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COOPERATION, IMITATION AND CORRELATED MATCHING

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Cooperation, Imitation and Correlated Matching

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Abstract

We study a setting where imitative players are matched into pairs to play a Prisoners' Dilemma game. A well know result in such setting is that under random matching cooperation vanishes for any interior initial condition. The novelty of this paper is that we add a certain correlation to the matching process: players that belong to a pair were both parties cooperate repeat partner next period whilst all other players are randomly matched into pairs. This intuitive correlation introduced in the matching process makes cooperation the unique outcome in the long run under some conditions. Furthermore, we show that no assortative equilibrium exits.

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1 Introduction

Individuals learn by imitation when their choices are based on the alternatives they observe others choose. The inability of real life subjects to correctly understand and process all the information they dispose of is a common justification for the use of imitation in some economic models. For example, in a situation where many players interact with each other, correctly anticipating other agents' actions can be a massive computational burden.

A well known property of imitation is that, under some conditions, it rules out dominated actions[1]. Thus, if every period imitative players are randomly matched to play a Prisoner's Dilemma game, cooperation vanishes. Given the importance of cooperation and its constant presence in societies and the relevance of imitation for modeling bounded rational behavior (see, for example, Axelrod (1984), Banerjee (1992), Eshel et al (1998) or Ellison and Fudenberg (1995)), the question we raise is: can cooperation survive when players learn by imitation?

We answer this question by exploring the mechanism by which players are matched to play a Prisoners' Dilemma game. The novelty of this paper is that a certain correlation is introduced in the matching process: players who cooperated with each other last period meet again in the next period whilst the rest of players are randomly matched into pairs. This matching mechanism is inspired by a simple rule of thumb: no player should have incentives to keep a non-cooperative partner. Examples of situations where this matching mechanisms seems plausible range from dealing with business suppliers to academic co-authorship or dating.

In the results of this paper, three main conclusions are achieved: First, under some conditions and for any interior initial condition, the survival of cooperation is guaranteed. That is, the situation where no player cooperates is not stable if some conditions on the payoff matrix and/or the specific imitative rule employed are satisfied. Second, no assortative equilibrium exists. This means that, apart from the equilibria on the boundaries, a situation where cooperative players do not face non-cooperative ones is not an equilibrium. Finally, we find that cooperation is more likely to prevail if imitation happens infrequently. In the limit this means that for all payoff matrices there exists a probability of imitating below which some level of cooperation is always present in the long run.

The reason behind the survival of cooperation lies in the fact that the matching mechanism considered in this paper adds a positive externality to playing cooperatively: in a situation where two players cooperate, switching action has the disadvantage that next period a new opponent, who might not be so keen on playing cooperatively, is faced. Thus, players that

[1] See, for instance, Schlag (1998) Remark 6.

cooperate may enjoy more payoff over time than these not cooperating. In this situation, non-cooperative players imitate cooperative ones, making the survival of cooperation possible.

To our knowledge, only Levine and Pesendorfer (2007), Bergstrom (2003) and Bergstrom and Stark (1993) study similar settings to the one considered in this paper. Levine and Pesendorfer (2007) show that cooperation can survive within a population who learns by imitation if each player holds some information about the strategy of the player with whom she is matched. Bergstrom (2003) and Bergstrom and Stark (1993) proves conditions under which cooperation survives in an evolutionary model where players are either cooperators or defectors, and are more likely to face a player of their same type.

The difference between this paper and Levine and Pesendorfer (2007) lies in that in our model there is a set of matches that are anonymous whilst in Levine and Pesendorfer (2007) all matches are non-anonymous to a certain degree. The present paper differs from Bergstrom and Stark (1993) and Bergstrom (2003) in that players can change their actions from one period to another. Thus, in our model, playing cooperatively in the present period is no guarantee of exhibiting a cooperative behavior in the next period.

The issue of partner selection in cooperative games has recently attracted attention from experimental economists. Duffy and Ochs (2009) conduct an experiment where a Prisoners Dilemma game with two treatments is considered. In the first treatment, matching is completely random whereas in the second one each player always repeats partner. The authors find that cooperation does not emerge in the random matching setting while it does in the fixed pairs treatment. Yang et al (2007) present an experiment where a Prisoner Dilemma game is played and individuals with similar histories are more likely to be matched together. Their results show that cooperation has a higher chance of survival when a history-dependent correlation is added to the matching process. Grimm and Mengel (2009) develop an experiment where players choose between two Prisoner's Dilemma games that differ in the gains from defection. Choosing the game with lower gains signals the player's willingness to cooperate. Grimm and Mengel find that this self selection significantly increases the amount of cooperation.

In order to get a better understanding on cooperation a preferential partner selection, we carry robustness checks and extensions to our main model. In particular, alternative matching processes are considered as extensions to the main model; apart from the matching mechanism whereby only cooperative pairs are maintained, we discuss the cases of complete assortative matching (cooperators only meet cooperators, defectors only meet defectors), all pairs are kept with some fixed probability and, finally, a setting where correlation is not perfect (only a fraction of cooperative pairs are maintained from one period to another). We argue in which of these settings cooperation is more likely to be sustained in the long run and in which ones cooperation does not survive.

The rest of the paper is organized as follows. In Section 2, we develop the model. Section 3 presents the main analysis and the results. In Section 4, we present a further comparison with the literature, a discussion on our assumptions, and some extensions. Finally, Section 5 concludes.

2 The Model

Consider a continuum of identical players uniformly distributed on the interval $[0, 1]$ with the standard Borel-Lebesgue measure[2]. At the beginning of each period $t = 0, 1, 2, \ldots$, every player is paired with another one and plays the following symmetric stage game against her partner:

Table 1: The Stage Game

	C	D
C	R, R	S, T
D	T, S	P, P

where C stands for cooperate and D stands for defect. The stage game above has the standard Prisoners' Dilemma structure: $T > R > P > S$ with $T, R, P, S \in \mathbb{R}$.

After the stage game is played, all pairs where at least one player chose D are broken while the rest of pairs are maintained. After that, unpaired players are randomly matched into pairs. The distribution of pairs at the beginning of period $t = 0$ is given.

Given the description above, at the beginning of each period $t \geq 1$ the population is divided into three sets: players who played C last period and faced an opponent who also played C, σ_{CC}, players who played C but faced an opponent who played D, σ_{CD}, and the rest, denoted by σ_D. We use σ_{CC}, σ_{CD} and σ_D to denote exchangeably both the sets and their respective measure. Thus, for instance, σ_{CC} is both the set of players who played C and faced an opponent who also chose C, and the measure (fraction) of players who played C and faced an opponent who also chose C.

Given the description above, we have that $\sigma_{CC} \in [0, 1]$, $\sigma_{CD} \in [0, 1)$ and $\sigma_D = 1 - \sigma_{CC} - \sigma_{CD}$. Evidently, $\sigma_{CD} + \sigma_{CC} \leq 1$ with equality only in the case when $\sigma_{CC} = 1$ (if $\sigma_{CD} + \sigma_{CC} = 1$ then all players chose C and, thus, all players faced another one playing C).

[2] All the results that follow are still valid if instead we consider a discreet but big population.

Notice that the fraction of players who maintain partner equals σ_{CC}. Furthermore, note that all players in σ_{CD} are matched with a player in σ_D and, thus, $\sigma_{CD} \leq \sigma_D$.

Define Ω as $\Omega = \{(\sigma_{CC}, \sigma_{CD}) \in \mathbb{R}^2_+ : \sigma_{CC} + \sigma_{CD} < 1 \ \cup \ (\sigma_{CC}, \sigma_{CD}) = (1, 0)\}$. Whenever we refer to interior points we mean $(\sigma_{CC}, \sigma_{CD}) \in \Omega$ with $\sigma_{CC} + \sigma_{CD} \in (0, 1)$ and $\sigma_{CD} \leq \sigma_D$. We denote the set of interior points by $\mathring{\Omega}$.

Players follow very simple decision rules. In particular, they observe the action and payoff of a random individual[3] and base their choice of action for the stage game on this information plus the information from own action and payoff. All players in the population are equally likely to be observed.

Let $A \in \{C, D\}$ be the action set and let $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) \in [0, 1]$ be the probability with which player $i \in [0, 1]$ changes action if she, who played action $a_i \in A$ and obtained payoff $\pi_i \in \mathbb{R}$, observes player $j \in [0, 1]$, who chose action $a_j \in A$ and achieved payoff $\pi_j \in \mathbb{R}$. Some assumptions on P are needed for the analysis:

Assumptions.

1. *If* $a_i = a_j$ *then* $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) = 0$,
2. $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) > 0$ *if and only if* $\pi_i < \pi_j$ *and,*
3. *for all* $i, j \in [0, 1]$ *and all* $a_i, a_j \in A$*:*
   - *if* $\pi_j > \pi'_j$ *then* $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) \geq P(\{i, a_i, \pi_i\}\{j, a_j, \pi'_j\})$*,*
   - *if* $\pi_i < \pi'_i$ *then* $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) \geq P(\{i, a_i, \pi'_i\}\{j, a_j, \pi_j\})$*.*

The first two assumptions are standard in imitation models (see, for instance, Schlag (1998)). Assumption 1 implies that players change their action only if the player they observe played a different action than the one they chose. Assumption 2 means that there is a positive probability of changing action if and only if observed action yielded more payoff than own action. The third assumption is a monotonicity condition that relates to reinforcement learning models (see, for example, Börgers et al (2004) and Rustichini (1999)). It means that the probability of changing action is weakly increasing in observed payoff and weakly decreasing in own payoff.

We simplify notation when using the function $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\})$ as follows: Denote by $P_{CC} : A^2 \times \mathbb{R}^2 \rightarrow [0, 1]$ the probability with which a player in σ_{CC} changes to D. Let $P_{CD} : A^2 \times \mathbb{R}^2 \rightarrow [0, 1]$ be the probability with which a player who belongs to σ_{CD} changes

[3]Since we are dealing with a continuous population, results presented in this paper do not depend on how many players are observed.

to D. Finally, denote by $P_D : A^2 \times \mathbb{R}^2 \to [0,1]$ the probability with which a player in σ_D changes to C.

Assumptions $1-3$ impose some restrictions on the functional forms of P_{CC}, P_{CD} and P_D. The function P_{CC} is only positive if the player in σ_{CC} observes a player in σ_D (assumption 1) who faced a player in σ_{CD} (assumption 2). In this case, the payoff of observed player equals T while own payoff equals R. Thus, we can write P_{CC} for $\sigma_{CC} < 1$ as

$$P_{CC} \quad = \quad \sigma_{CD} f(T,R) \tag{1}$$

for some function $f : \mathbb{R}^2 \to [0,1]$. The two arguments in f are observed payoff and own payoff respectively. The function f is weakly increasing in its first argument and weakly decreasing in its second argument by assumption 3. Furthermore, by assumption 2, $f(\pi', \pi) = 0$ for any $\pi' > \pi$.

The function P_{CD} is only positive if the player in σ_{CD} observes a player in σ_D. In this case, two different situations arise: If the player observed faced a player in σ_{CD}, then observed payoff equals T and own payoff equals S. On the other hand, if the observed player faced an opponent in σ_D, then observed payoff equals P and own payoff equals S. Therefore, we have that

$$P_{CD} \quad = \quad \sigma_{CD} f(T,S) + (\sigma_D - \sigma_{CD})\, f(P,S). \tag{2}$$

Finally, P_D is only positive if the player in σ_D faced a player also in σ_D and observed a player that belongs to σ_{CC}. In this case, observed payoff equals R while own payoff equals P. Hence, we have that

$$P_D \quad = \quad \frac{\sigma_D - \sigma_{CD}}{\sigma_D} \sigma_{CC} f(R,P) \tag{3}$$

if $\sigma_D > 0$, $P_D = 0$ otherwise.

Let σ_{CC}^t and σ_{CD}^t denote the values of σ_{CC} and σ_{CD} respectively at each point in time $t = 0, 1, 2, \ldots$ before the stage game is played with $(\sigma_{CC}^0, \sigma_{CD}^0) \in \Omega$ given. At $t = 0$ and prior to the starting of the game, all players not in σ_{CC}^0 are randomly and uniformly matched into pairs. For notational convenience the argument t in the functions P_{CC}, P_{CD} and P_D is omitted.

The model just described can be expressed with the following system of difference equations[4]:

$$\begin{aligned}\sigma_{CD}^{t+1} &= \sigma_{CC}^{t}\left(1-P_{CC}\right)+\sigma_{CD}^{t}\left(1-P_{CD}\right)+\sigma_{D}^{t}P_{D}\\ &\quad -\sigma_{CC}^{t}\left(1-P_{CC}\right)^{2}-\frac{\left(\sigma_{CD}^{t}\left(1-P_{CD}\right)+\sigma_{D}^{t}P_{D}\right)^{2}}{1-\sigma_{CC}^{t}}, \qquad (4)\\ \sigma_{CC}^{t+1} &= \sigma_{CC}^{t}\left(1-P_{CC}\right)^{2}+\frac{\left(\sigma_{CD}^{t}\left(1-P_{CD}\right)+\sigma_{D}^{t}P_{D}\right)^{2}}{1-\sigma_{CC}^{t}} \qquad (5)\end{aligned}$$

if $\sigma_D^t > 0$, $\sigma_{CD}^{t+1} = \sigma_{CC}^{t+1} = 0$ otherwise. Note that $\sigma_D^t > 0$ implies $\sigma_{CC}^t < 1$. Thus, σ_{CD}^{t+1} and σ_{CC}^{t+1} are both well defined in all Ω.

Equation (4) tells us the measure of players who played C in period t and faced a player who chose D in t. The value of σ_{CD}^{t+1} is computed as follows: The first three terms represent all players who played C in t (note that players in σ_{CC}^t and σ_{CD}^t played C in $t-1$ but may have played D in t). The fourth term subtracts the pairs in σ_{CC}^t where both players played C again in t. Finally, the fifth term subtracts the players not in σ_{CC}^t who chose C in t and faced a player one who also chose C in t.

Equation (5) is the measure of players who chose C in period t and faced an opponent playing C in t. The value of σ_{CC}^{t+1} is determined as follows: The first term adds the pairs in σ_{CC}^t where both players played C in t as well. The second term adds the players not in σ_{CC}^t who chose C in t and faced a player who also chose C in t.

Next, we define what an equilibrium of the model at hands is. Intuitively, an equilibrium is a situation where the measure of players belonging to each of the sets σ_{CC}, σ_{CD} and σ_D does not change. Formally:

Definition 1. *An equilibrium is a point* $(\sigma_{CC}, \sigma_{CD}) \in \Omega$ *such that* $\sigma_{CC}^{t+1} = \sigma_{CC}^t$ *and* $\sigma_{CD}^{t+1} = \sigma_{CD}^t$ *whenever* $\sigma_{CC}^t = \sigma_{CC}$ *and* $\sigma_{CD}^t = \sigma_{CD}$.

Definition 2. *An interior equilibrium is an equilibrium where* $(\sigma_{CC}, \sigma_{CD}) \in \mathring{\Omega}$.

Among all interior equilibria it is useful to single out the assortative equilibria. An assortative equilibrium is an interior equilibrium where a fraction of the population play C against themselves while all other players choose D. That is, in an assortative equilibrium $\sigma_{CD} = 0$ and the population is completely separated between cooperators and defectors.

Definition 3. *An assortative equilibrium is an interior equilibrium where* $\sigma_{CD} = 0$.

In order to illustrate the behavior of the model we present two simulations, both figures 1 and 2 show the evolution of σ_{CC}, σ_{CD}, and σ_D for certain parameter values where the

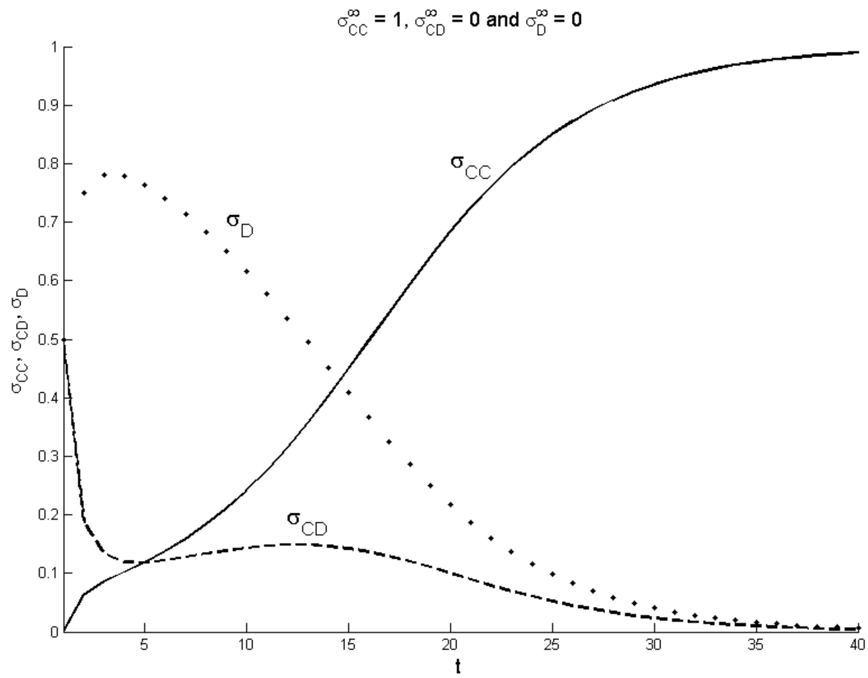
[4]Note that since we are dealing with a continuum population the system is deterministic.

function f is given by what is known as the Proportional Imitation Rule (PIR henceforth) with dominant switching rate (Schlag (1998))[5]:

$$f(\pi', \pi) \;=\; \frac{1}{R-S}\left(\pi' - \pi\right).$$

As it can be observed in figure 1, during the first periods the amount of cooperative players matched with non-cooperative ones, σ_{CD}, decreases. This is due to the fact that, during these first stages, most cooperative players enjoy less payoff than cooperative ones. However, as times evolves, more and more cooperative players meet each other. After this grouping stage is over, the payoff from cooperating is on average greater than that from not cooperating. This happens because most cooperative players face players that are also cooperative. The level of cooperation increases from there until all players have adapted the cooperative action.

Figure 1: Simulation: PIR with $T = 0.5$, $R = 0.4$, $P = 0$, $S = -0.1$ and $(\sigma^0_{CC}, \sigma^0_{CD}) = (0, 0.5)$

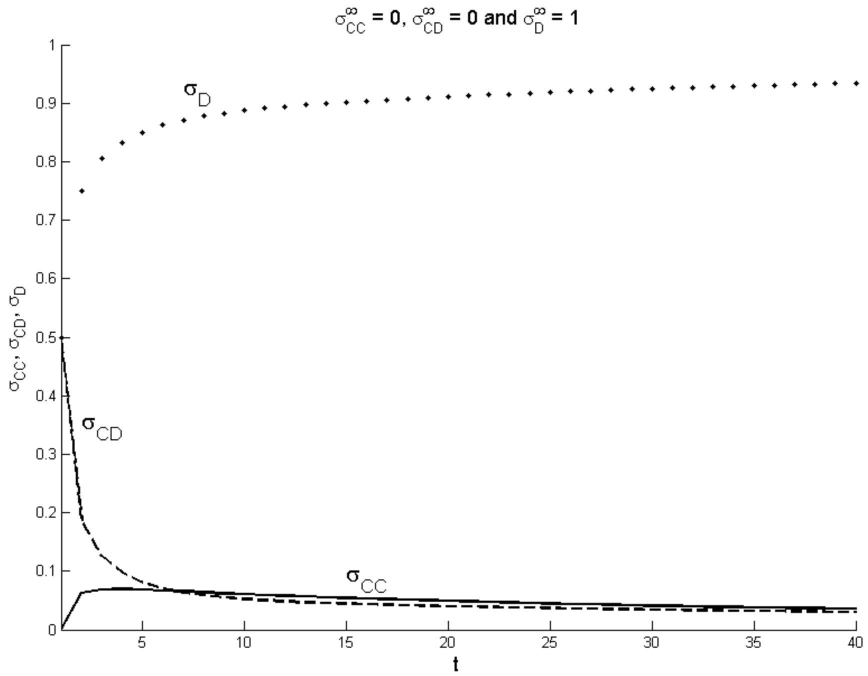


In figure 2, the payoff players in a cooperative pairs get is lower than in the previous simulation. This results in an environment where cooperation vanishes from the population. In figure 2 one can se that the number of players in σ_{CC} initially increases. This is simply due to the fact that some of the players that belong σ_{CD} are matched together and, if they do

[5]A deeper exposition of the relationship between this and other imitation rules and our model is presented in sections 3.3.1, 3.3.2 and 3.3.3.

not change their action immediately, they belong σ_{CC} next period. However, the aggregate level of cooperation decreases until cooperation eventually vanishes from the population.

Figure 2: Simulation: PIR with $T = 0.5$, $R = 0.1$, $P = 0$, $S = -0.1$ and $(\sigma^0_{CC}, \sigma^0_{CD}) = (0, 0.5)$



3 Results

3.1 Random Matching

In this subsection we consider the benchmark case of random matching. Under random matching, all pairs are broken after the stage game is played. We show that, under random matching, cooperation vanishes for any interior initial condition. The full analysis of the random matching case is presented in the appendix; here we restrict our attention to the main result from this analysis.

Proposition 1. *Under random matching, for any* $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$

$$\lim_{t\to\infty} \sigma^t_{CC} + \sigma^t_{CD} \quad = \quad 0.$$

Proof. See Lemma 1 in the appendix. □

As proposition 1 shows, under random matching cooperation does not survive in the population. This is the known result that under some monotonicity conditions (assumption 3) imitation rules out dominated actions.

With random matching, playing cooperatively is always dominated by the non-cooperative behavior. This is partly because under random matching cooperating has no effect in any period beyond the current one. As we shall see, once we add a certain correlation to the matching process, playing cooperatively may no longer be a dominated action.

3.2 Correlated Matching

We now revert back to the case explained in section 2 where there is correlated matching, i.e. pairs were both players cooperated are maintained in the next period. A first result is that there exist no assortative equilibrium.

Proposition 2. *No assortative equilibrium exists.*

Proof. First, note that assuming the necessary and sufficient equilibrium conditions $\sigma_{CC}^{t+1} = \sigma_{CC}^{t}$ and $\sigma_{CD}^{t+1} = \sigma_{CD}^{t}$ and adding equation (4) to equation (5) we obtain

$$\sigma_D P_D - \sigma_{CC} P_{CC} - \sigma_{CD} P_{CD} \quad = \quad 0. \tag{6}$$

The next step is to show that in an equilibrium with $\sigma_{CD} = 0$ no pairs in σ_{CC} are ever broken. Assume the contrary, this means that some players from σ_{CC} choose D. Hence, if we are at time t then $\sigma_{CC}^{t+1} < \sigma_{CC}^{t}$ unless a set of players in σ_D^t switch to C. If this happens, however, we have that some players will be matched against players who chose D in t. Therefore, if a pair is broken, either $\sigma_{CC}^{t+1} < \sigma_{CC}^{t}$ or $\sigma_{CD}^{t+1} > 0$, a contradiction to the definition of assortative equilibrium.

Given that in an assortative equilibrium no pairs are ever broken and that $\sigma_{CC} \in (0, 1)$, it follows that all players always choose the same action in the stage game. This implies that players in σ_{CC}^t obtain a payoff of R while players in σ_D^t obtain a payoff of P. Thus, from assumption 2, it follows that $P_{CC} = 0$ and $P_D > 0$. However, when $P_{CC} = 0$, equation (6) implies that

$$\sigma_D P_D \quad = \quad 0. \tag{7}$$

Since $\sigma_{CC} \in (0, 1)$, $\sigma_{CD} = 0$ and $P_D > 0$, we have that $\sigma_D P_D > 0$, a contradiction to (7). □

The intuition behind the result above is straightforward: In an assortative equilibrium, cooperative players, σ_{CC}, obtain a payoff of R whilst all the other players, $1 - \sigma_{CC}$, obtain a payoff of $P < R$. Hence, non-cooperative players imitate cooperative ones but cooperative players do not imitate non-cooperative ones. Therefore, the situation with complete separation between cooperators and defectors is not an equilibrium.

3.3 No Interior Equilibria

We continue the analysis focusing on situations where the payoff matrix and the function f are such that no interior equilibria exists. Interior equilibria are considered later in the paper. In the appendix we examine the conditions needed for the non-existence of interior equilibria. These conditions are not presented here simply for the ease of the exposition[6].

We are now ready to state one of the main results of this paper. Namely, if certain conditions on the payoff matrix and/or the specific imitative rule employed are satisfied, then in the long run all players in the population cooperate.

Proposition 3. *Assume no interior equilibria exist. If $f(R,P) > 2f(T,R)f(P,S)$, then for all $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$*

$$\lim_{t\to\infty} \sigma^t_{CC} \quad = \quad 1.$$

If $f(R,P) < 2f(T,R)f(P,S)$, then for all $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$

$$\lim_{t\to\infty} \sigma^t_{D} \quad = \quad 1.$$

Proof. Given that no interior equilibrium exists and that the equations for the dynamics of both σ_{CD} and σ_{CC}, equations (4) and (5), are continuous we have that no cycles can exist. Once this fact has been established the result of the proposition follows from proposition 4 in the next subsection. □

The idea behind the survival of cooperation is the following: Imagine a situation where only a small fraction of players cooperate. Some of these players will be matched together, thus, they repeat partner next period. This set of players playing cooperatively and that are matched together obtain the second-highest payoff, R. Since only very few players cooperate, there is almost no player obtaining the maximum payoff, T. Therefore, under certain conditions, more non-cooperative players imitate cooperative ones than cooperative players imitate non-cooperative ones.

[6] As one can see in the appendix, close form expressions of the conditions for the non-existence of interior equilibria are straightforward to compute for numerical examples. However, this is not the case when a general payoff matrix and function f are considered.

In what follows we seek a better understanding of the conditions in proposition 3 by exploring how the population behaves when different imitative rules are assumed. We focus our attention on three well known such rules: Proportional Imitation Rule, Imitate if Better and Proportional Reviewing Rule.

3.3.1 Proportional Imitation Rule

The general form of the PIR is given by

$$f(\pi', \pi) = s\left(\pi' - \pi\right)$$

where $s \in (0, 1/(T-S)]$ is called the switching rate. The simulation in figures 1 and 2 assumed $s = 1/(T-S)$. This value of the switching rate is known as the dominant switching rate as it leads to the imitation rule that yields a weakly higher expected increase than any other switching rate in the decision maker's payoff in a multi-armed bandit decision problem (Schlag (1998)).

From proposition 3, it is straightforward to show the following characterization for the PIR with dominant switching rate

Corollary 1. *Assume no interior equilibria exist. If the stage game is the one given in table 1 and players employ the PIR with switching rate* $s = 1/(T-S)$ *then for any* $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$*, if* $(R-P)(T-S) > 2(T-R)(P-S)$*, then* $\lim_{t\to\infty} \sigma^{\infty}_{CC} = 1$.

Proof. When players employ the PIR with $s = 1/(T-S)$ we can rewrite the condition for all player to cooperate in proposition 3 as

$$(R-P)(T-S) > 2(T-R)(P-S).$$

The result follows. □

To get a better understanding of the result above, we consider a particular case of the payoff matrix of the stage game.

Table 2: The Stage Game - Example

	C	D
C	$\pi_b - \pi_c, \pi_b - \pi_c$	$-\pi_c, \pi_b$
D	$\pi_b, -\pi_c$	$0, 0$

with $1 > \pi_b > \pi_c > 0$. We can interpret π_b as the benefit a player receives when her partner cooperates and π_c as the cost of cooperating. In this case, we have the following result.

Corollary 2. *Assume no interior equilibria exist. If the stage game is the one given in table 2 and players employ the PIR with switching rate $s = 1/(\pi_b + \pi_c)$ then for any $(\sigma_{CC}^0, \sigma_{CD}^0) \in \mathring{\Omega}$, if $\pi_b > \pi_c\sqrt{3}$, then $\lim_{t\to\infty} \sigma_{CC}^\infty = 1$.*

Proof. Comparing the stage games in tables 1 and 2 we have that $T = \pi_b$, $R = \pi_b - \pi_c$, $P = 0$ and $S = -\pi_c$. Using these values in corollary 1 gives the desired result. □

Although $s = 1/(T - S)$ is the dominant switching rate for multi-armed bandit problems, it is the switching rate that is less likely to make cooperation possible in the long run. This can be seen in the condition in proposition 3, the greater the value of f for any given payoff matrix, the less likely the condition for all players to cooperate holds. As a matter of fact, for any payoff matrix, if the switching rate is small enough then all players in the population cooperate in the long run for any interior initial condition.

Corollary 3. *Assume no interior equilibria exist. If the stage game is the one given in Table 1 and players employ the PIR then there exists a switching rate $\bar{s}$ such that for all $s < \bar{s}$ and any $(\sigma_{CC}^0, \sigma_{CD}^0) \in \mathring{\Omega}$, we have $\lim_{t\to\infty} \sigma_{CC}^\infty = 1$.*

Proof. When player employ the PIR we can rewrite the condition for all player to cooperate in proposition 3 as

$$s(R - P) \quad > \quad 2s^2(T - R)(P - S).$$

Thus, for any $T > R > P > S$ with $T, R, P, S \in \mathbb{R}$ we can choose s small enough so that the inequality above holds true. □

A conclusion that can be drawn from corollary 3 is that when players are more cautious in changing actions then cooperation is more likely to survive in the long run. As we shall see in the next subsection, this fact also holds in the limit.

3.3.2 Imitate if Better

Imitate if Better (IB) consists of simply imitating with probability one whenever the action observed yields more payoff than own action. That is, $f(\pi', \pi) = 1$ for all $\pi' > \pi$, $f(\pi', \pi) = 0$ otherwise. We have the following result.

Corollary 4. *Assume the stage game is the one given in table 1 and that players employ IB. For any* $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$*, we have* $\lim_{t\to\infty} \sigma^\infty_{CC} = 0$.

Proof. When player employ IB we can rewrite the condition for all player to cooperate in proposition 3 as $1 > 2$, which is evidently empty. □

As it was already hinted in the previous subsection when the PIR was considered, if the likelihood of imitation is higher, the chances of cooperation to arise are lower. In the limit, when the probability of imitating is one if observed payoff is higher than own payoff, cooperation vanishes for any interior initial condition.

The intuition for the fact that more cautious imitation makes cooperation more likely is that as the benefits from cooperating appear after cooperators repeat partner, if players are more likely to change their actions, then is much less likely that the benefits from repeating partner ever occur. Since these benefits are what makes cooperation possible in our setting, the fact that players change partner more often because of changing action more often makes cooperation harder to sustain. We extend this finding to a situation where interior equilibria are present and to any imitation rule in proposition 6 in the next subsection.

3.3.3 Proportional Reviewing Rule

The Proportional Reviewing Rule (PRR) is similar to the PIR except that own payoff is ignored. That is, the general form of the PRR is given by

$$f(\pi', \pi) \quad = \quad s\pi'$$

with $s \in (0, 1/(T-S)]$. Again, the parameter s is called the switching rate.

Corollary 5. *Assume the stage game is the one given in table 1 and that players employ the PRR with switching rate* $s = 1/(T-S)$*. For any* $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$*, if* $R(T-S) > 2TP$*, then* $\lim_{t\to\infty} \sigma^\infty_{CC} = 1$*.*

Proof. When player employ the PRR we can rewrite the condition for all players to cooperate in proposition 4 as

$$R(T-S) > 2TP.$$

□

An immediate consequence from the corollary above is that if players employ the PRR and that the stage game is the one in table 2, then in the long run all players cooperate for any interior initial condition. This is the case since in table 2, $P = 0$ and, thus, under the PRR players that belong to σ_{CD} only imitate those in σ_D that were matched with a player in σ_{CD}. Therefore, since players in σ_{CD} are less likely to change to the non cooperative action, their share in the population increases and eventually they are matched together and, hence, repeat partner next period, i.e. they belong σ_{CC}. This process continues until all players cooperate.

3.4 Interior Equilibria

We now consider situations where interior equilibria are present. The difficulty of dealing with these lie in the order of the system at hands. As the system in (4) and (5) is of order six, checking for the existence and/or stability of interior equilibria for a general payoff matrix and an arbitrary function f becomes a highly complex computational task. Ultimately, this means that we no longer present results about global converge. We proceed by restricting our attention first to local results and then we present a simulation where interior equilibria exist.

We need to define certain properties of the different equilibria when dealing with local results. The definitions below are based on Khalil (1995).

Definition 4. *Let $B_r(\sigma_{CC}, \sigma_{CD})$ be the ball of radius $r > 0$ around the point $(\sigma_{CC}, \sigma_{CD}) \in \Omega$. The equilibrium $(\sigma_{CC}, \sigma_{CD}) \in \Omega$ is*

- *stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(\sigma^0_{CC}, \sigma^0_{CD}) \in \Omega \cap B_\delta(\sigma_{CC}, \sigma_{CD})$ then $(\sigma^t_{CC}, \sigma^t_{CD}) \in \Omega \cap B_\varepsilon(\sigma_{CC}, \sigma_{CD})$ for all $t \geq 0$,*
- *unstable if it is not stable,*
- *asymptotically stable if it is stable and $\delta > 0$ can be chosen such that for any $\kappa < \varepsilon$ if $(\sigma^0_{CC}, \sigma^0_{CD}) \in \Omega \cap B_\delta(\sigma_{CC}, \sigma_{CD})$ then*

$$|| \lim_{t\to\infty} (\sigma^t_{CC}, \sigma^t_{CD}) - (\sigma_{CC}, \sigma_{CD})|| < \kappa,$$

- *a repeller if there exists a $\delta > 0$ such that if $(\sigma^0_{CC}, \sigma^0_{CD}) \in \Omega \cap B_\varepsilon(\sigma_{CC}, \sigma_{CD})$ for all $\varepsilon \in (0, \delta)$ then $(\sigma^t_{CC}, \sigma^t_{CD}) \notin \Omega \cap B_\delta(\sigma_{CC}, \sigma_{CD})$ for some $t \geq 0$,*

Cooperation in the long run

Even if interior equilibria exist and independently on whether they are stable or not, cooperation can survive in the population under the same conditions as those in proposition 3. Our second main result states this very fact. The reason for the survival of cooperation in the presence of interior equilibria is the same as the one behind the intuition of proposition 3.

Proposition 4. *If* $f(R,P) > 2f(T,R)f(P,S)$*, then the equilibrium* $(0,0) \in \Omega$ *is a repeller. On the other hand, if* $f(R,P) < 2f(T,R)f(P,S)$*, then the equilibrium* $(0,0) \in \Omega$ *is asymptotically stable.*

Proof. See the appendix. □

Note that most of the analysis carried out on the PIR, IB and PRR still applies in the presence of interior equilibria. The only difference is that if no interior equilibria are present then cooperation being sustainable implies that all players cooperate in the long run.

All cooperate asymptotically stable

Even if the condition in proposition 4 is not satisfied, cooperation may still survive if the initial amount of cooperators is high enough. This is proven in our next result.

Proposition 5. *If* $f(T,R)\left(2 + f(T,R)\left(\frac{2}{f(R,P)} - 1\right)\right) < 1$*, then the equilibrium* $(1,0) \in \Omega$ *is asymptotically stable. On the other hand, if* $f(T,R)\left(2 + f(T,R)\left(\frac{2}{f(R,P)} - 1\right)\right) > 1$*, then the equilibrium* $(1,0) \in \Omega$ *is a repeller*[7].

Proof. See the appendix. □

The relationship between the conditions in proposition 4 and proposition 5 are ambiguous as parameter values and imitation functions can be found such that all four possible combinations are possible.

To understand the result in proposition 5, imagine a situation where almost all players cooperate. In this case, if most defectors face other defectors, then cooperative players achieve higher payoff than non-cooperative ones. Thus, under certain conditions, the amount of cooperators increases until all players cooperate. Assume, on the other hand, that most defectors face cooperators. In this situation, defectors achieve higher payoff than cooperators and, thus, the total amount of cooperation decreases. However, the correlation in the matching process favors matches between cooperators and tends to leave defectors matched with other defectors. If the condition in proposition 5 is satisfied, the payoff from cooperating eventually surpasses that of non-cooperating and the amount of cooperation increases in the population until all players cooperate.

[7] If $f(R,P) = 0$ then $\sigma_D^t \geq \sigma_D^0$ for all t.

Cautious imitation

As already hinted when the different imitation rules where explored, we can deduce from proposition 4 that more cautions imitation makes cooperation more likely. Hence, in a setting where players are less likely to change actions cooperation is more likely to be present in the long run. If we consider the limit where players change actions very infrequently we have the following result:

Proposition 6. *Assume the stage game is the one given in table 1. There exists a function* $\bar{f}$ *such that for all* $f(x,y) < \bar{f}(x,y)$ *with* $x, y \in \mathbb{R}$ *and any* $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$*, we have* $\lim_{t\to\infty} \sigma^\infty_{CC} > 0$.

Proof. Take any function f that can be written as $f(\pi', \pi) = sg(\pi', \pi)$ for some $s > 0$ and some function g weakly increasing in its first argument and weakly decreasing in its second argument. Examples of such functions include the PIR, the PRR and $f(\pi', \pi) = s$.

The condition in proposition 4 means that cooperation does not vanish from the population if and only if $f(R,P) > 2f(T,R)f(P,S)$. This condition can be rewritten as $sg(R,P) > 2s^2 g(T,R)g(P,S)$. Thus, for any $T > R > P > S$ and any g we can find an $s > 0$ small enough so that cooperation survives in the long run. This gives the desired result. □

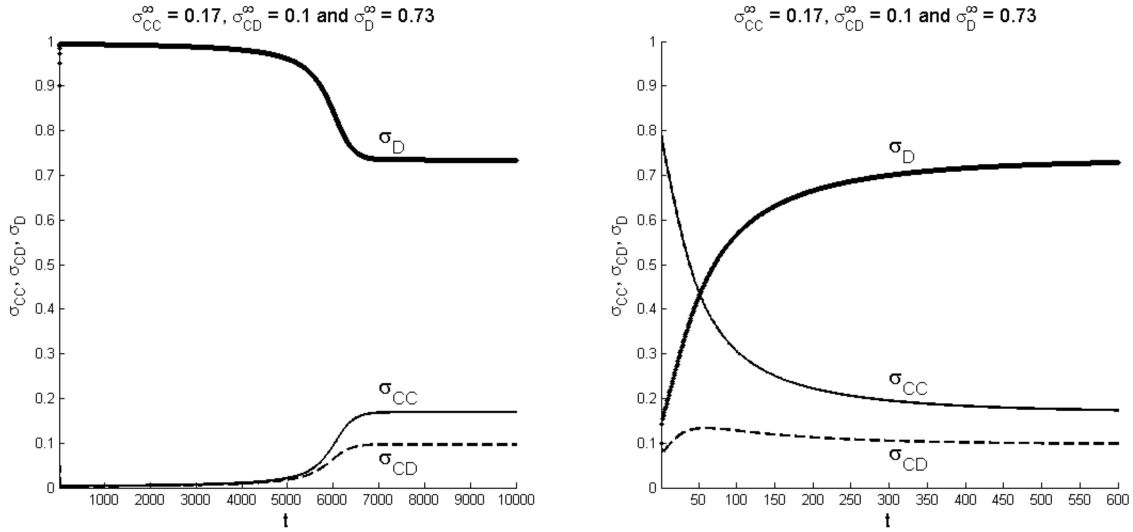
Note that proposition 6 above also means that if no interior equilibria are present then for all payoff matrixes we can find a function $\bar{f}$ such that for all $f < \bar{f}$ the unique stable equilibrium has all players cooperating.

Simulation

The behavior of the model when interior solutions are present is illustrated in figure 3. In the simulation performed on the left hand size the initial level of cooperation in relatively low, $(\sigma_{CC}, \sigma_{CD}) = (0, 0.1)$. On the contrary, in the simulation on the right hand side the initial level of cooperation in relatively high, $(\sigma_{CC}, \sigma_{CD}) = (0.8, 0.1)$. The parameters of the payoff matrix are set to the same values as those in figures 1 and 2 except that the value of R, the payoff a cooperative couple obtain, lies in between the one used in figure 1 and the one used figure 2. The imitation rule employed is again given by the PIR with dominant switching rate. One can check that the parameter values and imitation function used are such that both $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ and $(\sigma_{CC}, \sigma_{CD}) = (1, 0)$ are repellers.

As we can see on the left hand side of figure 3, the system converges to an interior equilibrium where slightly over 18% of the population cooperates. On the right hand side of figure 3, the initial level of cooperation is relatively high as 90% of the population initially cooperates yet this results in the same level of cooperation in the long run as before.

Figure 3: Simulation: PIR with $T = 0.5$, $R = 0.26$, $P = 0$, $S = -0.1$. Left hand side: $(\sigma^0_{CC}, \sigma^0_{CD}) = (0, 0.1)$, right hand side: $(\sigma^0_{CC}, \sigma^0_{CD}) = (0.8, 0.1)$.



4 Literature, Discussion and Other Matching Protocols

4.1 Literature: Bergstrom and Stark (1993) and Bergstrom (2003)

Bergstrom and Stark (1993) consider an evolutionary model where every player's behavior is hard wired to be either cooperate or defect. Each couples's offspring imitates either the behavior of their parents or the behavior of a random individual from the population and then plays a prisoner's dilemma game with each of her two siblings. The chances to survive to reproductive age depend on the payoff obtained and players that survive until reproductive age are then matched and reproduce. In Bergstrom and Stark (1993) cooperation does not survive if the offspring always imitate a random player from the population. This is the case as in their model the prisoner's dilemma game is played with one's siblings, whose behavior may not represent the average behavior in the population. That is, the prisoner's dilemma game is played locally with one's siblings yet the imitation takes place at a population level. In our model, imitation also takes place at a population level, all players in the population are equally likely to be observed. However, players who are not in a cooperative couple are randomly matched with another player from the entire population of non-cooperative couples. That is, as opposite to our model, in Bergstrom and Stark (1993) matching is local.

Bergstrom (2003) presents an evolutionary model where, as in Bergstrom and Stark (1993), players are hard wired to be either cooperators or defectors. In Bergstrom (2003) the shares of each of the two types of players in the population change according to their expected payoff. Thus, for instance, if cooperators get higher payoff than defectors then their share in the population increases whilst the share of defectors in the population decreases. In Bergstrom (2003) matching is assortative as the probability of meeting a player of the same type is different than the probability of meeting a player of a different type. The author shows conditions in the probability that matchings are assortative under which cooperation prevails in the long run. As we discuss below, if matching is completely assortative: cooperators only meet cooperators and defectors only meet defectors, then cooperation is more likely to arise than in the model presented in this paper, where assortative matching only occurs when the two players in a pair cooperate.

4.2 Discussion

The long run behavior of the population can be determined to a certain extend by the initial condition. For example, if no player cooperates initially, then no player ever cooperates. This fact disappears if, for example, mutations or mistakes are introduced in the model. Given that we are dealing with a continuum of population, introducing mistakes is straightforward.

Assume that at any given period with a small probability $\varepsilon > 0$ each player makes a mistake and chooses the action she intended not to. In this case and given that a continuum of population exists, each period exactly a fraction ε of players make mistakes. More specifically, a fraction $\varepsilon(\sigma_{CC} + \sigma_{CD})$ of players that intended to choose C play D, and a fraction $\varepsilon\sigma_D$ of players that intended to choose D play C.

Results presented are still valid if, in the model with mistakes, an equilibrium is defined as the situation where for any ε the change in σ_{CC} and σ_{CD} is always smaller or equal than $\varepsilon\sigma_{CC}$ and $\varepsilon\sigma_{CD}$ respectively. The convenience of adding mistakes is that unstable equilibria are eliminated. That is, in the model with mistakes, if $f(R,P) > 2f(T,R)f(P,S)$, then cooperation emerges independently of the initial conditions.

In the model presented, when it comes to imitating another player all agents in the population are equally likely to be observed. A sensible alternative is then to have correlation in sampling. For instance, one can consider a situation where cooperators are more likely to observe other cooperators and defectors are more likely to observe other non-cooperative players. In this case, if cooperation can be present in the model considered in this paper then cooperation is more likely in a setting where there is correlation in sampling. This is the case as if players are more prone to observe those who choose their same action, then chances of imitating are lower as a requirement for imitation is that a player choosing a different action should be observed. However, as it can be inferred from proposition 6, if the probability of imitating is lower, then cooperation is more likely to be present in the long run.

4.3 Other Matching Protocols

In this paper, we consider a matching mechanism whereby only the pairs where both players cooperate are maintained. This matching mechanism captures the simple idea that a player should have no incentives to repeat partner unless the partner played cooperatively last period. There are, however, other matching settings that could be considered. In this subsection we explore different matching protocols as well as justify why players may have incentives to keep cooperative partners only.

Assortative matching

An alternative matching protocol is such that matching is correlated for all the players who choose the same strategy as their partners, i.e. not only the couples where both players cooperate are maintained, pairs where both players do not cooperate are also maintained. In this case cooperation is possible for a bigger set of parameter values than in our main model. This is the case since the payoff of non-cooperative players is lower than in our original model as this players are less likely to be matched with a player being cooperative. In the model presented, this is as if players in σ_D who where matched with a player also in σ_D repeat partner and, thus, cannot be matched with a player in σ_{CD}, which is the matching that gives the highest payoff to defectors.

All pairs are kept

Another sensible option is to assume that players always keep their partners. In this case, cooperation is more likely to be present in the long run when compared to our main model as defectors are less likely to find a cooperator to take advantage of. That is, if all players repeat partner, then assortative matching tends to occur faster. This is the case as the cooperative players that are matched with a non-cooperative one are more likely to change to the non-cooperative action as they repeat couple and their partner does not change action (as she gets the highest possible payoff). However, a population that is separated between cooperators and defectors is not stable as the former always get more payoff than the latter. Thus, the share of cooperative pairs increases as gradually every two players in a non-cooperative pair switch simultaneously to the cooperative action.

Pairs are kept with some fixed probability

A further matching protocol is such that players keep their partner with some exogenous probability. This setting is a mixture between the case where players never keep their partner (random matching, section 3.1) and the case just described above. Therefore, one should expect the chances that cooperation survives to depend on the exogenous probability by which pairs are kept.

Correlation is not perfect

A fourth alternative has cooperative pairs maintained with a probability that is less than one. This imperfect correlation setting makes cooperation harder to be sustained as what makes cooperation possible in the main model are the benefits from repeating partner when both parties cooperate.

Why keeping only cooperative partners?

We argued in the introduction that it seems a reasonable rule of thumb not to keep a non-cooperative partner. A question is then to which extend this rule of thumb can appear if players rationally decide whether to keep their partner or not. Given that players prefer cooperative partners simply because facing a cooperative player strictly payoff dominates facing a non-cooperative one, a rational player chooses the option where the chances of finding a cooperative partner are highest.

If players in σ_{CC} keep their current couple then the chances of having a cooperative partner are $1 - P_{CC}$. If, however, they choose not to keep their couple, then they are matched with another player at random from the population of σ_{CD} and σ_D. In this case the chances of finding a cooperative partner are $\frac{1}{1-\sigma_{CC}} \left(\sigma_{CD}(1 - P_{CD}) + \sigma_D P_D\right)$. It is not hard to show that

$$1 - P_{CC} > \frac{1}{1 - \sigma_{CC}} \left(\sigma_{CD}(1 - P_{CD}) + \sigma_D P_D\right)$$

and, thus, chances of having a cooperative couple are highest for players in σ_{CC} if they keep their current partner.

Clearly, players in σ_{CD} want to change partner has their couple played D and obtained the maximum possible payoff and, furthermore, by changing partner there is some probability of facing a player also in σ_{CD} who cooperates with them. Finally, players in σ_D have incentive to change partners if their chances of meeting a cooperative player increase by facing a random partner. If the player in σ_D faced a player in σ_{CD} then she looses her couple as the player in σ_{CD} wants top change partner. On the other hand, if the player in σ_D faced a player also in σ_D then the chances that their current partner cooperates is given by $\sigma_{CC} f(R, P)$ whilst the chances of having a cooperative partner if they change their couple are given by $\frac{1}{1-\sigma_{CC}} \left(\sigma_{CD}(1 - P_{CD}) + \sigma_D P_D\right)$. It is not hard to show that if $f(R, P) < 0.5$ then

$$\sigma_{CC} f(R, P) < \frac{1}{1 - \sigma_{CC}} \left(\sigma_{CD}(1 - P_{CD}) + \sigma_D P_D\right)$$

and, thus, all players in σ_D have the highest chances of meeting a cooperative player if they change their partner. If $f(R, P) \geq 0.5$ then under some circumstances players in σ_D who faced a player also in σ_D prefer to keep their current partner. As already argued above when assortative matching was considered, if this was allowed then cooperation would be possible for a bigger set of parameter values than in our main model.

5 Conclusions

The present paper investigated cooperation in a setting where players who learn by imitation are matched to play a Prisoner's Dilemma game. Our contribution to the literature lies in the way matching takes place: players that belong to a pair were both parties cooperated repeat partner while the rest of players are randomly matched into pairs.

In the benchmark case with random matching, we showed that cooperation vanishes for any interior initial condition. When moving to the correlated matching setting, we proved that if some conditions on the payoff matrix and/or the specific way imitation takes place are satisfied, then a positive amount of cooperation appears from any interior initial condition. Furthermore, we found that no assortative equilibrium exists and that a situation where all players cooperate can be stable in the long run. Finally, we showed that if players change actions less frequently then cooperation has higher chances of surviving.

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Appendix

Random Matching

With random matching, there is no need to distinguish between players who cooperated and were paired with a player who also cooperated, σ_{CC}, and players who cooperated and faced a player who did not cooperate, σ_{CD}. Thus, these two sets of players are grouped into the same set σ_C.

Let σ_C^{t+1} the fraction of players who chose C at time t with $\sigma_C^0 \in [0,1]$ given. Let $1-\sigma_C$ be the fraction of players who chose D at time t. Furthermore, let $PR_C : A^2 \times \mathbb{R}^2 \to [0,1]$ be the probability with which a cooperative player switches to D and let $PR_D : A^2 \times \mathbb{R}^2 \to [0,1]$ be the probability with which a player who chose D switches to C. Assume PR_C and PR_D satisfy Assumptions $1-3$. The evolution of σ_C is then given by

$$\sigma_C^{t+1} \quad = \quad \sigma_C^t(1-PR_C)+(1-\sigma_C)PR_D. \tag{8}$$

Proceeding in a similar fashion as in equation (2), PR_C is positive only if the player in σ_C observed a player in $1-\sigma_C$. Three different situations can occur now: First, if the player in σ_C faced a player in σ_C and observed a player who faced another one in σ_C, then own payoff equals R while observed payoff equals T. Second, if the player in σ_C faced a player in $1-\sigma_C$ and observed a player who faced another one in σ_C, then own payoff equals S while observed payoff equals T. Finally, if the player in σ_C faced a player in $1-\sigma_C$ and observed a player who faced another one in $1-\sigma_C$, then own payoff equals S while observed payoff equals P. Therefore, we have that

$$PR_C \;=\; (1-\sigma_C)\left[\sigma_C^2 f(T,R) + (1-\sigma_C)\sigma_C f(T,S) + (1-\sigma_C)^2 f(P,S)\right]. \tag{9}$$

On the other hand, we have that P_D is positive only if the player in $1-\sigma_C$ faced another one who played D, and observes an individual choosing C that faced a player who also chose C. In this case, observed payoff equals R while own payoff equals P. Hence, we can write PR_D as follows:

$$PR_D \;=\; (1-\sigma_C)\sigma_C^2 f(R,P). \tag{10}$$

Lemma 1. *With random matching only $\sigma_C = 1$ and $\sigma_C = 0$ are equilibria. Furthermore, for any $\sigma_C^0 \in (0,1)$*

$$\lim_{t\to\infty} \sigma_C^t \;=\; 0.$$

Proof. We can see from equation (8) that both $\sigma_C = 1$ and $\sigma_C = 0$ are equilibria. The proof is completed by showing that from any point $\sigma_C \in (0,1)$ the system converges to $\sigma_C = 0$.

If $\sigma_C \in (0,1)$, using Assumptions 2 and 3 we obtain the following:

$$\begin{aligned}
\sigma_C^2 f(T,R) + (1-\sigma_C)\sigma_C f(T,S) + (1-\sigma_C)^2 f(P,S) \;&>\; (1-\sigma_C)\sigma_C f(T,S) \\
&\geq\; (1-\sigma_C)\sigma_C f(T,P) \\
&\geq\; (1-\sigma_C)\sigma_C f(R,P).
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
\sigma_C f(R,P) \;<\; & \sigma_C^2 f(R,P) + \sigma_C^2 f(T,R) + \\
& (1-\sigma_C)\sigma_C f(T,S) + (1-\sigma_C)^2 f(P,S).
\end{aligned}$$

Multiply both sides by $\sigma_C(1-\sigma_C)$ and use equations (10) and (9) to obtain

$$PR_D \;<\; \sigma_C\left(PR_C + PR_D\right). \tag{11}$$

From (8) we have that $\Delta\sigma_C = PR_D - \sigma_C(PR_C + PR_D)$. Hence, by equation (11), we know that whenever $\sigma_C \in (0,1)$, $\Delta\sigma_C < 0$. Thus, no point $\sigma_C \in (0,1)$ can be an equilibrium and the system cannot converge to $\sigma_C = 1$ from any initial condition $\sigma_C \in (0,1)$.

We still have to show that the system cannot converge to a point that is not an equilibrium. This is straightforward since $\Delta\sigma_C$ is a polynomial in σ_C and, hence, continuous for all $\sigma_C \in [0,1]$. $\square$

On the Existence of Interior Equilibria

In order to characterize the existence of interior equilibria, we have to examine the equilibria of the system given in (4) and (5). As already stated in the main text, a necessary condition for equilibrium is that equation (6) has to hold. If we substitute the values of P_{CC}, P_{CD} and P_D then equation (6) becomes

$$\begin{aligned} \sigma_{CC}\sigma_{CD}\left(f(T,R)+f(R,P)\right)+\sigma_{CD}\sigma_D f(P,S) & \\ -\sigma_{CC}\sigma_D f(R,P)+\sigma_{CD}^2\left(f(T,S)-f(P,S)\right) &= 0. \end{aligned} \tag{12}$$

Furthermore, if we impose the necessary equilibrium condition $\sigma_{CC}^{t+1} = \sigma_{CC}^t$ and substitute (12) in (5) we obtain

$$\sigma_{CC}P_{CC}\left(2-P_{CC}\right)-\frac{1}{1-\sigma_{CC}}\left(\sigma_{CD}+\sigma_{CC}P_{CC}\right)^2 = 0.$$

If we then substitute the value of P_{CC} we have

$$\begin{aligned} (1-\sigma_{CC})\left(\sigma_{CC}\sigma_{CD}^2 f(R,T)^2-2\sigma_{CC}\sigma_{CD}f(T,R)\right) & \\ +\sigma_{CD}^2\left(1+\sigma_{CC}f(T,R)\left(2+\sigma_{CC}\right)\right) &= 0. \end{aligned} \tag{13}$$

Equations (12) and (13) together with the fact that $\sigma_D = 1-\sigma_{CC}-\sigma_{CD}$ and $\sigma_{CD} \leq \sigma_D$ are necessary and sufficient conditions for equilibrium. Thus, an interior equilibrium exists if there is a $(\sigma_{CC}, \sigma_{CD}) \in \mathring{\Omega}$ such that both (12) and (13) are satisfied.

Proof of proposition 3

Proof. Define the set $\Sigma_r = (\sigma_{CC}, \sigma_{CD}) \in \mathring{\Omega} \cap B_r(0,0)$. For sufficiently small $\varepsilon > 0$ we can disregard terms of order $o(\varepsilon^2)$ and write the system (4) and (5) when $(\sigma_{CC}, \sigma_{CD}) \in \Sigma_\varepsilon$ as

$$\begin{aligned} \sigma_{CD}^{t+1}-\sigma_{CD}^t &= \sigma_D^t\left(f(R,P)\sigma_{CC}^t - f(P,S)\sigma_{CD}^t\right), \\ \sigma_{CC}^{t+1}-\sigma_{CC}^t &= 0. \end{aligned}$$

The approximation above is correct up to a term of order ε^2. Thus, when the process is arbitrarily close to $(0,0)$, the change in σ_{CC} with respect to the change in σ_{CD} is negligible.

The system above converges to $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$[8]. Hence, if we start in Σ_ε with ε small, the process converges to a situation where $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$. The system may hit the path $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$ outside the set Σ_ε. This poses no problem as the further away from $(0,0)$ the system can be in this case is within the set $\Sigma_{\varepsilon\frac{f(R,P)}{f(P,S)}}$, which is also arbitrarily close to $(0,0)$ when ε is small.

After starting in Σ_ε and once the system reaches $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$, we can rewrite (5) as

$$\sigma_{CC}^{t+1} - \sigma_{CC}^{t} \quad = \quad \left(\sigma_{CC}^{t}\right)^2 \frac{f(R,P)}{f(P,S)}\left(\frac{f(R,P)}{f(P,S)} - 2f(T,R)\right).$$

The equation of the motion of σ_{CD} is irrelevant because in the neighborhood of $(0,0)$ the system moves along the path $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$ as we just proved. To be more precise, the Center Manifold Theorem is being used here (see Sastry (1999) Section 7.8 or Khalil (1995) Section 8.1).

By Bézout's Theorem, the system (4) and (5) has a finite number of solutions (see Kirwan (1992)). Thus, we can fix $\varepsilon > 0$ such that no equilibrium points exists in $\Sigma_\varepsilon \setminus (0,0)$.

For any $\kappa < \varepsilon$, if $f(R,P) > 2f(T,R)f(P,S)$ then $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} > 0$. Thus, since $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} > 0$ and $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$, if the system starts in the boundary of Σ_κ, then it will leave that set. Assume that the system, after leaving Σ_κ, does not hit the boundary of the other bigger set Σ_ε. Since for any point in Σ_ε we have that $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} > 0$, by continuity of (5) and (4) if the process does not hit the boundary of Σ_ε then we must have that there exists a point $(\sigma_{CC}, \sigma_{CD}) \in \Sigma_\varepsilon \setminus (0,0)$ such that $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} = 0$ and, thus, $\sigma_{CD}^{t+1} - \sigma_{CD}^{t} = 0$. That is, there must exists at least one equilibrium point in $\Sigma_\varepsilon \setminus (0,0)$, which is a contradiction.

Thus, if the process starts in Σ_κ, then it must hit the boundary of Σ_ε. We know that for any point in Σ_ε, if $f(R,P) > 2f(T,R)f(P,S)$ then $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} > 0$ and $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$. Thus, starting in boundary of Σ_κ the process leaves Σ_ε, which is the condition for the point $(0,0) \in \Omega$ to be a repeller.

Assume now that $f(R,P) < 2f(T,R)f(P,S)$. By continuity, $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} < 0$, $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$ and the fact that no equilibrium point exists in $\Sigma_\varepsilon \setminus (0,0)$, if the system starts in $\Sigma_\varepsilon \setminus \Sigma_\kappa$ then it eventually enters the set Σ_κ for any $\kappa < \varepsilon$. This is the condition for asymptotic stability. $\square$

[8] If $f(P,S) = 0$ then the result in the lemma follows.

Proof of proposition 5

Proof. Proceeding in a similar fashion as above, define the set $\Sigma'_r = (\sigma_{CC}, \sigma_{CD}) \in \mathring{\Omega} \cap B_r(1,0)$. For sufficiently small $\varepsilon > 0$, we can disregard terms of order $o(\varepsilon^2)$ and write σ_D^{t+1} using the system (4) and (5) when $(\sigma_{CC}, \sigma_{CD}) \in \Sigma'_\varepsilon$ as

$$\sigma_D^{t+1} - \sigma_D^t = \sigma_{CD}^t \left(f(R,P) + f(T,R) \right) - \sigma_D^t f(R,P).$$

Assume ε is such that no equilibrium points exists in $\Sigma'_\varepsilon \setminus (1,0)$. Then the system above converges to $\sigma_D = \sigma_{CD}\left(1 + \frac{f(T,R)}{f(R,P)}\right)$[9]. Hence, if we start in Σ'_ε with ε small, the process converges to a situation where $\sigma_D = \sigma_{CD}\left(1 + \frac{f(T,R)}{f(R,P)}\right)$. The system may hit the path $\sigma_D = \sigma_{CD}\left(1 + \frac{f(T,R)}{f(R,P)}\right)$ outside the set Σ'_ε. This poses no problem as the further away from $(1,0)$ the system can be in this case is within the set $\Sigma'_{\varepsilon\left(1+\frac{f(T,R)}{f(R,P)}\right)}$, which is also arbitrarily close to $(1,0)$ when ε is small.

After starting in Σ'_ε and once the system reaches $\sigma_D = \sigma_{CD}\left(1 + \frac{f(T,R)}{f(R,P)}\right)$, we can rewrite (5) as

$$\begin{aligned}\sigma_{CC}^{t+1} - \sigma_{CC}^t &= \frac{(\sigma_{CD}^t)^2}{1-\sigma_{CC}^t}\left(1 - f(T,R)\left(2 + f(T,R)\left(\frac{2}{f(R,P)} - 1\right)\right)\right) \\ &= \frac{(\sigma_{CD}^t)^2}{1-\sigma_{CC}^t} A\end{aligned}$$

with $A = 1 - f(T,R)\left(2 + f(T,R)\left(\frac{2}{f(R,P)} - 1\right)\right)$.

For any $\kappa < \varepsilon$, if $A < 0$ then $\sigma_{CC}^{t+1} - \sigma_{CC}^t < 0$. Thus, since $\sigma_{CC}^{t+1} - \sigma_{CC}^t < 0$ and $\sigma_D = \sigma_{CD}\left(1 + \frac{f(T,R)}{f(R,P)}\right)$, if the system starts in the boundary of Σ'_κ, then it will leave that set. Assume that the system, after leaving Σ'_κ, does not hit the boundary of the other bigger set Σ'_ε. Since for any point in Σ'_ε we have that $\sigma_{CC}^{t+1} - \sigma_{CC}^t < 0$, by continuity of (5) and (4) if the process does not hit the boundary of Σ'_ε then we must have that there exists a point $(\sigma_{CC}, \sigma_{CD}) \in \Sigma_\varepsilon \setminus (1,0)$ such that $\sigma_{CC}^{t+1} - \sigma_{CC}^t = 0$ and, thus, $\sigma_{CD}^{t+1} - \sigma_{CD}^t = 0$. That is, there must exists at least one equilibrium point in $\Sigma_\varepsilon \setminus (1,0)$, a contradiction.

Thus, if the process starts in Σ'_κ, then it must hit the boundary of Σ'_ε. We know that for any point in Σ'_ε, if $A < 0$ then $\sigma_{CC}^{t+1} - \sigma_{CC}^t < 0$ and $\sigma_D = \sigma_{CD}\left(1 + \frac{f(T,R)}{f(R,P)}\right)$. Thus, starting in boundary of Σ'_κ the process leaves Σ'_ε, which is the condition for the point $(1,0) \in \Omega$ to be a repeller.

Assume now that $A > 0$. By continuity, $\sigma_{CC}^{t+1} - \sigma_{CC}^t > 0$, $\sigma_D = \sigma_{CD}\left(1 + \frac{f(T,R)}{f(R,P)}\right)$ and the fact that no equilibrium point exists in $\Sigma_\varepsilon \setminus (1,0)'$, if the system starts in $\Sigma'_\varepsilon \setminus \Sigma'_\kappa$ then it eventually enters the set Σ'_κ for any $\kappa < \varepsilon$. This is the condition for asymptotic stability. □

[9] If $f(R,P) = 0$ then $\sigma_D^t \geq \sigma_D^0$ for all t. Furthermore, convergence is guaranteed as no equilibrium point exists in $\Sigma'_\varepsilon \setminus (1,0)$ and, thus, $\Sigma'_\varepsilon \setminus (1,0)$ cannot contain any cycle.