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**MINIMAX REGRET AND STRATEGIC
UNCERTAINTY**

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Minimax regret and strategic uncertainty*

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Abstract

This paper introduces a new solution concept, a minimax regret equilibrium, which allows for the possibility that players are uncertain about the rationality and conjectures of their opponents. We provide several applications of our concept. In particular, we consider price-setting environments and show that optimal pricing policy follows a non-degenerate distribution. The induced price dispersion is consistent with experimental and empirical observations (Baye and Morgan (2004)).

Keywords: minimax regret, rationality, conjectures, price dispersion, auction.

JEL Classification Numbers: C7

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1 Introduction

Strategic situations confront individuals with the delicate tasks of conjecturing other individuals' decisions, that is, they face *strategic uncertainty*. Naturally, individuals might rely on their prior information or knowledge in forming their conjectures. For instance, if an individual knows that his opponents are rational, then he can infer that they will not play strictly dominated strategies.¹ Furthermore, common knowledge in rationality leads to rationalizable conjectures (Bernheim (1984) and Pearce (1984)). Likewise, common knowledge of conjectures, mutual knowledge of rationality and payoffs, and existence of a common prior imply that conjectures form a Nash equilibrium when viewed as mixed strategies (Aumann and Brandenburger (1995)). The aim of this paper is to introduce a new solution concept, called a *minimax regret equilibrium*, which postulates neither mutual or common knowledge in rationality nor common knowledge of conjectures.

Now, if an individual is uncertain about the rationality of his opponents, which conjectures about his opponents' actions should he form? This is a very intricate issue as there is at best little to guide the individual. Admittedly, he can form a subjective probabilistic assessment and play a best response to his assessment. However, any subjective assessment is largely arbitrary, and there is no obvious reasons to favor one assessment over another. Bayesian theory is silent on how to form initial probabilistic assessments (Morris (1995)). Moreover, experimental evidence such as the Ellsberg's paradox suggests that individuals frequently experience difficulties in forming a unique assessment. In this paper, we postulate that "regret" guides individuals in forming probabilistic assessments and, ultimately, in making choices. More precisely, we use the model of *minimax regret with multiple priors*, recently axiomatized by Hayashi (2007a) and Stoye (2007b), to represent the preferences of individuals. In essence, the minimax regret criterion captures the idea that individuals are concerned with foregone opportunities. Before proceeding, we wish to stress that the concern for minimizing maximal regret does not arise from any behavioral or emotional considerations. Rather, it is a consequence of relaxing some of the axioms of subjective

¹In this paper, "knowledge" refers to belief with probability 1 (certainty).

expected utility, in particular the axiom of independence to irrelevant alternatives. Furthermore, the behavior of an individual concerned with regret is indistinguishable from the behavior of an individual who has formed a unique probabilistic assessment (a Bayesian), provided that this assessment is one that leads to maximal regret. Therefore, we may say that minimax regret does indeed guide individuals in forming their probabilistic assessments.

While we include the case where an individual conjectures that any action profile of his opponents might be played, we allow, more generally, for conjectures to be constrained. For instance, conjectures might be constrained to be correct with some minimal probability (i.e., approximate common knowledge in conjectures) or consistent with almost mutual knowledge in rationality. We can now provide an informal definition of our solution concept. A profile of actions is a *minimax regret equilibrium* if the action of a player is optimal given his conjecture about his opponents' play. And his conjecture is consistent with the criterion of minimax regret and initial constraints on conjectures. A parametric variant of special interest is called an ε -*minimax regret equilibrium*. In an ε -minimax regret equilibrium, conjectures are directly related to the equilibrium actions as follows. With probability $1 - \varepsilon$, a player believes (or conjectures) that his opponents will play according to the equilibrium actions while, with probability ε , the player is completely uncertain about his opponents' play. The set of initial assessments is therefore the ε -contamination neighborhood around the equilibrium actions. It transpires that this parameterized version of minimax regret equilibrium is extremely simple, tractable and insightful for economic applications.

We provide several applications of our solution concept. In particular, we consider price-setting environments *à la* Bertrand and characterize their ε -minimax regret equilibria. In such environments, firms face two sources of regret. First, a firm's price might turned out to be lower than the lowest price of its competitors. Had the firm posted a higher price, then its profit would have been higher. Second, a firm's price might turned out to be higher than the lowest price of its competitors, and the regret arises from not serving the market at all. The exposure to these two sources of regret has important economic applications. In any ε -minimax regret equilibrium, firms price above marginal costs and make a positive profit. The intuition is simple.

Since a firm is concerned with foregone opportunities and, in particular, with the possibility that its competitors might price close to the monopoly price, its optimal pricing policy reflects these concerns and, consequently, the price posted is strictly above the marginal cost (in order to minimize (maximal) regret). Moreover, as the number of firms gets larger, the pricing policy converges to the competitive equilibrium. Furthermore, when there are at least three firms competing in the market or costs are heterogeneous, the equilibrium pricing policy exhibits a kink at a price close to the monopoly price. All these equilibrium predictions agree remarkably well with empirical and experimental observations, as documented by Baye and Morgan (2004).

Some related concepts have already appeared in the literature. The closest is Klibanoff's (1996) concept of equilibrium with uncertainty aversion. The essential difference between Klibanoff's concept and ours is that Klibanoff assumes that players conform with the maximin criterion (with multiple priors), whereas we assume that they conform with the minimax regret criterion. Neither we nor Klibanoff assume mutual knowledge in rationality. Consequently, equilibria with uncertainty aversion as well as minimax regret equilibria might not be rationalizable. While conceptually very similar, these two approaches might give very different predictions in games, as we will see. Another solution concept, which adopts the maximin criterion and which is called a belief equilibrium, is offered by Lo (1996). Lo's concept differs from Klibanoff's concept and ours in that it assumes common knowledge in rationality and, consequently, belief equilibria are rationalizable. It would be straightforward to adapt Lo's concept to the minimax regret criterion, but did not choose to do it. Indeed, in a wide range of experiments on the iterated deletion of strictly dominated strategies, a vast majority of subjects seems to be uncertain about the rationality of others (see Camerer (2003, Chapter 5) for a survey and Bayer and Renou (2008) for more recent evidence). Furthermore, a slight doubt about the rationality of others can yield very interesting predictions in economic models e.g., price dispersion, the existence of large and speculative trade (Neeman (1996)), just to name a few. Another closely related concept is the concept of an ambiguous equilibrium (Mukerji (1995)), which adopts the concept of Choquet expected utility and ε -ambiguous beliefs, a close relative to ε -contamination.

While all these approaches are largely complementary and, indeed, share similar axiomatic and epistemic foundations, we advocate in favor of the minimax regret equilibrium. Indeed, the maximin criterion frequently often leads to unsatisfactory predictions in strategic situations. In the price-setting environments mentioned above, the maximin solution implies that sellers price at the marginal cost and make zero profit (the Bertrand-Nash predictions). These predictions sharply contrast not only with our predictions, but also with empirical evidences. Bergemann and Schlag (2005, 2007), Linhart (2001) and Linhart and Radner (1989) make similar observations in other settings such as monopoly pricing and bilateral bargaining. Ultimately, a solution concept should be judged according to its merits in economic applications. We have written this paper with this perspective in mind and hope that its user-friendly exposition will help applied theorists to apply our solution concept fruitfully in future research.

The paper is organized as follows. Section 2 briefly summarizes the axiomatization of the minimax regret criterion and gives the definition of a minimax regret equilibrium. Section 3 proposes several examples to illustrate some interesting features of a minimax regret equilibrium, while Section 4 offers some general properties. Section 4 discusses two possible extensions. Lastly, Section 5 concludes.

2 Minimax Regret Equilibrium

2.1 Regret in Decision Theory

This section provides a brief review of “regret-type” decision rules. We refer the reader to Savage (1951), Milnor (1954), Hayashi (2007), Puppe and Schlag (2007) and Stoye (2007a,b) for in-depth treatments.

Consider a finite set Ω of states of the world and a finite set of outcomes A . For any finite set X , we denote $\Delta(X)$ the set of all probabilities over X , that is, $\Delta(X) = \{\sigma \in \mathbb{R}_+^{|X|} : \sum_{x \in X} \sigma(x) = 1\}$. An *act* f is a mapping from Ω to $\Delta(A)$, the set of lotteries over A , and we denote a *menu* of acts by \mathcal{F} . The primitive of the model is a preference relation $\succeq_{\mathcal{F}}$ over acts belonging to the menu \mathcal{F} .

Minimax regret theory departs from subjective expected utility theory (Anscombe and Aumann (1963)) in two important ways. First, it weakens the axiom of independence to irrelevant alternatives to the axiom of *independence to never-optimal alternatives*. In words, the axiom of independence to never-optimal alternatives states that the act f is preferred to the act g in the menu \mathcal{F} if and only if the act f is preferred to the act g in the menu \mathcal{F}' obtained by complementing \mathcal{F} with never-optimal acts.² Second, it imposes an axiom of *ambiguity aversion* as in Gilboa and Schmeidler (1989), that is, if an individual is indifferent between acts f and g , then he (weakly) prefers the mixing of f and g to either of them. An ambiguity averse individual prefers to hedge bets across states.³ Weakening menu independence together with ambiguity aversion leads to the following numerical representation of $\succeq_{\mathcal{F}}$: there exists a function $U : A \rightarrow \mathbb{R}$ such that for all $f \in \mathcal{F}$, $g \in \mathcal{F}$, $f \succeq_{\mathcal{F}} g$ if and only if

$$\max_{\omega \in \Omega} [\max_{h \in \mathcal{F}} u(h, \omega) - u(f, \omega)] \leq \max_{\omega \in \Omega} [\max_{h \in \mathcal{F}} u(h, \omega) - u(g, \omega)], \quad (1)$$

with $u(f, \omega)$ the expected payoff of the act f in state ω i.e., $u(f, \omega) = \sum_{a \in A} U(a) f(\omega)(a)$. In Eq. (1), the term “ $\max_{h \in \mathcal{F}} u(h, \omega) - u(f, \omega)$ ” is the difference between the highest payoff an individual would have got had he known the state was ω , and the payoff obtained by choosing f . We can therefore *interpret* this term as the regret an individual might experience by choosing f . Consequently, the act f is chosen over the act g if it minimizes the maximal regret, hence the term “minimax regret.” However, it is important to bear in mind that the axiomatization of minimax regret does not rely on any regret-led behaviors; it is rather “as if” individuals wish to minimize their maximal regret. It is also worth noting that no prior beliefs explicitly appear in Eq. (1). Or, more precisely, an individual considers all prior beliefs $\pi \in \Delta(\Omega)$ possible. The theory of minimax regret is therefore a theory of complete uncertainty. Another well-known theory of complete uncertainty is

²An act h added to menu \mathcal{F} is never-optimal if for all states $\omega \in \Omega$, there is some act $f' \in \mathcal{F}$ such that $f'(\omega)$ is preferred to $h(\omega)$ where $f'(\omega)$ and $h(\omega)$ are here identified with constant acts. (Stoye (2007a), p. 4.)

³The axiom of symmetry is also imposed. Loosely speaking, it states that “a preference ordering should not impose prior beliefs by implicitly assigning different likelihoods to different events.” (Stoye (2007, p. 11).)

maximin. Briefly, maximin differs from minimax regret in that it postulates the axiom of independence to irrelevant alternatives, but relaxes the axiom of independence. The axiom of independence states that the act f is preferred to the act g in the menu \mathcal{F} if and only if any mixture of the acts f and h is preferred to the same mixture of the acts g and h in the menu composed of all the mixtures of acts in \mathcal{F} and $\{h\}$.

To illustrate the above concepts, let us consider the example below with three states $\{\omega_1, \omega_2, \omega_3\}$ and $n > 4$.

| | ω_1 | ω_2 | ω_3 |
|-----|------------|------------|------------|
| f | 0 | n | $(n-1)/2$ |
| g | $1/n$ | 1 | $n/2$ |

For n large, both acts f and g are very similar in states ω_1 and ω_3 and, therefore, one might expect that f is preferred over g (since in state ω_2 , f gives a disproportionately larger payoff than g). Indeed, f is preferred over g according to the minimax regret theory. In contrast, g is preferred over f according to the maximin theory. The problem with maximin is the entire focus on the worst states of the world to compare acts, rather than to the states in which the choice of an act is the most consequential as with minimax regret. With minimax regret, the choice between two acts might depend on the menus considered, however. To see this, consider the act $h = (-n, -n, n)$. We have that f is preferred over g in the menu $\{f, g\}$, but g is preferred over f in the menu $\{f, g, h\}$. We do not find this violation of the axiom of independence to irrelevant alternatives disturbing. Experimental evidences indeed suggest that the choice between two acts does depend on the presence or absence of other options (see e.g., Simonson and Tversky (1992)).⁴

To capture the existence of partial prior information, consider a variant of minimax regret theory introduced by Hayashi (2007) (see also Stoye (2007b)) that allows for a restricted set of prior assessments. Relaxing the symmetry axiom, which captures the lack of prior information in the axiomatization of

⁴See Stoye (2007a) for more on this issue.

minimax regret given by (1), it follows that there exists a closed and convex set $\Pi \subseteq \Delta(\Omega)$ of prior beliefs such that $f \succeq_{\mathcal{F}} g$ if and only if

$$\max_{\pi \in \Pi} \sum_{\omega \in \Omega} [\max_{h \in \mathcal{F}} u(h, \omega) - u(f, \omega)] \pi(\omega) \leq \max_{\pi \in \Pi} \sum_{\omega \in \Omega} [\max_{h \in \mathcal{F}} u(h, \omega) - u(g, \omega)] \pi(\omega). \quad (2)$$

Note that when $\Pi = \Delta(\Omega)$, the minimax regret theory with multiple priors reduces to the standard minimax theory à la Savage and when $\Pi = \{\pi^*\}$, it reduces to standard subjective expected utility. In the next section, minimax regret with multiple priors is the cornerstone of our solution concept: minimax regret equilibrium.

2.2 Strategic-Form Games

Let $g := \langle N, (A_i, u_i)_{i \in N} \rangle$ be a strategic-form game with $N := \{1, \dots, n\}$ the set of players, A_i the finite set of actions available to player i , and $u_i : A := \times_i A_i \rightarrow \mathbb{R}$ the payoff function of player i . With a slight abuse of notation, we denote by $G := \langle N, (\Sigma_i, u_i)_{i \in N} \rangle$ the mixed extension of g , that is, $\Sigma_i = \Delta(A_i)$ is the set of mixed actions of player i and $u_i : \Sigma := \times_i \Sigma_i \rightarrow \mathbb{R}$ is the expected payoff function.⁵ Denote $\Sigma_{-i} := \times_{j \in N \setminus \{i\}} \Sigma_j$ and σ_{-i} a generic element of Σ_{-i} . Similarly, a_{-i} is a generic element of A_{-i} . A conjecture π_i for player i is a probability distribution on A_{-i} . We say that the action σ_i^* *dominates* the action σ_i if $u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Sigma_{-i}$, and $u_i(\sigma_i^*, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$ for some $\sigma_{-i} \in \Sigma_{-i}$. An action is *dominant* if it dominates all other actions. Similarly, we say that the action σ_i^* *strictly dominates* the action σ_i if $u_i(\sigma_i^*, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Sigma_{-i}$. And an action is *strictly dominant* if it strictly dominates all other actions. A Nash equilibrium of the game G is a profile of (mixed) actions σ^* such that for all a_i^* in the support of σ_i^* , $u_i(a_i^*, \sigma_{-i}^*) \geq u_i(a_i, \sigma_{-i}^*)$ for all $a_i \in A_i$, for all $i \in N$. In words, σ_i^* is the common belief (conjecture) of player i 's opponents about the pure actions player i will play. And rational players best-reply to their conjectures. This paper proposes a new solution concept for games, which presupposes neither mutual knowledge of rationality nor common knowledge of conjectures. We call this solution concept a minimax regret equilibrium.

⁵Precisely, $u_i(\sigma_1, \dots, \sigma_n) = \sum_{a_1 \in A_1} \dots \sum_{a_n \in A_n} \sigma_1(a_1) \dots \sigma_n(a_n) u_i(a_1, \dots, a_n)$.

We first define player i 's *ex-post regret* associated with any profile of **pure** actions (a_i, a_{-i}) as

$$r_i(a_i, a_{-i}) := \sup_{\hat{a}_i \in A_i} u_i(\hat{a}_i, a_{-i}) - u_i(a_i, a_{-i}), \quad (3)$$

that is, this is the difference between the payoff player i obtains when the profile of actions (a_i, a_{-i}) is played and the highest payoff he might have obtained had he known that his opponents were playing a_{-i} . The regret associated with the profile of mixed strategies (σ_i, σ_{-i}) is then given by:

$$\begin{aligned} R_i(\sigma_i, \sigma_{-i}) &:= \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \sigma_i(a_i) \sigma_{-i}(a_{-i}) r_i(a_i, a_{-i}) \\ &= \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i}) \sup_{\hat{a}_i \in A_i} u_i(\hat{a}_i, a_{-i}) - u_i(\sigma_i, \sigma_{-i}). \end{aligned} \quad (4)$$

Before defining the concept of a minimax regret equilibrium, let us discuss in more details the concept of regret. Regret as axiomatized in decision theory (see above) is defined with respect to a set of states of the world. Extending this framework to choices in strategic situations, we identify the profile of actions a_{-i} chosen by the other players with such a state. The alternative of identifying the opponents' profile of mixed actions with this state of the world is problematic. Indeed, it is fundamental for the motivation of the axioms that the states are not related: changing an outcome in one state should not have an impact on the outcomes in other states. If states were identified with mixed actions, this would not be the case.

For each player i , let $\Pi_i \subseteq \Delta(A_{-i})$ be some compact and convex set of player i 's beliefs about the play of his opponents. It is important to note that although conjectures are about opponents' mixed actions and, therefore, incorporate the fact that players play independently, player i 's belief $\pi_i \in \Pi_i$ about opponents' pure actions might be correlated. To see this, suppose that players 2 and 3 have two actions each, a and b , and player 1 conjectures that they play the mixed action $\sigma_2(a) = \sigma_3(a) = 1$ with probability $1/4$ and the mixed action $\sigma_2(a) = \sigma_2(b) = 1/3$ with probability $3/4$. Although player 1's conjecture puts strictly positive probability to independent mixing only, his belief π_1 over the play of his opponents is given by $\pi_1(a, a) = \pi_1(b, b) = 1/3$

and $\pi_1(a, b) = \pi_1(b, a) = 1/6$, a correlated distribution.⁶ A special case is where player i cannot rule out any mixed action profiles i.e., $\Pi_i = \Delta(A_{-i})$. In this case, we speak of *complete uncertainty*.

A convenient parametrization of the belief sets is the so-called ε -contamination neighborhood around some given profile σ_{-i}^* , with $\varepsilon \in [0, 1]$. In this case, with probability $1 - \varepsilon$, a player believes that his opponents will play $\sigma_{-i}^* \in \Sigma_{-i}$ and, with probability ε , is completely uncertain about the play of his opponents. Formally, we have that $\Pi_i = \Pi_{i\varepsilon}(\sigma_{-i}^*) := \{(1 - \varepsilon)\sigma_{-i}^* + \varepsilon\sigma_{-i}, \sigma_{-i} \in \Delta(A_{-i})\}$. An alternative is to consider the Cartesian product of *independent* ε -contamination neighborhoods that is, $\Pi_i = \Pi_{i\varepsilon}(\sigma_{-i}^*) := \{\times_{j \neq i} \{(1 - \varepsilon)\sigma_j^* + \varepsilon\sigma_j\}, \sigma_{-i} \in \Sigma_{-i}\}$.⁷ Both approaches have their merits and might give very different predictions in economic applications, as we will see later. To maintain focus and simplicity, we assume that ε is the same for each player; this can be easily relaxed. We are now ready to define the concept of a minimax regret equilibrium.

Definition 1 *A profile of strategies $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is a minimax regret equilibrium relative to (Π_1, \dots, Π_N) if for each player $i \in N$, $\sigma_{-i}^* \in \Pi_i$, and*

$$\max_{\sigma_{-i} \in \Pi_i} R_i(\sigma_i^*, \sigma_{-i}) \leq \max_{\sigma_{-i} \in \Pi_i} R_i(\sigma_i, \sigma_{-i}), \quad (5)$$

for all $\sigma_i \in \Sigma_i$.

Several remarks are worth doing. First, a Nash equilibrium $(\sigma_1^*, \dots, \sigma_n^*)$ is a minimax regret equilibrium relative to $\{\sigma_{-1}^*\}, \dots, \{\sigma_{-n}^*\}$. Second, for given belief sets $(\Pi_i)_{i \in N}$, a minimax regret equilibrium might not exist. For instance, consider the prisoner dilemma game, below.

| | | |
|-----|------|------|
| | a | b |
| a | 3, 3 | 0, 4 |
| b | 4, 0 | 1, 1 |

$G0$

There is clearly no minimax regret equilibrium relative to $(\{\delta_a\}, \{\delta_a\})$, where δ_a is the Dirac mass on a . However, we can prove the existence of

⁶See Fudenberg and Levine (1993) for more on this.

⁷This two formulations of ε -contamination correspond to two versions of the concept of independence in multiple prior models.

a minimax regret equilibrium $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ relative to the ε -contamination neighborhoods $(\Pi_{i\varepsilon}(\sigma_{-i}^*))_{i \in N}$, whether neighborhoods are product of independent neighborhoods or not. We call a minimax regret equilibrium relative to ε -contamination neighborhoods an ε -*minimax regret equilibrium*.⁸ Third, an alternative definition would include the belief sets $(\Pi_i)_{i \in N}$ as part of the equilibrium. Since we do not assume explicit randomization, we do not find this alternative definition compelling, however. Indeed, it would amount to define two separate sets of beliefs for each player, which might be inconsistent with equilibrium reasoning. To see this, suppose there are only two players and let $(\sigma_i^*, \Pi_i^*)_i$ be a minimax regret equilibrium (with the alternative definition). Then for player i , not only σ_j^* represents his belief about player j 's play, but also Π_i^* . And these two sets might not coincide. How can player i endogenously entertain two different sets of beliefs? In other words, if equilibrium reasoning leads player i to the belief σ_j^* , why does the same equilibrium reasoning lead him to the beliefs Π_i^* , which might be different from σ_j^* ? Lastly, Theorem 1 gives a saddle-point interpretation of a minimax regret equilibrium, which will prove extremely useful in applications.

Theorem 1 (Saddle point) *The profile $(\sigma_i^*, \sigma_{-i}^*)$ is a minimax regret equilibrium relative to $(\Pi_i)_{i \in N}$ if and only if there exists $\pi^* \in \times_{i \in N} \Pi_i$ such that $R_i(\sigma_i^*, \pi_i^*) \geq R_i(\sigma_i^*, \pi_i)$ for all $\pi_i \in \Pi_i$ and $R_i(\sigma_i^*, \pi_i^*) \leq R_i(\sigma_i, \pi_i^*)$ for all $\sigma_i \in \Sigma_i$ for all $i \in N$.*

Proof (\Leftarrow). For any $i \in N$, let $(\sigma_i^*, \pi_i^*) \in \Sigma_i \times \Pi_i$ be a saddle-point of R_i i.e., for all $\sigma_i \in \Sigma_i$ and $\pi_i \in \Pi_i$:

$$R_i(\sigma_i, \pi_i^*) \geq R_i(\sigma_i^*, \pi_i^*) \geq R_i(\sigma_i^*, \pi_i).$$

It follows that

$$R_i(\sigma_i^*, \pi_i^*) = \max_{\pi_i \in \Pi_i} R_i(\sigma_i^*, \pi_i) \leq R_i(\sigma_i, \pi_i^*) \leq \max_{\pi_i \in \Pi_i} R_i(\sigma_i, \pi_i),$$

⁸The existence of an ε -minimax regret equilibrium $(\sigma_i^*, \sigma_{-i}^*)$ follows from the fact that $\sigma_{-i}^* \in \Pi_i(\sigma_{-i}^*)$ and standard fixed-point arguments. Furthermore, note that in an ε -minimax regret equilibrium, the support of σ_{-i}^* is included in the support of π_i^* for all $\pi_i^* \in \Pi_i^*$, a requirement imposed by Marinacci (2000).

for all $\sigma_i \in \Sigma_i$. Henceforth, $(\sigma_i^*)_{i \in N}$ is a minimax regret equilibrium relative to $(\Pi_i)_i$.

(\Rightarrow). Let $(\sigma_i^*, \sigma_{-i}^*)$ be a minimax regret equilibrium relative to $(\Pi_i)_i$. In particular, this implies that $\sigma_{-i}^* \in \Pi_i$ and there exists a $\pi_i^* \in \Pi_i$ such that

$$R_i(\sigma_i^*, \pi_i^*) = \min_{\sigma_i \in \Sigma_i} \max_{\pi_i \in \Pi_i} R_i(\sigma_i, \pi_i).$$

Since the belief sets $(\Pi_i)_{i \in N}$ are compact and convex and the regret functions are bilinear, it follows from the Minimax Theorem (Von Neumann (1928)) that

$$R(\sigma_i, \pi_i^*) \geq \min_{\sigma_i' \in \Sigma_i} R_i(\sigma_i', \pi_i^*) = \max_{\pi_i' \in \Pi_i} \min_{\sigma_i' \in \Sigma_i} R_i(\sigma_i', \pi_i') \quad (6)$$

$$= \min_{\sigma_i' \in \Sigma_i} \max_{\pi_i' \in \Pi_i} R_i(\sigma_i', \pi_i') = \max_{\pi_i' \in \Pi_i} R_i(\sigma_i^*, \pi_i') \geq R_i(\sigma_i^*, \pi_i^*), \quad (7)$$

for all $\sigma_i \in \Sigma_i$ and $\pi_i \in \Pi_i$. Henceforth, (σ_i^*, π_i^*) is a saddle point, which completes the proof. \square

It follows from Theorem 1 that finding a minimax regret equilibrium σ^* relative to belief sets $(\Pi_i)_i$ is equivalent to checking whether (σ_i^*, π_i^*) is a Nash equilibrium of a two-player zero-sum game between player i and a fictitious player, i 's "Nature," in which player i 's action set is Σ_i , Nature's action set is Π_i , and the payoff function to Nature is R_i . In the next section, we repeatedly use this important observation to analyze various examples. Before presenting the examples, three further remarks are worth making. First, the requirement in Theorem 1 that $R_i(\sigma_i^*, \pi_i^*) \leq R_i(\sigma_i, \pi_i^*)$ for all $\sigma_i \in \Sigma_i$ is equivalent to σ_i^* being a best-reply of player i to the belief π_i^* that is, $u_i(\sigma_i^*, \pi_i^*) \geq u_i(\sigma_i, \pi_i^*)$ for all $\sigma_i \in \Sigma_i$. This is a rationality requirement. Thus, it is as if the player is selecting a belief about the play of his opponents according to the minimax regret criterion, and best replies to it. Furthermore, for ε small enough, conjectures are almost common knowledge and, consequently, ε -minimax regret equilibria are approximate Nash equilibria. Indeed, in an ε -minimax regret equilibrium σ^* , each player i assigns probability at least $1 - \varepsilon$ to the mixed action σ_{-i}^* being played.⁹ Second,

⁹The solution concept shares similar epistemic foundations as the concept of ε -ambiguous equilibrium of Mukerji (1995).

our framework is easily generalized to games with infinite strategy spaces. For instance, if A_i is a compact Hausdorff space, $\Delta(A_i)$ the set of (regular, countably additive) probability measures on the Borel subsets of A_i , then a sufficient condition for Theorem 1 to hold is that the zero-sum game between each player i and i 's Nature is payoff secure (see Reny (1999)).¹⁰ The game of price competition studied in Section 3 is payoff secure.

3 Applications

3.1 How to find ε -minimax regret equilibria?

This first example carefully spells out all steps necessary to find the ε -minimax regret equilibria of a game. Following examples will be treated as a faster pace. Consider the mixed extension $G1$ of the finite game below.

| | | |
|-----|------|--------|
| | a | b |
| a | 3, 3 | 0, 5 |
| b | 5, 0 | -3, -3 |

$G1$

The game $G1$ has two pure Nash equilibria (a, b) and (b, a) and one mixed equilibrium $((3/5, 2/5)(3/5, 2/5))$. Let us first consider large uncertainty, so that $\varepsilon = 1$. To find the ε -minimax regret equilibria with $\varepsilon = 1$, we first construct the (ex-post) regret table for player 1 as follows:

| | | |
|-----|-----|-----|
| | a | b |
| a | 2 | 0 |
| b | 0 | 3 |

¹⁰More precisely, Reny requires that the mixed extension of the zero-sum game between player i and i 's Nature to be better-reply secure. Since the game is zero-sum, its mixed extension is reciprocally upper semi-continuous. Therefore, payoff security insures that the mixed extension is better-reply secure.

For instance, if both players play a , player 1 experiences an ex-post regret of 2 as the best action would have been b , had he known that player 2 was playing a . Second, we use Theorem 1 to search for an equilibrium of the zero-sum game between player 1 and 1's Nature, represented below.

| | | |
|-----|-------|-------|
| | a | b |
| a | -2, 2 | 0, 0 |
| b | 0, 0 | -3, 3 |

This game has a unique Nash equilibrium, in which player 1 chooses the mixed action $(3/5, 2/5)$. It guarantees a maximal regret of $6/5$. Similarly, for player 2. Therefore, the totally mixed Nash equilibrium $((3/5, 2/5), (3/5, 2/5))$ is the unique ε -minimax regret equilibrium with $\varepsilon = 1$. We now show that it is an ε -minimax regret equilibrium for any ε .

Let $(\sigma_1(a), \sigma_1(b)), (\sigma_2(a), \sigma_2(b))$ be a mixed action profile. In the zero-sum game between player 1 and 1's "Nature," the payoff to player 1 if he plays $a_1 \in \{a, b\}$ and Nature plays $a_2 \in \{a, b\}$ is

$$-(1 - \varepsilon)(\sigma_2(a)r_1(a_1, a) + \sigma_2(b)r_1(a_1, b)) - \varepsilon r_1(a_1, a_2).$$

Since $((3/5, 2/5), (3/5, 2/5))$ is an ε -minimax regret equilibrium with $\varepsilon = 1$, we have that $(3/5)r_1(a, a) + (2/5)r_1(a, b) = (3/5)r_1(b, a) + (2/5)r_1(b, b)$. Thus, when player 1 conjectures that player 2 is playing the mixed action $(\sigma_2(a), \sigma_2(b)) = (3/5, 2/5)$ with probability at least $1 - \varepsilon$, player 1's payoff in the zero-sum game is $-(1 - \varepsilon)(6/5) - \varepsilon r_1(a_1, a_2)$. The zero-sum game between player 1 and 1's "Nature" is therefore given by:

| | | |
|-----|---|---|
| | a | b |
| a | $-(1 - \varepsilon)\frac{6}{5} - \varepsilon 2, (1 - \varepsilon)\frac{6}{5} + \varepsilon 2$ | $-(1 - \varepsilon)\frac{6}{5}, (1 - \varepsilon)\frac{6}{5}$ |
| b | $-(1 - \varepsilon)\frac{6}{5}, (1 - \varepsilon)\frac{6}{5}$ | $-(1 - \varepsilon)\frac{6}{5} - \varepsilon 3, (1 - \varepsilon)\frac{6}{5} + \varepsilon 3$ |

Clearly, $((3/5, 2/5)(3/5, 2/5))$ is an equilibrium of this game and, consequently, is an ε -minimax regret for all ε . The (symmetric) mixed Nash equilibrium survives any uncertainty from small to large. We can apply

the exact same arguments to show that $((3/5, 2/5)(3/5, 2/5))$ is the only non-degenerate ε -minimax regret equilibrium for any $\varepsilon > 0$.

Lastly, we can similarly check that (a, b) and (b, a) are ε -minimax regret equilibria for $\varepsilon \leq 2/5$. Note that (a, a) is not a minimax regret equilibrium for any ε . It is, however, an ε -maximin equilibrium if ε is large enough. Besides spelling out the working of ε -minimax regret equilibria, this first example gives the impression that only rationalizable action profiles are in the support of ε -minimax regret equilibria. The next example shows that this impression is false.

3.2 Fairness and social efficiency

Consider the mixed extension $G2$ of the finite game below.

| | | |
|-----|--------|--------|
| | a | b |
| a | $4, 2$ | $4, 5$ |
| b | $2, 0$ | $5, 1$ |

$G2$

The game $G2$ is dominance-solvable and has a unique rationalizable profile (b, b) . The set of ε -minimax regret equilibria is $\{((2/3, 1/3)(0, 1))\}$ for $\varepsilon > 1/3$, $\{((2/3, 1/3)(0, 1)), ((0, 1)(0, 1))\}$ for $\varepsilon = 1/3$ and $\{((0, 1)(0, 1))\}$ for $\varepsilon < 1/3$. In any ε -minimax regret equilibrium, player 1 believes that player 2 might play b with a probability ranging from $1 - \varepsilon$ to 1, and a with the complementary probability. Indeed, b is strictly dominant for player 2, and player 1 believes that player 2 is rational with probability $1 - \varepsilon$ in an ε -minimax regret equilibrium. Furthermore, player 1 is indifferent between playing a or b if he believes that player 2 is playing a with probability $1/3$. Therefore, for $\varepsilon \geq 1/3$, player 1 can rationally play either a or b as both actions are best-replies to the admissible belief $(1/3, 2/3)$. For $\varepsilon < 1/3$, the probability assigned to b is at least $2/3$ and, consequently, player 1 cannot rationally play a : there are no admissible beliefs that make a a best-reply.

This example highlights two interesting features of minimax regret equilibrium. First, an ε -minimax regret equilibrium might not be rationalizable. This is not surprising as rationalizability relies on common knowledge in

rationality, while minimax regret equilibrium does not even assume mutual knowledge in rationality. It is worth noting that the concept of conjectural equilibrium also shares this feature (Battigalli (1987)).¹¹ However, for ε small enough, any ε -minimax regret equilibrium survives iterated deletion of strictly dominated strategies. Second, for $\varepsilon \geq 1/3$, the profile (a, b) is in the support of ε -minimax regret equilibrium and, therefore, can be observed by an outside observer, say an experimenter. And not only is (a, b) socially efficient (Charness and Rabin (2002)), but it also minimizes inequality (Fehr and Schmidt (1999)).

3.3 Dominated strategies

This next example shows that a Nash equilibrium in weakly dominated strategies can be an ε -minimax regret equilibrium for any ε . Consider the game $G3$ below.

| | | | |
|----------|----------|----------|----------|
| | <i>a</i> | <i>b</i> | <i>c</i> |
| <i>a</i> | 0, 0 | 6, 6 | 0, 6 |
| <i>b</i> | 6, 6 | 0, 0 | 0, 0 |
| <i>c</i> | 6, 0 | 0, 0 | 1, 1 |

$G3$

The profile of strategies $((1/2, 1/2, 0), (1/2, 1/2, 0))$ is a Nash equilibrium in weakly dominated strategies.¹² Let us show that it is an ε -minimax regret equilibrium for any ε . To do so, we construct the table representing the zero-sum game between player 1 and 1's "Nature" (the table is similar for player 2) as follows.

| | | | |
|----------|--------------------------------------|--------------------------------------|--------------------------------------|
| | <i>a</i> | <i>b</i> | <i>c</i> |
| <i>a</i> | $(1 - \varepsilon)3 + \varepsilon 6$ | $(1 - \varepsilon)3 + \varepsilon 0$ | $(1 - \varepsilon)3 + \varepsilon 1$ |
| <i>b</i> | $(1 - \varepsilon)3 + \varepsilon 0$ | $(1 - \varepsilon)3 + \varepsilon 6$ | $(1 - \varepsilon)3 + \varepsilon 1$ |
| <i>c</i> | $(1 - \varepsilon)3 + \varepsilon 0$ | $(1 - \varepsilon)3 + \varepsilon 6$ | $(1 - \varepsilon)3 + \varepsilon 0$ |

¹¹Both concepts do not coincide, however. For instance, (a, b) is a conjectural equilibrium.

¹²For instance, $(1/2, 1/2, 0)$ is weakly dominated by $(1/2, 1/4, 1/4)$.

Note that the payoff in each cell is the payoff of 1's Nature. It is then easy to check that for any ε , $((1/2, 1/2, 0), (1/2, 1/2, 0))$ is a Nash equilibrium of this zero-sum game. A similar argument for player 2 shows that $((1/2, 1/2, 0), (1/2, 1/2, 0))$, albeit weakly dominated, is indeed an ε -minimax regret equilibrium for any ε . Although it is not our primary aim, the concept of an ε -minimax regret equilibrium might be the basis for a refinement of the concept of Nash equilibrium as $\varepsilon \rightarrow 0$. We can already note, however, that the Nash equilibria surviving our refinement might be neither perfect nor proper.¹³ Section 5 discusses this issue in more details.

3.4 A three-player example

In this example, we show that the representation of belief sets as either the product of independent ε -contaminations or a (correlated) ε -contamination has important consequences for the equilibrium characterization. Consider the game $G4$ below taken from van Damme (1991, p. 29).

| | | | | | |
|-----|---------|---------|--|---------|---------|
| | a | b | | a | b |
| a | 1, 1, 1 | 1, 0, 1 | | 1, 1, 0 | 0, 0, 0 |
| b | 1, 1, 1 | 0, 0, 1 | | 0, 1, 0 | 1, 0, 0 |
| | a | | | b | |

Note that action a is strictly dominant for both players 2 and 3. The game has a continuum of Nash equilibria, in which player 1 randomizes between a and b with any probability and players 2 and 3 play a . For n -player games ($n \geq 3$), the representation of belief sets as ε -contaminations entails a choice: either we model it as the product of independent ε -contaminations or as a (correlated) ε -contamination. First, consider the former. Let us check whether (a, a, a) is an ε -minimax regret equilibrium for some $\varepsilon > 0$. For players 2 and 3, since a is a strictly dominant action, it is clearly part of an ε -minimax regret equilibrium. Turning to player 1, we construct his regret table:

¹³Weakly dominated Nash equilibria are neither perfect nor proper.

| | | | | |
|-----|--------|--------|--------|--------|
| | a, a | a, b | b, a | b, b |
| a | 0 | 0 | 0 | 1 |
| b | 0 | 1 | 1 | 0 |

and then consider the zero-sum game between player 1 and his “Nature”:

| | | | | |
|-----|--------|-----------------------------|-----------------------------|-----------------------------|
| | a, a | a, b | b, a | b, b |
| a | 0, 0 | 0, 0 | 0, 0 | $-\varepsilon, \varepsilon$ |
| b | 0, 0 | $-\varepsilon, \varepsilon$ | $-\varepsilon, \varepsilon$ | 0, 0 |

Clearly, a is not part of any equilibrium of the above game. If player 1 considers playing a , 1’s Nature maximizes his regret by playing (b, b) , in which case a deviation to b is profitable for player 1. Henceforth, (a, a, a) is not an ε -minimax regret equilibrium for any $\varepsilon > 0$. Similarly, for (b, a, a) . In fact, we can show that the only ε -minimax regret equilibrium for any $\varepsilon > 0$ is $((1/2, 1/2), (1, 0), (1, 0))$.

Second, suppose that belief sets are represented by the product of independent contaminations. Let us check whether (a, a, a) is an ε -minimax regret equilibrium. The zero-sum game between player 1 and 1’s “Nature” is now:

| | | | | |
|-----|--------|-----------------------------|-----------------------------|---|
| | a, a | a, b | b, a | b, b |
| a | 0, 0 | 0, 0 | 0, 0 | $-\varepsilon^2, \varepsilon^2$ |
| b | 0, 0 | $-\varepsilon, \varepsilon$ | $-\varepsilon, \varepsilon$ | $-2(\varepsilon - \varepsilon^2), 2(\varepsilon - \varepsilon^2)$ |

For instance, if 1’s “Nature” plays (b, b) , player 1’s regret of playing a is

$$(1 - \varepsilon)^2 r_1(a, a, a) + (1 - \varepsilon)\varepsilon r_1(a, a, b) + \varepsilon(1 - \varepsilon)r_1(a, b, a) + \varepsilon^2 r_1(a, b, b).$$

It follows that (a, a, a) is an ε -minimax regret equilibrium (with independent contaminations) if $\varepsilon \leq 2/3$. The intuition is the following. If player 1 considers playing a , 1’s “Nature maximizes” his regret by playing (b, b) , which

occurs with probability ε^2 . If (b, b) is played, player 1's regret is 1. However, if player 1 considers deviating to b , his regret is 1 when the profile of actions (a, b) or (b, a) is played. And this occurs with probability $2\varepsilon(1 - \varepsilon)$. Since player 1 is indifferent between a and b when (a, a) is played by his opponents, it follows that player 1 is better off playing a if $\varepsilon^2 \leq 2\varepsilon(1 - \varepsilon)$ i.e., if $\varepsilon \leq 2/3$. Both approaches might therefore lead to very different predictions in games. The following economic application reinforces this point.

3.5 Bertrand competition and price dispersion

Consider a market in which n firms compete in prices to sell a homogeneous product. Each firm i posts a price p_i and the firm posting the lowest price wins the entire market. In the event of a tie, each firm charging the lowest price has an equal chance of serving the entire market. The market demand is unitary if the lowest price is smaller than 1, the monopoly price, and zero otherwise (the choke price is 1). Each firm marginal cost of production is c_i . The profit to firm i if it posts the price p_i and its competitors post the price p_{-i} is therefore:¹⁴

$$u_i(p_i, p_{-i}) = \begin{cases} (p_i - c_i) \delta_{\{i \in \arg \min_j \{(p_j)_{j \in N}\}\}} \frac{1}{|\arg \min_j \{(p_j)_{j \in N}\}|}, & \text{if } p_i \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

A firm faces two sources of regret. First, firm i 's price might turned out to be lower than the lowest price of its competitors. Had firm i posted a higher price, it would also have served the entire market and made a higher profit. Second, firm i 's price might turned to be higher than the lowest price of its competitors, and the regret arises from not serving the market at all. The exposure to these two sources of regret has important economic implications, as we will see. Formally, the regret to firm i is:

$$r_i(p_i, p_{-i}) = (\min(\{(p_j)_{j \in N}\} \cup \{1\}) - c_i) - u_i(p_i, p_{-i}).$$

¹⁴For any set X , δ_X is the Dirac mass on X .

Two firms and identical costs. For simplicity, we start by considering the case of two firms and identical marginal costs, normalized to zero. Let us show that each firm charging a price $p_i \in [\varepsilon, 1]$ according to the distribution

$$G(p_i) = \frac{1}{1-\varepsilon}(1 - \varepsilon p_i^{-1})$$

constitutes an ε -minimax regret equilibrium. Let us conjecture that firm i 's regret is maximized at the monopoly price i.e., $p = 1$. Accordingly, with probability $(1 - \varepsilon)$, firm i 's competitor follows the pricing strategy G defined above while, with probability ε , firm i faces an “irrational” competitor, and conjectures that it prices at the monopoly price. Firm i 's regret of posting any price $p_i \in [\varepsilon, 1)$ is:

$$(1 - \varepsilon) \left(\int_{\varepsilon}^{p_i} p_j dG(p_j) + \int_{p_i}^1 (p_j - p_i) dG(p_j) \right) + \varepsilon(1 - p_i) = -\varepsilon \ln \varepsilon.$$

For almost any price $p_i \in [\varepsilon, 1]$, firm i 's regret is therefore constant and equal to $-\varepsilon \ln \varepsilon$.¹⁵ Moreover, if firm i prices below ε , its regret is $-\varepsilon \ln \varepsilon + \varepsilon - p_i$, a non-profitable deviation. The intuition is simple: if firm i prices below ε , it is sure to serve the entire market. However, its exposure to potential regret is substantial: both the “rational” and “irrational” incarnations of its competitor might price all the way up to the monopoly price. Similarly, if firm i prices above 1, his regret is $-\varepsilon \ln \varepsilon + \varepsilon$, a non-profitable deviation. Consequently, the mixed strategy G satisfies all the requirements to be a mixed Nash equilibrium in the zero-sum game between firm i and i 's “Nature.” Note that pricing according to G is indeed player i 's best-reply to the conjecture that his opponent prices according to G with probability $(1 - \varepsilon)$ and at the monopoly price with probability ε .¹⁶

Let us now turn to our conjecture that firm i 's “Nature” maximizes i 's regret at the monopoly price, that is, we have to check that $R_i(G, \hat{p})$ is

¹⁵Due to the tie-breaking rule, there is a discontinuity at 1: i 's regret if it posts the monopoly price is $-\varepsilon \ln \varepsilon + 0.5\varepsilon$, which is higher than $-\varepsilon \ln \varepsilon$.

¹⁶His expected payoff of pricing in $[\varepsilon, 1)$ is $p_i[(1 - \varepsilon)(1 - G(p_i)) + \varepsilon] = \varepsilon$. In fact, solving for $p_i[(1 - \varepsilon)(1 - G(p_i)) + \varepsilon] = k$ for some $k \geq 0$ and for all $p_i \in (a, b)$ along with the boundary conditions $G(a) = 0$ and $G(b) = 1$, and $u_i(p_i, G) \leq k$ for all $p_i \notin (a, b)$ gives us the functional form of G .

maximized at $\hat{p} = 1$. Clearly, 1's "Nature" will price neither strictly below ε nor strictly above 1. Firm i 's regret if "Nature" prices at $\hat{p} \in [\varepsilon, 1]$ is

$$R_i(G, \hat{p}) = \hat{p} - \int_{\varepsilon}^{\hat{p}} p_i dG(p_i),$$

which is strictly convex in \hat{p} . So all we have to check is that $R_i(G, \varepsilon) \leq R_i(G, 1)$, which is satisfied if

$$\varepsilon \leq 1 - \int_{\varepsilon}^1 p \frac{\varepsilon}{1 - \varepsilon p^2} dp = 1 + \frac{\varepsilon}{1 - \varepsilon} \ln \varepsilon.$$

This inequality holds if $\varepsilon < 0.39423$. Therefore, for small values of ε , firms post prices according to the randomized strategy G in an ε -minimax regret equilibrium.

In equilibrium, the expected profit to each firm is strictly positive and each firm prices strictly above the marginal cost with probability one. The average price is $-(\varepsilon \ln \varepsilon) / (1 - \varepsilon)$. For instance, when $\varepsilon = 0.1$, the average price is 0.25584. While the distribution G converges to $\delta_{\{p \geq 0\}}$ in distribution as ε goes to zero, the marginal increase of the average price at $\varepsilon = 0$ is equal to $+\infty$. Only small uncertainty regarding the strategy of one's competitor causes a dramatic increase in prices. Our predictions are not only differ with the Bertrand-Nash predictions, but also with the "maxmin" predictions. Indeed, the worst a firm can face is that its competitor prices at the marginal cost, hence $(0, 0)$ is the unique ε -maximin equilibrium for any ε . Lastly, we can equivalently show that the Nash equilibrium $(0, 0)$ is not an ε -minimax regret equilibrium for any $\varepsilon > 0$.

Next, we prove that the ε -minimax regret equilibrium is almost surely unique, i.e., that the pricing strategies are unique up to how prices are set on a set of measure 0. To do this, we first remind the reader of the following property of equilibria of zero-sum games.

Lemma 1 *Let σ^* and $\hat{\sigma}$ be two minimax regret equilibria relative to $(\Pi_i)_{i \in N}$. Assume there exists a saddle point, so that for all $i \in N$, $\pi_i \in \Pi_i$ and $\sigma_i \in \Sigma_i$, there exist $\pi^* \in \times_{i \in N} \Pi_i$ and $\hat{\pi} \in \times_{i \in N} \Pi_i$ such that*

$$\begin{aligned} R_i(\sigma_i^*, \pi_i) &\leq R_i(\sigma_i^*, \pi_i^*) \leq R_i(\sigma_i, \pi_i^*) \\ R_i(\hat{\sigma}_i, \pi_i) &\leq R_i(\hat{\sigma}_i, \hat{\pi}_i) \leq R_i(\sigma_i, \hat{\pi}_i), \end{aligned}$$

then

$$R_i(\hat{\sigma}_i, \pi_i^*) = R_i(\sigma_i^*, \pi_i^*) = R_i(\hat{\sigma}_i, \hat{\pi}_i) = R_i(\sigma_i^*, \hat{\pi}_i).$$

In words if (σ_i^*, π_i^*) and $(\hat{\sigma}_i, \hat{\pi}_i)$ are two equilibria of the zero-sum game between player i and i 's "Nature," then the equilibrium payoff is the same. We now return to our proof. Assume $\varepsilon < 0.39423$. Let (G^*, G^*) be the ε -minimax regret equilibrium characterized above and suppose that $\hat{\sigma}$ is another ε -minimax regret equilibrium. Denote $\hat{G}_j(p) = \Pr_{\hat{\sigma}_j}(p_j \leq p)$ the probability under $\hat{\sigma}_j$ that firm j prices below p . From the above arguments, we know that player i 's "Nature" maximizes i 's regret when player i follows G^* by pricing at 1 (see above). Therefore, the belief $\hat{\pi}_i = (1 - \varepsilon)\hat{G}_j + \varepsilon\delta_{\{p \geq 1\}}$ maximizes i 's regret among all beliefs that are ε -contaminations of \hat{G} . From Lemma 1, it follows that $p_i[(1 - \varepsilon)(1 - \hat{G}_j(p_i)) + \varepsilon] = \varepsilon$ for all $p_i \in [\varepsilon, 1)$ and, therefore, $\hat{G}_j(p_i) = (1 - \varepsilon p_i^{-1}) / (1 - \varepsilon) = G^*(p_i)$ for all $p_i \in [\varepsilon, 1)$. This proves uniqueness.

Two firms and different marginal costs. We now assume that the two firms have different marginal costs c_1 and c_2 with $0 \leq c_1 < c_2 < 1$. Firm 1 is the most efficient firm. For small ε , we can expect the efficient firm to undercut the inefficient firm, thus focusing on the event that its conjecture is correct (with probability $1 - \varepsilon$). Consequently, let us conjecture that firm 1 prices according to the distribution G_1 on $[a, b]$, firm 2 prices according to G_2 on $[a, 1]$, and G_2 first-order stochastically dominates G_1 . What might maximize the regret of a firm? Since we expect firm 1 to try to undercut firm 2, we can conjecture that firm 1's regret is maximized at the monopoly price (the highest price consistent with a positive demand). Thus, firm 1 will face the distribution $F_2 = \delta_{\{p_2: p_2 \geq 1\}}$ with probability ε . Regarding firm 2, the inefficient firm, we expect it to be mostly concerned with foregone profits when, with probability ε , its conjecture about firm 1 is incorrect. Since foregone profits are higher, the higher firm 1's price, we expect firm 2's regret to be maximized when it faces the distribution F_1 with support $[b, 1]$. Let us show that we can indeed construct such an ε -minimax regret equilibrium.

First, we consider the indifference conditions. To be indifferent between almost all prices in the support of G_i , the following equality for firm i has to

be satisfied

$$(p - c_i) (1 - (1 - \varepsilon) G_{-i}(p) - \varepsilon F_{-i}(p)) = a - c_i, \quad (8)$$

since $G_{-i}(a) = F_{-i}(a) = 0$. Moreover, since $F_{-i}(b) = 0$, we obtain that

$$G_{-i}(p) = \frac{1}{1 - \varepsilon} \left(1 - \frac{a - c_i}{p - c_i} \right),$$

for all $p \in [a, b]$. Let us focus on firm 1. Since we conjecture an equilibrium with $G_1(b) = 1$, we have

$$\frac{1}{1 - \varepsilon} \left(1 - \frac{a - c_2}{b - c_2} \right) = 1,$$

and, consequently, the parameters a and b have to satisfy:

$$a = c_2 + \varepsilon(b - c_2).$$

Furthermore, since $G_1(p) = 1$ for all $p \geq b$, firm 2's indifference condition (8) implies

$$F_1(p) = 1 - \frac{a - c_2}{\varepsilon(p - c_2)},$$

for (almost) all $p \in [b, 1]$. Let us now turn to firm 2.

For any $p \in [b, 1]$, the support of F_1 , the indifference condition for 2's "Nature" (i.e., the "irrational" incarnation of firm 1) implies that the regret of firm 2 when facing price p , denoted by $R_2(G_2, p)$, is constant, that is,

$$R_2(G_2, p) = (p - c_2) - \int_a^p (x - c_2) g_2(x) dx$$

is constant in p .¹⁷ Hence, $g_2(p) = 1/(p - c_2)$ for all $p \in [b, 1]$, and a simple integration gives

$$G_2(p) = \frac{1}{1 - \varepsilon} \left(1 - \frac{a - c_1}{b - c_1} \right) + \ln \frac{p - c_2}{b - c_2},$$

¹⁷Since R_2 is the payoff function of 2's "Nature" in the zero-sum game between firm 2 and 2's "Nature".

for all $p \in [b, 1]$. In particular, we need that $G_2(1) = 1$ which implies that

$$a = c_1 + \left(1 - (1 - \varepsilon) \left(1 - \ln \frac{1 - c_2}{b - c_2}\right)\right) (b - c_1).$$

Together with the above expression for a , we obtain

$$c_2 - c_1 = (b - c_1) \ln \frac{1 - c_2}{b - c_2}. \quad (9)$$

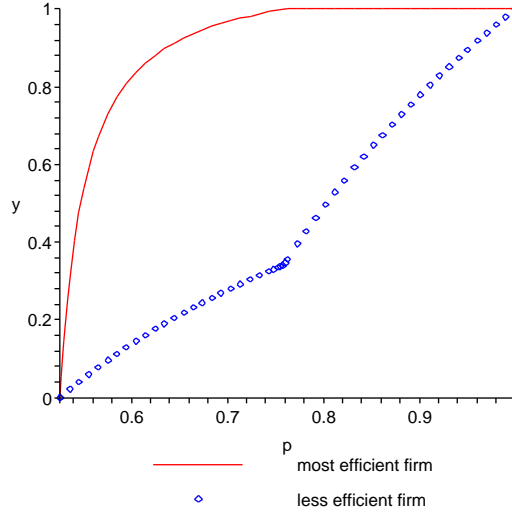
Note that the parameter b does not depend on ε . Clearly, the above equation has a solution with $b \geq c_2$. Furthermore, as ε goes to zero, the parameter a goes to c_2 , firm 1 prices at c_2 and firm 2 prices according to the distribution $\bar{G}_2(p) = 1 - \frac{c_2 - c_1}{p - c_1}$ for $p \in [c_2, b]$. By construction, firm 1 facing \bar{G}_2 is indifferent over all prices in $[c_2, b]$ and, therefore, these limit strategies constitute a mixed Nash equilibrium of the Bertrand game.

Second, we have to verify that no player has an incentive to deviate. Nature replacing firm 2 must be maximizing firm 1's regret at the monopoly price $p = 1$ (since we assume $F_2 = \delta_{\{p_2: p_2 \geq 1\}}$). As in the case with homogenous cost, the regret is strictly convex in p , and a necessary and sufficient condition for $R_1(G_1, a) \leq R_1(G_1, 1)$ is given by:

$$(1 - c_1) - \frac{a - c_2}{1 - \varepsilon} \int_a^b \frac{p - c_1}{(p - c_2)^2} dp \geq a - c_1. \quad (10)$$

Finally, it is easy to check that firm 1 has no incentive to price either above b or below a . Similarly, for firm 2 and 2's "Nature". Thus, if Eq. (10) holds, (G_1, G_2) is an ε -minimax regret equilibrium.

To illustrate our findings, let us consider a numerical example. Assume that $c_1 = 0$, $c_2 = 0.5$ and $\varepsilon = 0.1$. We have that $a = 0.52$, $b = 0.76$, and the pricing policies are illustrated in the figure below. Observe that firm 2's pricing policy exhibits a kink at b .



Several firms and identical costs. We now consider the case of $n \geq 3$ firms in order to illustrate, within an economic example, the differences between the two formulations of belief sets: correlated contaminations or product of independent contaminations. Costs are identical and normalized to zero. We first investigate the situation where the belief sets are product of independent contaminations. Each firm believes that each opponent independently chooses according to the distribution G with probability $1 - \varepsilon$ and is uncertain about the opponents' play, otherwise. Let us look for an ε -minimax regret equilibrium in which all firms price according to the distribution G on $[\varepsilon^{n-1}, 1]$ and each respective adversarial Nature maximizes regret at the monopoly price.

Let $G_\varepsilon = (1 - \varepsilon)G + \varepsilon\delta_{\{p \geq 1\}}$. For all $p \in [\varepsilon^{n-1}, 1)$, the profit to firm i is $p(1 - G_\varepsilon(p))^{n-1}$ and together with the fact that it goes to ε^{n-1} as p goes to

one, it follows that the distribution G is given by:¹⁸

$$G(p) = \frac{1}{1-\varepsilon} (1 - \varepsilon p^{-1/(n-1)}).$$

Denote g the density of G . Now, we show that each adversarial Nature indeed maximizes regret at the monopoly price. For this, we derive firm i 's regret by considering it as a sum of independent events. Consider the event in which $n - m$ firms out of $n - 1$ choose according to G while an adversarial Nature charges the prices of the remaining $m - 1$ firms. This event occurs with probability $(1 - \varepsilon)^{n-m} \varepsilon^{m-1} \binom{n-1}{n-m}$. Let q be the lowest price charged by Nature. Conditional on this event, firm i 's regret is

$$\int_0^q pg(p) (n - m) (1 - G(p))^{n-m-1} dp + q (1 - G(q))^{n-m} - \int_0^q p (1 - G(p))^{n-m} g(p) dp.$$

Differentiating this expression with respect to q , we obtain after simplifications:

$$(1 - G(q))^{n-m} (1 - qg(q)).$$

Lastly, note that

$$1 - qg(q) = 1 - \frac{\varepsilon}{(1-\varepsilon)(n-1)} q^{-\frac{1}{n-1}},$$

and is increasing in q . Moreover, at $q = \varepsilon^{n-1}$, the above expression is equal to $1 - \frac{1}{(1-\varepsilon)(n-1)}$, which is strictly positive if $\varepsilon < 1 - \frac{1}{n-1}$. Therefore, for $\varepsilon < 1 - 1/(n-1)$, the derivative of the regret with respect to q is strictly positive (except at the point $p = 1$), hence the regret is maximized at the monopoly price.

¹⁸Let \tilde{G}_ε be the distribution of the order statistics $\min(p_{-i})$. The regret to firm i of posting the price p_i is

$$\int_{\varepsilon^{n-1}}^{p_i} \min(p_{-i}) d\tilde{G}_\varepsilon(\min(p_{-i})) + \int_{p_i}^1 (\min(p_{-i}) - p_i) d\tilde{G}_\varepsilon(\min(p_{-i})),$$

and together with the boundary conditions give the distribution G .

The expected price is equal to $\varepsilon(1 - \varepsilon^{n-2})/[(1 - \varepsilon)(n - 2)]$ and is increasing in the degree of confidences about opponents' conjectures, ε , as in the case with two firms. However, unlike the two-firm case, the marginal increase in the expected price at $\varepsilon = 0$ is now finite (equal to $1/(n - 2)$). Furthermore, the expected price is decreasing in n i.e., the more intense the competition, the lower the expected price. As $n \rightarrow +\infty$, the ε -minimax regret equilibrium converges to the competitive equilibrium.

Let us now turn to the situation where a firm expects with probability $1 - \varepsilon$ that *all* other firms independently price according to G , and is completely uncertain otherwise, that is, we consider the case of "correlated" contaminations. In that situation, adversarial Natures have a larger impact on the pricing policy of firms. Following the same logic as before, we have the following. Firms price according to the pricing distribution G with:

$$G(p) = \begin{cases} 1 - \left(\frac{\varepsilon(1-p)}{(1-\varepsilon)p}\right)^{1/(n-1)} & \text{for } p \in [\varepsilon, b] \\ 1 - \left(\frac{\varepsilon(1-b)}{(1-\varepsilon)b}\right)^{1/(n-1)} + \ln \frac{p}{b} & \text{for } p \in [b, 1], \end{cases}$$

with b satisfying the equality

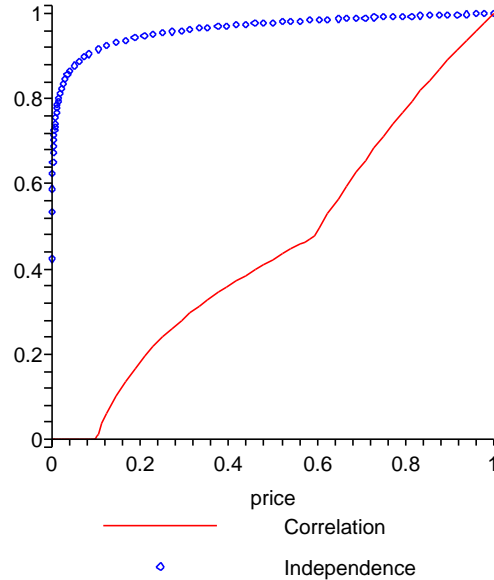
$$-\ln b = \left(\frac{\varepsilon(1-b)}{(1-\varepsilon)b}\right)^{1/(n-1)}.$$

Such a b exists for $\varepsilon < 1/e$ where $b \in (1/e, 1)$. For completeness, let us give the strategy (distribution) F followed by each adversarial Nature:

$$F(p) = 1 - \frac{1}{p} + \frac{1 - \varepsilon}{\varepsilon} \left(\left(\frac{\varepsilon(1-b)}{(1-\varepsilon)b}\right)^{1/(n-1)} - \ln \frac{p}{b} \right)^{n-1}$$

for $p \in [b, 1]$. As in the preceding cases, the reader can check that this is indeed an ε -minimax regret equilibrium for ε small enough i.e., $\varepsilon < (n - 2)/(n - 1)$.

A simple numerical example helps to illustrate the differences between both formulations. With 5 firms and $\varepsilon = 0.1$, the pricing policies in the case of independent and correlated contaminations are represented in the graph below ($b = 0.59$).



Both policies are strikingly different. With independent contaminations, a firm is mainly concerned with facing “rational” competitors who price close to the marginal cost; the likelihood to face only “irrational” firms is extremely small (10^{-5}). Consequently, firms price very close to the marginal cost (see the curve in dots). In contrast, with correlated contaminations, the likelihood to face only “irrational” firms is disproportionately higher 10^{-1} and, therefore, firms are more concerned with the possibility of foregoing opportunities. Their pricing policy reflects this concern and accordingly prices are more likely to be substantially above marginal costs.

To summarize, our results are that firms make positive expected profit, price above marginal costs and there is price dispersion in all cases. Moreover, with several firms and correlated contaminations, the pricing policy exhibits a kink at b , which is close to the monopoly price for ε small enough. All these findings agree remarkably well with experimental and field data (see Baye and Morgan (2004).)

Finally, let us note that although our results are stated for price competition, they readily translate to models of first-price auction (with complete information). To do so, if $v_i \in [0, 1]$ is the valuation for the object of bidder

i , then we can set $c_i = 1 - v_i$ and use the equivalence relation $b \sim 1 - p$ where b is the bid of a player to obtain the equilibrium bidding policy \hat{G}_i . Formally, $\hat{G}_i(b) = 1 - G_i(1 - p)$ if G_i is the pricing policy in an ε -minimax regret equilibrium of the Bertrand game. For instance, if both bidders have the same valuation 1, their bidding strategy follows the distribution $\varepsilon b / ((1 - \varepsilon)(1 - b))$ with support $[0, 1 - \varepsilon]$.

4 Some properties of ε -minimax regret equilibria

This section presents some properties of ε -minimax regret equilibria of finite games. Recall from example *G3* that a minimax regret equilibrium can be in weakly dominated strategies. Our first result shows that this cannot be the case when players have a dominant action.

Proposition 1 *Let σ^* be a minimax regret equilibria relative to $(\Pi_i)_{i \in N}$. (i) If σ_i^{**} is a dominant strategy for player i , then $\sigma_i^* = \sigma_i^{**}$. (ii) If a_i is a strictly dominated action, then $\sigma_i^*(a_i) = 0$.*

Proof Part (i). Let σ^* be a minimax regret equilibria relative to $(\Pi_i)_{i \in N}$ and σ_i^{**} a dominant strategy for player i . Since σ_i^{**} is dominant, we have that $u_i(\sigma_i^{**}, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$ for all $\sigma_i \in \Sigma_i$, for all $\sigma_{-i} \in \Sigma_{-i}$, and for all $\sigma_i \in \Sigma_i$, there exists a $\sigma'_{-i} \in \Sigma_{-i}$ such that $u_i(\sigma_i^{**}, \sigma'_{-i}) > u_i(\sigma_i, \sigma'_{-i})$. Therefore, $0 = R_i(\sigma_i^{**}, \sigma_{-i}) \leq R_i(\sigma_i, \sigma_{-i})$ for all $\sigma_i \in \Sigma_i$, for all $\sigma_{-i} \in \Sigma_{-i}$. This implies that $0 = R_i(\sigma_i^{**}, \pi_i) \leq R_i(\sigma_i, \pi_i)$ for all $\sigma_i \in \Sigma_i$, for all $\pi_i \in \Pi_i$. Since σ^* is a minimax regret equilibrium and $R_i(\cdot, \cdot) \geq 0$, it follows that $R_i(\sigma_i^*, \pi_i) = 0$ for all $\pi_i \in \Pi_i$. Consequently, $\sigma_i^* = \sigma_i^{**}$. For otherwise, there exists a σ'_{-i} such that $R_i(\sigma_i^{**}, \sigma'_{-i}) < R_i(\sigma_i^*, \sigma'_{-i})$, a contradiction with σ^* being a minimax regret equilibrium.

Part (ii). By contradiction. Suppose that there exists a strictly dominated action a_i such that $\sigma_i^*(a_i) > 0$ for some player i . This implies that σ_i^* is not a best-reply to any conjecture $\pi_i \in \Pi_i \subseteq \Delta(A_{-i})$ of player i . However, from Theorem 1, σ_i^* is a best-reply to the conjecture π_i^* , a contradiction. \square

Proposition 1 implies that Nash equilibria in dominant actions are robust to the introduction of uncertainty about the rationality and conjectures of opponents. This result is not surprising since players are assumed to be rational in our model. Part (ii) also shows that players do not play strictly dominated strategies in an ε -minimax regret equilibrium. In fact, for ε small enough, any pure action in the support of an ε -minimax regret equilibrium survives iterated deletion of strictly dominated actions.

Proposition 2 *There exists an ε^* such that for any $\varepsilon < \varepsilon^*$, if σ^* is an ε -minimax regret equilibrium and a_i does not survive iterated deletion of strictly dominated actions, then $\sigma_i^*(a_i) = 0$.*

Proof We present the proof for the case of correlated ε -contaminations. The case of independent ε -contaminations is similar. Let $(\varepsilon^m)_{m \in \mathbb{N}}$ be any sequence converging to 0 and, for each $m \in \mathbb{N}$, let σ^m be an ε^m -minimax regret equilibrium of G . Set $A_i^0 := A_i$ and define recursively:

$$A_i^k := \{a_i \in A_i^{k-1} : \text{there is no } a'_i \in A_i^{k-1} \text{ such that } u_i(a'_i, a_{-i}) > u_i(a_i, a_{-i}) \forall a_{-i} \in A_{-i}^{k-1}\},$$

that is, A_i^k is the set of player i 's pure strategies that survives k rounds of iterated deletion of strictly dominated pure strategies. We want to show that there exists an M^* such that for all $m > M^*$, the support of σ^m is included in $\bigcap_{k=0}^{\infty} A_i^k$.

First, we show that the support of σ_i^m is in A_i^1 for each player $i \in N$, for each $m \in \mathbb{N}$. By contradiction, suppose that a_i^* is in the support of σ_i^m , but $a_i^* \notin A_i^1$ for some player i , for some m . This implies that there exists an action \bar{a}_i such that $r_i(\bar{a}_i, a_{-i}) < r_i(a_i^*, a_{-i})$ for all $a_{-i} \in A_{-i}$. Consider the mixed strategy $\bar{\sigma}_i$ with $\bar{\sigma}_i(a_i^*) = 0$, $\bar{\sigma}_i(\bar{a}_i) = \sigma_i^m(a_i^*) + \sigma_i^m(\bar{a}_i)$, and $\bar{\sigma}_i(a_i) = \sigma_i^m(a_i)$ for all $a_i \neq a_i^*, \bar{a}_i$. It follows that $R_i(\bar{\sigma}_i, a_{-i}) < R_i(\sigma_i^m, a_{-i})$ for all $a_{-i} \in A_{-i}$ since

$$R_i(\bar{\sigma}_i, a_{-i}) - R_i(\sigma_i^m, a_{-i}) = \sigma_i^m(a_i^*)(r_i(\bar{a}_i, a_{-i}) - r_i(a_i^*, a_{-i})).$$

Consequently, σ_i^m cannot be an ε^m minimax regret equilibrium, a contradiction. Hence, the support of σ_i^m is included in A_i^1 for each player $i \in N$, for any $m \in \mathbb{N}$.

Second, we show that there exists an M^* such that the support of σ_i^m is included A_i^2 for each player $i \in N$, for all $m > M^*$. By contradiction, suppose that for any M , there exists an action a_i^m in the support of σ_i^m with $a_i^m \notin A_i^2$ for some player i , for some $m > M$ (hence, $A_i^2 \subset A_i^1$). Without loss of generality, assume it is for all $m > M$. This implies that there exists an action \bar{a}_i^m such that $r_i(\bar{a}_i^m, a_{-i}) < r_i(a_i^m, a_{-i})$ for all $a_{-i} \in A_{-i}^1$, for all $m > M$. For each $m > M$, construct the mixed strategy $\bar{\sigma}_i^m$ as $\bar{\sigma}_i$ above.

Since σ^m is an ε^m -minimax regret equilibrium and the expected regret is bi-linear, we must have that:

$$(1 - \varepsilon^m)\sigma_i^m(a_i^m) \sum_{a_{-i}} (r_i(\bar{a}_i^m, a_{-i}) - r_i(a_i^m, a_{-i}))\sigma_{-i}^m(a_{-i}) + \varepsilon^m \sigma_i^m(a_i^m) [\max_{a_{-i}} r_i(\bar{a}_i^m, a_{-i}) - \max_{a_{-i}} r_i(a_i^m, a_{-i})] \geq 0.$$

By the preceding arguments, the first term is strictly negative since the support of σ_{-i}^m is in A_{-i}^1 . As for the second term, it is bounded from above by:

$$K := \max_{(a'_i, a''_i) \in (A_i^1 \setminus A_i^2) \times A_i^2} [\max_{a_{-i}} r_i(a'_i, a_{-i}) - \max_{a_{-i}} r_i(a''_i, a_{-i})].$$

Note that K is well-defined by finiteness of the action spaces. Furthermore, there exists a \bar{M} such for all $m > \bar{M}$,

$$(1 - \varepsilon^m)\sigma_i^m(a_i^m) \sum_{a_{-i}} (r_i(\bar{a}_i^m, a_{-i}) - r_i(a_i^m, a_{-i}))\sigma_{-i}^m(a_{-i}) + \varepsilon^m K < 0.$$

Consequently, there exists a $M^{(2)}$ such that σ^m cannot simultaneously be an ε^m -minimax regret equilibrium for $m > M^{(2)}$ and the support of σ_i^m not included in A_i^2 for some player i . By induction, we can find such a $M^{(k)}$ for any $k > 2$.

Lastly, since we consider finite strategic-form games, there exists a K such that the iterated deletion of strictly dominated strategies stops after K rounds, and set M^* equal to the minimum of the $M^{(k)}$ for $k = 1, \dots, K$. \square

Proposition 2 thus states that for ε small enough, the pure actions in the support of an ε -minimax regret equilibrium survive iterated deletion of strictly dominated pure actions. However, we hasten to stress that an ε -minimax regret equilibrium might *not* survive iterated deletion of strictly

dominated (mixed and pure) strategies even for small $\varepsilon > 0$, as the following example illustrates.

| | | | |
|-----|-------|-------|-------|
| | a | b | c |
| a | -1, 1 | 1, -1 | 4, -2 |
| b | 1, -1 | -1, 1 | 0, -2 |

Observe that the pure action c is strictly dominated for player 2. Deleting c , the game is a game of matching pennies with $((1/2, 1/2)(1/2, 1/2))$ as the unique ε -minimax regret equilibrium for all ε (with a regret of 1 to each player). However, it is not an ε -minimax regret equilibrium of the game with action c . The reason is simple. If player 1 uniformly randomizes between a and b , 1's "Nature" maximizes player 1's regret by playing c , which gives a maximal regret of $1 + 3\varepsilon > 1$. However, if player 1 faces the mixed strategy $(1 - \varepsilon)(1/2, 1/2, 0) + \varepsilon(0, 0, 1)$, a is the unique best-reply. The unique ε -minimax regret equilibrium is $((2/3, 1/3)(1/2, 1/2, 0))$ for any $\varepsilon > 0$.

The next proposition states that ε -minimax regret equilibria are ε' -Nash equilibria for ε' appropriately chosen. Intuitively, Theorem 1 implies that each player is best replying to a mixture between the mixed strategy of his opponents (with probability $1 - \varepsilon$) and a mixed strategy of his "Nature" (with probability ε), and thus a player's payoff cannot differ too much from his optimal payoff had he known the strategy of his opponents.

Proposition 3 *Let σ^* be an ε -minimax regret equilibrium with $\varepsilon < 1$. There exists a $\gamma > 0$ such that σ^* is an ε' -Nash equilibrium with $\varepsilon' \geq \gamma\varepsilon/(1 - \varepsilon)$.*

Proof Suppose that σ^* is an ε -minimax regret equilibrium (with correlated ε -contamination). If $\varepsilon = 0$, then σ^* is a Nash equilibrium, hence an ε' -Nash equilibrium with $\varepsilon' = 0$. Assume that $\varepsilon > 0$. Since σ^* is an ε -minimax regret equilibrium, we have

$$(1 - \varepsilon)R_i(\sigma_i^*, \sigma_{-i}^*) + \varepsilon \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i^*, \hat{\sigma}_{-i}) \leq (1 - \varepsilon)R_i(\sigma_i, \sigma_{-i}^*) + \varepsilon \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i, \hat{\sigma}_{-i}),$$

for all $\sigma_i \in \Sigma_i$, for all $i \in N$. This is equivalent to

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) + \frac{\varepsilon}{1 - \varepsilon} \left(\max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i^*, \hat{\sigma}_{-i}) - \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i, \hat{\sigma}_{-i}) \right),$$

for all $\sigma_i \in \Sigma_i$, for all $i \in N$. Consider the set $S_i(\sigma^*) := \{\sigma_i \in \Sigma_i : u_i(\sigma_i^*, \sigma_{-i}^*) \leq u_i(\sigma_i, \sigma_{-i}^*)\}$ of player i ' strategies that improve upon $u_i(\sigma^*)$. We have

$$\gamma_i(\sigma^*) := \min_{\hat{\sigma}_i \in S_i(\sigma^*)} \left(\max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\sigma_i^*, \hat{\sigma}_{-i}) - \max_{\hat{\sigma}_{-i} \in \Delta(A_{-i})} R_i(\hat{\sigma}_i, \hat{\sigma}_{-i}) \right) \leq 0,$$

since σ^* is an ε -minimax regret equilibrium, $S_i(\sigma^*)$ is compact, and R_i is bi-continuous. Let $\varepsilon' := \varepsilon \max_{i \in N} |\gamma_i(\sigma^*)| / (1 - \varepsilon)$. Then, we have for all $i \in N$, for all $\sigma_i \in \Sigma_i$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) - \varepsilon',$$

which is the desired result. Note that ε' depends on σ^* . To get a uniform bound, consider

$$\bar{\gamma}_i = \min_{\hat{\sigma}_i} \max_{\hat{\sigma}_{-i}} R_i(\hat{\sigma}_i, \hat{\sigma}_{-i}) - \max_{\hat{\sigma}_i} \max_{\hat{\sigma}_{-i}} R_i(\hat{\sigma}_i, \hat{\sigma}_{-i}),$$

and let $\varepsilon' := \varepsilon \max_{i \in N} |\bar{\gamma}_i| / (1 - \varepsilon)$.

With the product of independent ε -contaminations, the proof is similar and left to the reader. \square

In an ε -minimax regret equilibrium, the parameter ε relates to the uncertainty about rationality and conjectures and, consequently, to the (expected) ex-post regret. In contrast, the parameter ε' relates to the ex-ante regret in an ε' -Nash equilibrium. Proposition 3 links these two parameters: we may therefore think of an ε -minimax regret equilibrium as an ε' -Nash equilibrium, in which players might not maximize their payoff even though they have correct conjectures about their opponents' strategies. Furthermore, ε' goes to 0 as ε goes to zero, and ε' is monotone increasing in ε . As a consequence, the set of ε -minimax regret equilibria converges to the set of Nash equilibria as ε goes to zero. The converse of Proposition 3 does not hold, however. For instance, in example G4, not all Nash equilibria, albeit ε' -equilibrium with $\varepsilon' = 0$, are ε -minimax regret equilibria, even with infinitesimally small uncertainty. However, any strict Nash equilibrium is an ε -minimax regret equilibrium for ε small enough.

Proposition 4 *Let σ^* be a strict Nash equilibrium. There exists a $\varepsilon^* > 0$ such that σ^* is an ε -minimax regret equilibrium for any $\varepsilon < \varepsilon^*$.*

The proof of Proposition 4 is similar to the proof of Proposition 3 and left to the reader. Note that not all ε -minimax regret equilibria are strict Nash equilibria, even with infinitesimally small uncertainty. For instance, in the game below, (a, a) is an ε -minimax regret equilibrium for any $\varepsilon \in [0, 1]$, but is not strict. And the strict Nash equilibrium (c, c) is an ε -minimax regret equilibrium for $\varepsilon \leq 1/3$.

| | | | |
|-----|------|------|----------|
| | a | b | c |
| a | 1, 1 | 1, 1 | 0, 0 |
| b | 1, 1 | 0, 0 | 0, 0 |
| c | 0, 0 | 0, 0 | 0.5, 0.5 |

5 Extensions

We now discuss two possible extensions. The first extension regards games with incomplete information.

Incomplete information. Let $\langle N, (A_i, \Theta_i, u_i, \Pi_i)_{i \in N} \rangle$ be a Bayesian game with multiple priors where $N := \{1, \dots, n\}$ is the set of players, A_i the set of pure actions of player i , Θ_i the finite set of possible types of player i , $u_i : X \times \Theta \rightarrow \mathbb{R}$ the payoff function of player i with $A := \times_i A_i$ and $\Theta := \times_i \Theta_i$, and Π_i is the (compact and convex) set of possible priors over Θ of player i .

A pure strategy for player i is a mapping $s_i : \Theta_i \rightarrow A_i$. Given a profile of pure strategies s and a profile of type θ , the *ex-post* regret of player i of type θ_i is:

$$r_i((s_i(\theta_i), s_{-i}(\theta_{-i})), (\theta_i, \theta_{-i})) = \sup_{a_i \in A_i} u_i(a_i, s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})) - u_i((s_i(\theta_i), s_{-i}(\theta_{-i})), (\theta_i, \theta_{-i})).$$

In words, if players follow the strategy s and the type profile turns out to be θ , the ex-post regret of player i measures his foregone payoff. The next definition extends our previous definition to games with incomplete information. For simplicity, we state the definition for pure strategies.

Definition 2 A profile of pure strategies $s^* = (s_i^*, s_{-i}^*)$ is a minimax regret equilibrium if for each player $i \in N$, for each type θ_i of player i ,

$$\begin{aligned} \max_{\pi_i \in \Pi_i} \sum_{\theta_{-i} \in \Theta_{-i}} r_i((s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), (\theta_i, \theta_{-i})) \pi_i(\theta_{-i} | \theta_i) \leq \\ \max_{\pi_i \in \Pi_i} \sum_{\theta_{-i} \in \Theta_{-i}} r_i((a_i, s_{-i}^*(\theta_{-i})), (\theta_i, \theta_{-i})) \pi_i(\theta_{-i} | \theta_i) \end{aligned}$$

for all $a_i \in A_i$.

In the above formulation, the assumption that payoff-relevant types are private information drives the strategic uncertainty. Nonetheless, it is not difficult to amend our formulation to account for incomplete information about rationality as well. For this, it suffices to extend the type space. It is also worth noting that an *ex-post equilibrium* is a minimax regret equilibrium with the particularity that the regret is nil. In a work in progress, Renou (2007) considers the problem of implementing social choice sets in minimax regret equilibrium and uses the former observation to provide conditions for implementing social choice sets in ex-post equilibrium.¹⁹ Along the same lines, Hayashi (2007b) studies minimax regret equilibria of auction games and shows the possibility of over and under bidding.²⁰ A slight variant of Definition 2 already appeared in computer science e.g., Hyafil and Boutilier (2004). Lastly, let us consider a simple example. There are two players 1 and 2. Player 2 has two types θ_2 and θ'_2 , player 1 has no type and $\Pi_1 = \Delta(\{\theta_2, \theta'_2\})$ i.e., player 1 is completely uncertain about the type of player 2. The payoffs are represented below.

| | | | | | | |
|-----|------------|------|--|-------------|------|------|
| | a | b | | a | b | |
| a | 2, 2 | 0, 0 | | a | 0, 0 | 0, 0 |
| b | 0, 0 | 0, 0 | | b | 0, 0 | 1, 1 |
| | θ_2 | | | θ'_2 | | |

¹⁹Bergeman and Morris (2007) provide sufficient conditions for full implementation in ex-post equilibrium.

²⁰In Hayashi, Π_i is a singleton for each player. However, since payoffs are discontinuous, minimax regret equilibria differ from Bayesian-Nash equilibria.

It is easy to see that $(s_1 = a, (s_2(\theta_2) = a, s_2(\theta'_2)))$ is a minimax regret equilibrium, while it is not a Bayesian-Nash equilibrium if the probability of the type θ_2 is smaller than $1/4$.

Minimal regret equilibrium. As already alluded, the concept of an ε -minimax regret equilibrium might be the basis for equilibrium refinement. Indeed, from Proposition 3, letting ε going to zero makes it possible to select among Nash equilibria. Equilibrium selection is not our primary aim, but the following offers such an equilibrium selection along with a short discussion for the interested readers.

Definition 3 *A profile of strategy σ^* is a minimal regret equilibrium if there exist some sequences $(\varepsilon_k)_{k \in \mathbb{N}}$ and $(\sigma_k^*)_{k \in \mathbb{N}}$ such that*

- For all $k \in \mathbb{N}$, $\varepsilon_k > 0$ and $\lim_{k \rightarrow +\infty} \varepsilon_k = 0$.
- For all $k \in \mathbb{N}$, σ_k^* is an ε_k -minimax regret equilibrium, and
- $\lim_{k \rightarrow +\infty} \sigma_k^* = \sigma^*$.

It follows from the existence of an ε -minimax regret equilibrium for each ε and the compactness of Σ that a minimal regret equilibrium exists. Moreover, we have seen in example *G4* that this concept helps to select even among undominated Nash equilibria. From example *G3*, we also have that dominated Nash equilibria might be minimal regret equilibria. The reader might find this property rather unsatisfactory, and we would have agreed before working on this project. Indeed, if players *cautiously* believe that all their opponents' actions can be played, say because of trembles, then it is not optimal to play dominated actions. However, players are also excessively cautious, albeit differently, in our formulation. They believe that the worst possible trembles would materialize i.e., the trembles that maximize a player regret. There is no clear justification for one form of cautiousness over another and, therefore, do not feel troubled with this feature of a minimal regret equilibrium. Hence, a minimal regret equilibrium might be neither perfect nor proper. The converse also holds true. For instance, consider the game *G5*.

| | | | |
|-----|--------|-------|--------|
| | a | b | c |
| a | 1, 1 | 0, 0 | -1, -2 |
| b | 0, 0 | 0, 0 | 0, -2 |
| c | -2, -1 | -2, 0 | -2, -2 |

$G5$

The action profile (b, b) is a perfect and proper equilibrium of $G5$ (van Damme p15), but is not a minimal regret equilibrium. Consequently, the set of minimal regret equilibria is neither a subset nor a superset of the set of perfect (or proper) equilibria.

6 Concluding Remarks

In this paper, we have proposed a solution concept, *minimax regret equilibrium*, where players are uncertain about the conjectures of their opponents and rationality. A parametric variant of our solution concept, an ε -minimax regret equilibrium, where uncertainty is modeled as ε -contaminations, is particularly appealing and intuitive. In an ε -minimax regret equilibrium, players have, with probability $1 - \varepsilon$, correct conjectures about their opponents' play. This specification greatly simplifies the computation of equilibria and most importantly for applications, one can study how the equilibrium predictions change as the degree of uncertainty varies. Nash equilibria are ε -minimax regret equilibria with $\varepsilon = 0$. Remarkably, however, our predictions are more than a simple perturbation of the standard game theoretic outcomes when $\varepsilon = 0$. For instance, in the model of price competition with two identical firms, we show that even the slightest shadow of doubt about the behavior of a competitor creates a dramatic increase in equilibrium prices. Moreover, our prediction is essentially unique. Relaxing the assumptions of common knowledge in conjectures and mutual knowledge in rationality does not necessarily imply a loss of predictive power. We therefore believe that relaxing these assumptions might prove particularly important in explaining economic and social phenomena. Although further research is still required, our model explains price dispersion in price-setting environments, which is largely consistent with empirical and experimental observations (Baye and

Morgan (2004)).

Finally, we like to mention two issues, which we believe deserves further research. First, it would be nice to extend the concept of minimax regret equilibrium to extensive-form games. Hayashi (2007c) seems to be a good starting point. Second, we like to have a theory of learning of minimax regret equilibrium.

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