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# Leadership cycles

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**Abstract**

We study a quality-ladder model of endogenous growth that produces stochastic leadership cycles. Over a cycle, industry leaders can innovate several successive times in the same industry, gradually increasing the magnitude of their technological lead before being replaced by a new entrant. Initially, new leaders are eager to enlarge their lead and do much of the research, but if they innovate repeatedly, their propensity to invest in R&D decreases. Eventually they stop doing research altogether, and as they are overtaken a new cycle starts. The model generates a skewed firm size distribution and a deviation from Gibrat's law that accord with the empirical evidence. We also consider various policy measures, showing that in some cases policy should favour R&D by incumbents, not outsiders, and that stronger patent protection may reduce innovation and growth.

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# 1 Introduction

We propose a Schumpeterian model of endogenous growth that endogenously produces stochastic leadership cycles. In each industry, both the industry leader and outsiders simultaneously invest in R&D. Initially, new leaders are eager to enlarge their lead and do much of the research. However, if they are lucky and innovate repeatedly, after each successive innovation their profits increase and their share in the R&D done decreases. The process continues until incumbents stop doing research altogether and are inevitably overtaken. A new cycle then starts with a new leader.

This model can help reconcile endogenous growth theory with several stylized facts not accounted for in early contributions. First, it is consistent with ample empirical evidence that while outsiders are responsible for many innovations, incumbents account for a sizeable share of the research done and often innovate repeatedly in the same industry.[1] Second, it endogenously produces a skewed firm size distribution that resembles the upper tail of the distributions typically found in empirical work.[2] Finally, it implies that as leaders grow bigger and older, they tend to rest on their laurels and do less R&D. This generates a negative correlation between the incumbents' size and age and their expected rate of growth, a deviation from Gibrat's law confirmed by the empirical evidence.[3]

To the best of our knowledge, this is the first endogenous growth model that provides a unified explanation for these stylized facts. Previous models in which neither leaders nor outsiders are precluded from innovating in the same industry either generate an "entrenchment-of-monopoly" effect whereby the risk

[1] Segerstrom and Zolniereck (1999) and Segerstrom (2007) provide a nice summary of the relevant empirical evidence.

[2] A wide literature documents the skewness of the firm size distribution. For an excellent contribution that provides also a survey of the previous literature, see Cabral and Mata (2003).

[3] See, among others, Evans (1987), Hall (1987), Rossi-Hansberg and Wright (2005), and Clementi and Hopenhayn (2006).

that the incumbent is replaced decreases with the duration of its leadership, as in Stein (1997), or predict that the probability that the incumbent is replaced is constant, as in Segerstrom and Zolnierek (1999) and Segerstrom (2007).

Other models have followed different routes. Klette and Kortum (2004) assume that incumbents can grow only by diversification, i.e., innovating in other industries. In their model, which has subsequently been developed by Lentz and Mortensen (2005, 2008) and others, incumbents that innovate repeatedly are active in several industries, but in each of them they lead by only one step. An alternative approach, followed by Aghion et al. (2001, 2005) and Acemoglu and Akcigit (2008), among others, posits that in each industry there are two incumbents, which alone can invest in R&D. These models allow the size of the leader's advantage to vary, as we do, but they do not capture the process of firm entry and exit, which plays a crucial role in many innovative industries.

Our model sticks more closely to the original papers by Aghion and Howitt (1992), Segerstrom et al. (1990), and Grossman and Helpman (1991a). The only change is that incumbents are assumed to be more efficient than outsiders in conducting research. This assumption seems natural when innovation is cumulative, since innovative technological knowledge that may be useful to search for the next innovation is often disclosed only partially to outsiders. The assumption has in fact been made by several papers in the growth literature, but previous contributions have added ancillary assumptions that prevent leadership cycles.[4]

Making no special assumptions, we simply focus on the case where innovations are incremental (or non drastic), so that new innovators are constrained by

[4] Grossman and Helpman (1991b), for instance, assume that the leader's advantage is small enough that all the research is conducted by outsiders. In their model, the leader invests in R&D only if the innovation has been perfectly imitated: in this case, which we rule out in this paper, Arrow's effect vanishes, and the leader ends up conducting all the research. Barro and Sala-i-Martin (1994) assume that the leader has not only an R&D advantage, but also a first-mover advantage. This leads to a preemption equilibrium in which the leader again conducts all research.

outside competition in the product market and cannot engage in monopoly pricing.[5] Then, for a range of parameter values the leader's advantage in conducting research can be exactly offset by Arrow's replacement effect.[6] As a result, neither leaders nor outsiders are precluded from innovating, and leadership cycles can arise as an equilibrium phenomenon.

In our model's equilibrium, the probability that an incumbent is replaced is one when a certain critical lead size is reached, which corresponds to the maximum "length" of leadership cycles. The economy's rate of growth depends only on the profits obtained by incumbents that have reached the maximal lead size. Intuitively, with constant returns to scale in research all "intermediate" profits (i.e., the profits obtained by incumbents that enjoy a smaller competitive advantage) are dissipated in the patent races which they participate in. The size of these intermediate profits determines only the division of the total R&D done among the incumbent and outsiders.

The model is tractable enough to yield a simple closed-form solution, so it could lend itself (in future research) to various extensions. Here we use it to address several policy issues. We are especially interested in ascertaining whether policy should favour R&D investment by leaders or by outsiders, a question that has broad policy implications (for example, in competition policy).

[5] Segerstrom and Zolnierek (1999) show that with constant returns to R&D and drastic innovations, in equilibrium either the leader or outsiders may conduct the research, but not both. To obtain an equilibrium where both the leader and the outsiders simultaneously invest in R&D, they then posit decreasing returns to R&D at the firm level (see also Segerstrom, 2007). Here, by contrast, we focus on non drastic innovations, retaining the assumption of constant returns to scale in research. This assumption is standard in the endogenous growth literature and seems to be well grounded both empirically and theoretically. For example, surveying the empirical literature Griliches (1990, p. 1677) notes that

> in the major range of the data [...] there is little evidence for diminishing returns, at least in terms of patents per R&D dollar. That is not surprising, after all. If there were such diminishing returns, firms could split themselves into divisions or separate enterprises and escape them.

At the industry level, by contrast, the existence of decreasing returns to scale in research is well documented. For simplicity, we make the assumption of constant return also at the industry level, but this serves only to obtain a closed-form solution.

[6] That is, leaders have a lower incentive to innovate than outsiders because the incentive for them is only the *incremental* profit, i. e., the difference over the current profit.

We analyze R&D taxes and subsidies and various patent policies. We show that the economy's rate of growth increases when the outsiders' R&D expenditure is taxed and the revenue is used to subsidize the leaders' R&D. As for patent policy, we argue that when patent protection is state-dependent, our model produces a trickle-down effect, as in Acemoglu and Akcigit (2008): the incumbents' investment in R&D increases if the level of patent protection in the early stages of a leadership cycle is reduced. However, in our model this policy does not affect the equilibrium rate of growth, as the incumbents' greater R&D investment crowds out the outsiders' on a one-to-one basis.

We consider also the possibility that follow-on innovations may infringe on the patents that protect previous innovators, as in O'Donoghue and Zweimuller (2004) and Chu (2009). In this case, successful outsiders must pay a licensing fee to the old incumbent, but no payment is due if the incumbent innovates repeatedly. In a leapfrogging equilibrium where incumbents do not invest in R&D, this form of patent protection is always bad for growth. We show that, on the contrary, its effect can be positive when leaders can innovate repeatedly.

The rest of the paper is organized as follows. Section 2 outlines the model. Section 3 derives the conditions that must hold in a steady state equilibrium. Section 4 characterizes the equilibrium where the total R&D investment is constant over a leadership cycle. Section 5 analyzes several innovation policies. Section 6 discusses other equilibria, where the total R&D varies over a cycle, and explains why the equilibrium where total R&D is constant is a natural focal point. Section 7 summarizes and concludes. Proofs are collected in the Appendix.

## 2 The model

For ease of comparison, we build on the textbook quality-ladder model of Barro and Sala-i-Martin (2004).

### 2.1 Preferences

The economy is populated by $L$ identical, infinitely-lived individuals. Time is continuous. Each individual inelastically supplies one unit of labour and has linear intertemporal preferences:

$$u(c(t)) = \int_0^\infty c(t)e^{-rt}dt, \tag{1}$$

where $c(t)$ is consumption at time $t$. With linear intertemporal preferences, the equilibrium rate of interest is fixed and coincides with the rate of time preference $r$.[7] Each individual maximizes (1) subject to the instantaneous budget constraint:

$$c(t) + \dot{a}(t) \leq w(t) + ra(t), \tag{2}$$

where $w(t)$ is the wage rate and $a(t)$ is the individual's wealth. Individuals are risk neutral, so in equilibrium by arbitrage all assets must yield the same instantaneous expected net rate of return $r$.

### 2.2 Final and intermediate goods

There is a unique final good in the economy that can be consumed, used to produce intermediate goods, or used in research. This good is taken as the numeraire. It is produced in a perfectly competitive market using labour (which is in fixed supply) and a continuum of intermediate goods $\omega \in [0, 1]$, the quality of which increases with technical progress. We normalize the quality of all intermediate goods at time 0 to unity and denote by $\lambda > 1$ the size of each

[7] One can easily allow for more general preferences (see footnote 15 below).

innovation. Thus, the quality of intermediate good $\omega$ of vintage $j(\omega,t)$ is $\lambda^{j(\omega,t)}$, where $j(\omega,t)$ denotes the number of innovations that have been achieved in industry $\omega$ by time $t$.

The final good can be produced according to the following constant-returns production function:

$$y(t) = \int_0^1 L^{1-\alpha} \left[ \sum_{k=0}^{j(\omega,t)} \lambda^{j(\omega,t)-k} q(j-k,\omega,t) \right]^\alpha d\omega, \qquad 0 < \alpha < 1,$$

where $L$ is labour input, $(1-\alpha)$ is the share of labour's income, and $q(j-k,\omega,t)$ denotes the input of the intermediate good of type $\omega$ and vintage $j-k$, so that $\sum_{k=0}^{j(\omega,t)} \lambda^{j(\omega,t)-k} q(j-k,\omega,t)$ is a quality-adjusted index of composite good $\omega$ that combines all past generations of intermediate goods of type $\omega$.

Independently of its type $\omega$ and vintage $j$, each unit of intermediate good can be produced using one unit of final good. Producers of intermediate good $\omega$ compete in price, so in equilibrium only the latest vintage of intermediate good $\omega$ is produced and employed in the production of the final good. Normalizing $L$ to one, the production function of the final good can then be re-written as:

$$y(t) = \int_0^1 \left[ \lambda^{j(\omega,t)} q(j,\omega,t) \right]^\alpha d\omega. \tag{3}$$

The amount of the final good used in the production of intermediate goods is

$$Q(t) = \int_0^1 q(j,\omega,t) d\omega.$$

### 2.3 The R&D sector

In each industry $\omega$, there is a sequence of patent races. As soon as innovation $j$ is achieved, a free-entry, simultaneous-move race to achieve innovation $j+1$ starts. Both innovator $j$ and a mass of outsiders can race to achieve innovation $j+1$; that is, outsiders need not duplicate innovation $j$ before starting to search for innovation $j+1$. Thus, our model differs from step-by-step models *à la* Aghion et al. (2001). However, we assume that the current leader is more efficient than outsiders in conducting research.[8]

The firms' R&D investment determines the arrival of the innovation according to a Poisson stochastic process. The instantaneous probability that a firm $s = o, \ell_i$ (where $o$ stands for outsiders and $\ell_i$ for a leader that leads by $i$ steps) achieves innovation $j+1$ in sector $\omega$ by time $t$ is

$$x_s(j, \omega, t) = \frac{R_s(j, \omega, t)}{c_s g^j}, \tag{4}$$

where $R_s(j, \omega, t)$ is firm $s$'s R&D investment in units of the final good and $g \equiv \lambda^{\frac{\alpha}{1-\alpha}} > 1$. The term $g^j$ in the denominator of (4) means that research becomes increasingly difficult as new innovations arrive, an assumption that serves to guarantee the existence of a steady state.[9] The parameters $c_{\ell_i}$ and $c_o$ measure the unit cost of research done by leaders and outsiders, respectively. All outsiders, including past innovators, have the same R&D cost, but the leader is more productive: $c_{\ell_i} < c_o$. For simplicity, we assume that a leader's productivity in R&D is independent of the size of its lead $i$: $c_{\ell_i} = c_\ell$. Relaxing this assumption complicates the calculations, but does not affect our qualitative results as long as $c_{\ell_i}$ does not decrease with $i$ too fast.

Notice that the R&D technology (4) exhibits constant returns to scale both at the firm level and at the industry level. While the assumption that there are

[8] With symmetric unit R&D costs, models of step-by-step innovations differ radically from models of leapfrogging. The former implicitly assume that there are no technological spillovers, so laggards must independently duplicate all past innovations before moving up the quality ladder. The latter are best suited to describe situations in which all innovative technological knowledge is publicly disclosed, but patent protection allows only the patent holder to practice the innovation. The truth probably lies somewhere in between, as technological spillovers are ubiquitous but typically far from perfect. In step-by-step models, such partial spillovers can be captured assuming that duplication is easier than innovation. In models of leapfrogging such as ours, one can instead assume that leaders have an R&D cost advantage over outsiders. Both modeling strategies essentially capture the same economic effect, and the choice of one or the other is largely a matter of analytical convenience.

[9] In a steady state, the expected waiting time to discovery must be constant. Since $g$ is the growth factor between successive innovations, as will be seen below, R&D investment grows at rate $g$ from one race to the next. Then in order for the aggregate hazard rate to be constant the productivity of R&D must decline at rate $g$. This requires the knife-edge assumption that the productivity of R&D expenditure decreases at rate $g$, which is standard in quality-ladder endogenous growth models (see e.g. Barro and Sala-i-Martin, 2004).

constant returns to R&D at the firm level seems well grounded both empirically and theoretically,[10] the assumption of constant return at the industry level is made only to obtain a closed-form solution. Later, we shall argue that the equilibrium we analyze is robust to the introduction of decreasing returns at the industry level.

R&D projects are independent, so the aggregate hazard rate $X_i(j, \omega, t)$ equals the sum of the individual rates:

$$X_i(j, \omega, t) = x_{\ell_i}(j, \omega, t) + X_{O_i}(j, \omega, t),$$

where $X_{O_i}$ denotes the outsiders' aggregate hazard rate. The total amount of final good used in research is

$$R(t) = \int_0^1 \left[c_\ell x_{\ell_i}(j, \omega, t) + c_o X_{O_i}(j, \omega, t)\right] g^{j(\omega,t)} d\omega.$$

## 2.4 Patent policy

Initially, we assume that each innovation is patentable and there is perfect, infinitely-long patent protection, meaning that nobody can imitate an innovation without infringing the patent that covers it. We also assume that no innovation infringes on previous patents, so innovators need not obtain any licence from previous patent holders to practice their innovations. These assumptions will be further discussed and relaxed in section 5.

## 2.5 Equilibrium concept

In our model, the labour market and the final good market are perfectly competitive, while firms may hold market power in the intermediate good markets and behave strategically in patent races. We assume that perfectly competitive markets clear at any point in time. When firms behave strategically, we focus

[10] See Griliches (1990).

on Markov Perfect Equilibria (MPE) where strategies depend only on payoff-relevant state variables.

### 2.6 Steady state

As new innovations arrive, the productivity of any one intermediate good jumps up discretely by a factor $g > 1$ at random intervals, but since there is a continuum of intermediate goods, by the law of large numbers, the economy can grow smoothly. A steady state is defined as a situation in which:

$(i)$ the output of the final good, consumption, aggregate R&D expenditure, the aggregate output of intermediate goods, and the wage rate all grow at a constant rate, denoted by $\gamma$;

$(ii)$ the fraction of industries in which the leader leads by $i = 1, 2, ...$ steps, denoted by $\kappa_i$, is constant;

$(iii)$ the average hazard rate and expected waiting time for innovations are constant.

We are interested in steady state MPE.

## 3 Equilibrium growth

In this section we derive the conditions that must hold in a steady state MPE, analyze its stability, and calculate the equilibrium rate of growth as a function of R&D investments.

### 3.1 Labour and goods market equilibrium

At any point in time, the wage rate adjusts so as to clear the labour market:

$$w(t) = (1 - \alpha)y(t).$$

The market clearing condition in the final good market is

$$y(t) = c(t) + Q(t) + R(t),$$

and by Walras' law it is equivalent to the labour market equilibrium condition, as we shall confirm below.

Now consider the intermediate goods markets. Profit maximization by perfectly competitive firms in the final good sector implies the following demand for the last vintage of the intermediate good of type $\omega$:

$$q(j,\omega,t) = \frac{\alpha^{\frac{1}{1-\alpha}}}{p(j,\omega,t)^{\frac{1}{1-\alpha}}} g^{j(\omega,t)},$$

where $p(j,\omega,t)$ is its price. Notice that the demand for intermediate good $\omega$ does not depend on the prices of the other intermediate goods (this follows from the linear specification (3)). The demand function has a constant elasticity $\frac{1}{1-\alpha}$, and each innovation shifts demand up by the constant factor $g = \lambda^{\frac{\alpha}{1-\alpha}}$.

Since different vintages of a given intermediate good $\omega$ are perfect substitutes at constant rates, firms producing different vintages can be treated as if all were producing the same good, measured in *efficiency units*, but with different costs that reflect the quantity of final good needed to make one efficiency unit. Specifically, the effective marginal production cost of vintage $k-i$, in efficiency units relative to the last vintage, is $\lambda^i$, since one unit of the intermediate good of vintage $k$ is as productive as $\lambda^i$ units of the good of vintage $k-i$. With Bertrand competition, in each intermediate good sector only the current leader will be active in equilibrium. However, the equilibrium price depends on what technology is available to its most efficient rival (i.e., the penultimate innovator), which can supply the next most productive vintage.

The monopoly price is $p_M = \dfrac{1}{\alpha}$. If $\lambda^i > \dfrac{1}{\alpha}$, the cost of the leader's most efficient competitor is greater than the monopoly price. The leader then is effectively unconstrained by outside competition and can charge the monopoly price. If instead $\lambda^i < \dfrac{1}{\alpha}$, the leader's cumulated cost advantage is too small to allow it to charge the monopoly price. In the ensuing Bertrand equilibrium, the leader engages in limit pricing: $p_L = \lambda^i$.

Define $m$ as the minimum lead size that allows the leader to charge the monopoly price; that is, the minimum value of $i$ such that $\lambda^i \geq \frac{1}{\alpha}$. Accounting for the integer constraint, $m$ is implicitly given by the inequalities

$$\lambda^m \geq \frac{1}{\alpha} > \lambda^{m-1}.$$

Then, if $i \geq m$, the leader charges the monopoly price and obtains the monopoly profit

$$\pi_m(j, \omega, t) = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} g^{j(\omega,t)},$$

which is independent of the size of its lead. If $i < m$, the leader engages in limit pricing, and its profit is:

$$\pi_i(j, \omega, t) = (\lambda^i - 1)\alpha^{\frac{1}{1-\alpha}} \lambda^{-\frac{i}{1-\alpha}} g^{j(\omega,t)}.$$

In this case the leader's profit depends on $i$, the size of its technological advantage. To simplify the notation, denote

$$\pi_i \equiv \begin{cases} (\lambda^i - 1)\alpha^{\frac{1}{1-\alpha}} \lambda^{-\frac{i}{1-\alpha}} & \text{for } i = 1, 2, ..., m-1 \\ (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} & \text{for } i = m, m+1, ... \end{cases}$$

so that $\pi_i(j, \omega, t) = \pi_i g^{j(\omega,t)}$.

The equilibrium price is independent of $j, \omega$ and $t$, but depends on $i$, the number of consecutive innovations achieved by the current leader. This is determined endogenously, which marks a key difference from standard quality-ladder models where the current leader never invests in R&D and so $i$ is always equal to one.

## 3.2 The patent race equilibrium

With a memory-less discovery process like Poisson, the payoff to any firm that participates in a patent race does not depend on how much time has passed since the start of the race, so in a MPE firms will choose a constant level of R&D expenditure until someone succeeds and the next race starts. As we proceed, it will become clear that in a MPE firms' strategies must be independent also of $\omega$ and $j$.

Let $V_1(j,\omega,t)$ be the expected value of leading by one step in industry $\omega$ at time $t$ if $j$ innovations have already been made in that industry. This is given by the following Bellman equation (to simplify the notation, we suppress the indices $\omega$ and $t$ when there is no risk of confusion):

$$rV_1(j) = \max_{x_{\ell_1}} \left[ \pi_1(j) - X_1 V_1(j) + x_{\ell_1} V_2(j+1) - c_\ell g^j x_{\ell_1} \right], \tag{5}$$

where $V_2(j+1)$ is the expected value of leading by two steps if $j+1$ innovations have been previously achieved. The interpretation of equation (5) is simple. The right-hand side is the expected flow value of leading by one step. A one-step leader earns the flow profit $\pi_1(j)$ and incurs the flow cost $c_\ell g^j x_{\ell_1}$ until innovation $j+1$ arrives. When innovation $j+1$ is achieved, which occurs with an instantaneous aggregate probability $X_1$, the leader incurs a capital loss $V_1(j)$, but in case it itself succeeds, an event whose probability is $x_{\ell_1}$, it obtains $V_2(j+1)$. That is, $V_2(j+1) - V_1(j)$ is the net capital gain obtained by a one-step leader that innovates again. The leader chooses $x_{\ell_1}$ to maximize its present expected profits. Equation (5) states that such maximized profits must guarantee an expected rate of return on the leader's asset, $V_1(j)$, equal to the equilibrium interest rate $r$.

The expected value of leading by two steps, $V_2(j+1)$, is in turn determined by the following Bellman equation

$$rV_2(j+1) = \max_{x_{\ell_2}} \left[ \pi_2(j+1) - X_2 V_2(j+1) + x_{\ell_2} V_3(j+2) - c_\ell g^{j+1} x_{\ell_2} \right],$$

where $V_3(j+2)$ is the value of leading by three steps, and so on. After $m$ successive innovations, the leader becomes an unconstrained monopolist in the product market. This implies that

$$V_{m+1}(j+m) = V_m(j+m),$$

since a firm leading by $m$ or more steps will earn monopoly profits, irrespective of the magnitude of its lead.

Since in a steady state the profit earned by a firm leading by $i$ steps increases by a constant factor $g$ from one period to the next,[11] this property must be inherited by the value functions. Hence, we must have $V_i(j) = g^j V_i$, where $V_i = V_i(0)$. We can then rewrite the Bellman equations as follows:

$$\begin{aligned} rV_1 &= \max_{x_{\ell_1}} \left[\pi_1 - X_1 V_1 + g x_{\ell_1} V_2 - c_\ell x_{\ell_1}\right], \\ &\quad ... \\ rV_i &= \max_{x_{\ell_i}} \left[\pi_i - X_i V_i + g x_{\ell_i} V_{i+1} - c_\ell x_{\ell_i}\right], \\ &\quad ... \\ rV_m &= \max_{x_{\ell_m}} \left[\pi_M - X_m V_m + g x_{\ell_m} V_m - c_\ell x_{\ell_m}\right]. \end{aligned} \tag{6}$$

Consider next a generic outsider that participates in a patent race in an industry where the leader leads by $i$ steps. If it wins, it obtains a one-step leadership, the value of which is $V_1$. Thus, the expected discounted profit of any individual outsider that invests $c_o g^j x_{o_i}$ units of the final good to obtain innovation $j+1$ is

$$\frac{x_{o_i} V_1(j+1) - c_o g^j x_{o_i}}{r + X_i}.$$

By the free entry condition, this cannot be positive,[12] and if it is negative then $x_{o_i}$ must vanish (and hence $X_{O_i} \equiv \sum x_{o_i}$ must also vanish). Since this property

[11] Here by period we mean the random time interval between two successive innovations.

[12] The zero-profit condition in patent races implies that the increase in the net value of the individuals' assets must equal aggregate R&D investment:

$$\dot{a}(t) = R(t).$$

Proprietary technological knowledge is the only asset in our model economy. The return to holding this asset, $ra(t)$, is the extra-profits earned by firms holding market power. Aggregating across firms, this equals

$$ra(t) = \alpha y(t) - Q(t).$$

Plugging these equations into the budget constraint (2) one sees that the labour market and final good market equilibrium conditions are equivalent (Walras' law), as was claimed earlier.

must hold for all $i$, we have

$$gV_1 - c_o \leq 0 \text{ and } \left(\sum_{i=1}^{m} X_{O_i}\right)(gV_1 - c_o) = 0.$$

A Markov Perfect Equilibrium in the patent race is a list of non-negative variables $(X_1, X_2,..., X_m, x_{\ell_1}, x_{\ell_2},..., x_{\ell_m}, V_1, V_2,..., V_m)$ with $x_{\ell_i} \leq X_i$ that satisfy the Bellman equations (6) and the free entry condition.

Since the variables $\omega$ and $j$ do not enter the Bellman equations nor the free entry condition, in a MPE firms' strategies cannot depend on $\omega$ and $j$. To see why this restriction is important, notice that the profitability of each innovation depends on the expected R&D efforts in the subsequent patent races. Innovators correctly anticipate such future efforts, but the forward-looking nature of the equilibrium means that expectations could be conditioned on "extraneous" variables. For example, if firms expect high levels of R&D effort when $j$ is even and low levels when $j$ is odd, this may turn out to be a self-fulfilling prophecy that generates two-period cycles, as in Aghion and Howitt (1992). Firms' expectations may depend also on $\omega$: for example, an arbitrarily large fraction of industries may be trapped in a no-growth trap (which is itself a degenerate two-period cycle) where nobody invests for fear that its profits will be terminated soon by the occurrence of the next innovation, as in Cozzi (2007). We have nothing to add to the analysis of these phenomena here, so we assume Markov perfection. By requiring that only variables that directly appear in the Bellman equations can affect firms' strategies, this assumption rules out cyclical growth and the possibility that an indeterminate fraction of industries may be trapped in a no-growth trap.

The next Lemma provides a useful characterization of the steady state MPE in terms of a set of inequalities and complementary slackness conditions.

**Lemma 1** *A list of non-negative variables* $(X_1, X_2,..., X_m, x_{\ell_1}, x_{\ell_2},..., x_{\ell_m}, V_1, V_2,..., V_m)$ *such that* $X_{O_i} = X_i - x_{\ell_i} \geq 0$ *for* $i = 1,...,m$ *is a Markov perfect* *patent-race equilibrium if and only if they satisfy the free entry condition*

$$gV_1 - c_o \leq 0 \text{ and } \left(\sum_{i=1}^{m} X_{O_i}\right)(gV_1 - c_o) = 0 \tag{7}$$

*and the following inequalities, with the associated complementary slackness conditions :*

$$gV_{i+1} - V_i - c_\ell \leq 0 \text{ and } x_{\ell_i}\left(gV_{i+1} - V_i - c_\ell\right) = 0 \ \forall i = 1, ..., m-1 \tag{8}$$

$$(g-1)V_m - c_\ell \leq 0 \text{ and } x_{\ell_m}\left[(g-1)V_m - c_\ell\right] = 0. \tag{9}$$

*Moreover, in any equilibrium*

$$V_i = \frac{\pi_i}{r + X_{O_i}} \qquad i = 1, 2, ..., m. \tag{10}$$

With constant returns to scale in R&D, in equilibrium either leaders do not invest altogether, or they must be indifferent between any level of R&D investment – conditions (8) and (9). These conditions in turn imply that the value of leading by $i$ steps must equal the expected present value of the corresponding flow of profits, $\pi_i$, where the discount rate $r$ is augmented by the probability that the current leader is replaced. This is condition (10).

## 3.3 Stability

Partition the set of industries $[0, 1]$ into $m$ sub-sets $\varkappa_i$ with $i = 1, 2, ..., m$, where the leader leads by 1, 2, ..., and $m$ *or more* steps, respectively. (Later, we shall show that no leader ever leads by more than $m$ steps in equilibrium.) The measure of $\varkappa_i$ is the fraction of industries in which the leader leads by $i$ steps, which we denote by $\kappa_i$. As new innovations arrive, the sets $\varkappa_i$ change continuously. The outflow from state $i$ is the probability that someone innovates, $X_i$; the inflow is, for $i > 1$, the probability that the leader innovates in state $i - 1$ (and also in state $m$ for $i = m$) and, for $i = 1$, the probability that an outsider innovates in any state. Hence we have:

$$\begin{aligned}
\dot{\kappa}_1 &= \kappa_1 X_{O_1} + \kappa_2 X_{O_2} + ... + \kappa_m X_{O_m} - \kappa_1 X_1 \\
&\quad ... \\
\dot{\kappa}_i &= \kappa_{i-1} x_{\ell_{i-1}} - \kappa_i X_i \qquad i = 2, ..., m-1 \\
&\quad ...... \\
\dot{\kappa}_m &= \kappa_{m-1} x_{\ell_{m-1}} + \kappa_m x_{\ell_m} - \kappa_m X_m
\end{aligned} \tag{11}$$

In a steady state, the $\kappa_i$'s are constant:

$$\dot{\kappa}_i = 0 \qquad \text{for } i = 1, 2, ..., m.$$

This system provides $m-1$ independent equations, as the first equation can be obtained from the others. Together with the adding-up condition $\sum_{i=1}^{m} \kappa_i = 1$, these equations can be solved to get:

$$\begin{aligned}
\kappa_1 &= \frac{1}{1 + \frac{x_{\ell_1}}{X_2} + \frac{x_{\ell_1} x_{\ell_2}}{X_2 X_3} ... + \frac{x_{\ell_1} ... x_{\ell_{m-1}}}{X_2 ... X_{m-1}(X_m - x_{\ell_m})}} \\
\kappa_i &= \frac{x_{\ell_1} ... x_{\ell_{i-1}}}{X_2 ... X_i} \kappa_1 \qquad i = 2, ..., m-1 \\
\kappa_m &= \frac{x_{\ell_1} ... x_{\ell_{m-1}}}{X_2 ... X_{m-1}(X_m - x_{\ell_m})} \kappa_1.
\end{aligned} \tag{12}$$

Since in our model there is no capital accumulation, and strategic variables such as prices and R&D investments can immediately jump to their equilibrium values, only the $\kappa_i$s adjust gradually after any parameter change, or starting from arbitrarily given initial conditions. The next result guarantees the stability of this adjustment process for any possible MPE of the patent race:

**Lemma 2** *The dynamical system (11) is globally stable.*

### 3.4 The growth rate

Now we calculate the equilibrium rate of growth as a function of R&D investments. Since the equilibrium price depends only on $i$, substituting (4) into (3) we get:

$$\begin{aligned} y(t) &= \int_0^1 \left[\lambda^{j(\omega,t)} \alpha^{\frac{1}{1-\alpha}} g^{j(\omega,t)} p(j,\omega,t)^{-\frac{1}{1-\alpha}}\right]^{\alpha} d\omega \\ &= \alpha^{\frac{\alpha}{1-\alpha}} \sum_{i=1}^{m} \left[ p_i^{-\frac{\alpha}{1-\alpha}} \int_{\varkappa_i} g^{j(\omega,t)} d\omega \right]. \end{aligned}$$

By the law of large numbers, the probability that the leader has an $i$-step advantage is the same across industries. Hence, the variable $g^{j(\omega,t)}$ will be identically distributed over any subset $\varkappa_i$, implying:

$$\int_{\varkappa_i} g^{j(\omega,t)} d\omega = \kappa_i \int_0^1 g^{j(\omega,t)} d\omega.$$

Substituting into the preceding expression we get:

$$y(t) = \alpha^{\frac{\alpha}{1-\alpha}} \left( \sum_{i=1}^{m} \kappa_i p_i^{-\frac{\alpha}{1-\alpha}} \right) G(t), \tag{13}$$

where $G(t) \equiv \int_0^1 g^{j(\omega,t)} d\omega$ is an intermediate good aggregate quality index that increases over time with technical progress.

Since the term $\sum_{i=1}^{m} \kappa_i p_i^{-\frac{\alpha}{1-\alpha}}$ is constant in a steady state, equation (13) implies that the rate of growth of output is the rate of growth of the average quality of the intermediate goods, $G(t)$. To calculate it, notice that in an industry where the leader has an $i$-step advantage, $j(\omega,t)$ jumps up to the next higher integer with a constant instantaneous probability $X_i$. Hence:

$$\dot{G}(t) = \int_0^1 \left[ g^{j(\omega,t)+1} - g^{j(\omega,t)} \right] X(j+1,\omega,t) d\omega.$$

Proceeding as before, we obtain

$$\begin{aligned} \dot{G}(t) &= \sum_{i=1}^{m} \left[ (g-1) X_i \int_{\varkappa_i} g^{j(\omega,t)} d\omega \right] \\ &= \sum_{i=1}^{m} \left[ (g-1) \kappa_i X_i \int_0^1 g^{j(\omega,t)} d\omega \right] \\ &= (g-1) X G. \end{aligned}$$

where $X = \sum_{i=1}^{m} \kappa_i X_i$. It follows that the economy's rate of growth is

$$\gamma = (g-1)X. \tag{14}$$

In what follows we shall assume that the "transversality" condition

$$r > \gamma \tag{15}$$

holds; if this inequality is violated, the utility $u$ is unbounded (Barro and Sala-i-Martin, 2004).

# 4 Leadership cycles

In this and the following section we focus on the equilibrium where the *total* R&D effort in an industry is constant over a leadership cycle, i.e., $X_i$ is independent of $i$:

$$X_i = X \qquad \text{for all } i = 1, 2, ..., m; \tag{16}$$

however, the division of $X$ between the leader and outsiders can vary with $i$. Condition (16) may be viewed as imposing a strong form of Markov perfection, so we call the equilibria that satisfy (16) *Strong* MPE (SMPE). After characterizing the unique SMPE, we discuss condition (16) more fully in section 6.

To avoid proliferation of cases, before proceeding we impose some parameter restrictions that guarantee the existence of an equilibrium with positive R&D investment by outsiders. First, we rule out persistent leadership equilibria where only leaders invest in R&D. Lemma 3 shows that these equilibria emerge if the leader's R&D advantage is large enough:

**Lemma 3** *If $\frac{c_o}{c_\ell} > \frac{g}{g-1}$, there is no MPE in which outsiders invest in R&D.*

Thus, we assume:[13]

[13] For a discussion of the special case $\frac{c_o}{c_\ell} = \frac{g}{g-1}$, see Segerstrom and Zolnierek (1999).

$$\frac{c_o}{c_\ell} < \frac{g}{g-1}. \tag{17}$$

Second, we assume that innovation is sufficiently profitable that some research is conducted at equilibrium:

$$\frac{\pi_1}{r} > \frac{c_o}{g}. \tag{18}$$

If this inequality is violated, the economy stagnates indefinitely.

To state our characterization result we need some more definitions. Let us recursively calculate the solution to (8), taken as an equality, starting from $V_1 = \frac{c_o}{g}$. We obtain:

$$\tilde{V}_i \equiv \frac{c_o + c_\ell(g + g^2 + ... + g^{i-1})}{g^i}.$$

$\tilde{V}_i$ increases with $c_o$, $c_\ell$ and $i$ and decreases with $g$. It is also useful to note the following Lemma:

**Lemma 4** *The ratio* $\frac{\pi_i}{\tilde{V}_i}$ *is either decreasing, or first increasing and then decreasing in* $i$*.*

Finally, define

$$i^* = \arg\max_{i=1,2,\ldots,m}\left[\frac{\pi_i}{\tilde{V}_i}\right].$$

We are now ready to state:

**Proposition 1** *There is a unique Strong Markov Perfect Equilibrium outcome. In equilibrium, outsiders and all leaders* $\ell_i$ *with* $i < i^*$ *invest in R&D simultaneously, whereas* $\ell_{i^*}$ *does not invest, so no leader ever leads by more than* $i^*$ *steps. The aggregate R&D effort is:*

$$X = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - r, \tag{19}$$

*and the leaders' R&D efforts are*

$$x_{\ell_i} = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - \frac{\pi_i}{\tilde{V}_i} \quad \textit{for } i \leq i^*, \tag{20}$$

*so that* $V_i = \tilde{V}_i$ *for* $i \leq i^*$.

The variable $i^*$ can be thought of as the maximum "length" of leadership cycles. When $i^* = 1$, leadership cycles are degenerate, and the SMPE actually reproduces the familiar leapfrogging equilibrium. Obviously, $i^*$ cannot be greater than one when innovations are drastic, i.e., $m = 1$. This confirms that with drastic innovations there is no equilibrium in which the leader and outsiders simultaneously invest in R&D – a point made by Segerstrom and Zolnierek (1999). However, when innovations are non drastic ($m \geq 2$) there always exists a non empty region of parameters values where leadership cycles are non degenerate. From the definition of $i^*$ it follows immediately that for any given $m \geq 2$, a necessary and sufficient condition for $i^*$ to be greater than one is $\frac{g\pi_1}{g\pi_2 - \pi_1} < \frac{c_o}{c_\ell} < \frac{g}{g-1}$. More generally, we have (the proof is in the Appendix):

**Corollary 1**. *The maximum length of leadership cycles,* $i^*$, *increases with* $\frac{c_o}{c_\ell}$ *and approaches* $m$ *as* $\frac{c_o}{c_\ell}$ *goes to* $\frac{g}{g-1}$.

The intuitive reason why both the leader and outsiders can simultaneously invest in R&D is as follows. Leaders are more efficient than outsiders in conducting research. With non drastic innovations, they also obtain greater profits than outsiders if they innovate repeatedly, as they are less severely constrained by outside competition in the product market. However, leaders have a lower incentive to innovate since only the incremental profit (i.e., the difference over the current profit) matters for them. This is Arrow's replacement effect.

In our model, the magnitute of Arrow's effect is endogenous, as it depends on the division of the total R&D among the leader and outsiders. Thus, if the leader's cost advantage is neither too large nor too small, it can be exaclty offset by Arrow's effect. However, if the leader innovates repeatedly, its profits increase and thus Arrow's effect tends to strengthen, reducing the leader's incentive to invest in R&D. As a result, the leader's share in the total R&D done must decrease. This is proved formally in the following:

**Corollary 2**. *In the SMPE,* $x_{\ell_{i-1}} > x_{\ell_i}$ for $i \leq i^*$.

The Corollary follows immediately from (20) noting that $\frac{\pi_i}{\tilde{V}_i}$ increases with $i$ for $i \leq i^*$ (this follows by Lemma 4 and the definition of $i^*$). After $i^*$ successive innovations Arrow's effect becomes so strong as to prevail over the R&D cost effect. The leader then stops investing in R&D altogether and is overtaken with probability one, and a new cycle starts.

The model is consistent with the empirical evidence that while a significant share of innovations is achieved by outsiders, incumbents account for much of the research done and often innovate repeatedly in the same industry. Scherer (1980), for instance, discusses survey evidence that industry leaders make significant R&D investment to improve their existing products. By innovating repeatedly over time, these incumbents may obtain a large competitive advantage over outsiders. Segerstrom (2007) amply documents the phenomenon.

A remarkable property of the equilibrium is that the hazard rate $X$ depends only on the profit accruing to leaders that lead by $i^*$ steps and is independent of whatever profits leaders may earn before. This seems surprising at first, since the incentive to innovate should reflect the expected present value of all profits earned over an innovator's life cycle. The intuitive explanation is that with constant returns to scale in research all "intermediate" profits (i.e., the profits obtained by incumbents that lead by less than $i^*$ steps) are dissipated in the patent races which they participate in.

To see this point more clearly, recall that with constant returns to scale in research incumbents are indifferent between any amount of R&D investment as long as $i < i^*$ (equation (8)). Imagine, then, an hypothetical outsider that, after having innovated for the first time, keeps investing in R&D at an arbitrarily large rate so as to achieve the subsequent $i^* - 1$ innovations almost instantaneously, and then stops investing. The expected present profit guaranteed by such a strategy is

$$\frac{g^{i^*}\pi_{i^*}}{X+r},$$

i.e., the discounted profit of leading by $i^*$ steps until somebody else innovates. Intermediate profits do not matter here, as they are earned for arbitrarily short periods of time. On the other hand, the effective unit R&D cost of achieving $i^*$ consecutive innovations (the first as an outsider and the others, in infinitely rapid succession, as a leader) is[14]

$$C_{i^*} \equiv c_o + c_\ell(g + g^2 + ... + g^{i^*-1}).$$

The standard zero-profit condition in a patent race with free entry then becomes

$$\frac{g^{i^*}\pi_{i^*}}{X+r} = C_{i^*}.$$

Noting that $C_{i^*} = g^{i^*}\tilde{V}_{i^*}$, one sees immediately that this condition is equivalent to (19). Of course, in equilibrium incumbents do not follow the hypothetical strategy described above, but their payoff must be the same as if they did, since they must be indifferent between any amount of R&D investment.

Another remarkable property of the leadership cycle dynamics is that, in innovative industries, the quality-adjusted price stays constant after each successive innovation by the leader, but jumps down when an outsider innovates. This implies that the leader alone benefits from technical progress, reaping higher and

[14]The total discounted cost is

$$c_o + \frac{x_{\ell_1} g c_\ell}{x_{\ell_1}+r} + \frac{x_{\ell_1} x_{\ell_2} g^2 c_\ell}{(x_{\ell_1}+r)(x_{\ell_2}+r)} + ... + \frac{x_{\ell_1} ... x_{\ell_{i^*-1}} g^{i^*-1} c_\ell}{(x_{\ell_1}+r) ... \left(x_{\ell_{i^*-1}}+r\right)}.$$

Letting the $x_{\ell_i}$ go to infinity, the expression in the text follows.

higher profits, as long as it continues to innovate. But when it is replaced by a new entrant, the cumulated benefits from all its past innovations eventually accrue to consumers. With a continuum of industries, however, this effect is smoothed out in the aggregate, so consumption grows smoothly.[15]

The model endogenously generates firms heterogeneity, as incumbents may lead by different numbers of steps. More precisely, the modal lead size is 1, and greater leads are less frequent:

**Corollary 3**. *In the SMPE,* $\kappa_1 > \kappa_2 > ... > \kappa_{i^*} > 0 = \kappa_{i^*+1} = ... = \kappa_m$.

This follows immediately from (12) and Corollary 2. Measuring firm size by its market value, it appears that the firm size distribution depends not only on the distribution of $i$ but also on the distribution of $j$, the number of innovations achieved in sector $\omega$ by time $t$, since $V_i(j, \omega, t) = V_i g^{j(\omega,t)}$. The variable $j(\omega, t)$ is distributed identically across sectors, with density

$$\frac{e^{-Xt}(Xt)^j}{j!},$$

which eventually decreases with $j$. Taken together, this observation and Corollary 3 imply a skewed firm size distribution, which again is consistent with empirical evidence.[16]

[15] It is now easy to see that the assumption of a linear utility function can be relaxed without affecting the main results. For example, with a concave instantaneous utility function like

$$u(c) = \int_0^\infty \left[\frac{c(t)^{1-\theta} - 1}{1-\theta}\right] e^{-\rho t} dt,$$

where $\rho$ is the rate of time preference and $1/\theta$ is the intertemporal elasticity of substitution, the Euler equation

$$\gamma = \frac{\dot{c}(t)}{c(t)} = \frac{r - \rho}{\theta}$$

provides an increasing relationship between the interest rate and the economy's rate of growth $\gamma$. Proposition 1 provides another, decreasing relationship – which can be obtained by plugging (19) into (14). These two equations can then be simultaneously solved to determine the equilibrium interest rate and growth rate.

[16] See, for instance, the classical work of Steindl (1965), or the more recent contribution by Cabral and Mata (2003). One must caution, however, that in real world much of firm heterogeneity seems to be associated with differences in productivity across firms that are active in the same industry, whereas in our model only incumbents are active.

While other models can explain these first stylized facts, a specific prediction of ours is the negative correlation between firm size (or age) and growth. To see how this emerges in our model, notice that both firm size (measured again by its value $V_i$) and firm average time from birth (i.e., from its first innovation) are positively correlated with $i$. The negative correlation with the growth rate then follows by the next result:

**Corollary 4**. *An incumbent firm's expected rate of growth decreases with the size of its lead, $i$.*

The proof is simple: the expected rate of growth of a firm's value is

$$\frac{x_{\ell_i} g\tilde{V}_{i+1} - \tilde{V}_i}{\tilde{V}_i}.$$

This decreases with $i$ because both $\dfrac{\tilde{V}_{i+1}}{\tilde{V}_i}$ and $x_{\ell_i}$ decrease with $i$. Corollary 4 means that Gibrat's law does not hold in our model, and the pattern of deviation seems consistent with empirical evidence: see for instance Hall (1987), Evans (1987) and, more recently, Rossi-Hansberg and Wright (2005) and Clementi and Hopenhayn (2006).

# 5 Innovation policy and growth

In this section we use the model to address several policy issues. We are especially interested in ascertaining whether policy should favour R&D investment by leaders or by outsiders. Although we focus on R&D taxes and subsidies and on certain specific patent policies, this question may have broader policy implications (for example, in competition policy).

## 5.1 Backward patent protection

In the baseline model we have assumed that nobody can imitate an innovation without infringing the patent that covers it (complete backward protection).

In real life, backward protection is limited both in length and in breadth. To avoid the technical difficulties of modeling a finite patent life in continuous time, we continue to assume that patent life is infinite,[17] focusing on patent breadth. Following Gilbert and Shapiro (1990), we model patent breadth as a cap on the price that the patent-holder is allowed to charge.[18] This captures the notion that when imitation is, to some extent, tolerated, the leader faces stronger competitive pressure from outsiders.

The profit accruing to a leader $\ell_i$ that faces a price cap $\hat{p}_i < \min[\lambda^i, \frac{1}{\alpha}]$ is

$$\hat{\pi}_i = (\hat{p}_i - 1)\alpha^{\frac{1}{1-\alpha}} \hat{p}_i^{-\frac{1}{1-\alpha}}.$$

One can then distinguish between two cases. Backward patent protection is *uniform* when the ratio between the profit collected by the patent-holder, $\hat{\pi}_i$, and the profit with complete protection, $\pi_i$, is constant:

$$\hat{\pi}_i = \mu\pi_i \qquad \text{for all } i \text{ with } \mu \in [0, 1].$$

In this case, the policy-maker chooses $\mu$, and the $\hat{p}_i$s then are uniquely determined. Backward protection is *state-dependent* when the policy-maker chooses the prices $\hat{p}_i$ independently of each other, so that the strength of protection can depend on the size of the leader's technological advantage, as in Acemoglu and Akcigit (2008).

In both cases, the equilibrium is still given by Proposition 1 after replacing the $\pi_i$s with the $\hat{\pi}_i$s. It follows immediately that with uniform patents, strengthening backward protection (i.e., increasing $\mu$) always increases $X$, and hence the economy's rate of growth.[19] Since $X$ increases faster than $\mu$ while

[17] To circumvent these difficulties, one could model patent life as a constant probability that the patent expires, as in Acemoglu and Akcigit (2008). However, this requires the additional assumption that the patents covering all past innovations expire when the patent on the latest innovation does.

[18] For a discussion of alternative interpretations of patent breadth see Denicolò (1996).

[19] However, strengthening backward patent protection clearly worsens static efficiency. Here we do not analyze the optimal resolution to the trade off between static and dynamic efficiency,

the $x_{\ell_i}$s are proportional to $\mu$, it follows also that strengthening backward protection decreases the share of R&D done by leaders. Thus, in this case the growth-enhancing policy favours outsiders more than leaders.

The case of state-dependent patents is different. Since the equilibrium total R&D effort $X$ depends only on $\pi_{i^*}$, only changes in $\hat{p}_{i^*}$ affect the economy's rate of growth, and the qualitative effect is the same as for uniform patents. A decrease in any $\hat{p}_i$ with $i < i^*$ now has no effect on the rate of growth (as long as $i^*$ does not change). However, such a decrease reduces $\hat{\pi}_i$ and hence increases $x_{\ell_i}$, as is clear from (20). This *trickle-down effect*, which has been first identified by Acemoglu and Akcigit (2008), is due to Arrow's replacement effect: all else equal, the lower is the leader's current profit, the greater is its incentive to innovate. Differently from Acemoglu and Akcigit, however, in our model such greater R&D investment by leader $\ell_i$ (when $i < i^*$) is exactly offset by lower R&D investment by outsiders. This crowding-out effect, which operates on a one-to-one basis, means that reducing patent protection granted to leaders that lead by less than $i^*$ steps has no impact on growth.[20]

## 5.2 Forward protection

The baseline model assumes that no innovation infringes on previous patents, so innovators need not obtain any licence from previous patent holders to practice their innovations (no forward patent protection). The patent literature has highlighted the possibility that granting some forward protection may be desirable when innovation is cumulative and intertemporal externalities arise (see Scotchmer, 2004, for an excellent discussion). Thus, let us now assume that

as it inevitably requires an estimate of the elasticity of the supply of inventions, which is notoriously difficult and controversial. In our framework, the analysis is further complicated by the model's transitional dynamics.

[20]However, it can impact social welfare. More precisely, there are three effects of lower $\hat{p}_i$'s on welfare. First, lower prices directly improve static allocative efficiency. However, by increasing the share of R&D done by leaders, they increase the fraction of industries in which the leader leads by more than $i$ steps, which is bad for allocative efficiency. Finally, the increase in the share of R&D done by leaders improves aggregate R&D efficiency.

innovation $j$ in industry $\omega$ infringes on the patent covering innovation $j-1$ in that industry (but, for simplicity, not on that covering innovation $j-2$). This implies that any successful outsider must pay a licensing fee to the previous incumbent, whereas no payment is due if the leader innovates repeatedly.

We take the size of the licensing fee as our policy variable.[21] More precisely, we assume that the licensing fee to be paid by innovator $j$ is $\rho g^j$, where $\rho$ is set by policy (e.g., by the courts in case of patent infringement), whereas the term $g^j$ guarantees that the fee is a constant share of the value of the innovation – a necessary condition for the existence of a steady state.[22]

With forward patent protection, the Bellman equation for leader $\ell_i$ becomes

$$rV_i = \max_{x_{\ell_i}} \left[\pi_i - X_i V_i + \rho X_{O_i} + g x_{\ell_i} V_{i+1} - c_\ell x_{\ell_i}\right],$$

since the current leader obtains a payment of $\rho g^j$ when it is replaced. Likewise, the free entry condition becomes

$$gV_1 - \rho - c_o \leq 0,$$

since a successful outsider that obtains innovation $j$ must pay the licensing fee $\rho g^j$. Lemma 3 continues to hold with $c_o + \rho$ replacing $c_o$, so condition (17) now becomes

$$\frac{c_o + \rho}{c_\ell} < \frac{g}{g-1}.$$

This places an upper bound on $\rho$: when $\rho$ exceeds this threshold, outsiders do not invest in R&D. Focusing on values of $\rho$ that lie strictly below the threshold,

[21] In a model of variable innovation size, O'Donoghue and Zweimuller (2004) posit that the policymaker sets the size of the innovation beyond which there is no patent infringement. Another difference between their analysis and ours is that in case of infringement they allow the successive patent holders to collude, whereas we assume that after paying the licensing fee, the latest innovator continues to compete against the penultimate one.

[22] Like backward protection, forward patent protection could be state-dependent in our model, in which case $\rho$ would depend on $i$. However, this complicates the analysis considerably, since the free-entry condition would now depend on $i$. Therefore, we leave this extension for future work. See Acemoglu and Akcigit (2008) for an analysis of state-dependent forward patent protection in a different framework.

it follows as in Proposition 1 that in equilibrium outsiders must invest in R&D, so the free entry condition must hold as an equality:

$$V_1 = \frac{c_o + \rho}{g} (\equiv \hat{V}_1).$$

Since conditions (8) do not change, we can again calculate the solution to (8) recursively, obtaining now

$$\hat{V}_i = \tilde{V}_i + \frac{\rho}{g^i}.$$

Finally, equations (10) now become

$$V_i = \frac{\pi_i + \rho X_{O_i}}{r + X_{O_i}}.$$

From the above discussion it is clear that Proposition 1 continues to hold with the following changes (the structure of the proof is identical to that of Proposition 1 and thus the proof is omitted).

**Proposition 2** *For each* $0 \leq \rho < \frac{g}{g-1} c_\ell - c_o$ *there is a unique Strong Markov Perfect Equilibrium outcome. In equilibrium,*

$$X = \frac{\pi_{i^*} - r\hat{V}_{i^*}}{\hat{V}_{i^*} - \rho},$$

*where now*

$$i^* = \arg \max_{i=1,2,\ldots,m} \left[ \frac{\pi_i - r\hat{V}_i}{\hat{V}_i - \rho} \right].$$

*Leader* $\ell_{i^*}$ *does not invest, so no leader ever leads by more than* $i^*$ *steps. For* $i \leq i^*$ *the leaders' R&D efforts are*

$$x_{\ell_i} = \frac{\pi_{i^*} - r\hat{V}_{i^*}}{\hat{V}_{i^*} - \rho} - \frac{\pi_i - r\hat{V}_i}{\hat{V}_i - \rho},$$

*so that* $V_i = \hat{V}_i$ *for* $i = 1, ..., i^*$.

Using Proposition 2, we can easily trace out the effects of an increase in forward patent protection $\rho$. It can be checked that increasing $\rho$ increases the length of leadership cycles $i^*$ and the share of R&D conducted by leaders. The effect on the growth rate, however, is ambiguous. To be precise, we have

**Corollary 4**. *An increase in forward protection $\rho$ increases the economy's rate of growth if and only if*

$$\frac{X}{X+r} > \frac{1}{g^{i^*}}, \tag{21}$$

*or, in terms of exogenous variables only, if and only if*

$$\frac{\pi_{i^*}}{\tilde{V}_{i^*}} > \frac{r}{1-g^{-i^*}}.$$

When $i^* = 1$, condition (21) can never be met. In other words, in the familiar leapfrogging equilibrium granting forward patent protection is always bad for innovation and growth.[23] The economic intuition is simple: each innovator obtains $\rho g$ when it is replaced by the occurrence of the next innovation, but it must pay the licensing fee $\rho$ when it succeeds. The net effect on the incentive to innovate is unambiguously negative because the two payments are the same in magnitude, except that the later term is higher because of growth of the economy, but must be discounted. The transversality condition (15) implies that discounting prevails over growth, so the effect of granting forward protection is negative.

When $i^* > 1$, however, increasing $\rho$ can be good for growth. The intuitive reason is that in equilibrium the incentive to innovate is the same as if a new leader reached the subsequent $i^* - 1$ innovations instantaneously, as we have seen above. When such leader is eventually replaced, it now obtains $\rho g^{i^*}$, but this payoff is discounted only for one period (as the length of the other $i^* - 1$ periods is arbitrarily small). This magnifies the positive effect of $\rho$ on growth,

[23] This result accords with O'Donoghue and Zweimuller (2004). They actually find that forward protection can be good for growth if it facilitates collusion. Ruling out collusion, however, the effect is unambiguously negative in their model.

without changing the negative effect. As a result, the net effect can now be positive even if the transversality condition holds. The greater is the maximum length of leadership cycles, the more likely the effect of forward protection on growth is positive.

### 5.3 R&D policy

Consider now R&D policy. We analyze a balanced-budget policy move: the policymaker taxes R&D expenditures by the leaders at rate $\tau$ and uses the revenue to subsidize R&D expenditures by outsiders at rate $\sigma$. When $\tau$ and $\sigma$ are negative, the leaders' R&D is subsidized, the outsiders' taxed.

The effective unit R&D costs of leaders and outsiders then become $c'_\ell = c_\ell(1+\tau)$ and $c'_o = c_o(1-\sigma)$. The government budget constraint is

$$\tau c_\ell \sum_{i=1}^{i^*} \kappa_i x_{\ell_i} = \sigma c_o \left( X - \sum_{i=1}^{i^*} \kappa_i x_{\ell_i} \right).$$

Notice that R&D taxes and subsidies affect only the unit R&D costs. Hence, replacing $c_\ell$ and $c_o$ with $c'_\ell$ and $c'_o$, respectively, all of our results hold *verbatim*. Consider, then, the effect of an increase in $\tau$ on the economy's rate of growth, assuming that

$$\frac{c_o(1-\sigma)}{c_\ell(1+\tau)} < \frac{g}{g-1}$$

so that Proposition 1 applies. We have:

**Proposition 3** *Taxing R&D expenditure by leaders to subsidize R&D expenditure by outsiders reduces the economy's rate of growth.*

Proposition 3 suggests that R&D policy should favour leaders, not outsiders.[24] This objective could be achieved also with other policy tools, such as competition policy.

[24] While reducing $\tau$ is good for growth, it is however bad for allocative efficiency. The reason is that decreasing $\tau$ increases the $x_{\ell_i}$s, increasing the fraction of industries in which the leader leads by more than one step, and hence prices are higher. However, the increase in the share of R&D done by the leaders improves aggregate R&D efficiency.

# 6 Indeterminacy and equilibrium selection

So far we have focused on the unique SMPE, where the total R&D effort $X_i$ is independent of the size of the incumbent's lead $i$. Dropping condition (16), however, other MPE may arise. In this section we demonstrate that there is a continuum of MPE and discuss the source of this indeterminacy. We also argue that all the MPE exhibit a pattern of leadership cycling, but the SMPE is the most natural solution.

To illustrate the indeterminacy, let us focus on the case of "quasi-drastic" innovations, where it takes a two-step lead to engage in monopoly pricing (that is, $m = 2$). Then, one can easily see that there exists a continuum of MPE. In all of these equilibria, we have $x_{\ell_2} = 0$, $X_{O_1} = \frac{g\pi_1}{c_o} - r \geq 0$ and $X_{O_2} = X_2 = \frac{g^2\pi_M}{c_o+gc_\ell} - r \geq 0$. However, $x_{\ell_1}$ can range from 0 to $\infty$, and, correspondingly, $X = \kappa_1 X_1 + \kappa_2 X_2$ can range from $X_{O_1}$ to $2X_{O_2}$. (The reason why $X$ stays finite in the limit is that $\kappa_1$ tends to zero, and $\kappa_1 X_1$ tends to $X_{O_2}$).[25]

A similar problem arises when $m > 2$, but now several variables (i.e., all the $x_{\ell_i}$'s for $i < m$) can be indeterminate for certain parameter values. In all MPE, however, the outsiders' aggregate R&D effort is fully determined.[26] Hence, the indeterminacy arises only when, with constant returns to R&D, both leaders and outsiders simultaneously invest in R&D.

Mathematically, the reason why multiple MPE equilibria exist is that for each $i < m$ the model has two unknowns, $X_{O_i}$ and $x_{\ell_i}$, but while there is a

[25] To show that these are, indeed, MPE's, it suffices to set $V_1 = \frac{c_o}{g}$ and $V_2 = \frac{c_o+gc_\ell}{g^2}$ and check that all the necessary and sufficient conditions in Lemma 1 hold. In particular, $V_1 = \frac{c_o}{g}$ implies that outsiders are indifferent between investing and not; $V_2 = \frac{c_o+gc_\ell}{g^2}$ then implies that condition (8) holds as an equality for $\ell_1$, so $\ell_1$ too is indifferent between investing in R&D or not. Thus, $x_{\ell_1}$ can take any non-negative value. For example, the familiar leapfrogging solution where $x_{\ell_1} = 0$ is an MPE. As $x_{\ell_1}$ increases, $\kappa_2$ increases and hence $X = \kappa_1 X_1 + \kappa_2 X_2$ can change without impairing the equilibrium conditions. Thus, we have a continuum of MPE.

[26] That is, greater investment in R&D by leaders does not crowd out investment by outsiders. This property depends on the assumption of linear utility: with a concave instantaneous utility function there would be the indirect crowding out caused by the increase in the equilibrium interest rate associated with the increase in $X$ (a general equilibrium effect).

separate equilibrium condition for each leader $\ell_i$, there is only one free-entry condition for all outsiders, since the reward to a successful outsider is always $V_1$. This implies that there are not enough equilibrium conditions to pin down a unique equilibrium.

From an economic point of view, the indeterminacy is due to the forward-looking nature of the equilibrium. This creates the possibility of self-fulfilling prophecies conditioned on extraneous variables, as discussed in subsection 3.2. By focusing on MPE, we have ruled out the possibility that the equilibrium may be conditioned on $t$, $j$ or $\omega$. But in our framework $i$ is a payoff-relevant state variable, so the MPE can depend on $i$.

However, we contend that there are good reasons to focus on the (unique) SMPE, and that, at any rate, all the MPE exhibit a pattern of leadership cycling. One reason why the SMPE is more likely to prevail is that the other MPE are implicitly based on "over-pessimistic" or "over-optimistic" expectations. To see why, consider for instance the leapfrogging solution where $X_1 = X = \frac{g\pi_1}{c_o} - r$ and $x_{\ell_1} = 0$. This is *always* a MPE. The intuitive reason is that if $X_{O_2}$ is expected to be sufficiently large, $V_2$ will be small enough that $gV_2 - V_1 - c_\ell$ is negative. This makes it optimal for $\ell_1$ not to invest in R&D and supports the leapfrogging solution as a MPE. In a Strong MPE, however, $X_{O_2}$ cannot be expected to be larger than $X = \frac{g\pi_1}{c_o} - r$,[27] so the leapfrogging solution can satisfy condition (16) only for a certain range of parameter values (to be precise, when $\frac{c_o}{c_\ell} < \frac{g\pi_1}{g\pi_2 - \pi_1}$). From this viewpoint, the role of condition (16) is to rule out over-pessimistic (as well as over-optimistic) beliefs that are conditioned on the payoff relevant variable $i$, and thus are not ruled out by Markov perfection, but are economically arbitrary.

Another reason for focusing on SMPE is based on the idea that the equi-

[27] Notice that condition (16) requires not only that the aggregate R&D effort $X_i$ is independent of $i$ for values of $i$ that are actually reached in equilibrium, but also for those values of $i$ that cannot ever be reached because $x_{\ell_s} = 0$ for some $s < i$.

librium should be robust to the introduction of a small degree of decreasing returns into the R&D technology at the industry level. To see why this robustness criterion yields condition (16), assume that the instantaneous probability of success by a generic firm $s$ is $\frac{x_s}{X_i}h(X_i)$, where $h(.)$ is an increasing and concave function that represents the aggregate probability of success. The case of constant returns corresponds to $h(X_i) = X_i$. With this R&D technology, the free-entry condition by outsiders requires that

$$\frac{\frac{x_{o_i}}{X_i}h(X_i)V_1(j) - c_o g^{j-1}x_{o_i}}{r + h(X_i)}$$

is non-positive and is zero if outsiders invest in R&D. Thus, in a free entry equilibrium the following condition must hold:

$$\frac{h(X_i)}{X_i} = \frac{c_o}{gV_1}.$$

Since the right-hand side of this equation is constant, $X_i$ must be independent of $i$. This property holds for any concave function $h(X_i)$, and hence it must hold also in the equilibrium that is the limit of the decreasing returns equilibria as $h(X_i) \to X_i$.

In any case, the next Proposition shows that in all MPE leaders necessarily stop investing in R&D altogether after a finite number of successive innovations that cannot be greater than $m$, so all MPE exhibit a pattern of leadership cycling:

**Proposition 4** *In any MPE, $x_{\ell_i} = 0$ for some $i \leq m$. Thus, no leader ever leads by more than $m$ steps.*

# 7 Conclusion

We have analyzed a tractable quality-ladder model of endogenous growth where the latest innovator is more efficient than any outsider in conducting the research for the next innovation. The model generates a rich set of predictions that can help reconcile theory and empirical evidence. It generates stochastic leadership cycles where both the industry leader and outsiders simultaneously invest in R&D in the same industry. Leaders can innovate several successive times, but their share in the total R&D done decreases as the size of their technological lead increases. This dynamics endogenously generates a skewed firms size distribution and implies a deviation from Gibrat's law that is confirmed by the empirical evidence. We have also used the model to analyze the effect of various patent and R&D policies on growth, showing that in some cases the growth-enhancing policy must favour R&D investment by the leader, not outsiders.

To isolate the source of these new predictions, we have retained the same basic modelling structure as first-generation quality-ladder models, assuming only that leaders have an R&D cost advantage over outsiders. But we believe that leadership cycles can be reproduced in the many models that have extended early Schumpeterian theories of endogenous growth in all sorts of ways. For example, scale effects can be eliminated as in Segerstrom (2007). By modeling innovations of variable size, one could account for an additional important source of firms heterogeneity (Minniti et al., 2009). Assuming that competition is less intense than Bertrand competition, one could produce equilibria in which not only incumbents can innovate repeatedly, but they are also displaced only gradually when they stop innovating, as in Denicolò and Zanchettin (2009).

It would also be interesting to complement our policy analysis by studying the effect of R&D and patent policy on social welfare, not only on growth. This requires finding the optimal resolution to the trade off between static and dynamic efficiency, which depends on the elasticity of the supply of inventions. To address this issue, it is therefore essential to allow for decreasing returns to R&D at the industry level, which however complicates the analysis. An additional complication is the model's transitional dynamics. Although analytical results seem to be out of reach, it should be possible to calibrate the model and solve for the optimal policy numerically. This analysis is left for future work.

# Appendix

*Proof of Lemma 1.* Sufficiency is obvious, so we focus on necessity. Notice that for all $i = 1, 2, ..., m-1$ the maximand in the Bellman equation can be rewritten as

$$\pi_i - (X_{O_i} + x_{\ell_i}) V_i + g x_{\ell_i} V_{i+1} - c_\ell x_{\ell_i},$$

and hence is linear in $x_{\ell_i}$ for any given $X_{O_i}$. As a consequence, the following inequality must hold

$$gV_{i+1} - V_i - c_\ell \leq 0,$$

for otherwise $V_i$ is unbounded, which cannot happen equilibrium. Moreover, $x_{\ell_i}$ can be positive only if condition $gV_{i+1} - V_i - c_\ell \leq 0$ holds as an equality, whence the complementary slackness conditions

$$x_{\ell_i} (gV_{i+1} - V_i - c_\ell) = 0$$

follow. This proves equation (8).

For $i = m$, the analogous condition is

$$(g-1)V_m - c_\ell \leq 0,$$

with the associated complementary slackness condition $x_{\ell_m} [(g-1)V_m - c_\ell] = 0$, whence (9) follows.

Finally, from the complementary slackness conditions it follows immediately that the system of Bellman equations (6) reduces to

$$\begin{aligned} rV_1 &= \pi_1 - X_{O_1} V_1 \\ rV_2 &= \pi_2 - X_{O_2} V_2 \\ &\ldots \\ rV_m &= \pi_M - X_{O_m} V_m. \end{aligned}$$

These equations then can be solved to get (10). ■

*Proof of Lemma 2.* Neglecting the first dynamic equation, which is implied by the others, the dynamical system becomes

$$
\begin{aligned}
\dot{\kappa}_2 &= (1-\kappa_2-\kappa_3-...-\kappa_m)x_{\ell_1}-\kappa_2 X_2 \\
\dot{\kappa}_3 &= \kappa_2 x_{\ell_2}-\kappa_3 X_3 \\
& ... \\
\dot{\kappa}_m &= \kappa_{m-1}x_{\ell_{m-1}}-\kappa_m X_m
\end{aligned}
$$

where we have used the fact that $x_{\ell_m}=0$ (see Proposition 1 below). The Lemma could be proved by applying the Routh-Hurwitz conditions to the characteristic equation of the associated autonomous system, but since this mechanical proof is very tedious we provide a more intuitive proof. This starts from the observation that since $\dot{\kappa}_i \geq 0$ when $\kappa_i = 0$, none of the $\kappa_i$s can become negative. Moreover, since $\kappa_2+...+\kappa_m = 1-\kappa_1$, all the $\kappa_i$ are necessarily bounded, so the characteristic equation cannot have roots with positive real parts. To complete the proof, it then suffices to show that there cannot be zero or imaginary roots. To show this, notice that the characteristic polynomial $C_m$ is recursively defined as $C_2 = \lambda + (X_2 + x_{\ell_1})$ and

$$C_{i+1}(\lambda) = (\lambda + X_{i+1})C_i(\lambda) + x_{\ell_i}x_{\ell_{i-1}}...x_{\ell_1}.$$

This implies that all the coefficients of the characteristic polynomial are strictly positive. As is well know, this suffices to rule out zero or imaginary roots, implying that all roots of the characteristic equation have negative real parts. ■

*Proof of Lemma 3.* Suppose to the contrary that outsiders invest in R&D. By (8), this requires that

$$V_1 = \frac{c_o}{g}.$$

Clearly, in equilibrium we must have $V_2 \geq V_1$, since a firm leading by two steps can always mimic a firm leading by one step only. This condition implies

$$\begin{aligned} gV_2 - V_1 - c_\ell &\geq (g-1)V_1 - c_\ell \\ &= \frac{g-1}{g}c_o - c_\ell. \end{aligned}$$

It follows that $\frac{c_o}{c_\ell} > \frac{g}{g-1}$ implies that $gV_2 - V_1 - c_\ell > 0$. But this violates (8). This contradiction establishes that when $\frac{c_o}{c_\ell} > \frac{g}{g-1}$ there is no equilibrium in which outsiders invest in R&D. ■

*Proof of Lemma 4.* For simplicity we treat $i$ as a continuous variable. Direct calculation shows that the derivative of the ratio $\frac{\pi_i}{\tilde{V}_i}$ with respect to $i$ has the same sign as

$$H = (1-\alpha)\left[\frac{c_o}{c_\ell}(g-1) - g\right] + g^i - \alpha g^{\frac{i}{\alpha}}.$$

When $i = 0$, $H$ reduces to $(1-\alpha)\left(\frac{c_o}{c_\ell} - 1\right)(g-1)$ and hence is positive. Thus, the ratio $\frac{\pi_i}{\tilde{V}_i}$ initially increases with $i$. But $\frac{dH}{di} = \left(g^i - g^{\frac{i}{\alpha}}\right)\log g < 0$, implying that $\frac{\pi_i}{\tilde{V}_i}$ is quasi-concave in $i$ if $i$ is treated as a continuous variable. The Lemma then follows immediately. ■

*Proof of Proposition 1.* To prove the proposition, we show that the list of variables

$$\begin{aligned} X &= \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - r, \\ x_{\ell_i} &= \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - \frac{\pi_i}{\tilde{V}_i} \text{ for } i = 1, ..., i^* \\ x_{\ell_i} &= 0 \text{ for } i = i^*, ..., m \\ V_i &= \tilde{V}_i \text{ for } i = 1, ..., i^* \\ V_i &= \frac{\pi_i}{\pi_{i^*}}\tilde{V}_{i^*} \text{ for } i = i^*, ..., m \end{aligned}$$

satisfies the conditions stated in Lemma 1; condition (16) is obviously met.

Notice first of all that since by definition of $i^*$ we have $\frac{\pi_{i^*}}{\tilde{V}_{i^*}} \geq \frac{\tilde{\pi}_1}{\tilde{V}_1}$, it follows

$$X \geq \frac{\tilde{\pi}_1}{\tilde{V}_1} - r > 0$$

where the last inequality holds by condition (18). Next, it is clear that $x_{\ell_i} \geq 0$ for all $i = 1, ..., m$, again by definition of $i^*$. Finally, notice that

$$X_{O_i} = \frac{\pi_i}{\tilde{V}_i} - r \geq \frac{\tilde{\pi}_1}{\tilde{V}_1} - r > 0 \text{ for } i = 1, ..., i^*,$$

by quasi-concavity of $\frac{\pi_i}{\tilde{V}_i}$ and (17), while obviously $X_{O_i} = X > 0$ for $i = i^*, ..., m$. Thus, all R&D investments are non negative.

By construction, at the candidate equilibrium the outsiders' zero-profit condition (7) and condition (8) for $i = 1, ..., i^* - 1$ hold as an equality. It is also immediate to check that condition (10), which becomes

$$V_i = \frac{\pi_i}{r + \frac{\pi_i}{\tilde{V}_i} - r} = \tilde{V}_i,$$

is met for $i = 1, ..., i^*$. Thus, it remains to check conditions (8) for $i = i^*, ..., m-1$, condition (9), and condition (10) for $i = i^* + 1, ..., m$.

Since $X_{O_i} = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - r$ for $i = i^*, ..., m$, condition (10) is immediate. Next, we show that conditions (8) hold as strict inequalities. It suffices to show that

$$gV_{i+1} - V_i < g\tilde{V}_{i+1} - \tilde{V}_i = c_\ell$$

when $i = i^*, ..., m - 1$. At the candidate equilibrium, this inequality becomes

$$\tilde{V}_{i^*} \left( g\frac{\pi_{i+1}}{\pi_{i^*}} - \frac{\pi_i}{\pi_{i^*}} \right) < g\tilde{V}_{i+1} - \tilde{V}_i,$$

which rewrites as

$$g \left( \frac{\pi_{i+1}}{\pi_{i^*}} - \frac{\tilde{V}_{i+1}}{\tilde{V}_{i^*}} \right) < \frac{\pi_i}{\pi_{i^*}} - \frac{\tilde{V}_i}{\tilde{V}_{i^*}},$$

or

$$g\pi_{i+1} \left( \frac{\tilde{V}_{i+1}}{\pi_{i+1}} - \frac{\tilde{V}_{i^*}}{\pi_{i^*}} \right) > \pi_i \left( \frac{\tilde{V}_i}{\pi_i} - \frac{\tilde{V}_{i^*}}{\pi_{i^*}} \right).$$

This inequality always holds since $g > 1$, $\pi_{i+1} > \pi_i$, and $\frac{\tilde{V}_{i+1}}{\pi_{i+1}} > \frac{\tilde{V}_i}{\pi_i}$ by quasi-concavity of $\frac{\pi_i}{\tilde{V}_i}$ (see the proof of Lemma 4) and the fact that $i \geq i^*$.

Finally, consider condition (9). We must show that

$$(g-1)\frac{\pi_m}{\pi_{i^*}}\tilde{V}_{i^*} < c_\ell.$$

Notice first of all that

$$(g-1)\tilde{V}_m < c_\ell,$$

since this inequality rewrites as

$$1 - \left[g - (g-1)\frac{c_o}{c_\ell}\right] g^{-m} < 1$$

which is obviously true when $\frac{c_o}{c_\ell} < \frac{g}{g-1}$. Since

$$\frac{\pi_m}{\pi_{i^*}}\tilde{V}_{i^*} \leq \tilde{V}_m$$

by definition of $i^*$, if it clear that condition (14) also holds as a strict inequality. This completes the proof that the candidate equilibrium is, indeed, a Strong MPE.

Now we turn to uniqueness. Proposition 4 below shows that in any MPE, $x_{\ell_m} = 0$. Denote by $\bar{\imath}$ the smallest integer such that $x_{\ell_{\bar{\imath}}} = 0$. Proposition 1 implies also that outsiders must do some research, so condition $V_1 = \tilde{V}_1 = \frac{c_o}{g}$ must hold. One can then iteratively solve conditions (8), which must then hold as equalities for $i = 1, ..., \bar{\imath} - 1$, obtaining

$$V_i = \tilde{V}_i \text{ for } i = 1, ..., \bar{\imath}.$$

Since $x_{\ell_{\bar{\imath}}} = 0$, equation (10) at $i = \bar{\imath}$ yields:

$$X_{\bar{\imath}} = \frac{\pi_{\bar{\imath}}}{\tilde{V}_{\bar{\imath}}} - r.$$

Then, in a SMPE we must have

$$X_i = \frac{\pi_{\bar{\imath}}}{\tilde{V}_{\bar{\imath}}} - r$$

for all $i = 1, ..., m$. Now suppose that there is an equilibrium in which $\bar{\imath} \neq i^*$. We show that this assumption leads to a contradiction.

We must distinguish between two cases. If $\bar{\imath} > i^*$, then $V_{i^*} = \tilde{V}_{i^*}$. Equation (10) at $i = i^*$ gives

$$X_{O_{i^*}} = \frac{\pi_{i^*}}{\tilde{V}_{i^*}} - r > X_{i^*},$$

which is impossible as it would imply $x_{\ell_{i^*}} < 0$. If $\bar{\imath} < i^*$, since $X_{O_{\bar{\imath}+1}} \leq X_{\bar{\imath}+1} = \frac{\pi_{\bar{\imath}}}{\tilde{V}_{\bar{\imath}}} - r$, from equation (10) we obtain the following inequality

$$V_{\bar{\imath}+1} \geq \frac{\pi_{\bar{\imath}+1}}{\pi_{\bar{\imath}}} \tilde{V}_{\bar{\imath}}.$$

On the other hand by quasi-concavity of $\frac{\pi_i}{\tilde{V}_i}$ and $\bar{\imath} < i^*$ we get

$$\frac{\pi_{\bar{\imath}}}{\tilde{V}_{\bar{\imath}}} < \frac{\pi_{\bar{\imath}+1}}{\tilde{V}_{\bar{\imath}+1}}.$$

Combining these two inequalities we finally get

$$V_{\bar{\imath}+1} > \tilde{V}_{\bar{\imath}+1},$$

violating condition (8) for $i = \bar{\imath}$.

This contradiction shows that $\bar{\imath}$ must be equal to $i^*$, which implies that the equilibrium is unique. ■

*Proof of Corollary 1.* Again, we treat $i$ as a continuous variable. The ratio $\frac{\pi_i}{\tilde{V}_i}$ achieves its maximum at $i^*$ where

$$H = (1-\alpha)\left[\frac{c_o}{c_\ell}(g-1) - g\right] + g^i - \alpha g^{\frac{i}{\alpha}} = 0.$$

Since $\frac{dH}{di} = \left(g^i - g^{\frac{i}{\alpha}}\right) \log g < 0$, by implicit differentiation we have that $\frac{\partial i^*}{\partial \frac{c_o}{c_\ell}}$ has the same sign as $\frac{\partial H}{\partial \frac{c_o}{c_\ell}} = (1-\alpha)(g-1) > 0$. Next, notice that when $\frac{c_o}{c_\ell} \to \frac{g}{g-1}$, $H \to g^i - \alpha g^{\frac{i}{\alpha}}$, so equation $H = 0$ becomes

$$g^{\frac{1-\alpha}{\alpha} i^*} = \frac{1}{\alpha},$$

or

$$\lambda^{i^*} = \frac{1}{\alpha},$$

which implies $i^* = m$ by the definition of $m$. ∎

*Proof of Proposition 3.* Suppose for the time being that a change in $\tau$ does not change $i^*$. Since R&D policy does not affect the $\pi_i$s, we have

$$\text{sign}\left[\frac{dX}{d\tau}\right] = \text{sign}\left[-\frac{d\tilde{V}_{i^*}}{d\tau}\right].$$

Now $\tilde{V}_{i^*}$ is given by

$$\tilde{V}_{i^*} = \frac{c_o(1-\sigma) + c_\ell(1+\tau)(g + g^2 + ... + g^{i^*-1})}{g^{i^*}}.$$

To proceed, we now show that in equilibrium

$$\frac{\sum_{i=1}^{i^*} \kappa_i x_{\ell_i}}{X} = 1 - \kappa_1,$$

i.e., the share of R&D conducted by leaders is equal to the share of industries in which the leader leads by more than one step. To show this, notice that $x_{\ell_{i^*}} = 0$, so from (12) we get

$$\frac{\sum_{i=1}^{i^*} \kappa_i x_{\ell_i}}{X} = \kappa_1 \frac{x_{\ell_1}}{X} + \kappa_2 \frac{x_{\ell_2}}{X} + ... + \kappa_{i^*-1} \frac{x_{\ell_{i^*-1}}}{X}.$$

From (12) one gets also $\kappa_{i+1} = \frac{x_{\ell_i}}{X}\kappa_i$, so the above equation rewrites as

$$\begin{aligned} \frac{\sum_{i=1}^{i^*} \kappa_i x_{\ell_i}}{X} &= \kappa_2 + \kappa_3 + ... + \kappa_{i^*} \\ &= 1 - \kappa_1. \end{aligned}$$

Using this result, the government's budget constraint can be rewritten as

$$\sigma = \frac{c_\ell}{c_o} \frac{1-\kappa_1}{\kappa_1} \tau.$$

Totally differentiating $\tilde{V}_{i^*}$ with respect to $\tau$, taking into account that $s$ is given by the above expression, we get

$$\frac{d\tilde{V}_{i^*}}{d\tau} = \frac{c_\ell g^{-i^*}}{\kappa_1} \left\{ \left[\kappa_1 \left(1 + g + ... + g^{i^*-1}\right) - 1\right] + \frac{t}{\kappa_1} \frac{d\kappa_1}{d\tau} \right\}$$

The term inside square brackets is positive since

$$\kappa_1 = \frac{1}{1 + \frac{x_{\ell_1}}{X} + \frac{x_{\ell_1} x_{\ell_2}}{X^2} + ... + \frac{x_{\ell_1} ... x_{\ell_{i^*-1}}}{X^{i^*-1}}} \geq \frac{1}{i^*}$$

and $\left(1 + g + ... + g^{i^*-1}\right) \geq i^*$, whence it is clear that

$$\kappa_1 \left(1 + g + ... + g^{i^*-1}\right) \geq 1,$$

with a strict inequality if $i^* \geq 2$. The second term inside curly brackets is also obviously positive, since an increase in $\tau$ with a corresponding increase in $\sigma$ raises the leaders' R&D cost and reduces the outsiders', and this must increase the share of R&D done by outsiders. Thus, the direct effect of an increase in $\tau$ on the rate of growth, holding $i^*$ constant, is negative.

To complete the proof, notice that $X$ depends on $\tau$ also through $i^*$, as $i^*$ jumps down at certain critical points where

$$\frac{\pi_{i^*}}{\tilde{V}_{i^*}} = \frac{\pi_{i^*-1}}{\tilde{V}_{i^*-1}}$$

as $\tau$ increases. Thus, $X$ is differentiable with respect to $\tau$ only piecewise. But $X$ is continuous in $\tau$ at those critical points. As a result, if the partial derivative of $\tilde{V}_{i^*}$ with respect to $\tau$ holding $i^*$ constant is positive, as it indeed is, the aggregate hazard rate is necessarily monotonically decreasing in $\tau$. ■

*Proof of Proposition 4.* We first show that in equilibrium outsiders invest in R&D. Assume to the contrary that they do not. Then, from (10) we get

$$V_1 = \frac{\pi_1}{r}.$$

Given inequality (18), this implies

$$V_1 > \frac{c_o}{g},$$

which however violates the free entry condition (7). This contradiction proves the claim. Next we show that there is no equilibrium in which outsiders and all leaders $\ell_i$ with $i = 1, ..., m$ simultaneously invest in R&D. (This generalizes a result originally obtained by Segerstrom and Zolnierek (1999), who showed that when $m = 1$ there is no equilibrium in which the leader and outsiders simultaneously invest in R&D.) To show this, notice that $x_{\ell_m} > 0$ implies $V_m = \frac{c_\ell}{g-1}$ by condition (9). Similarly, under the assumption that $x_{\ell_i} > 0$ for all $i = 1, 2, ...m-1$ one can solve the complementary slackness conditions in (8) recursively, obtaining $V_1 = V_2 = ... = V_m = \frac{c_\ell}{g-1}$. On the other hand, if outsiders's R&D investment is positive, one must have $V_1 = \frac{c_o}{g}$. But equations $V_1 = \frac{c_\ell}{g-1}$ and $V_1 = \frac{c_o}{g}$ cannot simultaneously hold if $\frac{c_o}{c_\ell} < \frac{g}{g-1}$. Since outsiders invest in R&D, there is at least an $i$ such that $x_{\ell_i} = 0$, whence the result immediately follows. ■

# References

[1] Acemoglu, D. and U. Akcigit, 2008, State-dependent intellectual property rights policy, mimeo, MIT.

[2] Aghion, P., R.W. Blundell, N. Bloom and R. Griffith, 2005, Competition and innovation: An inverted-U relationship, *Quarterly Journal of Economics,* 120, 701-728.

[3] Aghion, P., C. Harris, P. Howitt and J. Vickers, 2001, Competition and growth with step-by-step innovation, *Review of Economic Studies,* 68, 467-492.

[4] Aghion, P. and P. Howitt, 1992, A model of growth through creative destruction, *Econometrica,* 60, 323-351.

[5] Barro, R. J. and X. Sala-i-Martin, 1994, Quality improvements in models of growth, NBER Working Paper No. 4610.

[6] Barro, R. J. and X. Sala-i-Martin, 2004, *Economic growth*, 2nd edition, Cambridge, MA, MIT Press.

[7] Cabral, L. and J. Mata, 2003, On the evolution of the firm size distribution: facts and theory, *American Economic Review* 93, 1075-1090.

[8] Chu, A. 2009, Effects of blocking patents on R&D: a quantitative DGE analysis, *Journal of Economic Growth*, 14, 55-78.

[9] Clementi, G. and H. Hopenhayn, 2006, A theory of financing constraints and firm dynamics, *Quarterly Journal of Economics*, 121, 229-265.

[10] Cozzi, G., 2007, Self-fulfilling prophecies in the quality ladders economy, *Journal of Development Economics*, 84, 445-464.

[11] Denicolò, V., 1996, Patent races and the optimal combination of patent length and breadth, *Journal of Industrial Economics*, 44, 249-265.

[12] Denicolò, V. and P. Zanchettin, 2009, Competition, market selection and growth, *Economic Journal*, forthcoming.

[13] Evans, D., 1987, Tests of alternative theories of firm growth, *Journal of Political Economy*, 95, 657-674,

[14] Hall, B., 1987, The relationship between firm size and firm growth in the US manufacturing sector, *Journal of Industrial Economics*, 583-606.

[15] Klette, T. and S. Kortum, 2004, Innovating firms and aggregate innovation, *Journal of Political Economy*, 112, 986-1018.

[16] Gilbert, R. and C. Shapiro, 1990, Optimal patent breadth and length, *RAND Journal of Economics*, 21, 106-112.

[17] Griliches, Z., 1990, Patent statistics as economic indicators: A survey, *Journal of Economic Literature*, 28, 1661-1707.

[18] Grossman, G. M. and E. Helpman, 1991a, Quality ladders in the theory of growth, *Review of Economic Studies*, 58, 43-61.

[19] Grossman, G. M. and E. Helpman, 1991b, Quality ladders and product cycles, *Quarterly Journal of Economics*, 106, 557-586.

[20] Lentz R. and D. Mortensen, 2005, Productivity growth and worker reallocation, *International Economic Review*, 46, 731-751.

[21] Lentz R. and D. Mortensen, 2008, An empirical model of growth through product innovation, *Econometrica*, 76, 1317-1373.

[22] Minniti, A., C. Parello and P. Segerstrom, 2008, A Schumpeterian growth model with heterogenous firms, mimeo, University of Bologna.

[23] O' Donoghue, T. and J. Zweimuller, 2004, Patents in a model of endogenous growth, *Journal of Growth Economics*, 9, 81-123.

[24] Rossi-Hansberg, E. and M. Wright, 2005, Firm size dynamics in the aggregate economy, NBER WP 11261.

[25] Scotchmer, S. 2004, *Innovation and incentives*, Cambridge (Mass.), MIT Press.

[26] Segerstrom, P., 2007, Intel economics, *International Economic Review*, 48, 247-280.

[27] Segerstrom, P., Anant, C. and E. Dinopoulos, 1990, A Schumpeterian model of the product cycle, *American Economic Review,* 80, 1077–1091.

[28] Segerstrom, P. and J. Zolnierek, 1999, The R&D Incentives of Industry Leaders, *International Economic Review*, 40, 745-766.

[29] Scherer F.M., 1980, *Industrial market structure and economic performance*, 2nd edition, Chicago, Rand McNally.

[30] Stein, J.C., 1997. Waves of creative destruction: Firm-specific learning-by-doing and the dynamics of innovation, *Review of Economic Studies* 64, 265–288.

[31] J. Steindl, 1965, *Random processes and the growth of firms: A study of the Pareto law*, London, Griffin.