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Abstract

In a major contributions to behavioral economics, Loewenstein and Prelec (1992) set the foundations for the behavioral approach to decision making over time and derive the generalized hyperbolic discounting formula. Here we show that their assumption 'common difference effect with quadratic delay' cannot be weakened to 'common difference effect'.

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1. Introduction

In one of the major contributions to behavioral economics, Loewenstein and Prelec (1992) (henceforth LP) set the foundations for the behavioral approach to decision making over time. Furthermore, LP give the first axiomatic derivation of the *generalized hyperbolic discounting function.* This function has been the main, but not the only, alternative to the exponential discounting function[1]. Their derivation is based on two assumptions: *impatience* and *common difference effect with quadratic delay.* The latter is a strengthened version of another of their assumptions: *common difference effect.*

The question naturally arises whether the generalized hyperbolic discounting function can be derived from the weaker set: *impatience* and *common difference effect.* Here we give a negative answer to this question. We give a discounting function that satisfies *impatience* and *common difference effect* but not *common difference effect with quadratic delay.* Hence, the generalized hyperbolic discounting function cannot be derived from *impatience* and *common difference effect* alone.

2. Model and results

LP introduce five assumptions, all with good experimental bases (LP, II pp574-578). The two assumptions relevant to the derivation of the generalized hyperbolic discounting function, φ, are:

A1 (*impatience*) $\varphi : [0, \infty) \rightarrow (0, 1]$ is strictly decreasing[2] and $\varphi(0) = 1$.

A2 (*common difference effect*) If $s > 0$ and $t > 0$, then $\varphi(s)\varphi(t) < \varphi(s + t)$.

A1 is only implicit in LP, however, it is essential for Theorem 1 (below). Our formulation of A2 is equivalent to that of LP.

To derive the formula for generalized hyperbolic discounting (LP (15), p580), LP used a stronger form of A2:

A2a (*common difference effect with quadratic delay*) There exists $\alpha > 0$ such that, if $s > 0$ and $t > 0$, then $\varphi(s)\varphi(t) = \varphi(s + t + \alpha st)$.

Note that A2a $\Rightarrow$ A2 and that $\alpha = 0$ gives exponential discounting. LP call this axiom "common difference effect with linear delay". However, because of the presence of the term αst, "quadratic" may be more descriptive than "linear".

[1] The simpler *quasi-hyperbolic* formulation, due to Phelps and Pollack (1968) and later popularized by Laibson (1997) often tends to be used in applied theoretical work on account of its tractability. However, LP's formulation is the most general form of the *hyperbolic* discounting function. See LP for a brief history of the generalized hyperbolic discounting function.

[2] It is sufficient that φ be strictly decreasing in some interval: $(a, a + \delta)$, $a \geq 0, \delta > 0$.

Theorem 1 : *A1 and A2a hold if, and only if, the discount function is a generalized hyperbola:*

$$\varphi(t) = (1+\alpha t)^{-\frac{\beta}{\alpha}}, t \geq 0; \beta > 0 \text{ (α is as in A2a).} \tag{2.1}$$

Proof: See LP for the original proof. See al-Nowaihi and Dhami (2006) for a corrected version.

Theorem 2 : *The generalized hyperbolic discounting function (2.1) does not follow from A1 and A2.*

Proof: Define the class of functions:

$$\psi(t) = \left(1 + t^{\frac{1}{2}} + \alpha t\right)^{-\frac{\beta}{\alpha}}, t \geq 0; \alpha > 0, \beta > 0. \tag{2.2}$$

It is clear that (2.2) satisfies A1. To show that A2 is also satisfied, let $s > 0$ and $t > 0$. Starting with the obviously true inequality $s^{\frac{1}{2}}t^{\frac{1}{2}} > 0$, derive, in successive steps, the following inequalities:

$$s + t + 2s^{\frac{1}{2}}t^{\frac{1}{2}} > s + t, \tag{2.3}$$

$$\left(s^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^2 > s + t, \tag{2.4}$$

$$s^{\frac{1}{2}} + t^{\frac{1}{2}} > (s+t)^{\frac{1}{2}}, \tag{2.5}$$

$$s^{\frac{1}{2}} + t^{\frac{1}{2}} + s^{\frac{1}{2}}t^{\frac{1}{2}} + \alpha s^{\frac{1}{2}}t + \alpha s t^{\frac{1}{2}} + \alpha^2 st > (s+t)^{\frac{1}{2}}, \tag{2.6}$$

$$s^{\frac{1}{2}} + t^{\frac{1}{2}} + s^{\frac{1}{2}}t^{\frac{1}{2}} + \alpha s^{\frac{1}{2}}t + \alpha s t^{\frac{1}{2}} + \alpha^2 st + 1 + \alpha s + \alpha t > (s+t)^{\frac{1}{2}} + 1 + \alpha s + \alpha t, \tag{2.7}$$

$$\left(1 + s^{\frac{1}{2}} + \alpha s\right)\left(1 + t^{\frac{1}{2}} + \alpha t\right) > 1 + (s+t)^{\frac{1}{2}} + \alpha(s+t), \tag{2.8}$$

$$\left[\left(1 + s^{\frac{1}{2}} + \alpha s\right)\left(1 + t^{\frac{1}{2}} + \alpha t\right)\right]^{\frac{\beta}{\alpha}} > \left[1 + (s+t)^{\frac{1}{2}} + \alpha(s+t)\right]^{\frac{\beta}{\alpha}}, \tag{2.9}$$

$$\left[\left(1 + s^{\frac{1}{2}} + \alpha s\right)\left(1 + t^{\frac{1}{2}} + \alpha t\right)\right]^{-\frac{\beta}{\alpha}} < \left[1 + (s+t)^{\frac{1}{2}} + \alpha(s+t)\right]^{-\frac{\beta}{\alpha}}, \tag{2.10}$$

$$\left(1 + s^{\frac{1}{2}} + \alpha s\right)^{-\frac{\beta}{\alpha}}\left(1 + t^{\frac{1}{2}} + \alpha t\right)^{-\frac{\beta}{\alpha}} < \left[1 + (s+t)^{\frac{1}{2}} + \alpha(s+t)\right]^{-\frac{\beta}{\alpha}}, \tag{2.11}$$

$$\psi(s)\psi(t) < \psi(s+t). \tag{2.12}$$

Hence, (2.2) satisfies A1 and A2.

We now show that (2.2) does not satisfy A2a. Suppose ψ does satisfy A2a. Then a simple calculation gives:

$$st + 2s^{\frac{1}{2}}t^{\frac{1}{2}} + 2st^{\frac{1}{2}} + 2s^{\frac{1}{2}}t + 3\alpha st + 2\alpha s^{\frac{2}{3}}t^{\frac{1}{2}} + 2\alpha s^{\frac{1}{2}}t^{\frac{2}{3}} + 2\alpha st^{\frac{2}{3}} + 2\alpha s^{\frac{2}{3}}t + \alpha^2 st^2 + 2\alpha^2 s^{\frac{2}{3}}t^{\frac{2}{3}} + \alpha^2 s^2t^2 = 0. \tag{2.13}$$

which is absurd, since all terms on the left hand side are positive.

Hence the set of assumptions *impatience* and *common difference effect* is *strictly* weaker than the set of assumptions *impatience* and *common difference effect with quadratic delay.*

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