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On the relation between (C, E, P)–algebras and asymptotic algebras

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Abstract

On several occasions, the question has been asked whether (C, E, P)–algebras as introduced by Marti (1999), go beyond the framework of asymptotic algebras as defined by Delcroix and Scarpalezos (1998). This note summarizes the constructions and clarifies the relation between the corresponding algebras.

1 Introduction

In 1998, Delcroix and Scarpalezos have formulated the following interesting generalization of Colombeau's (simplified) "New generalized functions":

Definition 1 *The **asymptotic algebra** based on the semi-normed space* (E, P) *and the asymptotic scale* $\mathbf{a}$ *on* Λ*, is the factor space*

$$\mathcal{A}_{\mathbf{a}}(E,P) = \mathcal{F}_{\mathbf{a}}(E,P) \Big/ \mathcal{J}_{\mathbf{a}}(E,P) \tag{1}$$

where[1]

$$\mathcal{F}_{\mathbf{a}}(E,P) = \{f \in E^\Lambda \mid \forall p \in P \quad \exists m \in \mathbb{Z} : p(f) = O(a_m)\} \tag{2}$$

$$\mathcal{J}_{\mathbf{a}}(E,P) = \{f \in E^\Lambda \mid \forall p \in P \quad \forall m \in \mathbb{Z} : p(f) = O(a_m)\} \tag{3}$$

where $p(f) = (p(f_\lambda))_{\lambda\in\Lambda} \subset {\mathbb{R}_+}^\Lambda \subset \mathbb{K}^\Lambda$, $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$.

We recall the standard

Definition 2 *An **asymptotic scale** on a (filtered) set* Λ *is a family of functions* $\mathbf{a} = (a_m : \Lambda \to \mathbb{R}_+^*)_{m\in\mathbb{Z}}$ *with values in* $\mathbb{R}_+^* =]0, \infty[$ *such that*

$$\forall m \in \mathbb{Z} : \quad a_{m+1} = o(a_m) \ , \quad a_{-m} = 1/a_m \ , \quad \exists M \in \mathbb{Z} : \ a_M = O(a_m^2)$$

Example 3 *One recovers Colombeau's New Generalized Functions on* $\Omega \subset \mathbb{R}^d$ *for* $a_m = (\varepsilon^m)_{\varepsilon\in\Lambda=]0,1]}$, $E = C^\infty(\Omega)$ *and* $P = \left\{p_{K,\alpha} = \|\partial^\alpha\cdot\|_{L^\infty(K)}\right\}_{K\Subset\Omega,\alpha\in\mathbb{N}^d}$.

[1] Given the properties of $\mathbf{a}$, one could replace $O(a_m)$ by $o(a_m)$ in the definition of $\mathcal{J}$, which may be a little more convenient in some proofs.

Example 4 *Delcroix and Scarpalezos have introduced the "exponential algebra" of generalized functions stable under exponentiation, based on the scale with* $a_{-m-1} = \exp \circ a_{-m}$ *for* $m \in \mathbb{N}^*$, $a_{-1} = (e^{1/\varepsilon})_\varepsilon$. *Obviously this scale (and thus generalized algebra) is not of "Colombeau type" in the sense that one does not have* $a_m = (a_1)^m$ *and there is no equivalence with an algebra of such type, where the scale is given by* powers *of one or a finite number of given elements.*

Remark 5 *When* (E, P) *is a presheaf of topological* $\mathbb{K}$*–algebras or vector spaces, then* $\mathcal{A}_{\mathbf{a}}(E, P)$ *is again a presheaf of topological algebras or modules over the ring of* generalized numbers $\mathbb{K}_{\mathbf{a}} = \mathcal{A}_{\mathbf{a}}(\mathbb{K}, |\cdot|)$.

On the other hand, Marti's (C, E, P)–algebras are defined in terms of a semi-normed space (E, P) and a ring (of "generalized numbers") $C = A/I$, where I is an ideal of the subring $A \subset \mathbb{K}^\Lambda$, both being solid:

Definition 6 *We say that* $A \subset \mathbb{K}^\Lambda$ *is solid, iff for all* $x \in \mathbb{K}^\Lambda$ *and* $a \in A$, $|x| \leq | a|$ *implies* $x \in A$ *(where* $x \leq y \iff \forall \lambda \in \Lambda : x_\lambda \leq y_\lambda$*).*

Definition 7 *The* (C, E, P)*–algebra associated to the ring of generalized numbers* $C = A/I$ *(*$I \subset A \subset \mathbb{K}^\Lambda$ *both solid) and the semi-normed space* (E, P) *is the factor space*

$$\mathcal{A}_{C,E,P} = \mathcal{H}_{A,E,P} \big/ \mathcal{H}_{I,E,P} \ , \quad \mathcal{H}_{X,E,P} = \{ f \in E^\Lambda \mid \forall p \in P \ \ p(f) \in X \} \ . \tag{4}$$

Remark 8 *Apart from the very concise way of writing, this definition yields the nice property that* $\mathcal{H}_{X,\mathbb{K},|.|} = X$ *for any solid* $X \subset \mathbb{K}^\Lambda$ *and therefore* $\mathcal{A}_{C,\mathbb{K},|.|} = C$. *This would not necessarily hold if the definition would involve an "asymptotic relation" (*e.g.*, "for* λ *small enough").*
In practical applications, however, one usually does not want the values for "big λ*" to matter. Therefore, already* A *and* I *should contain all sequences that are zero for "*λ *small enough" (and thus, using stability under* $+$*, all sequences that differ from a sequence in* A *only for "large* λ*"). This can be obtained by requiring a stronger type of ("asymptotic") solidness:*

Definition 9 *We say that* $A \subset \mathbb{K}^\Lambda$ *is (asymptotically) solid, iff for all* $a \in A$ *and* $x \in \mathbb{K}^\Lambda$, $x = O(a)$ *implies* $x \in A$.

But the main interest, in practice, is not in arbitrary (C, E, P)–algebras, but those generated by a set of sequences, as defined below, which will automatically satisfy this property.

2 B–generated (C, E, P)–algebras.

Definition 10 *Let* $B \subset (\mathbb{R}_+^*)^\Lambda$ *be a set of strictly positive valued families. Then we denote by* $\langle B \rangle$ *the smallest subset of* $\mathbb{K}^\Lambda$ *containing* B *and stable under addition, multiplication and taking the inverse. It is easy to see that* $\langle B \rangle$ *is the set of all rational fractions whose numerator and denominator are linear combinations of products (or powers) of elements of* B*, with positive integer (or equivalently rational) coefficients,*

$$\langle B \rangle = \left\{ \Big(\sum_{\mathbf{b} \in B^n} \alpha_{\mathbf{b}} \, b_1 \cdots b_n \Big) \Big/ \sum_{\mathbf{b}' \in B^m} \beta_{\mathbf{b}'} \, b'_1 \cdots b'_m \ ; \ \ n, m \in \mathbb{N}^*, \ \alpha_{\mathbf{b}}, \beta_{\mathbf{b}'} \in \mathbb{N} \right\} \ ,$$

*where the sums are finite (*i.e.*, $\alpha_{\mathbf{b}} = \beta_{\mathbf{b}'} = 0$ except for a finite number of $\mathbf{b}, \mathbf{b}'$) but always have at least one nonzero term.*

Definition 11 *Given a subset $B \subset (\mathbb{R}_+^*)^\Lambda$, we let*

$$A_B = \{x \in \mathbb{K}^\Lambda \mid \exists b \in \langle B \rangle : x = O(b)\} \ , \tag{5}$$
$$I_{(B)} = \{x \in \mathbb{K}^\Lambda \mid \forall b \in \langle B \rangle : x = O(b)\} \ . \tag{6}$$

Lemma 12 *With the above definitions, A_B is a solid subring of $\mathbb{K}^\Lambda$ and $I_{(B)}$ is a solid ideal of A_B.*

Proof. It is easy to see from the very definition that the sets A_B and $I_{(B)}$ are solid, using transitivity of the $O(\cdot)$ relation.
Stability of A_B under addition and multiplication follows from the construction of $\langle B \rangle$ which is also closed under these operations. Concerning stability of $I_{(B)}$ under addition, for $x, y \in I_{(B)}$ and $b \in \langle B \rangle$, we have $x + y = O(b)$ because $\frac{1}{2}\, b \in \langle B \rangle$ and $x = O(\frac{1}{2}b)$, $y = O(\frac{1}{2}b)$.
To show that the product of $y \in I_{(B)}$ and $x \in A_B$ is again in $I_{(B)}$, let be given an arbitrary $b \in \langle B \rangle$, and $b' \in \langle B \rangle$ such that $x = O(b')$. But then b/b' is again in $\langle B \rangle$, thus $y = O(b/b')$ and $x \cdot y = O(b' \cdot b/b') = O(b)$. □

Remark 13 *Having shown that A_B is a subring, it is easy to see that it is the smallest solid subring to contain $\langle B \rangle$, $A_B = \mathrm{ssr}(\langle B \rangle)$. We observe that this may be larger than the smallest solid rubring which contains B, $A_B \supset \mathrm{ssr}(B)$, due to the fact that $\langle B \rangle$ also contains the inverse of the elements of B.*

Lemma 14 (and Definition.) *To any solid subring $A \subset \mathbb{K}^\Lambda$ (with unit), we canonically associate the (solid) ideal*

$$I_A = \{x \in \mathbb{K}^\Lambda \mid \forall a \in A^* : x = O(a)\} \ , \tag{7}$$

*where A^* denotes the invertible elements of A (*i.e.*, having an inverse in A).*

Proof. If a, b are invertible in A, then so is $|a| + |b|$, which yields stability of I_A under $-$. If $x \in A$, $y \in I_A$, $a \in A^*$, then $x = O(b)$ with $b = |x| + 1 \in A^*$, and $y = O(a/b)$ yields $x \cdot y = O(b \cdot a/b) = O(a)$. □

Lemma 15 *For $A = A_B$ defined in* (5)*, the ideal I_A defined by* (7)*, equals $I_{(B)}$ defined in* (6)*.*

Proof. This follows from the fact that for all $b \in \langle B \rangle$, we have $b^{-1} \in \langle B \rangle$, *i.e.*, $\langle B \rangle \subset A^*$ which entails $I_A \subset I_{(B)}$. On the other hand, if $a \in A^*$, then $a^{-1} \in A^* \subset A_B$ is dominated by some $b \in \langle B \rangle$, $a^{-1} = O(b)$, and $x = O(b^{-1})$ for any $x \in I_{(B)}$ since $b^{-1} \in \langle B \rangle$, so $x = x \cdot a^{-1} \cdot a = O(b^{-1})\, O(b)\, a = O(a)$, whence $I_{(B)} \subset I_A$. □

Remark 16 *It is easy to see that $I_{(B)} = I_{\langle B \rangle}$ as defined by equation* (7)*, but of course $\langle B \rangle = \langle B \rangle^*$ is neither solid nor a ring.*

3 Relation between aymptotic and B–generated (C, E, P)–algebras.

Now we will establish a relation between aymptotic algebras and B–generated (C, E, P)–algebras, defined as follows:

Definition 17 *For $B \subset (\mathbb{R}_+^*)^\Lambda$, we call B–**generated** any (C, E, P)–algebra associated to $C = A_B/I_{(B)}$.*

Example 18 *For $B = \left\{(\varepsilon)_{\varepsilon\in]0,1]}\right\}$ we get Colombeau's algebras, which are the asymptotic algebras corresponding to the scale $(a_m = (\varepsilon^m)_\varepsilon)_{m\in\mathbb{Z}}$.*

Remark 19 *Although $\langle B\rangle$ is not an asymptotic scale, we obviously have, for $C = A_B/I_{(B)}$, that $\mathcal{A}_{C,E,P} = \mathcal{A}_{\langle B\rangle}(E, P)$ as defined in equation* (1), i.e.*, with $\{a_m; m \in \mathbb{Z}\}$ replaced by $\langle B\rangle$ in equations* (2)*–*(3)*.*

We will prove the

Theorem 20 *For finite or countable sets B which contain at least one sequence with zero or infinite limit, we can extract from $\langle B\rangle$ a family $\mathbf{a} = (a_m; m \in \mathbb{Z})$ which is an asymptotic scale such that $\mathcal{A}_{A_B/I_{(B)},E,P} = \mathcal{A}_{\mathbf{a}}(E, P)$.*

Remark 21 *The reason for the requirement of having at least one sequence of zero or infinite limit is obvious: If B contains only sequences bounded from below or above, then A_B also contains only bounded sequences, and no sequence with zero limit can be invertible; therefore there may be sequences in $I_{(B)}$ which don't have zero limit, and this cannot be the case for an ideal defined by an asymptotic scale, in view of $a_{m+1} = o(a_m)$.*

Lemma 22 *If B is finite or countable, then there is a subset $\{r_m; m \in \mathbb{Z}\} \subset \langle B\rangle$ with $r_{-m} = 1/r_m$ such that for any finite subset $B' \subset \langle B\rangle$*

$$\exists m \in \mathbb{N} \;\; \forall b \in B' \;\; \forall \lambda \in \Lambda : \quad r_m(\lambda) < b(\lambda) < r_{-m}(\lambda) \; . \tag{8}$$

Proof. One key point in the proof is the observation that, although $\langle B\rangle$ does not necessarily contain $\min(x, y) := (\min\{x_\lambda, y_\lambda\})_{\lambda\in\Lambda}$ nor $\max(x, y) := (\max\{x_\lambda, y_\lambda\})_{\lambda\in\Lambda}$, we have

$$\forall x, y \in \langle B\rangle : \quad \max(x, y) < x + y \in \langle B\rangle$$

and

$$\min(x, y) < x \parallel y := (x^{-1} + y^{-1})^{-1} \in \langle B\rangle \; .$$

Now it is straightforward to construct the sequence (r_m). First we observe that in view of its definition, the set $\langle B\rangle$ is countable whenever B is at most countable, *i.e.*, we can write $\langle B\rangle = \{b_0, b_1, b_2, ...\}$. Now let $r_0 = 1 \in \langle B\rangle$ and for $m \in \mathbb{N}$,

$$r'_m = r_m \parallel b_m < \min(r_m, b_m) \; , \quad r''_m = r_{-m} + b_m > \max(r_{-m}, b_m) \; ,$$

and finally $r_{m+1} := r'_m \parallel 1/r''_m < \min(r'_m, 1/r''_m)$, $r_{-m-1} := 1/r_{m+1} > r''_m$. This way we obviously have a subset $\{r_m; m \in \mathbb{Z}\} \subset \langle B\rangle$ with the property

$$\forall m \in \mathbb{N} \;\; \forall k < n : \quad r_{m+1} < r_m < b_k < r_{-m} = 1/r_m < r_{-m-1} \; .$$

□

Proof of the Theorem. Now it remains to extract from $(r_m)_{m\in\mathbb{Z}}$ a subsequence $(a_m)_{m\in\mathbb{Z}}$ which verifies the additional requirements of an asymptotic scale, namely $a_{m+1} = o(a_m)$ and $\forall m : \exists M : a_M = O(a_m^2)$. This is obviously possible whenever B contains a sequence with zero (or infinite) limit: Indeed, this sequence (or its inverse) will appear at a given moment as b_m in the proof of the preceding Lemma. From then on, all $r_{m'},\ m' > m$ will have zero limit, and we can let $a_1 = r_{m+1}$. Furthermore, for each a_m, its square $(a_m)^2$ will also appear eventually as some $b_{m''}$, and the associated $r_{m''+1} = O(a_m^2) = o(a_m)$, so we can let $a_{m+1} = r_{m''+1}$ and have all of the required properties. □

4 Colombeau type asymptotic algebras

From the construction given in the preceding proof we can see that we have

Theorem 23 *A (C, E, P)–algebra generated by a finite set $B = \{b_1, ..., b_n\}$ (of which at least one element has zero or infinite limit) is a Colombeau type algebra generated by a single element $t_B \in \langle B\rangle$ (*i.e.*, an asymptotic algebra corresponding to the scale $\{a_m = t^m;\ m \in \mathbb{Z}\}$), given by*

$$t_B = b_1 + \cdots + b_n + (b_1)^{-1} + \cdots + (b_n)^{-1} .$$

Proof. The element t_B is obviously strictly larger, and its inverse is strictly smaller, than any $b \in B$. So any polynomial $\sum_{m\in\mathbb{N}^n} \alpha_m b^m \in \mathbb{N}[B] \setminus \{0\}$ is majorated and minorated by some power of t_B, and the same applies therefore to the set $\langle B\rangle$ of rational fractions of such polynomials. From there it is easy to see that

$$A_B = \left\{x \in \mathbb{K}^\Lambda \mid \exists m \in \mathbb{Z} : x = O(t_B^m)\right\} ,$$

$$I_{(B)} = \left\{x \in \mathbb{K}^\Lambda \mid \forall m \in \mathbb{Z} : x = O(t_B^m)\right\} ,$$

i.e., we have a Colombeau type asymptotic algebra with the scale $\{a_m = t_B^m\}_{m\in\mathbb{Z}}$. □

5 Conclusion

To conclude, we can say that (C, E, P)–algebras defined by some completely arbitrary $C = A/I$, *e.g.*, with $I = \{o\}$ or $I = A$ or I containing sequences which do not have zero limit, can certainly not be written as an asymptotic algebra. Moveover, this also seems impossible when A cannot be written as $A = A_B$ with some countable set B. However, in all practical applications we are aware of, it is sufficient to consider B–generated (C, E, P)–algebras with a finite or countable set B containing at least one sequence with zero or infinite limit. As proved in the present note, in this case the resulting algebra can also be interpreted as asymptotic algebra in the sense of Delcroix and Scarpalezos, with an asymptotic scale which can be constructed explicitely as shown above.

Moreover, when the algebra is generated by a finite set, then it is equivalent to a Colombeau type algeba. This is however not the case, in general, for asymptotic scales obtained, *e.g.*, by composition of function, as shown by the example of Delcroix and Scarpalezos' exponential algebra.

Algebras of this type can be useful in a setting where one wants to construct solutions by successive approximations. Then the stability of the algebra under iteration of the relevant map will imply that the iterative process yields and element of the algebra after each step. These ideas will be developed in more detail in a separate forthcoming paper.

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