Centre d'Etude et de Recherche en Economie,
Gestion, Modélisation et Informatique Appliquée

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Document de travail 2011-06

Décembre 2011

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Ahmed AINOUCHE

Université des Antilles et de la Guyane. Faculté de Droit et d'Economie de la Martinique. Campus de Schoelcher - Martinique FWI
B.P. 7209 - 97275 Schoelcher Cedex - Tél. : 0596. 72.74.00 - Fax. : 0596. 72.74.03

$\alpha-$Degree Closures for Graphs

Ahmed Ainouche
UAG - CEREGMIA
Campus de Schoelcher - B.P. 7209
97275 Schoelcher Cedex. Martinique (FRANCE)
a.ainouche@martinique.univ-ag.fr

September 27, 2011

Abstract

Bondy and Chvátal [7] introduced a general and unified approach to a variety of graph-theoretic problems. They defined the k-closure $C_k(G)$, where k is a positive integer, of a graph G of order n as the graph obtained from G by recursively joining pairs of nonadjacent vertices a, b satisfying the condition $C(a, b) : d(a) + d(b) \geq k$. For many properties P, they found a suitable k (depending on P and n) such that $C_k(G)$ has property P if and only if G does. For instance, if P is the hamiltonian property, then $k = n$.

In [3], we proved that $C(a, b)$ can be replaced by $d(a) + d(b) + |Q(G)| \geq k$, where $Q(G)$ is a well-defined subset of vertices nonadjacent to a, b.

In [4], we proved that, for a $(2 + k - n)$-connected graph, $C(a, b)$ can be replaced by $|N(a) \cup N(b)| + \delta_{ab} + \varepsilon_{ab} \geq k$, where ε_{ab} is a well defined binary variable and δ_{ab} is the minimum degree over all vertices distinct from a, b and non adjacent to them. The condition on connectivity is a necessary one.

In this paper we show that $C(a, b)$ can be replaced by the condition $d(a) + d(b) + (\overline{\alpha}_{ab} - \alpha_{ab}) \geq k$, where $\overline{\alpha}_{ab}$ and α_{ab} are respectively the order and the independence number of the subgraph $G - N(a) \cup N(b)$.

All these three last conditions are uncomparable, unique and well defined. Moreover any hamiltonian cycle in $C_n(G)$ can be transformed into a hamiltonian cycle in the original graph within a polynomial time. However, unlike the conditions given in [3] and [4], the condition $(\overline{\alpha}_{ab} - \alpha_{ab})$ cannot be computed in polynomial time. By giving suitable upper bounds of α_{ab} (or lower bounds of $(\overline{\alpha}_{ab} - \alpha_{ab})$) we satisfy this last nice property. In doing so, we surprisingly obtain a result of [8] as an easy Corollary.

Key words: *Closure*, *Degree Closure*, *Neighborhood Closure*, *Dual Closure*, *Stability, Hamiltonicity, Cyclability, Degree Sequence, Matching Number, k-Leaf-Connected.*

1 Introduction

Let $G = (V, E)$ be a finite simple graph of order n, connectivity $\kappa(G)$. Ore [7] proved that G is hamiltonian if the condition $d(a) + d(b) \geq n$ is satisfied by any pair (a, b) of nonadjacent vertices. Later, Bondy and Chvátal [7] observed that G is in fact hamiltonian if and only if $G + ab$ is hamiltonian. This observation motivated the introduction of the concept of the $k-$closure $C_k(G)$ of G, for a given positive integer k. The graph $C_k(G)$ is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least k. This graph is unique and polynomialy obtained from G. For a number of various properties of a graph G on n vertices, they showed that it is possible to find a suitable integer k, such that if G has property $P(k)$, so does $C_k(G)$. For instance, if P is the hamiltonian property, then $k = n$.

Starting from the main result obtained in [2] we improved the condition $P(k) : d(a) + d(b) \geq k$ in two directions:

In [3], $P(k)$ becomes: $d(a) + d(b) + |Q(G)| \geq k$, where $Q(G)$ is a well-defined subset of vertices nonadjacent to a, b. The corresponding condition is named "$\boldsymbol{\beta}-$ **dcc**" for $\beta-$*degree closure condition.*

In [4], for a $(2 + k - n)$-connected graph, $P(k)$ becomes: $|N(a) \cup N(b)| + \delta_{ab} + \varepsilon_{ab} \geq k$, where ε_{ab} is a well defined binary variable and δ_{ab} is the minimum degree over all vertices distinct from a, b and non adjacent to them. The corresponding condition is named "$\boldsymbol{\beta}-$ **ncc**" for $\beta-$*neighborhood closure condition.* The condition on connectivity is not a real constraint since it is a necessary condition.

In this paper, we use a relaxation of the main result given in [1] to obtain another improvement of $P(k)$. The new condition is*:* $d(a) + d(b) + (\overline{\alpha}_{ab} - \alpha_{ab}) \geq k$, where $\overline{\alpha}_{ab}$ and α_{ab} are respectively the order and the independence number of the subgraph $G - N(a) \cup N(b)$. We shall refer to it as "$\boldsymbol{\alpha}-$ **dcc**" for $\alpha-$*degree closure condition.*

To state the new results and to relate them to existing ones, we need some preliminary definitions and notations.

2 Definitions and notations

We use Bondy and Murty [9] for terminology and notation not defined here and consider simple graphs only. Let $G = (V, E)$ be a graph of order $n \geq 3$. The set of neighbors of a vertex $v \in V$ is denoted $N_G(v)$ and $d_G(v) = |N_G(v)|$ is the degree of v. If A is a subset of V, $G[A]$ will denote the subgraph induced by A.

Let C be a cycle in G, in which a direction of traversing it is given. For $u \in V(C), u^+$ (resp. u^-) denotes its successor (resp. predecessor) on C. More generally, if $A \subseteq V$ then $A^+ := \{u \in C \mid u^- \in A\}$ and $A^- := \{u \in C \mid u^+ \in A\}$. Given vertices a, b of C let $C[a, b]$ denote the subgraph of C from a to b in the chosen direction. We shall write $C(a, b]$, $C[a, b)$ or $C(a, b)$ if a, b or (a and b) are respectively excluded. The same notation will be adopted if we consider a path P (where the direction of traversing it is assumed) instead of a cycle C.

Paths and cycles in $G = (V, E)$ are considered as subgraphs and for simplicity we use the same notation to mean a subgraph, its vertex set or its edge set.

The concept of vine [6] plays a central role for our proofs. A vine on a path $\pi := \pi[a, b]$ is a set $\mathcal{P} := \{\pi_i[x_i, y_i] \mid 1 \le i \le m\}$ of internally disjoint paths such that

(a) $\pi_i \cap \pi = \{x_i, y_i\}$

(b) $a = x_1 \prec x_2 \prec y_1 \preceq x_3 \prec y_2 \preceq x_4 \prec ... \preceq x_m \prec y_{m-1} \prec y_m = b$ on π

(here $u \prec v$ (resp. $u \preceq v$) means that u precedes v on P (resp. possibly $u = v$) where π is oriented from a to b).

With each vine $\mathcal{P}$ on a path $\pi[a, b]$ is associated a *constrained cycle* $C_{ab} := \sum_{i=1}^{m} C_i$, where $C_i := \pi[x_i, y_i] . \overleftarrow{\pi}_i$, $1 \le i \le m$ and the addition of edges in $\sum_{i=1}^{m} C_i$ is taken modulo 2. In this paper the paths $\pi_i[x_i, y_i]$ are in fact edges because we shall only focusing on a hamiltonian path π instead of any path.

Let (a, b) be a pair of nonadjacent vertices, x be any vertex not adjacent to a and b and k be a positive integer. Then we associate

$$\begin{cases} (a) & G_{ab} := G - N_G(a) \cup N_G(b),\ T_{ab}(G) := V(G_{ab}) \setminus \{a, b\} \\ (b) & \overline{\alpha}_{ab}(G) := |G_{ab}| = 2 + |T_{ab}(G)|\ ,\ \alpha_{ab} := \alpha(G_{ab}),\ \nu_{ab} := \nu(G_{ab}) \\ (c) & \Delta_{ab}(G) := \max\{d_G(x) | x \in T_{ab}(G)\},\ \delta_{ab}(G) := \min\{d_G(x) | x \in T_{ab}(G)\} \\ (d) & \sigma_{ab}(G) := d_G(a) + d_G(b),\ \gamma_{ab}(G) := |N_G(a) \cup N_G(b)| \\ (e) & \lambda_{ab}(G) := |N_G(a) \cap N_G(b)|. \end{cases}$$

Note that G_{ab} is disconnected since a, b are isolated vertices and $\nu(G_{ab})$ is the matching number of G_{ab}. For a given hamiltonian path μ, let the vertices be ordered so that $i < j$ implies that i appears before j on the path μ traversed from a to b. Let a directed graph $\overrightarrow{G}$ be produced from G by designing a direction to arc ij of G from i to j whenever $i < j$. The a to b vertex connectivity of $\overrightarrow{G}$ is denoted $h^{\mu}_{ab}(G)$. Dirac [10] proved that a vine with two paths exists on any path in a two-connected graph. In that case, these two paths satisfy the constraint on $h^{\mu}_{ab}(G)$. In other words, $h^{\mu}_{ab}(G) \ge 2$ holds for any 2-connected graph .

In [1], we proved:

Theorem 1 *Let G be a graph of order $n \ge 3$. If $\alpha_{ab} \le h^{\mu}_{ab}(G)$ then G is hamiltonian if and only if $G + ab$ is hamiltonian.*

In [2] we conjectured the following.

Conjecture 1 *Let G be a κ-connected graph of order $n \ge 3$. If $\alpha_{ab} \le \max(\kappa, \lambda_{ab})$ then G is hamiltonian if and only if $G + ab$ is hamiltonian.*

The condition $\alpha_{ab} \le \max(\kappa, \lambda_{ab})$ will be referred to as the "$\alpha - cc$" for $\alpha-$ *closure condition.* This new condition admits two incomparable relaxations.

- Going beyond our result in [1], we treat the case $\max(\kappa, \lambda_{ab}) = \lambda_{ab}$ in this paper. In particular we get the main result of [8] as an easy corollary. The corresponding condition of this case, that is, $\alpha_{ab} \leq \lambda_{ab}$, will be referred as the $\alpha-$degree closure condition ($\alpha - dcc$). This condition involves the degree sum of (a, b) since $\alpha_{ab} \leq \lambda_{ab} \Leftrightarrow \sigma_{ab} + (\overline{\alpha}_{ab} - \alpha_{ab}) \geq n$.

- In another paper in preparation [5] we consider the conjectured part of the condition, that is $\max(\kappa, \lambda_{ab}) = \kappa$. This will be referred as the alpha-neighborhood closure condition ($\alpha - ncc$). Particular cases $\kappa = 2, 3$ will be proved and a particular condition treated in [8] will be improved.

Following Bondy and Chvátal ([7]) , we define:

Definition 1 *Let P be a property defined for all graphs G of order n and let k be an integer. Let a, b be two nonadjacent vertices satisfying the condition*

$$P(k) : \alpha_{ab}(G) \leq \lambda_{ab} + n - k \Leftrightarrow \sigma_{ab}(G) + (\overline{\alpha}_{ab} - \alpha_{ab}) \geq k. \qquad (*)$$

Then P is k-alpha degree stable if whenever $G + ab$ has property P and $P(k)$ holds then G itself has property P. We simply denote by $dC_k(G)$ the associated (α-degree closure).

The graph $dC_k(G)$ is then obtained from G by recursively joining pairs of nonadjacent vertices a, b for which (*) holds until no such pair remains. The equivalence in (*) comes from the equalities $\sigma_{ab} = d(a) + d(b) = \gamma_{ab} + \lambda_{ab}$ and $\overline{\alpha}_{ab} + \gamma_{ab} = n$. For the very particular case where T_{ab} is an independent set, (*) reduces to Bondy-Chvátal's known closure condition.The statement below is an easy adaptation of Proposition 2.1 in [7].

Proposition 1 *If P is $k-$alpha degree stable and $dC_k(G)$ has property P then G itself has property P.*

In this paper, we investigate the stability of a number of properties of graphs which remain in any super-graph of G (a graph obtained from G by addition of edges). Most of these properties are studied in [7]. We also provide new properties. Throughout let (a, b) be a pair of nonadjacent vertices of a graph G satisfying the condition (*) for a given positive integer k. For each one of the considered properties P we fix k so that G has properties P whenever $G + ab$ does. Below is a key-lemma for the remaining of the paper.

Lemma 1 *Let $\pi[a, b]$ be a hamiltonian $a - b$ path. If $\alpha_{ab} \leq \lambda_{ab}$ then G is hamiltonian.*

Proof. A proof by induction is already given in [1]. Here we provide an alternative constructive one which has its own interest. By contradiction we assume G nonhamiltonian.

Set $W := \{w_i \in N(a) \cap N(b) | i = 1, .., \lambda_{ab}\}$ and $W_j := \pi(w_j, w_{j+1})$ for $j = 1, .., \lambda_{ab} - 1$. It is clear that the vertices of W cannot be consecutive on π and

$W_i \subset T$ holds for all i since otherwise we have an obvious hamiltonian cycle. This implies that $T \neq \emptyset$ and $\alpha_{ab} \geq 3$. This in turn implies $\lambda_{ab} \geq 3$. Therefore we have $w_1 \prec w_2 \prec ... \prec w_{\lambda_{ab}}$. Within each W_i, choose a subinterval $D_i := \pi(b_i, a_i)$ such that $b_i \in N(b)$, $a_i \in N(a)$ and $D_i \subseteq T$. Such a subpath exists as we always may choose $b_i = w_i$, $a_i = w_{i+1}$. Otherwise we have $w_i \preceq b_i \prec a_i \preceq w_{i+1}$ for $i = 1, .., \lambda_{ab}$.

We now define a set $R := \{r_1, .., r_{\lambda_{ab}-1}\}$ as follows:

(i) $r_1 := a_1^-$. Clearly $r_1 \in W_1 \neq \emptyset$ exists.

(ii) $r_2 \in W_2 \backslash [N(r_1)]$ is chosen so that $r_2^+ \in N(a) \cup N(r_1)$. If $W_2 \cap N(r_1) = \emptyset$ we set $r_2 := a_2^-$. Note that $r_1 b_2^+ \notin E$ since otherwise the constrained cycle of the vine $\mathcal{P} := \left\{aa_1, r_1 b_2^+, b_2 b\right\}$ is hamiltonian. Thus $r_2 \neq b_2$ exists and $\{r_1, r_2\}$ is an independent set.

(iii) $r_3 \in W_3 \backslash [N(r_1) \cup N(r_2)]$ is chosen so that $r_3^+ \in N(a) \cup N(r_1) \cup N(r_2)$. If $W_3 \cap [N(r_1) \cup N(r_2)] = \emptyset$ we set $r_3 = a_3^-$. Note that $r_1 b_3^+ \notin E$ since otherwise the constrained cycle of the vine $\mathcal{P} := \left\{aa_1, r_1 b_3^+, b_3 b\right\}$ is hamiltonian. Similarly $r_2 b_3^+ \notin E$ since otherwise the constrained cycle of either the vine $\mathcal{P} := \left\{aa_2, r_2 b_3^+, b_3 b\right\}$ is hamiltonian if $r_2 = a_2^-$ or the vine $\mathcal{P} := \left\{aa_1, r_1 r_2^+, r_2 b_3^+, b_3 b\right\}$ is hamiltonian if $r_2 \prec a_2^-$ and $r_1 = a_1^-$. We observe that r_3 exists and $\{r_1, r_2, r_3\}$ is an independent set

(iv) we continue this way and for $3 < i \leq \lambda_{ab} - 1$ we choose $r_i \in W_i \backslash \left[\cup_{j=1}^{i-1} N(r_j)\right]$ such that $r_i^+ \in N(a) \cup \left[\cup_{j=1}^{i-1} N(r_j)\right]$. If $W_i \cap \left[\cup_{j=1}^{i-1} N(r_j)\right] = \emptyset$ we set $r_i = a_i^-$. Following the above method we reach the conclusion that $R := \{r_1, ..., r_{\lambda_{ab}}\}$ is an independent set. This is a contradiction to the hypothesis since then $\alpha_{ab} \geq |\{a, b\} \cup R| > \lambda_{ab}$. ■

We would like to point out that the proof of the above Lemma shows that one can find a hamiltonian cycle in G in polynomial time if we know one in $dC_k(G)$. However the construction of the closure itself cannot be found in polynomial time as it is well known to be a hard problem to compute the independence number α_{ab}. This is why we provide in section 4 an alternative closure condition which is a relaxation of $P(k)$.

Throughout, $S \subset V$ denotes a subset with s vertices.

3 Main results

Theorem 2 *The property of being hamiltonian is $n-$degree stable.*

Proof. Consequence of Lemma 1. ■

The graph G is $S-$hamiltonian, $s \leq n-3$, if it remains hamiltonian whenever some or all vertices of S are removed. We simply say that it is $s-$hamiltonian if we are only interested by the number s instead of the set S of vertices.

Theorem 3 *The property of being S- hamiltonian is $(n+s)$-degree stable.*

Proof. For some $W \subseteq S$, set $H := G - W$. By the hypothesis $\alpha_{ab}(G) \leq \lambda_{ab}(G) - s$. Clearly $\alpha_{ab}(H) \leq \alpha_{ab}(G)$ and $\lambda_{ab}(G) \leq \lambda_{ab}(H) + |W|$. Thus $\alpha_{ab}(H) \leq \alpha_{ab}(G) \leq \lambda_{ab}(G) - s \leq \lambda_{ab}(H) + |W| - s$. It follows that $\alpha_{ab}(H) \leq \lambda_{ab}(H)$ since $|W| - s \leq 0$. Therefore H is hamiltonian by Theorem 2. Note that this property is $(n + s - 1)$ - degree stable if S is not an independent set, in which case $\overline{\alpha}_{ab} - \alpha_{ab} \geq 1$ The proof is now complete. ■

The subgraph $G[S]$ is hamiltonian if G is $V \backslash S-$hamiltonian. Applying Theorem 3 we obtain:

Theorem 4 *The property "$G[S]$ is hamiltonian" is $(2n - s)$-degree stable.*

We say that G is $S-$cyclable ($S-$ traceable resp.) if it contains a cycle C (a path resp.) with all vertices of S.

Theorem 5 *The property "G is S-cyclable" is $n-$degree stable.*

Proof. Suppose that $(G + ab)$ contains a cycle C such that $S \subset V(C)$ but G does not. Then a, b are connected by a path $\pi := a_1 ... a_p$ with $a = a_1$, $b = a_p$, $n \geq p \geq s$. Set $H := G[V(\pi)]$ and $W := V \backslash V(H)$. Clearly $N(a) \cap N(b) \subset V(\pi)$ since otherwise H is hamiltonian. Obviously $\alpha_{ab}(H) \leq \alpha_{ab}(G)$. By the hypothesis $\alpha_{ab}(G) \leq \lambda_{ab}(G)$. Thus $\alpha_{ab}(H) \leq \alpha_{ab}(G) \leq \lambda_{ab}(G) = \lambda_{ab}(H)$. Therefore H is hamiltonian by Theorem 2. ■

A *caterpillar* is a particular tree which results in a path when its leaves are removed. The *spine* of the caterpillar is the longest path of it. The graph G is called $S-$ *caterpillar spannable* if it has a spanning tree that is a caterpillar, whose leaves are the vertices of $S := \{x_1, ..., x_s\}$. Suppose that the spine is an $[x_1, x_2]-$path. Let G' be a graph obtained from G by adding a new vertex, v say, that is joined to x_1 and x_2. Then G is $S-$ *caterpillar spannable if* G' is $(S - \{x_1, x_2\})-$hamiltonian. Applying Theorem 3 to the graph G' we obtain

Theorem 6 *Let $S \subset V(G)$ with s vertices, $2 \leq s < n$. Then the property "G is S-caterpillar spannable" is $(n + s - 1)$-degree stable.*

A set $F \subset E$ of edges such that the components of the graph (V, F) are vertex disjoint paths is called F-cyclable (or $|F| -$*edge*$-$*hamilton*) if there exists a cycle that contains F. It is F-traceable if there exists a path that contains F. Applying Theorem 2 to the graph obtained from G by subdividing each edge in F into two, we obtain.

Theorem 7 *The property "G is $F-$cyclable with $|F| \leq n - 3$", is $(n + |F|)$-degree stable.*

A graph G is defined to be $|F|-$Hamilton-connected if for each pair (x, y) of vertices there is a hamiltonian path with endpoints x, y that contains F. Now G must be $(F \cup xy)-$cyclable and using Theorem 7 we obtain.

Theorem 8 *The property "G is $F-$Hamilton-connected with $|F| \leq n-4$", is $(n+|F|+1)$-degree stable.*

Let sK_2 be an s-matching, that is, a subgraph with s independent edges.

Theorem 9 *Let n, s be positive integers with $s \leq \frac{n}{2}$. Then the property of containing sK_2 is $(2s-1)$-degree stable.*

Proof. If $G+ab$ contains an sK_2 but G does not, then there exists an $(s-1)$-matching $\{a_1b_1, ..., a_{s-1}b_{s-1}\}$ in G and an $s-$matching in $G+ab$. For $i \in [1, s-1]$ we set

$$\left\{ \begin{array}{c} A := \{a_i\}, B := \{b_i\}, D := V \setminus (A \cup B \cup \{a,b\}) \\ M := \{a_ib_i | i \in [1, s-1]\}, \ M_i = \{a_i, b_i\} \text{ and } m_i := |N_{M_i}(a) \cup N_{M_i}(b)|. \end{array} \right.$$

We label the vertices of A, B so that $a_i \in N(a) \cup N(b)$ whenever $m_i \geq 1$. An M-augmenting path is a path with an even number of vertices, unsaturated endpoints in $D \cup \{a,b\}$ and whose edges are alternatively in $E-M$ and M. To avoid contradiction, we obviously assume that G contains no M-augmenting path. Moreover $D \cup \{a,b\}$ must be an independent set since otherwise there is an $s-$matching in G. We shall assume $\alpha_{ab} \neq \overline{\alpha}_{ab}$ (that is T_{ab} is not an independent set), by Bondy and Chvátal's result [7].

To distinguish the all possible configurations we define the following independent sets: $J_0 := \{i | m_i = 0\}$, $J_{11} := \{i | \ m_i = 1 \text{ and } |N(a_i) \cap \{a,b\}| = 1\}$, $J_{12} := \{i | m_i = 1 \text{ and } |N(a_i) \cap \{a,b\}| = 2\}$ and $J_2 := \{i | m_i = 2\}$. If $j \in J_2$ then $d_{M_j}(a) + d_{M_j}(b) = 2$ and either $d_{M_j}(a) = 2$ or $d_{M_j}(b) = 2$ for if $aa_j, bb_j \in E$ then aa_jb_jb is an M-augmenting path. These sets form a partition of $J := J_0 \cup J_{11} \cup J_{12} \cup J_2$. We note that ,

$$\begin{aligned} \sigma_{ab} &= |J_{11}| + 2(|J_{12}| + |J_2|), \ s = |J| + 1 & (1) \\ \text{and } \overline{\alpha}_{ab} &= 2 + |J_{11}| + |J_{12}| + 2|J_0| + |D|. & (2) \end{aligned}$$

By the hypothesis $\sigma_{ab} + \overline{\alpha}_{ab} = 2 + 2|J| + |J_{12}| + |D| \geq 2s - 1 + \alpha_{ab}$. On the other hand, we prove that $\alpha_{ab} \geq 2 + |J_{12}| + |D|$. It suffices to prove that $\{a,b\} \cup \{b_i | i \in J_{12}\} \cup D$ is an independent set. We already know that $\{a,b\} \cup D$ is independent. If $D \neq \emptyset$, choose $x \in D$ and suppose $xb_1 \in E$ with $1 \in J_{12}$. Then $a_1 \in N(a) \cap N(b)$ and aa_1b_1x is an M-augmenting path. It remains to prove that $b_1b_2 \notin E$ if $1, 2 \in J_{12}$.Otherwise $aa_1b_1b_2a_2b$ is an M-augmenting path. Finally we have $\sigma_{ab} + \overline{\alpha}_{ab} \geq 2s - 1 + \alpha_{ab} \geq 2s - 1 + 2 + |J_{12}| + |D|$,that is $2|J| \geq 2s - 1$.This is a contradiction since $|J| = 2(s-1)$. The proof is now complete. ■

Theorem 10 *Let n, s be positive integers with $s \leq n$. Then the property "$\alpha(G) \leq s$" is $(2n-2s-1)-$(degree stable).*

Proof. Suppose that $\alpha(G+ab) \leq s$ while $\alpha(G) > s$. Then there must exist an independent set $W \cup \{a,b\} \subset V$ with $(s+1) \geq 3$ vertices. More precisely

$W \subseteq T$. Now $d(a) + d(b) \leq 2\gamma_{ab} = 2(n - \overline{\alpha}_{ab})$. By the hypothesis $d(a) + d(b) + (\overline{\alpha}_{ab} - \alpha_{ab}) \geq (2n-2s-1)$. It follows that $2(n-\overline{\alpha}_{ab}) + (\overline{\alpha}_{ab} - \alpha_{ab}) \geq (2n-2s-1)$, that is $\overline{\alpha}_{ab} + \alpha_{ab} < 2\,(s+1)$. On the other hand $\alpha_{ab} \geq |W \cup \{a, b\}| = s+1$. Moreover $\overline{\alpha}_{ab} \geq \alpha_{ab} \geq s+1$. With this contradiction, Theorem 10 is proved. ■

Theorem 11 *Let n, s be positive integers with $s \leq n-2$. Then the property of being $s-$connected is $(n + s - \overline{\alpha}_{ab})-$(degree stable).*

Proof. Suppose that $G + ab$ is $s-$connected but G is not. Then there exists a set D of $(s-1)$ vertices such that a and b belong to two distinct components of $G - D$. It follows in particular that $\lambda_{ab} < s$. By the hypothesis $d(a) + d(b) + (\overline{\alpha}_{ab} - \alpha_{ab}) \geq (n + s - \overline{\alpha}_{ab})$. As $d(a) + d(b) = \gamma_{ab} + \lambda_{ab}$ and $\overline{\alpha}_{ab} + \gamma_{ab} = n$ we get $n - \overline{\alpha}_{ab} + \lambda_{ab} + (\overline{\alpha}_{ab} - \alpha_{ab}) \geq (n + s - \overline{\alpha}_{ab})$, that is $\lambda_{ab} - \alpha_{ab} \geq s - \overline{\alpha}_{ab}$. As $\lambda_{ab} < s$ we obtain $\alpha_{ab} \leq 1$, contradicting the fact that $\alpha_{ab} \geq 2$. This completes the proof. ■

Note that even if $(\overline{\alpha}_{ab} - \alpha_{ab}) = 0$, this result improves Bondy-Chvátal's result in [7].

Theorem 12 *Let n, s be positive integers with $s \leq n-2$. Then the property of being $s-$edge-connected is $(n + s - \overline{\alpha}_{ab})-$(degree stable).*

Proof. Suppose that $G + ab$ is $s-$edge-connected but G is not. Then there exists a set F of $(s-1)$ edges such that a and b belong to two distinct components of $G - F$. It follows in particular that $\lambda_{ab} < s$. The remaining of the proof follows that of the preceding Theorem. ■

4 Corollaries

The following results can be easily derived as Corollaries. Let G be a graph of order n, S be a subset of vertices and $s \leq |S|$ be an integer.

Let $c(G)$ denote the circumference of G. The first Corollary follows easily from Theorem 2.

Corollary 1 *The property $c(G) \geq s$ is $n-$degree stable.*

The graph G is S-pancyclable if, for every integer s, with $3 \leq s \leq n$, there exists a cycle C in G such that $|S \cap V(C)| = s$. As usual, G is pancyclic if it contains cycles of all lengths from 3 to n.

Corollary 2 *The property "G is $S-$pancyclable" is $(n + s - 3)$ - degree stable with $3 \leq s \leq n$.*

Proof. Let $R \subset S$ be a subset of r vertices, $3 \leq r \leq s$, which is not contained in any cycle C of G. This means that $G - (S \backslash R)$ is not hamiltonian, in other words G is not $(s-r)-$hamiltonian. By Theorem 3, $d_G(a) + d_G(b) + (\overline{\alpha}_{ab} - \alpha_{ab}) < n + s - r \leq n + s - 3$, a contradiction to the hypothesis. ■

Corollary 3 *The property "G is pancyclic" is* $(2n-3)$ *- degree stable.*

Proof. By identifying S and V in the preceding Corollary, we are done. ■

Corollary 4 *The property of being Hamiltonian-connected is* $(n+1)$*-degree stable.*

Proof. Follows from Theorem 6 with $s = 2$ or Theorem 8 with $F = \emptyset$. ■

The graph G is $S-$ vertex Hamiltonian-connected if it remains Hamiltonian-connected if s vertices of S or less are removed. Using similar arguments as for Theorem 6 we get.

Corollary 5 *The property "*$S-$ *vertex Hamiltonian-connected" is* $(n+s+1)$*-degree stable.*

Applying Theorem 7, we easily get:

Corollary 6 *The property of being s-edge-hamiltonian is* $(n+s)$*-degree stable.*

Let $\mu(G)$ be the number of paths that collectively contain the vertices of G.

Corollary 7 *The property* $\mu(G) \leq p$, $1 \leq p \leq n$ *is* $(n-p)$*-degree stable.*

Proof. Apply Theorem 2 for the graph $G + pK_1$. ■

The graph G is called $S-$leaf-connected if it has a spanning tree whose leaves are the vertices of S. Thus a graph is 2-leaf-connected if and only if it is Hamilton-connected.

Corollary 8 *Let* $S \subset V(G)$ *with s vertices,* $2 \leq s \leq n$*. Then the property "G is S-leaf-connected" is* $(n+s-1)$ *- stable.*

Proof. This is a particular case of Theorem 6. ■

4.1 Open Problem

We believe that the following must be true.

Problem 1 *Let* n, s *be positive integers with* $2 \leq s < n$*. Then the property of having an* $s-$*factor is* $(n+2s-4)-$*degree stable.*

5 A polynomial version of the α-degree closure

To improve Bondy-Chvàtal's closure condition we have added $(\overline{\alpha}_{ab} - \alpha_{ab})$ to σ_{ab} in order to define $P(k)$. This can be indeed a large number but α_{ab} is hard to compute. This motivates us to introduce some easy computable upper bounds of α_{ab} (or alternatively lower bounds of $(\overline{\alpha}_{ab} - \alpha_{ab})$).

The first lower bound is based on the matching number ν_{ab} of the graph G_{ab} as it is well known that $\alpha(H) \leq |H| - \nu(H)$ holds for any graph. In particular we have $\alpha_{ab} \leq \overline{\alpha}_{ab} - \nu_{ab}$ if $H = G_{ab}$. It is worth noting that for any subgraph H of G we have $\nu(H) \leq \nu(G)$.

For the second lower bound, we introduce a new invariant of a graph based on the degree sequence of that graph.

Definition 2 *Let H be any graph of order n and θ be a nonnegative integer. Set $D_\theta := \{x \in V(H) \mid d_H(x) \geq \theta\}$. The adjusted maximum degree $\Delta^\circ(H)$ is the maximum integer θ such that $|D_\theta| \geq \theta$.*

In fact, we are mainly interested by this invariant when applied to G_{ab}. Thus we have $\Delta^\circ_{ab} := \max\{\theta |\, |D_\theta| \geq \theta\}$ where $D_\theta := \{x \in T \mid d_T(x) \geq \theta\}$. The next Proposition precises some properties of this new invariant.

Proposition 2 *The invariant $\Delta^\circ(G)$ satisfies the following properties:*

1. $\Delta^\circ(G)$ does not necessarily correspond to a degree of some vertex of G,
2. $\Delta^\circ(G)$ is well-defined and $0 \leq \delta(G) \leq \Delta^\circ(G) \leq \Delta(G) \leq n-1$,
3. the invariants $\Delta^\circ(G)$ and $\nu(G)$ are incomparable,
4. for any subgraph H of G with $V(H) \subseteq V(G)$, $E(H) \subset E(G)$ we have $\Delta^\circ(H) \geq \Delta^\circ(G) - |V(G) \backslash V(H)|$. Similarly $\nu(H) \geq \nu(G) - |V(G) \backslash V(H)|$.
5. $\Delta^\circ(G) = |\{i | i + d_i > n\}|$ where $d_1 \leq d_2 \leq ... \leq d_n$ is the degree sequence of G.
6. $\alpha(G) \leq n - \max\{\nu, \Delta^\circ\}$ and hence $\alpha_{ab}(G) \leq \overline{\alpha}_{ab}(G) - \max\{\nu_{ab}, \Delta^\circ_{ab}\}$.

Proof. 1. For instance, if the degree sequence is $(2, 2, 2, 2, 4, 4, 4)$ then $\Delta^\circ = 3$ since $|D_3| \geq 3$, while $|D_4| < 4$.

2. Obvious.

3. For instance, if $G = pC_k$, $k \geq 3$, $p \geq 1$ then $\delta = \Delta^\circ = \Delta = 2$ and $\nu(G) = p \left\lfloor \frac{k}{2} \right\rfloor$. Similarly if $G = K_n$ then $\delta = \Delta^\circ = \Delta = n-1$ and $\nu(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

4. The inequalities are obvious if $V(H) = V(G)$. Otherwise, use a simple induction on $|V(H)|$.

5. Suppose that G is not trivial since otherwise $\Delta^\circ(G) = 0$. Choose $p \in [1, n]$ so that $p := \min\{i | i + d_i > n\}$ and set $\theta := |\{i | i + d_i > n\}|$. Clearly $\theta = n + 1 - p$. We claim that $d_i \geq \theta$ whenever $i \geq p$. Otherwise suppose $d_p < \theta$. Then $\theta = n + 1 - p > d_p$, that is $p + d_p \leq n$. This contradicts the definition of $p := \min\{i | i + d_i > n\}$.

6. Suppose first $\max\{\nu, \Delta^\circ\} = \Delta^\circ(G)$. Let H be any component of G and consider a maximum independent set $S := \{x_1, ..., x_{\alpha(H)}\}$ of H. We label the vertices of H so that $d(x_1) \le d(x_2) \le ... \le d(x_{\alpha(H)})$. Clearly $d(x_{\alpha(H)}) + |S| \le |H|$, that is

$$d(x_{\alpha(H)}) + \alpha(H) \le |H| .$$

So $d_H(x_i) + i \le n$ must be true for all $i \le \alpha(H)$.If, for some $j > \alpha(H)$ we have $d_H(x_j)+j > |H|$, then necessarily $x_j \in V(H)\backslash S$. Therefore $\{x_j | j + d_j > |H|\} \subseteq V(H)\backslash S$, that is $\Delta^\circ(H) = |\{x_j | j + d_j > |H|\}| \le |H| - \alpha(H)$ or $\alpha(H) \le |H| - \Delta^\circ(H)$. Applied to the graph G_{ab}, this inequality becomes $\alpha_{ab} + \Delta^\circ_{ab} \le \overline{\alpha}_{ab}$. It is well known that $\alpha(G) + \nu \le n$ holds for any graph. Therefore $\alpha(G) \le n - \max\{\nu, \Delta^\circ\}$ holds again if $\max\{\nu, \Delta^\circ\} = \nu$. ■

We note that the proof of 6 of Proposition 2 suggests an interesting result in itself, that is:

Proposition 3 *Let G with degree sequence $d_1 \le .... \le d_n$. Then $d_\alpha + \alpha \le n$.*

Moreover Proposition 2 suggests an alternative condition for $P(k)$, namely:

Definition 3 *Let P be a property defined for all graphs G of order n and let k be an integer. Let a, b be two nonadjacent vertices satisfying the condition*

$$P^*(k): \ \overline{\alpha}_{ab} \le \lambda_{ab} + \max(\nu_{ab}, \Delta^\circ_{ab}) + (n-k) \Leftrightarrow \ \sigma_{ab} + \max(\nu_{ab}, \Delta^\circ_{ab}) \ge k. \quad (**)$$

Then P is k-alpha degree stable if whenever $G + ab$ has property P and $P^*(k)$ holds then G itself has property P. We simply denote by $dC^*_k(G)$ the associated (α-degree closure).

The graph $dC^*_k(G)$ is then obtained from G by recursively joining pairs of nonadjacent vertices a, b for which (**) holds until no such pair remains.

Unlike $dC_k(G)$, the closure graph $dC^*_k(G)$ can be constructed in polynomial time. Obviously, the main results given in section 3 remain true with the new definition of $P(k)$.

Also with the following Proposition we obtain a surprising result involving the main closure condition of Broersma and Schiermeyer [8].

Proposition 4 *Let (a, b) be a pair of nonadjacent vertices of a graph G. Suppose $T \ne \emptyset$ and let $d_T(x)$ denote the degree of $x \in T$ with respect to $G[T_{ab}]$. Set $\gamma_{abx} := |N(a) \cup N(b) \cup N(x)|$. Then*

$$\begin{aligned} |\{x \in T | \gamma_{abx} \ge n - \lambda_{ab}\}| &\ge \overline{\alpha}_{ab} - \lambda_{ab} \Rightarrow && (3)\\ \overline{\alpha}_{ab} &\le \lambda_{ab} + \Delta^\circ_{ab} \Rightarrow && (4)\\ \alpha_{ab} &\le \lambda_{ab}. && (5)\end{aligned}$$

Note that $\overline{\alpha}_{ab} \le \lambda_{ab} + \Delta^\circ_{ab}$ is equivalent to $\sigma_{ab} + \Delta^\circ_{ab} \ge n$.

Proof. We first note that $\gamma_{abx} \geq n - \lambda_{ab} \Leftrightarrow d_T(x) \geq \overline{\alpha}_{ab} - \lambda_{ab}$ since $\overline{\alpha}_{ab} = n - \gamma_{ab}$ and $\gamma_{abx} = \gamma_{ab} + d_T(x)$. Therefore $|\{x \in T | \gamma_{abx} \geq n - \lambda_{ab}\}| \geq \overline{\alpha}_{ab} - \lambda_{ab}$ becomes $|\{x \in T | d_T(x) \geq \overline{\alpha}_{ab} - \lambda_{ab}\}| \geq \overline{\alpha}_{ab} - \lambda_{ab}$. By the definition, $\Delta^\circ_{ab} \geq |\{x \in T | d_T(x) \geq \overline{\alpha}_{ab} - \lambda_{ab}\}|$ and hence the Broersma-Shiermeyer's inequality becomes $\Delta^\circ_{ab} \geq \overline{\alpha}_{ab} - \lambda_{ab}$.This proves (4). By 6. of Proposition 2 $\alpha_{ab} + \Delta^\circ_{ab} \leq \overline{\alpha}_{ab}$. This completes the proof of the above Proposition. ■

In other words, the main part of Theorem 2.1 [8] can be restated as follows:

Theorem 13 *Let u and v be two nonadjacent vertices of a graph G of order $n \geq 3$. If $d(u) + d(v) + \Delta^\circ_{uv} \geq n$ then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Acknowledgement 1 *The author would like to thank the referees for their helpful comments.*

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