_S(x)g(\omega)(x)]\right).$$

We cancel $b_S(\omega)$, divide by $\sum_{\omega\in\Omega}a_S(\omega)$, and define

$$\varphi_S(\omega):=\frac{a_S(\omega)}{\sum_{\omega'\in\Omega}a_S(\omega')} \quad (46)$$

to obtain inequality (3).

Repeating this construction for every $S\in\mathcal{S}$ yields representations of Theorems 1 and 2, respectively. □

A.6 Proof of Proposition 2

Suppose that Properties 1 to 6 hold. We need to show that $\mu_S\in\Delta(S)$ for $S\not\succeq S(E)$ if and only if, for all events F such that $S(F)=S(E)$, we have $f_F\sim h_Fg$ for all $f,g,h\in\mathcal{A}$.

"⇒": If $\mu_S\in\Delta(S)$ with $S\not\succeq S(E)$, then for all events F with $S(F)=S(E)$,

$$\sum_{\omega\in S}\left(\sum_{x\in supp(f_Fg(\omega))}u_S(x)f_Fg(\omega)(x)\right)\mu_S(\omega)$$
$$=\sum_{\omega\in S}\left(\sum_{x\in supp(h_Fg(\omega))}u_S(x)h_Fg(\omega)(x)\right)\mu_S(\omega). \quad (47)$$

for all $f, g, h \in \mathcal{A}$. By Theorem 1, we have $f_F g \sim_S h_F g$ for all events F such that $S(F) = S(E)$ and all $f, g, h \in \mathcal{A}$.

"$\Leftarrow$": If, for all events F with $S(F) = S(E)$ we have $f_F g \sim h_F g$ for all $f, g, h \in \mathcal{A}$, then $\sim = \sim_S$ for $S \not\preceq S(E)$, otherwise we have a contradiction to Remark 6. By Theorem 1, there exists an awareness-dependent expected utility for which equation (47) holds for all $f, g, h \in \mathcal{A}$ and $F \in \Sigma$ such that $S(F) = S(E)$. Thus $\mu_S \in \Delta(S)$ with $S \not\preceq S(E)$. $\square$

References

[1] Ahn, D. and H. Ergin (2010). Framing contingencies, *Econometrica* **78**, 655-695.

[2] Anscombe, F. and R. Aumann (1963). A definition of subjective probability, *Annals of Mathematical Statistics* **34**, 199-205.

[3] Blume, L., Easley, D., and J.Y. Halpern (2009). Constructive decision theory, Cornell University.

[4] Board, O. and K.S. Chung (2009). Object-based unawareness: Theory and applications, University of Minnesota.

[5] Board, O., Chung, K.S., and B.C. Schipper (2011). Two Models of Unawareness: Comparing the Object-Based and Subjective-state space Approaches, *Synthese* **179**, 13-34.

[6] Dekel, E., Lipman, B., and A. Rustichini (1998). Standard state space models preclude unawareness, *Econometrica* **66**, 159-173.

[7] van Ditmarsch, H. and T. French (2011). Becoming aware of propositional variables, in: M. Banerjee and A. Seth (eds.), *Proceedings of ICLA 2011*, LNAI 6521, Springer, 204-218.

[8] Fagin, R. and J. Halpern (1988). Belief, awareness, and limited reasoning, *Artificial Intelligence* **34**, 39-76.

[9] Feinberg, Y. (2009). Games with unawareness, Stanford University.

[10] Fishburn, P. (1970). *Utility theory for decision making*, New York: John Wiley.

[11] Galanis, S. (2011a). Unawareness of theorems, *Economic Theory*, forthcoming.

[12] Galanis, S. (2011b). Syntactic foundations of unawareness of theorems, *Theory and Decision*, forthcoming.

[13] Grant, S. and J. Quiggin (2011). Inductive reasoning about unawareness, Rice University.

[14] Halpern, J. (2001). Alternative semantics for unawareness,*Games and Economic Behavior* **37**, 321-339.

[15] Halpern, J. and L. Rêgo (2008). Interactive unawareness revisited, *Games and Economic Behavior* **62**, 232-262.

[16] Heifetz, A., Meier, M., and B.C. Schipper (2006). Interactive unawareness, *Journal of Economic Theory* **130**, 78-94.

[17] Heifetz, A., Meier, M., and B.C. Schipper (2008). A canonical model for interactive unawareness, *Games and Economic Behavior* **62**, 304-324.

[18] Heifetz, A., Meier, M., and B.C. Schipper (2011a). Unawareness, beliefs, and speculative trade, University of California, Davis.

[19] Heifetz, A., Meier, M., and B.C. Schipper (2011b). Dynamic unawareness and rationalizable behavior, University of California, Davis.

[20] Heinsalu, S. (2011). Equivalence of the information structure with unawareness to the logic of awareness, Yale University.

[21] Herstein, I. and J. Milnor (1953). An axiomatic approach to measurable utility, *Econometrica* **21**, 291-297.

[22] Karni, E. and M.-L. Vierø (2011). "Reverse Bayesianism": A choice-based theory of growing awareness, Johns Hopkins University.

[23] Li J. (2009). Information structures with unawareness, *Journal of Economic Theory* **144**, 977-993.

[24] Li J. (2008). A note on unawareness and zero probability, University of Pennsylvania.

[25] Meier, M. and B.C. Schipper (2010). Speculative trade under unawareness: The infinite case, University of California, Davis.

[26] Meier, M., and B.C. Schipper (2011). Bayesian games with unawareness and unawareness perfection, University of California, Davis.

[27] Modica, S. and A. Rustichini (1999). Unawareness and partitional information structures, *Games and Economic Behavior* **27**, 265-298.

[28] Morris, S. (1996). The logic of belief and belief change: A decision theoretic approach, *Journal of Economic Theory* **69**, 1-23.

[29] Rubinstein, A. (1991). Comments on the interpretation of game theory, *Econometrica* **59**, 909-924.

[30] Savage, L.J. (1954). *The foundations of statistics*, New York: Dover.

[31] Schipper, B.C. (2011). Preference-based unawareness, University of California, Davis.