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FROM ORDIENTS TO OPTIMIZATION: SUBSTITUTION EFFECTS WITHOUT DIFFERENTIABILITY

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Ordients: Optimization and Comparative Statics without Utility Functions*

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Abstract

This paper introduces the concept of ordient for binary relations (preferences), a relative of the concept of gradient for functions (utilities). The main motivation for this study is to replace the binary relation at the center stage of economic analysis, rather than its representation (whenever it exists). Moreover, ordients have a natural economic interpretation as marginal rates of substitution. Some examples of ordientable binary relations include the lexicographic order, binary relations resulting from the sequential applications of multiple rationales or binary relations with differentiable representations. We characterize the constrained maxima of binary relations through ordients and provide an implicit function theorem and an envelope theorem.

Keywords: Binary relation, ordient, maxima, envelope theorem, implicit function theorem.

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1 Introduction

This paper introduces the concept of *ordient* for binary relations (preferences), a relative of the concept of gradient for functions (utilities). Ordients have a natural economic interpretation as *marginal rates of substitution* between alternatives. If a binary relation has an ordient at a given alternative (e.g., a bundle of goods, a profile of strategies, or an act), the ordient characterizes the possible trade-offs between alternatives that locally lead to strictly preferred and strictly worse alternatives. In other words, an increasing (respectively, decreasing) ordient of a binary relation at a given alternative characterizes the *improvement* (respectively, *worsening*) directions, and the binary relation is ordientable at the given alternative if it has both an increasing and decreasing ordient at that alternative. Importantly, the characterization of these local trade-offs does not necessitate the existence of a representation and, even less so, of a differentiable representation. The main motivation for this study is indeed to replace the binary relation at the center stage of economic analysis rather than its cardinal representation (if it exists). Our objective is to uncover what type of results can be generated with the minimal structure of ordientable binary relations.

Although the concept of ordient has not yet been formally defined, we next present important examples of ordientable binary relations so as to delineate the applicability of our concept.

Example 1: Multiple rationales. Suppose that the set of alternatives $\mathbf{X}$ is a subset of $\mathbb{R}^{nm}$, and let $\succcurlyeq$ be a binary relation on $\mathbf{X}$. For instance, an alternative x might represent the consumption bundles of n individuals over m goods or a strategy profile in a n-player game. An important class of preferences is the class of preferences resulting from the sequential applications of K rationales $(\succcurlyeq_1, \ldots, \succcurlyeq_K)$ such that $x \succ y$ if and only if $x \succ_1 y$ or there exists k^* with $x \sim_k y$ for all $k < k^*$ and $x \succ_k^* y$. A first example of such a binary relation is the classic lexicographic order. This corresponds to the case $n = 1$ and $\succ_k = >$, the "greater than" binary relation, for all k. Driffill and Rotondi (2004) argue that the preferences of the European Central Bank with respect to output gap and inflation is lexicographic, with inflation being the primary target. Another example is the efficiency-first relation of Tadenuma (2002), which results from the application of the Pareto rationale first and the "envy" rationale second. More precisely, suppose that each of the n individuals has a preference relation $\succcurlyeq_i^*$ on $\mathbb{R}_+^m$ and define the Pareto rationale $\succcurlyeq_1$ as $x \succ_1 y$ if and only if $x \succcurlyeq_i^* y$ for all i and $x \succ_i^* y$ for some i. The second rationale $\succcurlyeq_2$ Tadenuma considers is the envy relation.

For any allocation $x = (x_1, \ldots, x_n)$, define the envy set as $H(x) := \{(i, j) : x_j \succ_i^* x_i\}$: if $(i, j) \in H(x)$, individual i is envious of individual j's allocation at x. The envy relation $\succcurlyeq_2$ is then defined as follows: $x \succcurlyeq_2 y$ if and only if the cardinality of $H(x)$ is smaller than the cardinality of $H(y)$. The allocation x is preferred to the allocation y (according to $\succcurlyeq_2$) if the society is less envious at x than at y. Alternatively, the equity-first relation of Tadenuma results from the application of the envy rationale first and the Pareto rationale second. Yet, another example is given by the preferences of schools over students in schooling problems. Schools' preferences over students often result from the sequential application of several rationales (priorities) e.g., pupils in public care, catchment areas, number of siblings in the school, distance to the schools. For further examples, see among others, Apesteguia and Ballester (2008), Houy and Tadenuma (2009), Mariotti and Manzini (2007), Tadenuma (2005). Such binary relations often do not have representations and yet they might be ordientable. The lexicographic order is ordientable everywhere with the vector $(1, 0, \ldots, 0)$ as ordient. Similarly, the efficiency-first relation of Tadenuma, albeit incomplete, is ordientable everywhere provided that each individual preference relation is. Naturally, classic preferences are a special case: a single rational is needed.

Example 2: Decision under uncertainty. Let S be a finite set of states of the world and define $\mathbf{X} = \mathbb{R}_+^{|S|}$ as the set of all Savage acts over monetary prizes. If an individual's binary relation $\succcurlyeq$ over acts is complete, transitive, monotone and convex, then the binary relation $\succcurlyeq$ has a decreasing ordient everywhere (although the binary relation might have a representation with points of non-differentiability). Important examples include maxmin expected utility, minimax regret, Choquet expected utility, and variational preferences, among others. See Rigotti et al. (2008) for more examples. Moreover, the set of decreasing ordients at an act f coincides with the set of subjective beliefs at that act f and, thus, with the set of beliefs revealed by willingness and unwillingness to trade at f (Rigotti et al. (2008)).

Example 3: Differentiable utility functions. If a binary relation has a differentiable representation, the gradient of the representation at a non critical point is an ordient and the local trade-offs induced by any ordient coincide with the classical marginal rate of substitutions.

The core contributions of the paper are two-fold. Firstly, we derive necessary and sufficient conditions for constrained maximization problems. Again, it is not necessary to assume the existence of a representation and even less so of a differentiable representation. In particular, we provide two alternative characterizations: one characterization with increasing ordients and another one with decreasing ordients. These two characterizations parallel two equivalent formulations of the statement "x^* maximizes $\succcurlyeq$ on X." The first formulation states that x^* maximizes $\succcurlyeq$ on X if the strict upper contour set of $\succcurlyeq$ at x^* does not intersect X, while the second formulation states that the lower contour set of $\succcurlyeq$ at x^* is a superset of X. Naturally, the first formulation involves improvement directions and, thus, increasing ordients, while the second involves worsening directions and, thus, decreasing ordients. For instance, the second characterization is particularly important whenever preferences are assumed to be convex and monotone (since they have decreasing ordients everywhere). Secondly, we present ordinal versions of the envelope theorem and implicit function theorem; two important tools for comparative statics.

Our results have natural applications in economics. For instance, consider the classic problem of maximizing individual preferences over the budget set. At an interior maximum, the increasing (or decreasing) ordient of the preferences is collinear to the vector of prices. Moreover, the indirect preferences are ordientable with the vector composed of the Walrasian demand and the unit vector as an ordient. In turn, this result implies the celebrated Roy's identity. Again, since our approach only rests on the ordering of alternatives, it is more natural than the traditional (differential) approach. Moreover, our approach remains applicable even in situations where the traditional approach fails. As another example, consider an all-pay auction. For each profile of bids of a player's opponents, the payoff's function is discontinuous. Yet, the binary relation induced by the payoff function is ordientable. We can then apply our results to characterize the best-reply maps and, ultimately, the equilibria. Yet, another application is the maximization of incomplete preferences. For instance, Ok (2002) considers choice problems where an individual applies several binary relations to make his choice, and prove the existence of a multi-valued representation. Again, if the preferences are ordientable, our results make it possible to characterize the maxima without requiring the existence of a multi-valued representation. In sum, this paper provides tools for the optimization of binary relations, which dot not have to be representable, and thus makes it possible to study economic problems involving such binary relations. Naturally, if a binary relation has a differentiable representation, our characterizations coincide with the classical first-order necessary and sufficient conditions.

Rubinstein (2005) is an inspiration to this paper. Rubinstein introduces the concept of "differentiable" binary relations for continuous, convex and monotone binary relations. Unlike Rubinstein, we do not restrict our attention to continuous, convex and monotone binary relations. We consider general binary relations. However, our concept of ordientability coincides with Rubinstein's concept whenever the binary relation is continuous, convex and monotone.

Before presenting formal definitions, a caveat is in order. The term "ordient" is a contraction of the terms "order" and "gradient." This deliberate choice of non-standard terminology is an attempt to steer the reader away from the differential approach to economic theory. In fact, we initially used the term "ordinal gradient." Informal discussions with colleagues quickly taught us that the term "gradient" was inextricably associated with the concepts of functions, derivatives and their implications. The main message of this paper is that there is no impetuous need for utility functions and even less so for differential utility functions. In that respect, we fully concur with Rubinstein.

2 Ordients

This section defines the concept of *ordient* for binary relations, a relative to the concept of gradient for maps. Let $(\mathbf{X}, \succcurlyeq)$ be a totally pre-ordered set (i.e., $\succcurlyeq$ is complete, reflexive and transitive) with $\mathbf{X}$ an open and convex subset of $\mathbb{R}^n$.[1] In Section 6, we discuss how the concept of ordient generalizes if $\succcurlyeq$ is neither complete nor transitive. We denote by $\succ$ and $\sim$ the asymmetric and symmetric parts of $\succcurlyeq$. Let $x := (x_1, \ldots, x_i, \ldots, x_n)$ be a vector in $\mathbb{R}^n$. We write $x \geq 0$ if $x_i \geq 0$ for all $i \in \{1, \ldots, n\}$, $x >> 0$ if $x_i > 0$ for all $i \in \{1, \ldots, n\}$ and $x \neq 0$ if $x_i \neq 0$ for some $i \in \{1, \ldots, n\}$. For two vectors x and y, $x \cdot y$ denotes the inner product of x and y. For a set $X \subseteq \mathbf{X}$, we denote by $\text{int}X$ its interior, $\text{cl}X$ its closure, and let $\partial X := X \setminus \text{int}X$.[2] For a vector x, we denote $||x||$ its Euclidean norm. For concepts such as convex or locally non-satiated binary relations, we refer the reader to Mas-Colell et al. (1995) or Rubinstein (2005).

Directions. A direction d is a vector in $\mathbb{R}^n$. A direction d is an *improvement* direction of $\succcurlyeq$ at $x_0 \in \mathbf{X}$ if there exists $\varepsilon^* > 0$ such that any $\varepsilon \in (0, \varepsilon^*)$ with

[1] More generally, $\mathbf{X}$ might be an open and convex subset of a linear (vector) space endowed with a inner product.

[2] Note that ∂X does *not* denote the boundary of X. The boundary $\text{bd}X$ of X is $\text{cl}X \setminus \text{int}X$.

$x_0 + \varepsilon d \in \mathbf{X}$ implies $x_0 + \varepsilon d \succ x_0$. A direction d is a *worsening* direction of $\succcurlyeq$ at $x_0 \in \mathbf{X}$ if there exists $\varepsilon^* > 0$ such that any $\varepsilon \in (0, \varepsilon^*)$ with $x_0 + \varepsilon d \in \mathbf{X}$ implies $x_0 + \varepsilon d \prec x_0$.

Ordients. A vector g in $\mathbb{R}^n \backslash \{0\}$ is an *increasing ordient* of $\succcurlyeq$ at $x_0 \in \mathbf{X}$ if $d \cdot g > 0$ implies that d is an improvement direction of $\succcurlyeq$ at x_0. A vector g in $\mathbb{R}^n \backslash \{0\}$ is a *decreasing ordient* of $\succcurlyeq$ at $x_0 \in \mathbf{X}$ if $d \cdot g < 0$ implies that d is a worsening direction of $\succcurlyeq$ at x_0.

Definition 1 (Ordient) *A vector g in $\mathbb{R}^n \backslash \{0\}$ is an ordient of $\succcurlyeq$ at $x_0 \in \mathbf{X}$ if g is an increasing and decreasing ordient of $\succcurlyeq$ at x_0. The binary relation $\succcurlyeq$ is ordientable if it has an ordient at each vector $x \in \mathbf{X}$.*

Three preliminary remarks are worth making. Firstly, if there is an improvement direction of $\succcurlyeq$ at each $x_0 \in \mathbf{X}$, then $\succcurlyeq$ is locally non-satiated. In turn, this implies that if the binary relation $\succcurlyeq$ is ordientable, then it is locally non-satiated. To accommodate local satiation (i.e., "thick" indifference curves), a weaker concept of ordientability is needed. Section 6 introduces such a concept. Secondly, assume that g is an ordient of $\succcurlyeq$ at $x \in \mathbf{X}_1 \times \cdots \times \mathbf{X}_n$ and fix the first $m < n$ components of x. If the vector $(g_{m+1}, \ldots, g_n)$ is different from zero, then $(g_{m+1}, \ldots, g_n)$ is an ordient of the restriction of $\succcurlyeq$ to $\mathbf{X}_{m+1} \times \cdots \times \mathbf{X}_n$ at $(x_{m+1}, \ldots, x_n)$. Moreover, whenever the vector $(g_{m+1}, \ldots, g_n)$ is equal to zero, the subspace $\mathbf{X}_{m+1} \times \cdots \times \mathbf{X}_n$ is "less consequential" than the subspace $\mathbf{X}_1 \times \cdots \times \mathbf{X}_m$ in determining the improvement and worsening directions. Indeed, for any direction $d = (0, \ldots, 0, d_{m+1}, \ldots, d_n)$, we have $g \cdot d = 0$ and, thus, d does neither have to be an improvement direction nor a worsening direction. Thirdly, ordients are uniquely defined up to multiplication by a positive scalar: g and g' are ordients of $\succcurlyeq$ at x_0 if and only if $g = \lambda g'$ for some $\lambda > 0$. In other words, g and g' are collinear. However, two increasing/decreasing ordients need not be collinear. The set of increasing/decreasing ordients is convex. Furthermore, if $\succcurlyeq$ has an increasing ordient g at x_0 and a decreasing ordient g' at x_0, then g and g' must be collinear and, consequently, are ordients of $\succcurlyeq$ at x_0.

We now provide an economic interpretation of the concept of ordients. In consumer theory, ordients have a natural interpretation as *marginal rates of substitution between goods.* First of all, we need to define the marginal rate of substitution of good l for good k at x_0 without imposing unnecessary assumptions on the preferences and indifference sets. In particular, indifference sets might be singletons (e.g., lexicographic order) or non-differentiable manifolds (e.g., maxmin or minimax regret expected utility). We say that MRS_{lk} is the marginal rate of substitution of good l for good k at x_0 if for all d_k and d'_k with $d_k > MRS_{lk} > d'_k$, there exists $\varepsilon^* > 0$ such that $x_0 + \varepsilon(0, \ldots, -1, \ldots, d_k, \ldots, 0) \succ x_0 \succ x_0 + \varepsilon(0, \ldots, -1, \ldots, d'_k, \ldots, 0)$ for all $\varepsilon < \varepsilon^*$ (with -1 in the l-th component). In words, if we reduce the consumption of good l by a marginal unit, a marginal increase of good k makes the consumer strictly better off (resp., worse off) if $d_k > MRS_{lk}$ (resp., $d'_k < MRS_{lk}$). If the preference relation $\succcurlyeq$ is ordientable at x_0, then $MRS_{lk} = g_l/g_k$ provided that $g_k \neq 0$ since the direction $(0, \ldots, -1, \ldots, d_k, \ldots, 0)$ is an improvement (resp., worsening) direction whenever $-g_l + g_k d_k > 0$ (resp., $-g_l + g_k d_k < 0$). Naturally, if the preference $\succcurlyeq$ has a differentiable representation, this definition coincides with the textbook definition of marginal rates of substitution as ratio of marginal utilities.

This interpretation goes beyond consumer theory. For instance, in continuous games, an ordient captures the impact on a player's welfare of a marginal change of actions by some or all players. Similarly, if X is the set of Savage acts $a : S \to \mathbb{R}$ from a finite set of states of nature to monetary prizes, then an ordient of $\succcurlyeq$ represents the marginal rates of substitution between money in one state and money in another state.

We now present several examples. Our first example of an ordientable binary relation is the lexicographic order $\succcurlyeq$ on $\mathbb{R}^2$ where $x \succ x'$ if and only if $x_1 > x'_1$ or both $x_1 = x'_1$ and $x_2 > x'_2$. At each vector x, the vector $(1, 0)$ is an ordient of $\succcurlyeq$ at x. To see this, consider the point x^* in Figure 1 and the vector $(1, 0)$. Clearly, for all directions in the half plane on the right of x^* (vertical dashed lines), the inner product is strictly positive, while it is negative for all directions in the half plane on the left of x^* (horizontal dotted lines). Furthermore, the half plane on the right (resp., left) of x^* is included in the strict upper (resp., lower) contour set of $\succcurlyeq$ at x^*, as required. Therefore, and most importantly, the concept of ordientable binary relations does not imply the existence of representations. Moreover, this example illustrates the fact that marginal rates of substitution are not confined to the realm of differentiable representations. A binary relation might not be representable and yet we can meaningfully speak about marginal rates of substitution, provided that the binary relation is ordientable.

More generally, suppose that $\succcurlyeq$ results from the applications of K binary relations $(\succcurlyeq_1, \ldots, \succcurlyeq_K)$ such that $x \succ y$ if and only if either $x \succ_1 y$ or there exists $k^* \leq K$ such $x \sim_k y$ for all $k < k^*$ and $x \succ_{k^*} y$. Manzini and Mariotti (2007) call the choice correspondence resulting from the sequential application of binary relations

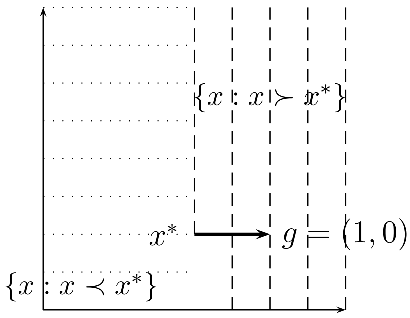


Figure 1: Lexicographic order

a *sequentially rationalizable choice correspondence.*[3] If $\succcurlyeq_1$ is ordientable, then $\succcurlyeq$ is ordientable: the ordient of $\succcurlyeq_1$ is an ordient of $\succcurlyeq$. For an example, consider the following binary relation on $\mathbb{R}_{++}$: $x \succ x'$ if and only if $x_1x_2 > x_1'x_2'$ or both $x_1x_2 = x_1'x_2'$ and $x_2 > x_2'$. The function $x \mapsto f(x) := x_1x_2$ represents the first rationale $\succcurlyeq_1$ and is ordientable: the vector (x_2, x_1) is an ordient of $\succcurlyeq_1$ at each point (x_1, x_2) and, consequently, is an ordient of $\succcurlyeq$.

The next series of examples deal with *representable* binary relations. A binary relation $\succcurlyeq$ on $\mathbf{X}$ is representable if there exists a real-valued function $f : \mathbf{X} \to \mathbb{R}$ such that $x \succcurlyeq x'$ if and only if $f(x) \geq f(x')$ for any pair (x, x'). Note that for any real-valued function f, there exists a binary relation $\succcurlyeq$ such that f represents $\succcurlyeq$.

The following example shows that a binary relation might be representable and yet it can fail to have an ordient at some points. Consider the Leontieff preferences represented by the function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ with $f(x) = \min\{x_1, x_2\}$. These preferences are not ordientable at points x with $x_1 = x_2$. Any $g \neq 0$ with $g \geq 0$ is a decreasing ordient, but there is no increasing ordient as improvement directions are within $\pi/2$ radian of each others. Note, however, that they are ordientable at any point with $x_1 \neq x_2$. For instance, if $x_1 > x_2$, then $(0, 1)$ is an ordient.

The next example shows that a binary relation might be ordientable and yet fail to have a representation that is differentiable everywhere. Consider the binary relation $\succcurlyeq$ on $(0, 3)$ induced by the function f with $f(x) = x$ if $x \leq 1$ and $f(x) = 3 - x$ if $x > 1$. The binary relation $\succcurlyeq$ is ordientable everywhere with ordient $g(x) = 1$ if $x \leq 1$ and $g(x) = -1$ if $x > 1$. Clearly, the binary relation $\succcurlyeq$ does not admit a differentiable representation at $x = 1$. Another example is given by the payoff function of an all-pay

[3] Note that Manzini and Mariotti do not assume the completeness and transitivity of each binary relation. See Section 6 for more on this.

auction.[4]

Finally, if a binary relation is representable by a differentiable function, then the gradient of the function is an ordient. The next proposition formally states this result.

Proposition 1 *Let f be a representation of $\succcurlyeq$. If f is differentiable at x with non-null derivative, then the gradient $\nabla f(x)$ of f at x is an ordient of $\succcurlyeq$ at x. Moreover, any ordient of $\succcurlyeq$ at x is collinear to $\nabla f(x)$.*

Proof Consider $d \in \mathbb{R}^n$. Since f is differentiable at x, the directional derivative $\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon d)-f(x)}{\varepsilon}$ exists and is equal to $d \cdot \nabla f(x)$. If $d \cdot \nabla f(x) > 0$, then there exists $\varepsilon^* > 0$ such that $f(x+\varepsilon d) > f(x)$ for all $\varepsilon \in (0, \varepsilon^*)$. Therefore, $\nabla f(x)$ is an increasing ordient of $\succcurlyeq$ at x. A similar argument shows that $\nabla f(x)$ is a decreasing ordient of $\succcurlyeq$ at x and, consequently, $\nabla f(x)$ is an ordient of $\succcurlyeq$ at x. The second statement follows from the fact that g and g' are ordients of $\succcurlyeq$ at x if and only if they are collinear. $\square$

The requirement of a non-null derivative (no critical points) cannot be dispensed with. For instance, consider the function $x \mapsto x - x^2$ on $(0, 1)$. At the point $x = 1/2$, the derivative is zero and no increasing ordient exists. However, any $g \neq 0$ is a decreasing ordient at $x = 1/2$. More generally, we have that x^* is a strict local maximum of $\succcurlyeq$ if and only if any $g \neq 0$ is a decreasing ordient at x^*. Section 6 contains a more detailled discussion of the relationship between ordients and unconstrained maxima. Furthermore, we like to stress that ordients and gradients are conceptually different. To see this, consider the function f on $\mathbb{R}$ with $f(x) = x^3$ and let $\succcurlyeq_f$ be the binary relation induced by f. The binary relation $\succcurlyeq_f$ is ordientable everywhere with $+1$ as an ordient, while 0 is a critical point of f. The gradient of f at 0 characterizes the best linear approximation of the function f at 0, while the ordient captures the improvement and worsening directions of $\succcurlyeq_f$. Naturally, the function f is only one out of the many possible representations of $\succcurlyeq_f$. In particular, the identity function is a representation of $\succcurlyeq_f$ for which the gradient coincides with the ordient.

We now provide further insights about ordients. The first insight is geometrical and links increasing (resp., decreasing) ordients to closest points to lower (resp., upper) contour sets. For any $x_0 \in \mathbf{X}$, denote $U(x_0) := \{x : x \succcurlyeq x_0\}$ the upper contour set of $\succcurlyeq$ at x_0 and $L(x_0) := \{x : x_0 \succcurlyeq x\}$ the lower contour set of $\succcurlyeq$ at x_0.

[4]Let b_{-i} be a profile of bids of player i's opponents and v_i player i's valuation. Assume that $\max\{b_j : j \neq i\} < v_i$. Then -1 is an ordient if $b_i \neq \max\{b_j : j \neq i\}$ and 1 is an ordient if $b_i = \max\{b_j : j \neq i\}$.

For any set X and $x \notin X$, denote $c_X(x)$ the set of closest points in X to x. If X is closed, then $c_X(x)$ is non-empty.

Proposition 2 *Let $x_0 \in \mathbf{X}$ and choose any $x_1 \notin U(x_0)$ and $x_2 \notin L(x_0)$. (i) For each $\overline{x} \in c_{U(x_0)}(x_1)$, the vector $(\overline{x} - x_1)$ is a decreasing ordient of $\succcurlyeq$ at $\bar{x}$. (ii) For each $\underline{x} \in c_{L(x_0)}(x_2)$, the vector $(x_2 - \underline{x})$ is an increasing ordient of $\succcurlyeq$ at $\underline{x}$.*

Proof (i) Let $\overline{x} \in c_{U(x_0)}(x_1)$. Since $\overline{x}$ minimizes the distance between x_1 and $U(x_0)$, we have that $||x - x_1||^2 \geq ||\overline{x} - x_1||^2$ for all $x \in U(x_0)$. Choose d such that $d \cdot (\overline{x} - x_1) < 0$. For $\varepsilon > 0$, we have that

$$0 > 2\varepsilon d \cdot (\overline{x} - x_1) = ||\overline{x} - x_1 + \varepsilon d||^2 - ||\overline{x} - x_1||^2 - \varepsilon^2 ||d||^2.$$

If $||\overline{x} + \varepsilon d - x_1||^2 \geq ||\overline{x} - x_1||^2$, it follows that

$$0 > 2d \cdot (\overline{x} - x_1) \geq -\varepsilon ||d||^2,$$

which cannot hold for all arbitrarily small $\varepsilon > 0$. Hence, there exists $\varepsilon^* > 0$ such that $\varepsilon \in (0, \varepsilon^*)$ implies $||\overline{x} + \varepsilon d - x_1||^2 < ||\overline{x} - x_1||^2$. This implies that $\overline{x} + \varepsilon d \prec \overline{x}$ for $\varepsilon \in (0, \varepsilon^*)$, which concludes the proof.

(ii) As part (i). $\square$

As an illustration of Proposition 2, consider the Leontieff preferences on $\mathbb{R}^2_+$. We have already argued that the Leontieff preferences have no increasing ordient on the 45° line. This is an immediate consequence of part (ii) of Proposition 2: For any x_0 on the 45° line, there is no x in the *strict* upper contour set of $\succcurlyeq$ at x_0 such that x_0 is the closest point to x. More generally, a locally non-satiated binary relation $\succcurlyeq$ has an increasing ordient at x_0 if there exists x in the *strict* upper contour set of $\succcurlyeq$ at x_0 such that x_0 is the closest point in the lower contour set of $\succcurlyeq$ at x_0 to x. A similar argument holds for decreasing ordients. This last observation is reminiscent of the concept of proximal normals in non-smooth analysis (see Clarke et al. (1998, Chapter 1)). Indeed, the vector $(x_2 - \underline{x})$ of Proposition 2(ii) is nothing else than a perpendicular to $U(x_0)$ at $\underline{x}$ (in the terminology of Clarke et al. (1998)). In turn, a proximal normal to $U(x_0)$ at $\underline{x}$ is any non-negative multiple of a perpendicular to $U(x_0)$ at $\underline{x}$. This implies the following relationship between proximal normal cones and ordients: If $\zeta \neq 0$ belongs to the proximal normal cone of $L(x_0)$ (resp., $U(x_0)$) at x_0, then ζ (resp., $-\zeta$) is an increasing (resp., decreasing) ordient of $\succcurlyeq$ at x_0. However, the converse is not true. For a counter-example, let $\mathbf{X} = \mathbb{R}^2$ and suppose that the function $|x_1|^{3/2} + x_2$ represents $\succcurlyeq$. At the point $(0,0)$, $(0,1)$ is a decreasing ordient of the binary relation $\succcurlyeq$. Yet, the point $(0,0)$ is clearly not the closest point in $U((0,0))$ to $(0,-1)$. In fact, $(0,0)$ is the unique proximal normal to $U((0,0))$ at $(0,0)$.[5]

The next insights we present regard convex binary relations, a common assumption in economics. In particular, we show that if a binary relation is convex, then it has a decreasing ordient at each boundary point of any upper contour set. Furthermore, we show that if a binary relation is continuous, convex and ordientable, then indifference sets are C^1-manifolds.

Proposition 3 *Let $x_0 \in \partial U(x_0)$ and assume that $U(x_0)$ is convex.*

(i) The binary relation $\succcurlyeq$ has a decreasing ordient at x_0.

(ii) Let g be a decreasing ordient of $\succcurlyeq$ at x_0. If $\succcurlyeq$ is continuous, then for any improvement direction d of $\succcurlyeq$ at x_0, we have $g \cdot d > 0$.

(iii) If any two decreasing ordients of $\succcurlyeq$ at x_0 are collinear and $\{x : x \sim x_0\} \subseteq \partial U(x_0)$, then $\succcurlyeq$ is ordientable at x_0.

(iv) If $\succcurlyeq$ is ordientable on $\partial U(x_0)$, then $\partial U(x_0)$ is a C^1-manifold.

Proof Part (i). Since $U(x_0)$ is convex, it follows from the supporting hyperplane theorem (see Theorem 5.3 in Aliprantis and Border (1999) p. 202) that there exists a half-space H_+ at x_0 supporting $U(x_0)$ at x_0 i.e., $U(x_0) \subseteq H_+$. Therefore, the complement H_- of H_+ is included in the strict lower contour set of $\succcurlyeq$ at x_0. The normal of the supporting hyperplane at x_0 in the direction of H_+ is thus a decreasing ordient of $\succcurlyeq$ at x_0.

Part (ii). From part (i), we know that there exists a decreasing ordient g of $\succcurlyeq$ at x_0. Consider an improvement direction d of $\succcurlyeq$ at x_0, i.e., there exists $\varepsilon^* > 0$ such that $x_0 + \varepsilon d \succ x_0$ for all $0 < \varepsilon < \varepsilon^*$. Since g is a decreasing ordient, we must have $g \cdot d \geq 0$. Assume that $g \cdot d = 0$. From the continuity of $\succcurlyeq$, for any $\varepsilon \in (0, \varepsilon^*)$, there exists ε' such that $x_0 + \varepsilon d - \varepsilon' g \succ x_0$. However, $g \cdot (x_0 + \varepsilon d - \varepsilon' g - x_0) = -\varepsilon' g \cdot g < 0$, a contradiction.

Part (iii). Since any two decreasing ordients of $\succcurlyeq$ at x_0 are collinear, it follows from part (i) that there exists a unique hyperplane supporting $\mathrm{cl}U(x_0)$ at x_0. Around x_0, the convex set $\mathrm{cl}U(x_0)$ can be expressed locally as the epigraph of a convex function F. Since there exists a unique hyperplane supporting $\mathrm{cl}U(x_0)$ at x_0, it follows that there is a unique subgradient g_0 of F at x_0 (see Rockafellar (1970) for the definition

[5] Note that if the lower (resp., upper) contour sets are closed, then the set of x such that $\succcurlyeq$ has an increasing (resp., decreasing) ordient is dense. (See Corally 6.2 p. 49 of Clarke et al. (1998).)

of subgradient). From Propositions 1.2 and 1.13 in Clarke (1975), it follows that F is differentiable at x_0 (with $\nabla F(x_0) = g_0$). Hence $g_0 \cdot d > 0$ implies that $x_0 + \varepsilon d \in \text{int}U(x_0)$ for ε sufficiently small. Lastly, since $\{x : x \sim x_0\} \subseteq \partial U(x_0) \subset \text{bd}U(x_0)$, we have that $x \succ x_0$ whenever $x \in \text{int}U(x_0)$ and, therefore, g_0 is an increasing ordient of $\succcurlyeq$ at x_0. From part (i), g_0 is a decreasing ordient and, therefore, g_0 is an ordient.

Part (iv). First, we show that $\partial U(x_0) = \{x : x \sim x_0\}$. Clearly, $\{x : x \sim x_0\} \subseteq U(x_0)$. Consider $x' \in \{x : x \sim x_0\}$ and assume that $x' \in \text{int}U(x_0)$. Since $\succcurlyeq$ is ordientable, there exists $x'' \in \text{int}U(x_0)$ such that $x'' \prec x_0$, a contradiction. Second, since $\succcurlyeq$ is ordientable at x_0, all decreasing ordients at x_0 are collinear. From part (iii) and Proposition 1.13 of Clarke (1975), it then follows that $\{x : x \sim x_0\}$ is a C^1-manifold. $\square$

Proposition 3(i) states that convex binary relations have decreasing ordients at each point x_0 such that x_0 is a boundary point of the upper contour set at x_0. In particular, if the binary relation is in addition continuous and monotone, then it has a decreasing ordient everywhere. This result follows from standard separating arguments (each decreasing ordient defines a supporting hyperplane) and relates to Rigotti et al. (2008). Rigotti et al. consider individuals with continuous, convex, and monotone preferences over acts and identify the set of subjective beliefs at an act x_0 with the set of hyperplanes that support the upper contour set of the binary relation at x_0. In turn, they show that the set of subjective beliefs at an act x_0 coincides with the set of beliefs revealed by willingness and unwillingness to trade at x_0. Since the set of subjective beliefs at an act x_0 coincides with the set of decreasing ordients at x_0, their results provide further interpretations of the concept of decreasing ordients in the context of decision making under uncertainty.

Proposition 3(iv) relates to Proposition 2.3.9 of Mas-Colell (1985, p. 64). Following Debreu (1972), Mas-Colell considers locally non-satiated binary relations with connected indifference sets. Mas-Colell's Proposition 2.3.9 then states that the binary relation admits a C^k representation with no critical points if and only if the boundary of the binary relation is a C^k-manifold. Furthermore, if the boundary of the binary relation is a C^k manifold, then each indifference set is a C^k manifold (Proposition 2.3.10). In contrast with Mas-Colell, our result does not rest on the boundary of the binary relation to be a C^1 manifold. Instead, we use arguments from convex analysis (e.g., Rockafellar (1970)). Moreover, Proposition 3(iv) together with Theorem 4 below implies that the ordient of $\succcurlyeq$ at x_0 gives the marginal rate of substitutions.

This result parallels Neilson (1991), who shows that if a binary relation is continuous and monotone and has smooth indifference sets (i.e., C^k manifolds), then marginal rate of substitutions are well-defined. Proposition 3(iv) also explains the failure of the Leontieff preferences to be ordientable everywhere: the indifference curves of the Leontieff preferences have kinks on the 45 degree line. The same argument applies to the maxmin and minimax regret criteria for decision making under uncertainty (since these preferences exhibit kinks on the certainty line) and the preferences of Fehr and Schmidt (1999). Yet, part (i) implies that these convex preferences have decreasing ordients everywhere. As we will see, this will make it possible to characterize the maxima of $\succcurlyeq$ on a set.

Let us now contrast our definition with Rubinstein's (2005) definition. First of all, Rubinstein confines his attention to continuous, convex and monotone binary relations. Secondly, according to his definition, a binary relation is *differentiable* at x_0 if there exists a vector $g \geq 0$ such that $d \cdot g > 0$ *if and only if* d is an improvement direction of $\succcurlyeq$ at x_0. From Proposition 3 (ii), the notions of ordientability and differentiability (à la Rubinstein) coincide with continuous, convex and monotone binary relations.

For convex binary relations, however, differentiability is stronger than ordientability. Clearly, if a binary relation is differentiable at x_0, then it has an increasing ordient g at x_0. Furthermore, it follows from Proposition 3(i) that if a convex binary relation $\succcurlyeq$ has an increasing ordient at x_0, then it is ordientable. So, if a convex binary relation is "differentiable" in the sense of Rubinstein, it is ordientable. The converse is not true. For instance, the lexicographic order on $\mathbb{R}^2$ is convex and ordientable, while it is not "differentiable". With the lexicographic order, $(0, 1)$ is an improvement direction at each x_0 but the inner product of the direction $(0, 1)$ and the ordient $(1, 0)$ is zero, a violation of Rubinstein's requirement. (See Figure 1.) Thus, for convex binary relations, the concept of ordientable binary relations is weaker than the concept of differentiable binary relations.

3 Optimality

This section characterizes the maximal elements of a set $X \subset \mathbf{X}$ according to the binary relation $\succcurlyeq$. Denote by $\max_{\succcurlyeq} X$ the set of maximal elements, i.e., $\max_{\succcurlyeq} X := \{x \in X : x \succcurlyeq x' \text{ for all } x' \in X\}$.

A simple observation helps to organize our results. There are two equivalent formulations of the statement "x^* maximizes $\succcurlyeq$ on X. The first formulation states that the strict upper contour set of $\succcurlyeq$ at x^* does not intersect X, while the second states that the lower contour set of $\succcurlyeq$ at x^* is a superset of X. Clearly, the first formulation relates to improvement directions and increasing ordients, while the second relates to worsening directions and decreasing ordients. In turn, this simple observation suggests two alternative, but equivalent, characterizations of maxima: a characterization with increasing ordients and another with decreasing ordients. Theorem 1 and Theorem 2 offer two such alternative characterizations.

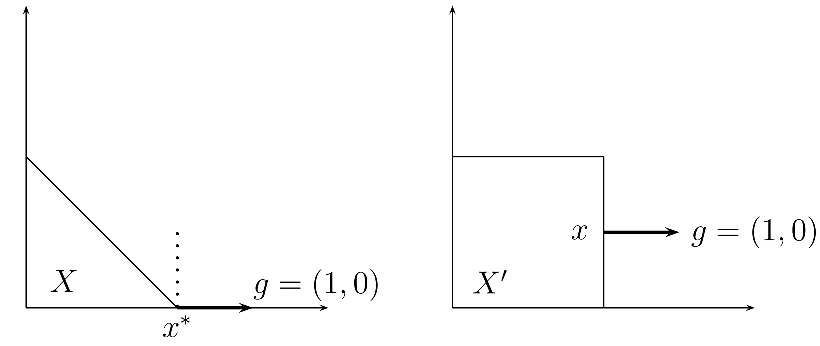
Before presenting our main results, note that for any non-empty set X with $\partial X \neq \emptyset$, there exists a complete and transitive binary relation $\succcurlyeq_X$ such that $X = \{x \in \mathbf{X} : x_0 \succcurlyeq_X x\}$ with $x_0 \in \partial X$. To see this, we construct the binary relation $\succcurlyeq_X$ as follows: we let $x' \succ_X x$ if $(x, x') \in \partial X \times \mathbf{X} \setminus X$ or $(x, x') \in \text{int}X \times \partial X$, and $x \sim_X x'$ if neither $x \succ_X x'$ nor $x' \succ_X x$. Therefore, if g is an increasing ordient of $\succcurlyeq_X$ at $x_0 \in \partial X$, there exists $\varepsilon^* > 0$ such that $d \cdot g > 0$ implies $x_0 + \varepsilon d \notin X$ for all $\varepsilon \in (0, \varepsilon^*)$: we leave the set by moving in direction d. Similarly, if g is a decreasing ordient of $\succcurlyeq_X$ at $x_0 \in \partial X$ and $g \cdot d < 0$, then we enter the interior of the set by moving in direction d. Again, it is worth emphasizing that no reference to functions is needed.

Theorem 1 *(i) If $x^* \in \max_{\succcurlyeq} X$ and g is an increasing ordient of $\succcurlyeq$ at x^*, then $x^* \in \partial X$ and g is an increasing ordient of $\succcurlyeq_X$ at x^*.*

(ii) Assume that $x^* \in \partial X$, that X and $\{x : x \succcurlyeq x^*\}$ are convex sets, and let g be an increasing ordient of both $\succcurlyeq$ and $\succcurlyeq_X$ at x^*. (a) If either X or $\{x : x \succcurlyeq x^*\}$ is strictly convex, then $\{x^*\} = \max_{\succcurlyeq} X$. (b) If $\succcurlyeq$ is continuous, then $x^* \in \max_{\succcurlyeq} X$.

To illustrate Theorem 1, consider the case of lexicographic preferences $\succcurlyeq$ on $\mathbb{R}^2_+$ and the sets X and X' in Figure 2. Part (i) of Theorem 1 simply states that the increasing ordients of $\succcurlyeq$ and $\succcurlyeq_X$ are collinear at a maximum. This condition resembles the classic first-order conditions and is illustrated in panel (i), where the point x^* maximizes $\succcurlyeq$ on X. We already know that $(1, 0)$ is an ordient of $\succcurlyeq$. Moreover, it is easy to check that $(1, 0)$ is an increasing ordient of X at x^*: for any direction d such that $(1, 0) \cdot d > 0$, $x + \varepsilon d \notin X$. Part (ii) of Theorem 1 provides sufficient conditions. As illustrated on panel (ii), the assumption of strict convexity cannot be dispensed with. The point x does not maximize $\succcurlyeq$ on X' and yet g is both an increasing ordient of $\succcurlyeq$ and $\succcurlyeq_{X'}$ at x.[6]

[6]Note that the convexity of $\{x : x \succcurlyeq x^*\}$ in part (ii) implies that the increasing ordient g of $\succcurlyeq$ at



(i) (ii)

Figure 2: Lexicographic preferences and maxima

Proof (i) Let $x^* \in \max_{\succcurlyeq} X$ and suppose that $x^* \notin \partial X$. Since g is an increasing ordient of $\succcurlyeq$ at x^* and x^* is interior, there exists an improving direction d and $\varepsilon > 0$ such that $x^* + \varepsilon d \in X$ and $x^* + \varepsilon d \succ x$, a contradiction with the maximality of x^*.

Assume now that g is not an increasing ordient of $\succcurlyeq_X$ at x^*. Hence, for every $\varepsilon' > 0$, there exists d and $\varepsilon < \varepsilon'$ such that $d \cdot g > 0$ and $x^* + \varepsilon d \in X$. Since g is an increasing ordient of $\succcurlyeq$ in x^*, it follows that $x^* + \varepsilon d \succ x^*$ for sufficiently small ε, a contradiction.

(ii) We show that the increasing ordient g at x^* defines a hyperplane $\{x : g \cdot (x - x^*) = 0\}$ that separates X from $\{x : x \succcurlyeq x^*\}$. Since g is an increasing ordient of $\succcurlyeq_X$ at $x^* \in \partial X$ and X convex, we have that $g \cdot (x - x^*) > 0$ implies that $x \notin X$. Therefore, $\{x : g \cdot (x - x^*) > 0\} \cap X = \emptyset$. It follows that $X \subseteq \{x : g \cdot (x - x^*) \leq 0\}$.

Since $\{x : x \succcurlyeq x^*\}$ is convex, it follows from the above arguments that $\{x : x \succcurlyeq x^*\} \subseteq \{x : g \cdot (x - x^*) \geq 0\}$.

Assume that there exists $x' \in X \cap \{x : x \succcurlyeq x^*\}$, with $x' \neq x^*$. It follows from the above that $g \cdot (x' - x^*) = 0$.

(a) Suppose that X is strictly convex and consider $z = \frac{1}{2}x' + \frac{1}{2}x^*$. Clearly, $g \cdot (z - x^*) = 0$ and z is an interior of point of X. From the continuity of the inner product, it follows that there exists $x'' \in X$ with $g \cdot (x'' - x^*) > 0$, a contradiction. It follows that x^* is the unique maximizer of $\succcurlyeq$ on X.

If $\{x : x \succcurlyeq x^*\}$ is strictly convex, a similar reasoning applies.

(b) Assume that $\succcurlyeq$ is continuous and that $x' \succ x^*$. There exists an open ball $B_\delta(x')$ around x' of radius δ such that $x \succ x^*$ for all $x \in B_\delta(x')$, therefore $g \cdot (x - x^*) \geq 0$ for all $x \in B_\delta(x')$. Furthermore, $g \cdot (x' - x^*) = 0$, together with the bilinearity of the inner

x^* is in fact an ordient of $\succcurlyeq$ at x^*. See Proposition 3(i).

product, implies that there exists $\hat{x} \in B_\delta(x')$ with $g \cdot (\hat{x} - x^*) < 0$, a contradiction. $\square$

For an application of Theorem 1, consider the classic problem of maximizing a preference relation $\succcurlyeq$ over the budget set $B(p, w) := \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$ where $p >> 0$ is the vector of prices and $w > 0$ the income. Let $x^* \in \max_{\succcurlyeq} B(p, w)$. If $\succcurlyeq$ has an increasing ordient $g(x^*)$ at x^*, then Theorem 1(i) implies that $g(x^*)$ is an increasing ordient of $B(p, w)$ at x^*. Moreover, if $x^* >> 0$, all increasing ordients of $B(p, w)$ (more precisely of $\succcurlyeq_{B(p,w)}$) are collinear to p. So, there exists $\lambda > 0$ such that $g(x^*) = \lambda p$. This "corresponds" to the classic first-order necessary conditions. Alternatively, consider $x^* >> 0$ with $p \cdot x^* = w$ and assume that $\succcurlyeq$ is convex. If p is an increasing ordient of $\succcurlyeq$ at x^* and either $\succcurlyeq$ is continuous or $\{x : x \succcurlyeq x^*\}$ is strictly convex, then Theorem 1(ii) implies that $x^* \in \max_{\succcurlyeq} B(p, w)$. This "corresponds" to the classic first-order sufficient conditions. This observation was originally made by Rubinstein (2005) for continuous, convex, monotone and "differentiable" binary relations.

For another application, consider a production economy with m agents and k firms. Agent i has preference $\succcurlyeq_i$ on $\mathbb{R}^n$, the set of commodities, and firm j production set is $Y_j \subseteq \mathbb{R}^n$. For any $x^i \in \mathbb{R}^n$, denote $x = \sum_i x^i$ the total consumption vector and $y = \sum_j y^j$ the total production, so that $z = x + y$ is the total net consumption of the economy. Assuming closed and convex upper contour sets and production sets, Debreu (1951) shows that if an allocation $\bar{z}$ is efficient, then there exists a vector of "prices" $\bar{p} \in \mathbb{R}^n_{++}$ such that $\bar{x}^i \succcurlyeq_i x^i$ for all x^i with $\bar{p} \cdot x^i \leq \bar{p} \cdot \bar{x}^i$, for all i, and $\bar{p} \cdot \bar{y}^j \leq \bar{p} \cdot y^j$ for all $y^j \in Y_j$, for all j. In words, agents maximize their preferences subject to budget constraint and firms maximize profits subject to production constraints. Most importantly, Debreu does not assume the existence of a representation u_i of $\succcurlyeq_i$. In fact, he does argue against such a route. He wrote (p. 277): *"First of all, such a correspondence [representation] need not exist, but even more important is the fact that the numerical value of this function has never any role to play, that only the ordering itself matters. The advisability of introducing such a function (always accompanied by the mention "defined but for an arbitrary monotonically increasing function"), which is useless and moreover might not exist at all, may be questionable."* However, what is the significance of the price vector $\bar{p}$ if no representation exists? Theorem 1 gives an answer. If $\succcurlyeq_i$ has an increasing ordient at $\bar{x}^i$ for all i, then $\bar{p}$ is an ordient of $\succcurlyeq_i$ at $\bar{x}^i$.[7] The price vector $\bar{p}$ has thus a concrete significance in terms

[7]This follows from the convexity of $\succcurlyeq_i$. Moreover, since upper contour sets are closed and convex,

of marginal rates of substitution.[8] An important message of this paper is that we can perfectly speak of marginal rates of substitutions or characterize constrained maxima without requiring the existence of a representation, provided that the preferences are ordientable.

The next result provides an alternative characterization of $\max_{\succcurlyeq} X$ in terms of decreasing ordients of $\succcurlyeq$. This result is particularly useful in economic applications with convex binary relations.[9]

Theorem 2 *Assume that X and $\{x : x \succcurlyeq x^*\}$ are convex sets.*

(i) If $\succcurlyeq$ is locally non-satiated and $x^* \in \max_{\succcurlyeq} X$, then there exists g that is both an increasing ordient of $\succcurlyeq_X$ at x^* and a decreasing ordient of $\succcurlyeq$ at x^*.

(ii) Assume that $x^* \in \partial X$ and let g be an decreasing ordient of $\succcurlyeq$ and an increasing ordient of $\succcurlyeq_X$ at x^*. (a) If either X or $\{x : x \succcurlyeq x^*\}$ is strictly convex, then $\{x^*\} = \max_{\succcurlyeq} X$. (b) If $\succcurlyeq$ is continuous, then $x^* \in \max_{\succcurlyeq} X$.

Proof Part (i). Assume that $x^* \in \max_{\succcurlyeq} X$, so that $X \cap \{x : x \succ x^*\} = \emptyset$. By our assumptions of convexity and a version of the separating hyperplane theorem (e.g. see Frenk and Kassay, Theorem 1.3), there exists $g \in \mathbb{R}^n \setminus \{0\}$ and $y \in \partial X$ such that $x^* + \varepsilon d \succ x^*$ implies $g \cdot (x^* + \varepsilon d) \geq g \cdot y$ and $x^* + \varepsilon d \in X$ implies $g \cdot (x^* + \varepsilon d) \leq g \cdot y$. Since $x^* \in X$, it follows that $g \cdot x^* \leq g \cdot y$. Local non-satiation of $\succcurlyeq$ implies that $x^* \in \mathrm{cl}\{x : x \succ x^*\}$ and, hence, that $g \cdot x^* \geq g \cdot y$. Together, this means that $g \cdot x^* = g \cdot y$. Hence, if $g \cdot d > 0$, then $x^* + \varepsilon d \notin X$, and if $g \cdot d < 0$, then $x^* + \varepsilon d \prec x^*$, which completes this part of the proof.

Part (ii). As in the proof of part (ii) of Theorem 1. □

To summarize, Theorems 1 and 2 provide two alternative but equivalent characterizations of constrained maxima. The characterization with increasing ordients might prove more useful in some applications, while other applications might require the characterization with decreasing ordients. In the sequel, we provide additional characterizations in terms of the increasing/decreasing ordients of the constraint set.

it follows from Proposition 3(i) that $\succcurlyeq_i$ has a decreasing ordient everywhere. From Theorem 2 below, it then follows that $\bar{p}$ is a decreasing ordient of $\succcurlyeq_i$.

[8]Moreover, even if the preferences have no increasing ordients at $\bar{x}$, it follows from Proposition 6 below that $\bar{p}$ is a decreasing ordient. Thus, it characterizes the relative prices that make consumers strictly worse off.

[9]Since they have decreasing ordients at each point x_0 such that x_0 is on the boundary of the upper contour set of $\succcurlyeq$ at x_0. If, in addition, $\succcurlyeq$ is monotone, then it has a decreasing ordient everywhere.

In Economics, the set X is often taken to be the intersection of several sets, e.g., budget constraints, technological constraints, positivity constraints, so that $X = \cap_{j=1}^m X_j$ with $X_j \subseteq \mathbb{R}^n$ for all j. For instance, the budget set $B(p, w)$ corresponds to $\cap_{j=1}^{n+1} X_j$ with $X_j = \{x \in \mathbb{R}^n : x_j \geq 0\}$ for $j = 1, ..., n$ (non-negative consumption) and $X_{n+1} = \{x \in \mathbb{R}^n : p \cdot x \leq w\}$. The following proposition characterizes the increasing ordients of $\succsim_X$ as a function of the decreasing/increasing ordients of each $\succsim_{X_j}$.

Proposition 4 *Let $X = \cap_{j=1}^m X_j$ with $X_j \subseteq R^n$ for all $j \in \{1, \ldots, m\}$.*

(i) Assume that g is an increasing ordient of $\succsim_X$ at $x^* \in \partial X$, h^j is a decreasing ordient of $\succsim_{X_j}$ at x^* for $j \in \{1, .., m\}$ and $\cap_{j=1}^m \{x : h^j \cdot (x - x^*) < 0\} \neq \emptyset$. There exists $\lambda \in \mathbb{R}_+^m \setminus \{0\}$ such that

$$g = \sum_{j : x^* \in \partial X_j} \lambda_j h^j. \tag{1}$$

(ii) If h^j is an increasing ordient of $\succsim_{X_j}$ at $x^* \in \partial X$ for $j \in \{1, ..., m\}$ and there exists $\lambda \in \mathbb{R}_+^m$ such that Equation (1) holds, then g is an increasing ordient of $\succsim_X$ at x^*.

Proof (i) Let $x^* \in \partial X$. We show that x^* is maximizer of $g \cdot x$ subject to $x \in \cap_{j=1}^m \{x : h^j \cdot (x - x^*) \leq 0\}$.

To the contrary, suppose that there exists $x' \in \cap_{j=1}^m \{x : h^j \cdot (x - x^*) \leq 0\}$ such that $g \cdot x' > g \cdot x^*$. From the bilinearity of the inner product, there exists $x'' \in \cap_{j=1}^m \{x : h^j \cdot (x - x^*) < 0\}$ such that $g \cdot x'' > g \cdot x^*$. Let $d = x'' - x^*$, so that $g \cdot d > 0$. Since g is an increasing ordient of $\succsim_X$ at x^*, it follows that $x^* + \varepsilon d \notin X$ for sufficiently small ε. However, since h^j is a decreasing ordient of $\succsim_{X_j}$ at x^*, it follows that $x^* + \varepsilon d \in X$ for sufficiently small ε, a contradiction.

The statement then follows from the classic necessary first-order conditions for the above linear maximization problem involving linear constraints.

(ii) Consider $g = \sum_{j : x^* \in \partial X_j} \lambda_j h^j$. Assume that $g \cdot d > 0$. It follows that $(\sum_{j : x^* \in \partial X_j} \lambda_j \cdot h^j) \cdot d > 0$, so that there exists j such that $x_j^* \in \partial X_j$ and $d \cdot h^j > 0$. This implies that $x^* + \varepsilon d$ will lie outside X_j and, hence, outside of X if ε is sufficiently small. Therefore, g is an increasing ordient of $\succsim_X$ at x^*. $\square$

As an illustration, consider the budget set in Figure 2 panel (i). The consumption bundle x^* is in the interior of the set $X_1 = \{x \in \mathbb{R}^2 : x_1 \geq 0\}$ and at the boundary of the sets $X_2 = \{x \in \mathbb{R}^2 : x_2 \geq 0\}$ and $X_3 = \{x \in \mathbb{R}^2 : p_1 x_1 + p_2 x_2 \leq w\}$ with

$p >> 0$. The decreasing ordient of $\succcurlyeq_{X_2}$ at x^* is $(0, -1)$ while the decreasing ordient of $\succcurlyeq_{X_3}$ at x^* is (p_1, p_2). Above, we have argued that $(1, 0)$ is an increasing ordient of $\succcurlyeq_{B(p,w)}$ at x^*, so that the vector λ in Proposition 4 is $(0, p_2/p_1, 1/p_1)$. The vector λ can be interpreted as a Kuhn-Tucker multiplier. Indeed, from Theorem 1, we have that the increasing ordient of $\succcurlyeq$ is collinear to the increasing ordient of $\succcurlyeq_{B(p,w)}$ at x^*. From Proposition 4, we have that the increasing ordient of $\succcurlyeq_{B(p,w)}$ is a non-negative linear combination of the decreasing ordient of each $\succcurlyeq_{X_j}$ at x^*. Combining both observations, we have that the increasing ordient of $\succcurlyeq$ at x^* is a non-negative linear combination of the decreasing ordient of each $\succcurlyeq_{X_j}$. This last observation is reminiscent of the classic Kuhn-Tucker conditions for constrained optimization. In fact, combining Theorem 1 and Proposition 4, we obtain "first-order" necessary and sufficient conditions for the maximization of a binary relation in terms of the increasing ordients of each of the constraint sets. As with the Kuhn-Tucker conditions, if x^* is a maximizer and x^* is interior to X_j (i.e., the j-th constraint is not binding), then we can set $\lambda_j = 0$.[10]

To continue our exploration of optimality conditions, the next proposition gives a classic duality theorem between minimizing $\succcurlyeq_X$ on $\{x : x \succcurlyeq x^*\}$ and maximizing $\succcurlyeq$ on X.[11]

Proposition 5 (Duality) *Assume that int$X \neq \emptyset$ and consider $x^* \in \partial X$.*

(i) If $\succcurlyeq$ is continuous, $x^* \in \min_{\succcurlyeq_X} \{x : x \succcurlyeq x^*\}$ and $\partial X \subseteq$ cl(int$X$), then $x^* \in \max_{\succcurlyeq} X$.

(ii) If $\succcurlyeq$ is locally non-satiated and $x^* \in \max_{\succcurlyeq} X$, then $x^* \in \min_{\succcurlyeq_X} \{x : x \succcurlyeq x^*\}$.

Proof (i) Assume that $x^* \notin \max_{\succcurlyeq} X$, so that there exists $x' \in X$ such that $x' \succ x^*$. If $x' \in \text{int}X$, it follows from the definition of $\succcurlyeq_X$ that $x^* \succ_X x'$, a contradiction. If $x' \in \partial X$, it follows that x' is on the boundary of X (since $\partial X = X \setminus \text{int}X \subseteq \text{cl}X \setminus \text{int}X = \text{bd}X$). Moreover, since $\partial X \subseteq \text{cl}(\text{int}X)$, there exists an open ball $B_\delta(x')$ around x' such that $B_\delta(x') \cap \text{int}X \neq \emptyset$. The continuity of $\succcurlyeq$ then implies that there exists $x'' \in B_\delta(x') \cap \text{int}X$ such that $x'' \succ x^*$ and $x'' \succ_X x^*$, a contradiction.

(ii) Assume that $x^* \notin \min_{\succcurlyeq_X} \{x : x \succcurlyeq x^*\}$, so that there exists $x' \in \{x : x \succcurlyeq x^*\}$ with $x^* \succ_X x'$. From the definition of $\succcurlyeq_X$, we have that $x' \in \text{int}X$. Hence, by local non-satiation, there exists $x'' \in X$ with $x'' \succ x^*$, a contradiction. $\square$

[10]Note that we can interpret the condition $\cap_{j=1}^m \{x : h^j \cdot (x - x^*) < 0\} \neq \emptyset$ as a qualification constraint.

[11]For the classic duality theorem, see Proposition 3.E.1 in Mas-Colell et al. (1995).

We can then combine Theorem 1 and Proposition 5 to obtain necessary and sufficient conditions for the maximization of $\succcurlyeq$ on X as a function of the ordient of the *constraint set*. The next proposition is useful in applications where one has more information about the constraint set than about the binary relation.

Proposition 6 *If $\succcurlyeq$ is locally non-satiated, $x^* \in \max_{\succcurlyeq} X$ and g is a decreasing ordient of $\succcurlyeq_X$ at x^*, then $x^* \in \partial X$ and g is a decreasing ordient of $\succcurlyeq$ at x^*.*

Proof The proof is essentially the same as the proof of Theorem 1. All we have to do is to insure that $\partial X \subseteq \mathrm{cl}(\mathrm{int} X)$ for Proposition 5 to apply. This holds true, however, as g is a decreasing ordient of $\succcurlyeq_X$ at x^*. □

To illustrate the scope of Proposition 6, consider again the problem of maximizing preferences over budget sets. If $\succcurlyeq$ is locally non-satiated and $x^* \in \max_{\succcurlyeq} B(p, w)$ (with $x^* >> 0$), then p is a decreasing ordient of $\succcurlyeq$ at x^*. In particular, these necessary conditions apply to Leontieff preferences, while Theorem 1 does not apply: the Leontieff preferences fail to have an increasing ordient everywhere. More generally, whenever a binary relation fails to have an increasing ordient everywhere, Proposition 6 still makes it possible to characterize the maxima, provided that the constraint set has a decreasing ordient at any point on its boundary. As already mentioned, important examples of binary relations failing to have an increasing ordient everywhere include the maxmin and minimax regret criteria for decision-making under uncertainty and the other-regarding preferences of Fehr and Schmidt (1999).[12] Yet, most constraint sets found in Economics have a decreasing ordient at any point on their boundary, so that Proposition 6 applies.

4 Envelope Theorem

The object of this section is to study how the set $\max_{\succcurlyeq} X$ of maximal elements of X changes as the "constraint" set X varies. Let T be a convex set of parameters and $\{X_t\}_{t \in T}$ a family of closed sets with $X_t \subseteq \mathbf{X}$ for all $t \in T$.[13]

Classic envelope theorems (see Milgrom and Segal (2002)) consider problems of the form $V(t) := \sup_{x \in \{x' \in \mathbf{X}: h(x',t) \leq 0\}} f(x,t)$ where f is a parameterized real-valued function (often the utility function) and h another parameterized function, and "quantify"

[12] Alternatively, we might apply Theorem 2 since these preferences are convex and continuous.

[13] The assumption of closeness is not essential: our results remain valid if we consider the closure of each X_t and the supremun of $\succcurlyeq$ on X instead of the maximum.

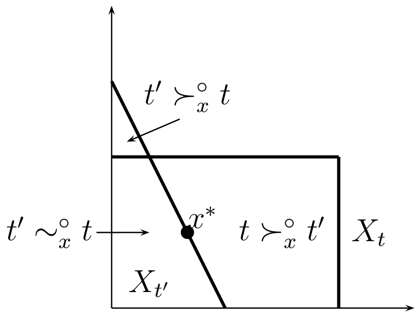


Figure 3: Comparison of X_t and X_t'.

the changes in the value function V as the parameter t varies. Unlike this classical approach, our approach does neither impose the existence of a representation of $\succcurlyeq$ (the function f) nor the representation of the constraint set X_t as functional inequalities. Consequently, our analysis has to be entirely casted in terms of binary relations, directions and ordients.

We first need to impose some structure on the family of sets $\{X_t\}_{t\in T}$. For any $x \in \mathbf{X}$, we define the complete and transitive binary relation $\succcurlyeq_x^\circ$ on T as follows: $t' \succ_x^\circ t$ if and only if either $x \in \partial X_t \cap \text{int}X_{t'}$ ($X_{t'}$ is "locally larger" than X_t at x) or $x \in X_{t'} \setminus X_t$. In particular, if $X_t \subset X_{t'}$, then $t' \succcurlyeq_x^\circ t$ for all $x \in X_t \cup X_{t'}$. A simple example helps to illustrate our definition. In Figure 3, X_t is the rectangle and $X_{t'}$ the triangle. The point x^* is on the boundary of $X_{t'}$ and interior to X_t, hence $t \succ_{x^*}^\circ t'$. Similarly, the points to the left of the downward-sloping line are in X_t and exterior to X_t', so that $t \succ_x^\circ t'$ for all such x. For another concrete example, consider the family of budget sets $\{B(p,w) = \{x : p\cdot x \le w\}\}_{(p,w)\in\mathbb{R}^{n+1}}$. If the bundle of goods x^* is affordable at (p_1, w_1) but not at (p_2, w_2), then we have $(p_1, w_1) \succ_{x^*}^\circ (p_2, w_2)$. Similarly, if x^* does not exhaust the budget at (p_1, w_1) but does exhaust the budget at (p_2, w_2), we have $(p_1, w_1) \succ_{x^*}^\circ (p_2, w_2)$.

Consequently, the vector d is an improvement direction of $\succcurlyeq_x^\circ$ at t if there exists $\varepsilon^* > 0$ such that $\varepsilon \in (0, \varepsilon^*)$ and $t+\varepsilon d \in T$ imply $t+\varepsilon d \succ_x^\circ t$. In particular, if $x \in \partial X_t$, this implies that $x \in \text{int}X_{t+\varepsilon d}$ for all $\varepsilon < \varepsilon^*$. This case is important since maxima of locally non-satiated binary relations are on the boundary.

Indeed, suppose that $\succcurlyeq$ is locally non-satiated and let $x^* \in \max_{\succcurlyeq} X_t$. It follows that $x^* \in \partial X_t$. Consider an improvement direction d of $\succcurlyeq_{x^*}^\circ$ at t. Therefore, for ε sufficiently small, $t' := t + \varepsilon d \succ_{x^*}^\circ t$ implies that $x^* \in \text{int}X_{t'}$ as $x^* \in \partial X_t$. As $\succcurlyeq$ is locally non-satiated, there exists $x' \in X_{t'}$ such that $x' \succ x^*$. It follows that for all $x^{**} \in \max_{\succcurlyeq} X_{t'}$, $x^{**} \succ x^*$. Intuitively, since $\succcurlyeq$ is locally non-satiated, and X_t'

is locally larger than X_t, we can find an element in $X_{t'}$ that is strictly preferred to x^*. However, can we "quantify" the change as we move from X_t to $X_{t'}$? This is the question we study next.

To answer this question, we need to introduce (yet) another binary relation $\succcurlyeq^*$ on T: the indirect binary relation. We say that $t' \succ^* t$ if there exists $x' \in X_{t'}$ such that $x' \succ x$ for all $x \in X_t$. The indirect binary relation is at the heart of this section as it implies direct comparisons between the set of maxima of $\succcurlyeq$ on X_t and $X_{t'}$. This is akin to the value function of classic envelope theorems.

Theorem 3 *Assume that $\succcurlyeq$ is locally non-satiated and let $x^* \in \max_{\succcurlyeq} X_t$. If g is an increasing ordient of $\succcurlyeq^{\circ}_{x^*}$ at t, then g is an increasing ordient of $\succcurlyeq^*$ at t.*

Theorem 3 states that the "first-order" effects of a change in t on the maxima are given by the direct effect, i.e., the change in the set X_t. This is an envelope theorem. More precisely, Theorem 3 states that $\max_{\succcurlyeq} X_{t+\varepsilon d} \succ \max_{\succcurlyeq} X_t \sim x^*$ whenever $g \cdot d > 0$, i.e., whenever $X_{t+\varepsilon d}$ is locally larger than X_t at x^*.

To get more intuition, let us return to the classic problem of maximizing $\succcurlyeq$ over the budget set $B(p, w) := \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$. In this example, the set T of parameters is $\mathbb{R}^{n+1}_{++}$: the set of prices p and income w. Let $x^* \in \max_{\succcurlyeq} B(p, w)$. If $\succcurlyeq$ is locally non-satiated, we have that $(-x^*, 1)$ is an increasing ordient of $\succcurlyeq^{\circ}_{x^*}$, hence of $\succcurlyeq^*$.[14] Furthermore, if $\succcurlyeq^*$ is representable by the indirect utility function $(p, w) \mapsto v(p, w)$ with v differentiable (hence, ∇v is an ordient of the indirect preference $\succcurlyeq^*$), then we have

$$x^* = -\frac{1}{\nabla_w v(p, w)} \nabla_p v(p, w).$$

This last equation is, of course, the celebrated Roy's identity.

Rubinstein (2005, p. 77) makes a related observation. Assuming that for each (p, w), the set $\max_{\succcurlyeq} B(p, w)$ is the singleton $\{x^*(p, w)\}$, he notes that the hyperplane $H := \{(p, w) : (-x^*(p^*, w^*), 1) \cdot (p, w) = 0\}$ for some (p^*, w^*) is tangent to the indifference sets of $\succcurlyeq^*$ at (p^*, w^*).[15] In other words, the vector $(-x^*(p^*, w^*), 1)$ is normal to H. Our result is of a different nature: it states that $(-x^*(p^*, w^*), 1)$ is an increasing ordient of $\succcurlyeq^*$. As we will see in Section 5, this does not imply that it is normal to the indifference curve of $\succcurlyeq^*$ at (p^*, w^*).

[14] To see this, note that the function $(p, w) \mapsto w - p \cdot x$ is differentiable with gradient $(-x, 1)$. The result then follows since local non-satiation implies $p \cdot x^* = w$ at the maximum.

[15] Remember that Rubinstein considers continuous, convex and monotone binary relations. In particular, this implies that the Walras's law holds.

Proof Choose any $x^* \in \sup_{\succsim} X_t$ so there $x^* \succsim x$ for all $x \in X_t$ and there exists $\left(x^k\right)_k$ such that $x^k \in X_t$ and $\lim_{k\to\infty} x^k = x^*$. Since $\succsim$ is locally non-satiated, we have that $x^* \in \partial X_t$. We first show that if d is an improvement direction of $\succsim^\circ_{x^*}$ at t, then d is an improvement direction of $\succsim^*$ at t. Assume that d is an improvement direction of $\succsim^\circ_{x^*}$ at t. Therefore, there exists $\varepsilon^* > 0$ such that $t + \varepsilon d \succ^\circ_{x^*} t$ for all $\varepsilon < \varepsilon^*$. Since $x^* \in \partial X_t$, this implies that x^* is in the interior of $X_{t+\varepsilon d}$ for all $0 < \varepsilon < \varepsilon^*$. Fix $\varepsilon < \varepsilon^*$. Since $\succsim$ is locally non-satiated, it follows that there is some $x'(\varepsilon) \in X_{t+\varepsilon d}$ such that $x'(\varepsilon) \succ x^*$. It follows that $t + \varepsilon d \succ^* t$: d is an improvement direction of $\succsim^*$ at t. Theorem 3 then follows directly from the definition of an increasing ordient. $\square$

Yet, Theorem 3 does not imply that g is an ordient of $\succsim^*$; it only states that g is an *increasing* ordient. A simple example from consumer theory illustrates that point. There are two perfectly substitutable goods with preferences given by the utility function $f(x) = x_1 + x_2$. The indirect utility function v is given by $v(p, w) = w/\min\{p_1, p_2\}$. The function v is differentiable in w with $\partial v/\partial w = 1/\min\{p_1, p_2\}$, while it is not differentiable with respect to p at any p such that $p_1 = p_2$. Consider $x^* = (w/2, w/2)$. Then, $x^* \in \max_{\succsim} B((1,1), w)$. Given $\varepsilon > 0$, we have that $v((1 + 2\varepsilon, 1 - \varepsilon), w) = w/(1-\varepsilon) > v((1,1), w)$, so that $(2, -1, 0)$ is an improvement direction at $(1, 1, w)$. It follows that $(-x^*, 1)$ is not a decreasing ordient as the inner product of $(-w/2, -w/2, 1)$ and $(2, -1, 0)$ is negative, while $(2, -1, 0)$ is an improvement direction. The next proposition provides sufficient conditions for g to be an ordient.

We need the following definition. The vector d is a local worsening direction of $\{\succsim^\circ_x, x \in \mathrm{bd}(X_t)\}$ at x^* if there exists $\varepsilon^* > 0$ and $\delta > 0$ such that $\varepsilon \in (0, \varepsilon^*)$ and $\|x - x^*\| < \delta$ with $x \in \mathrm{bd}(X_t)$ implies that $t \succ^\circ_x t + \varepsilon d$. Geometrically, this means that $B_\delta(x^*) \cap \mathrm{bd}(X_t) \cap X_{t+\varepsilon d} = \emptyset$ for all $\varepsilon < \varepsilon^*$. In words, for ε sufficiently small, any point on the boundary of X_t sufficiently close to x^* does not belong to $X_{t+\varepsilon d}$. For instance, in the example above, at $(w/2, w/2)$, the direction $(1 + 2\varepsilon, 1 - \varepsilon, w)$ is a local worsening direction of $\{\succsim^\circ_x, x \in \{x' : x'_1 + x'_2 = w\}\}$. Regardless of $\varepsilon > 0$, the budget line $(1 - 2\varepsilon)x_1 + (1 - \varepsilon)x_2 = w$ intersects the budget line $x_1 + x_2 = w$ at $(w/3, 2w/3)$ and, therefore, for all $\delta < w/6$, $B_\delta((w/2, w/2)) \cap \{x : x_1 + x_2 = w\} \cap \{x : (1 - 2\varepsilon)x_1 + (1 - \varepsilon)x_2 \le w\} = \emptyset$, as required.

Proposition 7 *Let $x^* \in \max_{\succsim} X_t$. Assume that $\succsim$ is locally non-satiated, that $X_{t'}$ is locally path-connected for all t' in a neighborhood of t, and that $\lim_{t_k \to t} X_{t_k} = X_t$ and $x^* \in \lim_{t_k\to t}\{X_{t_k} \cap X_t\}$. Assume that g is a decreasing ordient of $\succsim^\circ_{x^*}$ at*

t and that any worsening direction d of $\succcurlyeq^{\circ}_{x^*}$ at t is a local worsening direction of $\{\succcurlyeq^{\circ}_{x}, x \in bd(X_t)\}$ at x^*. If $\{x^*\} = cl\{x : x \succcurlyeq x^*\} \cap X_t$, then g is a decreasing ordient of $\succcurlyeq^*$ at t.

Before presenting the proof, we briefly discuss the assumptions made in Proposition 7. Firstly, if the constraint sets are convex, then there are locally path-connected, so that in most economic applications, this assumption is satisfied. Secondly, the assumption "$x^* \in \lim_{k\to\infty} \{X_{t_k} \cap X_t\}$" means that x^* has arbitrarily close points in X_t and X_{t_k} for k large enough. This assumption is satisfied in our example with two perfectly substitutable goods. Lastly, the proposition requires the existence of a unique maximum. This assumption is not satisfied in our example with two perfectly substitutable goods when the price of both goods is the same. This explains why $(-x^*, 1)$ is not an ordient of $\succcurlyeq^*$, albeit an increasing ordient.

Proof Since $\succcurlyeq$ is locally non-satiated, we have that $x^* \in \text{bd}(X_t)$.

Consider d such that $d \cdot g < 0$, i.e., d is a worsening direction of $\succcurlyeq^{\circ}_{x^*}$ at t. Assume that d is not a worsening direction of $\succcurlyeq^*$ at t. There exist sequences $(\varepsilon_k)_k$ and $(x_k)_k$ with $\varepsilon_k > 0$, $x_k \in X_{t+\varepsilon_k d}$ and $x_k \succcurlyeq x^*$ for all k, and $\lim_{k\to\infty} \varepsilon_k = 0$.

We have that $x_k \neq x^*$ for all k. This is because $x^* \in \text{bd}X_t \subset X_t$ and d is a worsening direction of $\succcurlyeq^{\circ}_{x^*}$ at t. Note that since $\{x^*\} = \text{cl}\{x : x \succcurlyeq x^*\} \cap X_t$, we have that $x_k \notin X_t$ for all k.

Assume without loss of generality that $\lim_{k\to\infty} x^k = x'$ holds for some $x' \in \mathbf{X}$. So $x' \in \text{cl}\{x : x \succcurlyeq x^*\}$. Moreover, since $\lim_{k\to\infty} X_{t+\varepsilon_k d} = X_t$, it follows that $x' \in X_t$. Hence, $x' \in \text{cl}\{x : x \succcurlyeq x^*\} \cap X_t$.

We now show that $x' \neq x^*$. This will contradict the assumption that $\{x^*\} = \text{cl}\{x : x \succcurlyeq x^*\} \cap X_t$ and, hence, proves that d is in fact a worsening direction of $\succcurlyeq^*$ at t.

Assume that $x' = x^*$, so that x_k is arbitrarily close to $x^*$for k large enough. To derive a contradiction, consider a sequence $(z_k)_k$ in $X_t \cap X_{t+\varepsilon_k d}$ such that $\lim_{k\to\infty} z_k = x^*$. Such a sequence exists as $x^* \in \lim_{k\to\infty} \{X_{t+\varepsilon_k d} \cap X_t\}$.

Connect z_k to x_k with a continuous function with range in $X_{t+\varepsilon_k d}$. This function exists by local path-connectedness of $X_{t+\varepsilon_k d}$. So, there exists $h_k : [0,1] \to \mathbb{R}^n$ such that h_k is continuous, $h_k(0) = z_k$, $h_k(1) = x_k$ and $h_k(\lambda) \in X_{t+\varepsilon_k d}$ for all $\lambda \in [0,1]$.

Moreover, there exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*)$, there exists k^* such that for all $k > k^*$, we have that $h_k(\lambda) \in B_\delta(x^*)$ for all $\lambda \in [0,1]$.

As $z_k \in X_t$, $x_k \notin X_t$ and h_k is continuous, there exists $\bar{\lambda}$ such that $h(\bar{\lambda}) \in$

$X_{t+\varepsilon_k d} \cap \mathrm{bd} X_t$. Hence, $B_\delta(x^*) \cap \mathrm{bd}(X_t) \cap X_{t+\varepsilon d} \neq \emptyset$. This contradicts the fact that d is a local worsening ordient of $\{\succcurlyeq_x^\circ, x \in \mathrm{bd}(X_t)\}$ at x^*. $\square$

As already suggested, an immediate corollary of Theorem 3 and Proposition 7 is the ordinal version of the celebrated Roy's identity.

Corollary 1 (Roy's Identity) *Consider $x^* \in \max_{\succcurlyeq} B(p,w)$. Then $(-x^*, 1)$ is an increasing ordient of $\succcurlyeq^*$ at (p,w). If $\succcurlyeq$ is continuous and $\{x^*\} = \arg\max_{\succcurlyeq} B(p,w)$ then $(-x^*, 1)$ is an ordient of $\succcurlyeq^*$ at (p,w).*

5 Implicit Function Theorem and Indifference Sets

This section presents an implicit function theorem for ordientable binary relations. Assume that $\mathbf{X}$ can be written as $\mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n$ and let $x^* \in \mathbf{X}$. We study the existence of two open neighborhoods U and V of $(x_1^*, \ldots, x_{n-1}^*)$ and x_n^*, respectively, and of an *implicit* function $f : U \to V$ such that $(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) \in \mathrm{bd}\{x : x \succcurlyeq x^*\}$ for all $(x_1, \ldots, x_{n-1}) \in U$. There are two main motivations for our formulation. Firstly, we do not want to assume either the continuity of the binary relation or the connectedness of the indifference sets. This approach is consistent with our definition of marginal rate of substitutions. In fact, our implicit function theorem shows that the gradient of the implicit function exists at x^* and gives the marginal rate of substitutions of good n for any good $i \neq n$ at x^*. Secondly, if the binary relation $\succcurlyeq$ is continuous, our formulation coincides with the classic formulation of the implicit function theorem, namely the existence of an implicit function f such that $(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) \sim x^*$ for all $(x_1, \ldots, x_{n-1})$ in a neighborhood of x^*. Theorem 4 formally states our implicit function theorem.

Theorem 4 *Let g be an ordient of $\succcurlyeq$ at $x^* \in \mathbf{X}$ with $g_n \neq 0$. There exist two open neighborhoods $U \subset \times_{i=1}^{n-1} \mathbf{X}_i$ and $V \subset \mathbf{X}_n$ with $(x_1^*, \ldots, x_{n-1}^*) \in U$ and $x_n^* \in V$, and a function $f : U \to V$ that satisfies the following:*

$$f(x_1^*, \ldots, x_{n-1}^*) = x_n^*,$$

and

$$(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) \in bd\{x : x \succcurlyeq x^*\},$$

for all $(x_1, \ldots, x_{n-1}) \in U$. Moreover, f is differentiable at $(x_1^*, \ldots, x_{n-1}^*)$ with gradient $\nabla f(x_1^*, \ldots, x_{n-1}^*) = (-g_1/g_n, \ldots, -g_{n-1}/g_n)$.

Proof We present the proof for $n = 2$. (The case of $n > 2$ is analogous.) Let g be the ordient of $\succcurlyeq$ at x^* with $g_2 \neq 0$. Without loss of generality, assume that $g_2 > 0$. Consider d_1 with $|d_1| = 1$. We first establish the existence of f that satisfies $(x_1^* + \varepsilon d_1, f(x_1^* + \varepsilon d_1)) \in \text{bd}\{x : x \succcurlyeq x^*\}$ for all ε sufficiently small. First of all, note that since $\succcurlyeq$ is ordientable at x^*, we have that $x^* \in \text{bd}\{x : x \succcurlyeq x^*\}$. Consider $\eta > 0$ and let $d_2' = d_1 \cdot (-g_1/g_2) - \eta/g_2$ and $d_2'' = d_1 \cdot (-g_1/g_2) + \eta/g_2$. It follows that $-\eta = d_1 g_1 + d_2' g_2 < 0 < d_1 g_1 + d_2'' g_2 = \eta$, so that (d_1, d_2') is a worsening direction and (d_1, d_2'') is an improvement direction of $\succcurlyeq$ at x^*. Hence, there exists $\varepsilon^* > 0$ such that

$$(x_1^* + \varepsilon d_1, x_2^* + \varepsilon d_2') \prec x^* \prec (x_1^* + \varepsilon d_1, x_2^* + \varepsilon d_2'') \tag{2}$$

for all $\varepsilon \in (0, \varepsilon^*)$.

We now claim that for each $\varepsilon \in (0, \varepsilon^*)$, there exists $f(x_1^* + \varepsilon d_1) \in [x_2^* + \varepsilon d_2', x_2^* + \varepsilon d_2'']$ such that $(x_1^* + \varepsilon d_1, f(x_1^* + \varepsilon d_1)) \in \text{bd}\{x : x \succcurlyeq x^*\}$. To see this, let Y be the set

$$\{(x_1^* + \varepsilon d_1, x_2) : x_2 \in [x_2^* + \varepsilon d_2', x_2^* + \varepsilon d_2'']\}, \tag{3}$$

and consider the two sets $A := Y \cap \{x : x \succcurlyeq x^*\}$ and $B := Y \cap \{x : x \prec x^*\}$. Note that Y is connected. From Eq. (2), both A and B are non-empty, disjoint and their union is Y.

By contradiction, assume that $Y \cap \text{bd}\{x : x \succcurlyeq x^*\} = \emptyset$. It follows that the boundary of A is empty. If the boundary of B is non-empty, then there exists x which belongs to the closure of B and the interior of A (since the closure of A is its interior from the previous line). Since A and B are disjoint, x must be a limit point of B. Since the interior of A is an open set, that is impossible, so that the boundary of B is empty. It follows that A and B are both open (and closed) and, consequently, form a separation of Y. This contradicts the fact that Y is connected.

To complete the proof, it suffices to show that

$$\lim_{\varepsilon \to 0} \frac{f(x_1^* + \varepsilon d_1) - f(x_1^*)}{\varepsilon} = -\frac{g_1}{g_2} \cdot d_1.$$

Assume not. There exists $\beta > 0$ such that

$$\left| \frac{f(x_1^* + \varepsilon_m d_1) - f(x_1^*)}{\varepsilon_m} - \left(-\frac{g_1}{g_2} \cdot d_1 \right) \right| > \beta,$$

for all converging sequences (ε_m) to zero. Choose some $\eta \in (0, \beta)$. From the above, we have that

$$\frac{f(x_1^* + \varepsilon d_1) - f(x_1^*)}{\varepsilon} \in [d_2', d_2''],$$

for all $\varepsilon < \varepsilon^*$ and, therefore,

$$\left| \frac{f(x_1^* + \varepsilon d_1) - f(x_1^*)}{\varepsilon} - \left(-\frac{g_1}{g_2} \cdot d_1 \right) \right| \leq \eta,$$

for all $\varepsilon < \varepsilon^*$, a contradiction. This completes the proof. $\square$

As already alluded, the gradient of the implicit function f at x^* gives the marginal rate of substitutions of good n for any good $i \neq n$. It is also worth noting that if $\succcurlyeq$ has a continuously differentiable representation, then Theorem 4 is nothing else than the classic implicit function theorem (see Krantz and Parks (2002) for a reference on implicit function theorems). Moreover, even if the binary relation $\succcurlyeq$ is continuous and, hence, has a continuous representation, it does not imply the existence of a differentiable representation. Naturally, there are versions of the implicit function theorem for non-differentiable functions. For instance, Kumagai (1980) shows that if $F : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ is continuous with $F(x^*) = 0$ and for each $(x_1, \ldots, x_{n-1})$ in a neighborhood of $(x_1^*, \ldots, x_{n-1}^*)$, $F((x_1, \ldots, x_{n-1}), \cdot)$ is locally one-to-one, then x_n can be uniquely expressed as a function of $(x_1, \ldots, x_{n-1})$ in neighborhoods of $(x_1^*, \ldots, x_{n-1}^*)$ and x_n^*. Theorem 4 differs from Kumagai's theorem in two main aspects. Firstly, Theorem 4 gives the gradient of the implicit function, while Kumagai's theorem only gives the existence of the implicit function. Secondly, suppose that F is a continuous representation of $\succcurlyeq$. Kumagai's theorem requires that $F((x_1, \ldots, x_{n-1}), \cdot)$ is locally one-to-one for each $(x_1, \ldots, x_{n-1})$ in a neighborhood of $(x_1^*, \ldots, x_{n-1}^*)$, which is not guaranteed when $\succcurlyeq$ is ordientable at x^*. Furthermore, unlike Kumagai, Theorem 4 does not guarantee the uniqueness of the implicit function. For an example, consider $\mathbf{X} = \mathbb{R}^2$ and suppose that the binary relation $\succcurlyeq$ is such that $I((0,0)) = \{(x_1, x_2) : 0 \leq x_2 \leq x_1^2\}$ and $\{x : x \succ 0\} = \{(x_1, x_2) : x_2 > x_1^2\}$. At $x^* = (0,0)$, the binary relation is ordientable with $(0,1)$ as an ordient, but there are two implicit functions: the functions $x_2 = x_1^2$ and $x_2 = 0$. Both implicit functions are differentiable at 0 with a zero gradient. The following proposition provides conditions so as to guarantee the *uniqueness* of the implicit function.

Proposition 8 *Assume that the binary relation $\succcurlyeq$ is continuously ordientable in a neighborhood of x^*. There exist two open neighborhoods $U \subset \times_{i=1}^{n-1} \mathbf{X}_i$ and $V \subset \mathbf{X}_n$ with $(x_1^*, \ldots, x_{n-1}^*) \in U$ and $x_n^* \in V$, and a unique function $f : U \to V$ that satisfying the properties as stated in Theorem 4.*

Proof Let $x^* \in \mathbf{X}$. From Theorem 4, there exist two open neighborhood U' and V' and an implicit function $f : U' \to V'$ which satisfies the above condition. We have to show that there exists $U \subseteq U'$ such that for each $(x_1, \ldots, x_{n-1}) \in U'$, there exists a unique $x_n \in f(U')$ with $(x_1, \ldots, x_{n-1}, x_n) \in \text{bd}\{x : x \succcurlyeq x^*\}$. Let $g(x)$ be the ordient of $\succcurlyeq$ at x and assume that $g_n(x^*) > 0$. Since g is continuous in a neighborhood of x^* and $e_n = (0, ..., 0, 1)$ is an improvement direction at x^*, there exists $\delta > 0$ such that e_n is an improvement direction at any x with $x \in B_\delta(x^*)$. It follows that $(x_1, \ldots, x_{n-1}, x'_n) = x' \prec x'' = (x_1, \ldots, x_{n-1}, x''_n)$ holds whenever $x'_n < x''_n$, $x' \in B_\delta(x^*)$ and $x'' \in B_\delta(x^*)$. Letting $U' = U \cap B_\delta(x^*)$ completes the proof. $\square$

Theorem 4 has implications for the normality of the ordient of a continuous binary relation $\succcurlyeq$ at x^* to the indifference set $I(x^*)$. If the binary relation is continuous, $\text{bd}\{x : x \succcurlyeq x^*\}$ is a subset of $I(x^*)$. From Theorem 4, it follows that the set $\{(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) : (x_1, \ldots, x_{n-1}) \in U\}$ is included in $I(x^*)$ and that $g(x^*) \cdot (x_1 - x_1^*, \ldots, x_n - x_n^*) \approx 0$ for all $(x_1, \ldots, x_n) \in U \times V$. Thus, the ordient of a continuous binary relation at x^* defines a hyperplane tangent to the indifference set at x^*: it is normal to $I(x^*)$. In the rest of this section, we study the relationships between ordients and normals to indifference sets.

We say that $g \neq 0$ is *normal* to $I(x^*)$ if for every $c > 0$, there exists $\delta > 0$ such that $\|x - x^*\| < \delta$ and $x \in I(x^*)$ imply $|g \cdot ((x - x^*)/\|x - x^*\|)| \leq c$. A normal is thus othogonal to *any* hyperplane tangent to the indifference set. To illustrate an extreme case, note that any $g \neq 0$ is normal to $I(x^*)$ whenever $I(x^*)$ is a singleton (as in the lexicographic case).

Proposition 9 *If (i) g is normal to $I(x^*)$ and $\succcurlyeq$ is continuous or (ii) g is normal to bd$\{x : x \succcurlyeq x^*\}$, then either g or $-g$ is either an increasing or a decreasing ordient of $\succcurlyeq$ at x^*.*

Proof Let g be normal to $I(x^*)$. Choose d such that $\|d\| = 1$ and $g \cdot d \neq 0$. As g is normal, given $c < |g \cdot d|$, there exists $\delta > 0$ such that $|g \cdot (x - x^*)| \leq c\|x - x^*\|$ for all $x \in I(x^*) \cap B_\delta(x^*)$, with $B_\delta(x^*)$ an open ball of radius δ around x^*. Consequently, $x^* + \varepsilon d \not\sim x^*$ for all $\varepsilon \in (0, \delta)$. From the continuity of $\succcurlyeq$, either $x^* + \varepsilon d \succ x^*$ for all $\varepsilon \in (0, \delta)$ or $x^* + \varepsilon d \prec x^*$ for all $\varepsilon \in (0, \delta)$. Hence, d is either an improvement direction at x^* or a worsening direction at x^*.

Assume that d is an improvement direction at x^*. Consider d' such that $\|d'\| = 1$ and $(g \cdot d)(g \cdot d') > 0$. The above arguments show that d' is either an improvement direction or a worsening direction. Evoking continuity of $\succcurlyeq$, it is easy to see that d'

is in fact also an improvement direction at x^*. Hence, g is an increasing ordient of $\succcurlyeq$ at x^* if $g \cdot d > 0$ and $-g$ is an increasing ordient of $\succcurlyeq$ at x^* if $g \cdot d < 0$.

Assume that d is a worsening direction at x^*. Then, analogous arguments as above show that g is a decreasing ordient of of $\succcurlyeq$ at x^* if $g \cdot d < 0$ and $-g$ is an increasing ordient of $\succcurlyeq$ at x^* if $g \cdot d > 0$. This completes the proof of part (i).

The proof of part (ii) is similar to the proofs of part (i) and Theorem 4. $\square$

The converse of Proposition 9 does not hold, however. For a counter-example, let $\mathbf{X} = \mathbb{R}^2$ and suppose that $I((0,0)) = \{x : x_2 = 0\} \cup \{x : x_2 = \sqrt{|x_1|}, x_1 \neq 0\}$, $\{x : x \succ 0\} = \{x : x_1 = 0, x_2 > 0\} \cup \{x : 0 < x_2 < \sqrt{|x_1|}, x_1 \neq 0\}$ and $\{x : x \prec 0\} \supseteq \{x : x_2 < 0\}$. At the point $(0,0)$, $(0,1)$ is an ordient of the binary relation $\succcurlyeq$. Yet, the ordient $(0,1)$ is clearly not normal to the indifference set.[16] To get normality, a stronger notion of ordient is necessary.

Definition 2 *The binary relation $\succcurlyeq$ has a uniform ordient g at x^* if for any $c > 0$, there exists $\varepsilon^* > 0$ such that $d \cdot g \geq c$ with $\|d\| = 1$ implies that $x^* + \varepsilon d \succ x^*$ and $d \cdot g \leq -c$ with $\|d\| = 1$ implies that $x^* + \varepsilon d \prec x^*$, for all $\varepsilon < \varepsilon^*$. The binary relation $\succcurlyeq$ is uniformly ordientable at x^* if it has a uniform ordient.*

The concept of uniform ordient is stronger than the concept of ordient as it requires that for any direction d with $|g \cdot d| \geq c$, there exists a *unique* $\varepsilon^* > 0$ such that $x^* + \varepsilon d \succ x^*$ or $x^* + \varepsilon d \prec x^*$ for all $\varepsilon < \varepsilon^*$. With the concept of ordient, the choice of $\varepsilon^* = \varepsilon^*(d)$ can depend on the direction d considered. For instance, in the counter-example above, the ordient $(0,1)$ of $\succcurlyeq$ at $(0,0)$ requires that $\varepsilon^*(d_1, 1)$ in direction $(d_1, 1)$ goes to zero as d_1 goes to zero. Uniform ordientability ensures that the indifference set is tangent to the hyperplane defined by the uniform ordient.

Proposition 10 *If $\succcurlyeq$ is uniformly ordientable at x^* with ordient g, then g is normal to $I(x^*)$.*

Proof Let g be the uniform ordient of $\succcurlyeq$ at x^* and consider a sequence $(x_n)_n$ converging to x^* with $x_n \in I(x^*)$ for all n. Fix $c > 0$. By contradiction, assume that g is not normal to $I(x^*)$. There exists n^* such that $|g \cdot (x_n - x^*)| > c\|x_n - x^*\|$ for all $n \geq n^*$. For all $n \geq n^*$, we can write x_n as $x^* + \varepsilon_n d_n$ with $\varepsilon_n = ||x_n - x^*|| > 0$

[16] Note that at $(0,0)$, there are two implicit functions f_1 and f_2: $f_1(x_1) = 0$ for all x_1, and $f_2(x_1) = \sqrt{|x_1|}$ if $x_1 \neq 0$ and $f_2(0) = 0$. Only the first implicit function f_1 is differentiable at 0 with the gradient given by $0 = -g_1/g_2$.

and $d_n = (x_n - x^*)/\|x_n - x^*\|$. It follows that either $g \cdot d_n > 0$ or $g \cdot d_n < 0$. Assume $g \cdot d_n > 0$. Since g is a uniform ordient of $\succcurlyeq$ at x^*, d_n is an improvement direction. Consequently, there exists an ε^* (independent of n) such that $x^* + \varepsilon d_n \succ x^*$ for all $\varepsilon < \varepsilon^*$. If $\varepsilon_n < \varepsilon^*$, we have a contradiction with the fact that $x_n \sim x^*$. If $\varepsilon_n \geq \varepsilon^*$ for all $n > n^*$, we have a contradiction with the fact that ε_n converges to 0. Lastly, if $I(x^*) = \{x^*\}$ or if there is no sequence $(x_n)_n$ converging to x^* with $x_n \in I(x^*)$ for all n, then there is nothing to prove. $\square$

To conclude, we mention another result about normality and ordients inspired by Mas-Colell (1985). Suppose that $\succcurlyeq$ is locally non-satiated on an open set with connected indifference set $I(x^*)$. If, furthermore, the boundary of the binary relation $\succcurlyeq$ is a C^1-manifold, then the normal of $I(x^*)$ exists and is an ordient of $\succcurlyeq$ at x^*. The result directly follows from Proposition 2.3.9 (p. 64) in Mas-Colell (1985) and Proposition 1. From Proposition 2.3.9 in Mas-Colell, we have that $\succcurlyeq$ has a differentiable representation f at x^*. Consequently, the gradient ∇f at x_0 exists and is normal to $I(x^*)$ at x^*. From Proposition 1, it is an ordient of $\succcurlyeq$ at x^*. This result follows the differential approach to economic theory. Our results suggest, yet again, that the differential approach obscures the importance of local trade-offs. Ordients do characterize these local trade-offs - without the need to assume the existence of representations.

6 Extensions

The object of this section is to provide further insights into the concept of ordients and to demonstrate that the concept of ordients is flexible enough so as to characterize unconstrained maxima or maxima of incomplete binary relations.

6.1 Unconstrained Maxima

An important implication of assuming an ordientable binary relation is local non-satiation. In some applications, e.g., consumer theory, this is a reasonable assumption. Yet, in other applications such as game theory, this assumption is problematic. We now discuss how a slight generalization of the concept of ordient can accommodate the existence of unconstrained local maxima.

We say that d is a weak improvement (resp., worsening) direction of $\succcurlyeq$ at x_0 if there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ with $x_0 + \varepsilon d \in \mathbf{X}$, we have $x_0 + \varepsilon d \succcurlyeq x_0$

(resp., $x_0 + \varepsilon d \preccurlyeq x_0$). Weakly increasing and weakly decreasing ordients are defined analogously as in Section 2. If g is a weakly increasing and weakly decreasing ordient of $\succcurlyeq$ at x_0, then g is called a *weak ordient*.

Two remarks are worth making. Firstly, if the binary relation $\succcurlyeq$ is ordientable, then $\succcurlyeq$ is weakly ordientable. The concept of weak ordientability is a slight generalization of the concept of ordientability. Secondly, it can accommodate the existence of binary relations with "thick" indifference curves. For instance, suppose that $\succcurlyeq$ is the binary relation on $(0, 3)$ induced by the function f with $f(x) = x$ if $x \leq 1$, $f(x) = 1$ if $x \in (1, 2)$ and $f(x) = x - 1$ if $x > 2$. With the binary relation $\succcurlyeq$, an individual is indifferent between any $x \in [1, 2]$. The binary relation $\succcurlyeq$ is not ordientable, but is weakly ordientable.

With this generalized definition, the vector x^* is a local maximum if and only if the set of weakly decreasing ordients at x^* is $\mathbb{R}^n \setminus \{0\}$ (since all directions are weakly worsening directions).

6.2 Incomplete Binary Relations

While we have casted our results for complete, reflexive and transitive binary relations, we believe that the concept of ordient naturally extends to more general binary relations. For instance, suppose that $\succcurlyeq$ is transitive and reflexive, but not necessarily complete. A possible solution is to complete $\succcurlyeq$.[17] However, this solution is not satisfactory. To see this, consider the component-wise order $\geq$ on $\mathbb{R}^2$ i.e., $x > y$ if and only if $x_i \geq y_i$ for all $i \in \{1, 2\}$ and $x_i > y_i$ for some $i \in \{1, 2\}$. We can extend $\geq$ to a complete order $\succcurlyeq$ by letting $x \sim y$ whenever x and y are not comparable under $\geq$. Unfortunately, the binary relation $\succcurlyeq$ has no ordient everywhere. A more satisfactory solution consists in slightly modifying the definition of improvement and worsening directions. For any $x \in \mathbf{X}$, denote by $C(x)$ the set of alternatives comparable to x i.e., $C(x) := \{y \in \mathbf{X} : y \succcurlyeq x \text{ or } x \succcurlyeq y\}$. We say that d is an improvement (resp., worsening) direction of $\succcurlyeq$ at x_0 if there exists $\varepsilon^* > 0$ such that $\varepsilon \in (0, \varepsilon^*)$ and $x_0 + \varepsilon d \in C(x_0)$ implies that $x_0 + \varepsilon d \succ x_0$ (resp., $x_0 + \varepsilon d \prec x_0$). Increasing and decreasing ordients are then naturally defined as in Section 2. Of course, if $\succcurlyeq$ is complete, these new definitions coincide with our original definitions.

With these new definitions, the component-wise order on $\mathbb{R}^2$ is ordientable: $(1, 1)$ is an ordient everywhere. For another example, assume that $\succcurlyeq$ is represented by m

[17] This is possible by Szpilrajn's Theorem (see Ok (2007, p. 17)).

real-valued functions $(f_1, \dots, f_m)$ such that $x \succ y$ if and only if $f_j(x) \geq f_j(y)$ for all $j \in \{1, \dots, m\}$ and $f_j(x) > f_j(y)$ for some $j \in \{1, \dots, m\}$. (See Ok (2002) for a representation theorem of incomplete preferences.) If the m functions are differentiable, then $\nabla f_1(x_0) + \cdots + \nabla f_m(x_0)$ is an ordient of $\succcurlyeq$ at x_0 (provided it is non zero). More generally, $\succcurlyeq$ might be given by m (complete) binary relations $(\succcurlyeq_1, \dots, \succcurlyeq_m)$ such that $x \succ y$ if and only if $x \succcurlyeq_j y$ for all $j \in \{1, \dots, m\}$ and $x \succ_j y$ for some $j \in \{1, \dots, m\}$. For instance, each binary relation $\succcurlyeq_j$ might represent the preferences of individual j and $\succcurlyeq$ is the Pareto order. It then easy to show that if each $\succcurlyeq_j$ is ordientable with ordient g^j, then $g^1(x_0) + \cdots + g^m(x_0)$ is an ordient of $\succcurlyeq$ at x_0.[18]

Turning to optimality conditions, we say that x^* maximizes $\succcurlyeq$ on X if there does not exist another $x \in X$ such that $x \succ x^*$. We write $x^* \in \overline{\max}_{\succcurlyeq} X$. Then g is an increasing ordient of the set X at x if for each d with $g \cdot d > 0$, there exists $\varepsilon^* > 0$ such that $x + \varepsilon d \notin X$ or $x + \varepsilon d \in X \setminus C(x)$ for all $\varepsilon \in (0, \varepsilon^*)$. It is then easy to see that if $x^* \in \overline{\max}_{\succcurlyeq} X$ and g is an increasing ordient of $\succcurlyeq$ at $x^* \in \partial X$, then g is an increasing ordient of $\succcurlyeq_X$at x^*. In other words, a version of Theorem 1(i) remains valid if suitable modifications of our definitions are made.[19] For an illustration, consider the maxima of the set $\{x \in \mathbb{R}^2_+ : x_1 + 2x_2 \leq 1\}$ according to the component-wise order $\geq$. We have that $\overline{\max}_{\geq}\{x \in \mathbb{R}^2_+ : x_1 + 2x_2 \leq 1\} = \{x \in \mathbb{R}^2_+ : x_1 + 2x_2 = 1\}$ and, clearly, $(1, 1)$ is both an increasing ordient of $\geq$ and of $\{x \in \mathbb{R}_+ : x_1 + 2x_2 \leq 1\}$ at each maximum.

In sum, we believe that most of our results can be suitably generalized to (almost) any binary relation. This is likely to be important as recent advances in decision theory, e.g., preferences with multiple rationales (among others, Mariotti and Manzini (2007) and Apesteguia and Ballester (2008)), often relax the assumption of completeness or transitivity.

7 Discussion

This paper introduces the concept of ordient for binary relations, a relative to the concept of gradient for functions. Ordients have a natural economic interpretation as "marginal rate of substitutions." In effect, ordients characterize the directions of improvement and worsening from any given alternative. Most importantly, there is

[18]Note that for each $\lambda \in \mathbb{R}^m_{++}$, $\sum_j^m \lambda^j g^j(x_0)$ is also an ordient of $\succcurlyeq$ since $\lambda^j g^j$ is an ordient of each $\succcurlyeq_j$. The parameter λ^j could be interpreted as a weight of a welfare function.

[19]Note that, unlike Theorem 1, $x^* \in \overline{\max}_{\succcurlyeq} X$ and g an increasing ordient of $\succcurlyeq$ at x^* does not imply that $x^* \in \partial X$ since any x^* maximizes $\succcurlyeq$ on X when $C(x^*) = \emptyset$.

no particular need for cardinal comparisons (utility functions) and even less so for comparisons based on local linear approximations (differentiable utility functions). The main message of this paper is that only the ordering of the alternatives and knowledge of the local trade-off between the alternatives do matter.[20]

We show how the concept ordient makes it possible to characterize the maximal elements of a set. We also derive an envelope theorem and implicit function theorem for binary relations with ordients.

We believe that this paper paves the way for future developments in Economics, without imposing the existence of (differentiable) representations. Recent developments in decision theory/behavioral economics seem to move away from the "representable and differentiable" approach to Economics. Equipped with the concept of ordient, the characterization of maxima or comparative statics exercises are possible, even in a world without utility functions.

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