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Mécanismes d'Echange en Présence d'Externalités

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Introduction

Les enchères sont des mécanismes d'échange très communs. Mais le fait que cette classe de mécanismes attire l'attention des économistes et soit un sujet important dans la littérature n'est pas seulement le résultat de son caractère répandu. Nous pouvons identifier deux propriétés typiques de cette classe de mécanismes qui peuvent expliquer la richesse de la littérature qui étudie les enchères. D'abord, il s'agit d'une situation économique dans laquelle plusieurs acheteurs potentiels interviennent, par opposition aux mécanismes bilatéraux, par exemple. De plus, le moment d'échange, autrement dit, le moment d'interaction entre les acheteurs, c'est-à-dire les enchérisseurs, et le vendeur, c'est-à-dire le commissaire-priseur, est très bien identifié. D'un point de vue stratégique, la combinaison de ces deux propriétés ouvre la porte à une étude très intéressante de la question de la collusion dans les enchères.

La collusion est un accord conclu entre tous les enchérisseurs, ou une partie d'entre eux, afin de former un cartel. Clairement, cet accord, ou engagement, peut prendre des formes différentes, mais il comporte normalement deux caractéristiques: l'identité de l'enchérisseur qui fait une offre compétitive à l'enchère, c'est-à-dire le représentant du cartel, alors que les autres font des offres non compétitives; le régime des paiements mis en œuvre entre les membres du cartel pour partager le gain si le cartel gagne à l'enchère, c'est-à-dire les transferts. L'intérêt des membres du cartel de participer à une telle collusion est clair. La formation du cartel réduit la concurrence dans l'enchère, qui se traduit par la baisse du prix gagnant et l'augmentation de la probabilité du cartel de gagner le bien.

Pour étudier la question de la collusion dans les enchères, nous partons dans deux axes différents. Dans le chapitre 1, nous commençons par une analyse du processus de négociation qui décrit la formation d'un cartel. En particulier, nous précisons les détails du protocole de communication entre les enchérisseurs, en spécifiant les offres, les réponses, etc. Le but de cette analyse est non seulement de percevoir quelle collusion émergera, mais aussi de décrire comment telle collusion émergera. Ensuite, dans les chapitres 2 et 3, nous prenons un point de vue plus distant, en analysant les propriétés d'une forme de collusion donnée, sans traiter la question de comment cette collusion a émergé. Plus précisément, nous essayons de répondre à la question: la partition des enchérisseurs en cartels est-elle faisable? Stable? Probable?

Avant d'être prêt à partir dans ces deux axes, il faut se poser la question des externalités. La notion d'externalité (directe) a été introduite dans le cadre des enchères, par exemple, par Jehiel et Moldovanu (1996) et Caillaud et Jehiel (1998). Un enchérisseur subit alors une externalité s'il ne gagne pas à l'enchère. C'est-à-dire, l'utilité finale de chaque enchérisseur est soit son évaluation (nette du prix du bien) s'il gagne à l'enchère, soit une externalité (qui peut être négative). Dans le cas de figure le plus simple (voir, par exemple, Caillaud et Jehiel (1998)), l'externalité subie par un enchérisseur qui ne gagne pas à l'enchère est un paramètre exogène et fixé, et ne dépend pas de l'identité de celui qui la subit, ni de l'identité du gagnant. Un cas de figure plus riche considère les externalités comme fonctions de l'identité du gagnant autant que de celui qui subit l'externalité (voir, par exemple, Jehiel et Moldovanu (1996)).

Évidemment, la situation économique est beaucoup plus complexe en présence d'externalités. Au lieu de regarder un "vecteur d'utilités" nous considérons maintenant une matrice d'utilités, ou plutôt une "matrice d'externalités". Les deux derniers articles précités insistent sur la richesse de la situation économique en présence d'externalités, et présentent plusieurs résultats intéressants en proposant une comparaison avec la situation économique analogue modélisée sans externalités. Jehiel et Moldovanu (1996) traitent les enchères sans collusion (avec information complète) en présence d'externalités, et montrent qu'un enchérisseur peut préférer ne pas participer à l'enchère. Ce scénario n'arrive jamais sans externalité parce qu'un enchérisseur perdant a une utilité nulle. En revanche, avec externalités, un enchérisseur perdant peut avoir une utilité négative. Étant donné que cette utilité est une fonction de l'identité du gagnant, un enchérisseur peut déclarer sa non-participation, afin de favoriser un gagnant alternatif que l'enchérisseur non participant préfère au gagnant original en terme d'externalités. Notons que le fait que la non-participation d'un agent puisse entraîner un changement de gagnant dépend crucialement de la présence d'externalités. La non-participation d'un enchérisseur élimine une menace potentielle qu'il fait peser sur un autre. Dès que la menace disparaît, cet autre enchérisseur peut réagir par une offre moins agressive à l'enchère. Cela peut se terminer par un gagnant alternatif.

Dans ce papier, Jehiel et Moldovanu (1996) identifient une classe particulière d'équilibres dans les enchères en présence d'externalités. Finalement, pour ce qui est de la collusion dans les enchères, ils discutent la faisabilité d'engagements entre les enchérisseurs et le commissaire-priseur. Ils démontrent que de tels engagements ne sont en général pas stables. Caillaud et Jehiel (1998) étudient les enchères avec information incomplète en présence d'externalités, et se concentrent sur la question de la collusion de la grande coalition, c'est-à-dire, un cartel qui inclut tous les enchérisseurs. Dans cette situation économique plus complexe, ils identifient eux aussi une classe spécifique d'équilibres, en insistant sur le fait qu'en présence d'externalités les enchérisseurs sont prêts à faire des offres plus agressives que dans les enchères sans externalités, car dans le second cas, un enchérisseur perdant risque de souffrir une utilité négative. Nous reviendrons à ce papier à la fin de cette introduction pour discuter un résultat intéressant sur la faisabilité d'une collusion ex-post efficace.

La présence d'externalités est en particulier intéressante pour la question de la collusion dans les enchères. Pour mieux comprendre comment les externalités interviennent dans la formation d'un cartel, commençons par examiner une enchère qui a lieu dans un marché sans externalités. En suivant une intuition assez simple, nous voyons qu'un cartel plus grand est plus "efficace", au sens où la concurrence dans l'enchère et le prix gagnant diminuent avec la taille du cartel. Nous concluons que si rien n'empêche la collusion de l'ensemble des agents, ce sera en effet la grande coalition qui se formera.

C'est exactement là que la présence d'externalités change les règles du jeu. Pour voir les choses plus clairement, imaginons qu'un enchérisseur qui ne gagne pas à l'enchère subisse une externalité négative. Le fait qu'un cartel "gagne" le bien veut dire que le représentant du cartel gagne le bien et obtient son évaluation (nette du prix payé), mais qu'en même temps, tous les autres membres du cartel subissent leurs externalités négatives! Cela veut dire que le cartel doit exécuter des transferts pour compenser les non gagnants. Autrement dit, le gagnant doit compenser les autres membres du cartel en partageant son gain. L'effet d'externalités accumulées est plus important dans les grands cartels. Nous identifions donc les externalités comme un obstacle à coopération. Naturellement, la question qui se pose est: la grande coalition se formera-t-elle en présence d'externalités?

Notons qu'en discutant la formation d'un cartel il faut se poser la question de l'information des agents. Un premier aspect de cette question concerne l'information que chaque enchérisseur possède sur son évaluation. Les différents modèles que nous couvrons dans cette thèse proposent des approches différentes à cet égard. Dans le chapitre 1, nous partons d'une situation économique où tous les agents connaissent tous les paramètres du marché, en particulier leurs évaluations propres, c'est-à-dire de l'information complète. Et nous finissons, dans une partie du chapitre 2 et dans le chapitre 3, par un marché dans lequel les agents sont informés en privé de leurs évaluations.

Un autre aspect de la question de l'information dans le processus de formation d'un cartel concerne le partage d'information entre les membres du cartel. Les questions naturelles qui se posent incluent, entre autres, les questions suivantes: Comment les membres du cartel partagent-ils leur information privée? Les membres du cartel ont-ils intérêt à mentir? Pouvons-nous identifier un mécanisme de collusion qui vérifie que les participants disent la vérité? Les décisions prises par un cartel dépendent-elles de l'information des agents hors du cartel? Le modèle d'information privée que nous couvrons insiste sur la subtilité qui se cache dans le fond de ce genre de questions.

Notre travail est lié à différents domaines couverts par la littérature qui étudie la collusion dans les enchères. La motivation principale fait l'objet de nombreux articles qui examinent des situations économiques de la vie réelle où la collusion dans les enchères a pu être identifiée. Commençons par un cas examiné par Porter et Zona (1999) dans le cadre de plusieurs appels d'offre dans le marché de la distribution du lait aux écoles à Cincinnati, dans l'état américain d'Ohio. En regardant de près les offres faites par les laiteries locales de la région de Cincinnati, Porter et Zona (1999) ont trouvé que celles-ci avaient fait des offres plus compétitives dans les appels d'offres visant des marchés distants. Étant donné que le coût augmente avec la distance de la distribution, ces données ont soulevé des soupçons. Et en effet c'est la collusion qui explique ce comportement non typique. Les laiteries locales de Cincinnati ont formé un cartel. En dehors du représentant, les membres du cartel ont voulu faire des offres non compétitives dans le marché local où le cartel était actif. En même temps les mêmes laiteries ont participé à des appels d'offre pour des marchés distants sans faire partie du cartel, de manière tout à fait compétitive. Le fait que ces laiteries n'aient pas été assez prudentes a révélé la collusion.

Ce dernier exemple est particulièrement intéressant pour deux raisons. D'abord il s'agit d'une collusion "partielle". Le cartel a inclus les laiteries locales seulement, et les autres enchérisseurs ont participé, apparemment, de manière compétitive. Cela veut dire qu'il existe une évidence empirique de cartels qui n'incluent pas tous les enchérisseurs. De plus, dans cet exemple, le contrat vendu était annuel. Le cartel a donc participé à plusieurs enchères, et surtout a signé à l'avance un engagement de coopération de long terme, bien avant de participer aux enchères. Il n'y a aucune raison de croire que les membres de cartels aient pu calculer leurs évaluations futures au moment de l'engagement. C'est donc une évidence d'accord de collusion à un moment où les membres du cartel ne sont pas encore informés.

Parmi les autres papiers qui considèrent des études de cas spécifiques, nous trouvons aussi Porter et Zona (1993) qui examinent la collusion dans le cadre de contrats de construction d'autoroutes à Long Island, Bajari et Ye (2003) qui analysent des données d'enchères dans le marché de la construction dans la région Midwest aux États Unis, et Hendricks, Porter et Tan (2008) qui se consacrent aux baux fédéraux du pétrole et du gaz naturel aux États Unis. Notons que ce dernier exemple concerne des enchères qui ont eu lieu dans les années 1954-1970 quand la coopération entre des enchérisseurs n'était pas considérée illégale dans ce marché. La législation qui interdit la collusion dans ce marché a été constituée vers la fin de 1975.

Comme dernier exemple nous voudrions rappeler l'appel d'offre organisé en 1992 par le gouvernement Sud-coréen pour la construction d'un réseau de train à grande vitesse entre Séoul et Pusan (voir, par exemple, Caillaud et Jehiel (1998)). Trois entreprises ont participé à cet appel d'offre: La Japonaise Mitsubishi, l'Allemande Siemens, et l'Anglo-française GEC-Alsthom. Les gouvernements européens ont essayé d'inciter les deux joueurs européens à former une entreprise commune, pour améliorer leur compétitivité. Les deux entreprises ont décliné cette initiative, et finalement c'est GEC-Alsthom qui a gagné le contrat. Dans le contexte de cet exemple, Caillaud et Jehiel (1998) insistent sur l'existence d'externalités. Comme le gouvernement Sud-coréen a spécifié que le gagnant transférerait la technologie en question aux entreprises locales, les participants ont bien conçu que le gagnant aurait l'occasion d'introduire le nouveau standard pour la construction de trains à grande vitesse. L'interprétation naturelle d'un externalité négative pour les perdants suit. De plus, cet exemple met en évidence l'existence de situations économiques où la coopération entre tous les enchérisseurs n'est pas envisageable (entreprises européennes face à une entreprise asiatique), mais où on peut considérer la collusion entre un sous-ensemble d'enchérisseurs, et examiner le lien entre l'existence d'externalités et une telle collusion (partielle).

Comme nous l'avons déjà dit, nous suivons deux approches complémentaires pour analyser la question de la collusion dans les enchères en présence d'externalités, et surtout pour étudier les externalités comme un obstacle à la coopération entre les enchérisseurs. La première approche (chapitre 1) se veut descriptive: elle détaille le protocole de négociation, et analyse la manière de laquelle un cartel se forme. Cette approche fait partie d'une littérature très vaste. Entre autres, nous notons l'article de Bloch (1996) qui étudie un jeu séquentiel de formation de coalitions, où le premier joueur propose la formation d'une coalition. Si l'un des membres potentiels rejette la proposition, il fait une contre offre. Une fois qu'un accord est obtenu, la coalition formée sort du jeu, et les autres continuent de la même façon. Nous notons aussi Ray et Vohra (1999) qui analysent un jeu de formation de coalitions où dans l'état initial tous les joueurs sont dans la même "salle de négociation". Pendant les négociations un groupe de joueurs peut décider de quitter la salle, un acte qui est interprété comme la formation d'une coalition. La question principale qu'ils posent concerne la nature d'engagements "stables" que les différentes coalitions peuvent atteindre. Nous notons encore Ray et Vohra (2001) qui considèrent un jeu où les joueurs sont des régions, chaque région produit une partie du bien public, interprété comme le contrôle de pollution, et le paiement pour chaque région est une fonction du coût de la production locale et de la valeur totale du bien public. Dans ce cadre ils regardent un jeu de formation de coalitions (des régions) où la première région propose la formation d'une coalition en spécifiant un plan de production et des transferts. Les membres potentiels du cartel répondent. En cas d'accord la coalition se forme et le jeu continue parmi les autres régions. En cas d'objection, une contre offre est considérée. Nous notons finalement Bloch et Gomes (2006) qui examinent le jeu de formation de coalitions suivant. A chaque période, étant donné une partition des joueurs en coalitions qui est le résultat de la période précédente, les coalitions participent aux deux étapes: L'étape des contrats et l'étape des actions. Dans la première, une coalition est choisie au hasard pour proposer la formation d'une nouvelle coalition (c'est-à-dire, unir des coalitions existantes pour créer une nouvelle coalition), en spécifiant des transferts. Si toutes les coalitions sollicitées acceptent l'offre la nouvelle coalition se forme. Dans la deuxième étape de la même période les coalitions choisissent soit de quitter le jeu, soit de rester actives et participer à la répartition des membres des coalitions à la période suivante.

L'approche qualitative que nous suivons (chapitre 2 et chapitre 3) se concentre plutôt sur la question: Quelles sont les propriétés d'un cartel donné, ou d'une partition donnée d'enchérisseurs en cartels? Plus précisément, étant donné un cartel ou une partition en cartels, nous essayons d'analyser la situation économique pour savoir si ce cartel ou cette partition est faisable, ou probable, au sens où aucun groupe d'enchérisseurs ne préférerait s'en aller, et quitter sa coalition pour former une coalition indépendante. Dans cette approche, nous ignorons complètement le processus qui a produit le cartel, ou partition, en question. Pour fixer les idées, supposons que nous essayons de déterminer si un cartel qui contient tous les enchérisseurs est probable, ou stable. Pour répondre à cette question nous voulons savoir si un groupe d'enchérisseurs a intérêt à s'en aller, en formant un cartel indépendant. Il faut donc comparer ces deux situations économiques.

La première question qu'un groupe d'enchérisseur qui veut former un cartel indépendant se pose concerne la réaction des autres à leur sécession. Dans les deux extrêmes nous pouvons considérer le cas où les autres vont continuer à coopérer malgré la sécession d'un groupe d'enchérisseurs, en formant le cartel complémentaire, et le cas où suite à telle sécession la coopération se casse complètement, et le cartel qui a dévié se trouverait à l'enchère face aux singletons. Maskin (2003) considère le premier cas de figure. Il parle de coalitions avec "attente de fusion". Pour une coalition donnée de joueurs, il compare leur situation, en termes d'utilité, en faisant partie de la grande coalition, avec la situation où cette coalition se sépare et se trouve en concurrence avec la coalition complémentaire. Le deuxième cas de figure est traité par Hafalir (2007). Il considère les coalitions avec "attente de singletons". Une coalition qui décide de dévier conjecture que les autres vont réagir en se séparant en singletons.

Ces deux approches nous apprennent l'importance de la réaction des autres membres de la grande coalition, en terme de répartition en coalitions, par rapport à la sécession d'un cartel. Nous suivons ces deux approches différentes, parmi d'autres possibles, et nous montrons qu'en analysant la sécession d'un cartel, et en analysant l'utilité d'un cartel qui s'en va, ou qui envisage de dévier, la conjecture faite par rapport à la réaction des autres joue un rôle très important. En termes techniques, cela veut dire qu'il faut étudier cette situation stratégique comme un jeu en forme de partition plutôt qu'un jeu en forme caractéristique (voir, par exemple, Lucas et Thrall (1963)).

La prochaine étape à suivre pour être capable de comparer la position d'un groupe d'enchérisseurs dans un cartel donné, en participant dans le cartel et en déviant pour former un nouveau cartel indépendant, est expliquée par la notion d'"équilibre coalitionnel" introduite par Ray et Vohra (1997) et Ray (2007). Pour chaque partition possible, ils proposent d'associer un jeu entre les coalitions qui appartiennent à cette partition. Puis, ils identifient la valeur de chaque coalition par rapport à la partition avec son utilité dans un équilibre de Nash pour ce jeu. Cette approche propose un lien entre la classe de jeux coopératifs et celle de jeux non coopératifs.

Avec "attente de fusion" Myerson (1997) propose une approche alternative. Il calcule la valeur d'une coalition en utilisant le "minmax". C'est-à-dire la coalition choisit une stratégie qui maximise son utilité sachant que la coalition complémentaire répondra avec une stratégie qui la minimise. Il propose encore une autre approche qui utilise les menaces rationnelles. En revanche, telle approche est plus naturelle dans le contexte de jeux séquentiels, notamment en utilisant les équilibres parfaits en sous-jeux. Elle est donc moins applicable aux jeux d'enchères.

Avec ces deux outils en main, nous pouvons commencer à avoir une idée plus claire sur la question de la stabilité d'un cartel. Prenons un cartel, et prenons un sous-ensemble d'enchérisseurs dans ce cartel. Imaginons deux scénarios économiques différents. Le premier consiste en la participation de ce sous-ensemble dans le cartel. Étant donné le plan de collusion de ce cartel, le gain total de ce sous-ensemble dans le cartel, autrement dit, son utilité, est bien défini. Dans un second scénario, supposons que ce sous-ensemble d'enchérisseurs considère l'option de dévier pour former un cartel indépendant. Nous pouvons utiliser les deux outils acquis pour estimer la position de ce nouveau cartel indépendant dans la situation économique correspondante. La conjecture de ce sous-ensemble d'enchérisseurs quant à la répartition des autres en coalitions s'il s'en va, définit les "joueurs" dans la nouvelle situation économique. Étant donné telle partition, nous implémentons la nouvelle notion d'équilibre pour déterminer la valeur du sous-ensemble déviant par rapport à la partition. En comparant les deux positions, nous pouvons estimer si le sous-ensemble d'enchérisseurs voudrait participer dans le cartel original, ou plutôt préférerait essayer de former un cartel indépendant, en entrant en compétition avec ses anciens camarades.

Bien entendu, la notion centrale dans le fond de ce genre de considération est connue dans la littérature de jeux coopératifs comme le "cœur". En reprenant la démarche qu'un sous-ensemble d'enchérisseurs suit pour déterminer s'il participe du cartel ou non, une position meilleure en partant est interprétée comme un blocage au cartel. C'est là que la question d'information, mentionnée toute à l'heure, revient. Dans une situation économique où les enchérisseurs ne connaissent pas leurs évaluations jusqu'à un instant spécifique, la question de quand les enchérisseurs vont être invités à s'engager au cartel est critique. De plus, dans les enchères avec information privée la question d'incitations joue également. Nous voulons vérifier qu'un enchérisseur invité à s'engager dans un cartel révèle honnêtement son information privée, afin de ne pas empêcher le cartel de fonctionner de façon efficace.

Forges, Mertens et Vohra (2002) répondent à ces deux difficultés en proposant le "cœur incitatif à l'étape ex-ante". Étant donné une coalition, ils considèrent tous les mécanismes incitatifs de collusion pour cette coalition, en calculant la somme des utilités des membres de la coalition pour chaque tel mécanisme, estimées à l'étape ex-ante, c'est-à-dire, avant d'apprendre son information privée. Finalement, ils définissent la valeur de la coalition en choisissant le mécanisme qui maximise cette somme. Le cœur correspondant est l'ensemble de vecteurs de paiements qui sont faisables pour la grande coalition, (c'est-à-dire, dont la somme des termes est inférieure à la valeur de la grand coalition), et par rapport auxquels aucune coalition n'est bloquante, (c'est-à-dire, la valeur d'aucune coalition n'est inférieure à son paiement). Dans les situations où le cœur "standard" est non vide, ils montrent que le cœur incitatif n'est pas vide non plus en utilisant des transferts qui vérifient que le cœur "standard" est inclus dans le cœur incitatif. Cela résulte d'une idée très importante, exploitée également dans notre travail, selon laquelle l'usage des transferts permet d'assurer les incitations.

L'analyse d'engagement de coalitions à l'étape ex-ante, surtout la formation du cartel d'enchérisseurs à cette étape, se trouve aussi chez Waehrer (1999), Bajari (2001) et Marshall et al. (1994), qui étudient les équilibres entre des joueurs qui sont asymétriques, au niveau de leur information, déjà à l'étape ex-ante. Dans le cadre de l'analyse de la collusion à cette étape, ce type de résultats est important car même dans un marché où les enchérisseurs individuels sont symétriques, dès que la formation de cartels s'éveille, la symétrie disparaît.

Finalement, il faut rappeler la littérature qui étudie la question de la collusion dans les enchères (en information incomplète) comme un problème de "conception de mécanisme". Un mécanisme de collusion pour une coalition détermine son profil d'actions (les offres, ou des fois juste l'identité de son représentant) et les transferts entre ses membres pour partager le gain. La question fondamentale qui se pose concerne les propriétés de ces mécanismes. En l'absence d'externalités, Mailath et Zemsky (1991) construisent un tel mécanisme pour la grande coalition qui est ex-post efficace, incitatif et vérifie des contraintes de la participation individuelle à l'étape interim. Cette dernière propriété veut dire qu'après avoir appris leur information privée, les individus ont (toujours !) intérêt à participer au mécanisme. Il est intéressant à rappeler, par opposition à ce dernier résultat, le théorème célèbre d'impossibilité de

Myerson et Satterthwaite (1983) concernant l'inexistence d'un mécanisme incitatif, ex-post efficace, qui vérifie des contraintes de participation individuelle à l'étape interim dans lequel le vendeur participe aussi. La tension entre ces deux résultats peut nous apprendre l'importance du rôle du vendeur pour la stabilité des contrats de collusion.

Concernant l'engagement ex-ante, Mailath et Zemsky (1991) montrent qu'il existe un mécanisme incitatif et ex-post efficace pour la grande coalition, qui vérifie des contraintes de participation pour les groupes à cette étape. C'est-à-dire, tous les sous-ensembles possibles d'enchérisseurs veulent participer au mécanisme, avant d'avoir appris leur information privée. D'autres exemples incluent: Graham et Marshall (1987) qui se concentrent sur la question de comment un cartel gagnant attribue l'objet à l'un de ses membres. Ils discutent aussi les manières de réaction que possède le commissaire-priseur pour combattre la collusion, et identifient la collusion de tous les enchérisseurs comme la collusion la plus efficace; McAffee et McMillan (1992) qui étudient les questions du partage d'information entre les membres du cartel et du partage du gain, en insistant sur l'impact des transferts sur le comportement du cartel; Marshall et Marx (2007) qui comparent la vulnérabilité de différents types d'enchère communs à la collusion, en termes d'existence de mécanismes de collusion pour ces classes d'enchères; Che et Kim (2008) qui étudient la non vulnérabilité des enchères à la collusion, au sens où, par exemple, le profit du commissaire-priseur ne baisse pas en raison de la présence de collusion. Ils établissent aussi des formes d'enchères qui résistent à la collusion, qui permettent notamment, de ne jamais vendre l'objet à aucun cartel. Contrairement aux articles précédents, ce dernier s'écarte donc des procédures d'enchères communes, au premier et au second prix.

Finalement, nous revenons à l'article de Caillaud et Jehiel (1998) qui étudient la collusion dans les enchères comme un problème de conception de mécanismes en présence d'externalités directes. Ils identifient une condition nécessaire et suffisante pour l'existence d'un mécanisme de collusion pour la grande coalition qui est ex-post efficace, incitatif, et qui vérifie des contraintes de participation individuelles à l'étape interim. Dans ce modèle nous étudions la stabilité de la grande coalition par rapport à la déviation de groupes d'enchérisseurs à l'étape ex-ante, en insistant sur la difficulté de comparer ces deux approches, c'est-à-dire, ex-ante et interim.

Les chapitres sont organisés de la façon suivante. Chapitre 1 présente un modèle d'enchère au premier prix en information complète et externalités directes non-symétriques. Nous suivons dans ce chapitre une approche non-coopérative en étudiant le processus de négociation qui décrit la formation d'un cartel. Nous montrons qu'en présence d'externalités directes la formation de la grande coalition n'est pas assurée, en proposant un exemple d'enchère dans lequel une petite coalition se forme à l'équilibre. Dans chapitre 2 nous étudions la stabilité (au sens du cœur) de coalitions dans les jeux bayésiens. Nous montrons que tout équilibre coalitionnel est sans perte de généralité incitatif. Nous appliquons ainsi la notion de stabilité aux procédures d'enchères communes sans externalités directes, en établissant (surtout) la stabilité de la grande coalition. Avec externalités directes en information complète nous montrons que la grande coalition (ainsi qu'une coalition plus petite) peut devenir instable. Chapitre 3 étudie la notion de stabilité dans les enchères au deuxième prix avec externalités directes en information incomplète. Nous identifions une classe d'équilibres maniables dans ces enchères pour toutes formes de collusion données. Finalement, dans ce modèle, nous démontrons l'instabilité de la grande coalition en présence d'externalités directes, en identifiant encore les externalités directes comme un obstacle au coopération.

Chapter 1

Strategic collusion in auctions with externalities

Abstract

We study a first price auction preceded by a negotiation stage with complete information, during which bidders may form a bidding ring. We prove that in the absence of externalities the grand cartel forms in equilibrium, allowing ring members to gain the auctioned object for a minimal price. However, identity dependent externalities may lead to the formation of small rings, as often observed in practice. Potential ring members may condition their participation on high transfer payments, as a compensation for their expected (negative) externalities if the ring forms. The cartel may therefore profitably exclude these bidders, although risking tougher competition in the auction. We also analyze ring (in)efficiency in the presence of externalities, showing that a ring may prefer sending an inefficient member to the auction, if the efficient member exerts threatening externalities on bidders outside the ring, which in turn leads to a higher winning price.

1.1 Introduction

Auctions are known as a common trading mechanism. In order to suppress competition, increase the chances of winning and reduce the winning price, bidders may try to collude, namely to form a cartel or a bidding ring. The grand cartel, in which all bidders participate, may seem as the efficient way to operate, since it totally eliminates competition and allows bidders to win the good for a minimal price. Nevertheless, the presence of externalities may introduce inefficiencies and disturb cooperation.

Given the intuition that a large bidding ring seems to be efficient collusion, we address the question of the formation of small bidding rings which are often observed in practice. We wish to identify the motivation of the grand cartel to exclude certain bidders, although risking tougher competition in the auction. We also examine the question of ring (in)efficiency, asking whether a ring always sends its efficient member to the auction.

1.1.1 Partial collusion example

Consider a market consisting of four competitive firms.[1] An auction is organized in order to issue a single valuable production license in this market. Externalities are due to the pollution level that the winning firm is anticipated to cause.

[1] Partial collusion can be demonstrated in a 3-player market as well. However, the discussed tension between excluding players and decreasing competition is more obvious as the number of players grows. Moreover, the symmetry which is considered in this example is not necessary for the formation of a small cartel. With respect to the notion of "externality matrix" which we introduce promptly, partial collusion can be demonstrated in the non-symmetric 3-player market given by:

Firms F_C and F'_C are Conservative players who operate in the market for quite some time. Their estimated profit if winning the auction is rather low due to the relatively old technology they possess, which also causes quite a great deal of pollution. Therefore, if either F_C or F'_C wins the production license, it is likely to exert significantly negative externalities on the others. Firm F_G is a young player in this market, that acts under the banner of conserving the environment, and may therefore exert no externalities on the others if winning. Due to the high costs of its "Green" technology its profit if winning the contract is anticipated to be rather low. The last player, F_H, is a dynamic High-tech firm with a rather high valuation for the contract in question, which is anticipated to exert some mild externalities on the others if winning.

If no cartel forms and all firms participate in the auction as individual bidders, F_G is assumed to win being ready to pay a high price for the license in order to avoid the externality it might suffer if either F_C or F'_C wins the production license. F_C and F'_C will fail to compete with an aggressive bid of F_G as they cannot afford paying a high price for the license, given their low valuations. F_H on its side would rather let F_G win instead of paying an expensive price, anticipating that the latter would exert no externalities.

Consider a state of nature where F_G is drawn to be the collusion designer in a negotiation stage which precedes the auction. Full collusion should designate F_H as the cartel bidder, since it maximizes the total welfare (high profit, relatively low externalities). However, if indeed such a cartel forms, both F_C and F'_C will demand a positive transfer as a compensation for the externality they are about to suffer, as opposed to the negotiation status-quo, where no cartel forms and F_G wins the license, exerting no externalities.

Excluding F_C and F'_C from the proposed cartel increases the competition in the auction. Hence, a narrower cartel, consisting of F_G and F_H only, is risking a high winning price in the auction. However, F_C and F'_C are both rather weak competitors due to their low profits, and therefore the threat each of them imposes on the narrow cartel is rather tolerable. In such a setup "partial collusion" may be more profitable than "full collusion". Example 1.4.2 demonstrates this scenario.

1.1.2 Related literature

Collusion in private value auctions without externalities was studied using the tools of mechanism design. McAfee and McMillan (1992) study first price auction with independent private values, showing that the grand cartel is feasible, and that if transfers between cartel members are allowed then the collusive mechanism is efficient. Graham and Marshall (1987) and Mailath and Zemsky (1991) study collusion in second price auction with private values, and find that partial collusion is possible. (The latter consider heterogeneous bidders.) Marshall and Marx (2007) and Robinson (1985)[2] compare the resistance of first and second price auctions to collusion, finding the second price auction more vulnerable.

In practice, collusion in auctions and in auction-like situations is widely observed. Examples include Long-Island highway construction contracts (Porter and Zona (1993)), Ohio school milk procurements (Porter and Zona (1999)), Midwest seal coat contracts (Bajari and Ye (2003)), and U.S. oil and gas leases federal auctions (Hendricks, Porter and Tan (2008)). We find the latter example particularly interesting as the considered auctions took place in the years 1954-1970, when joint bidding ventures were legal in this market. (In late 1975, however, Congress passed prohibiting legislation.)

We consider a bargaining stage before the auction during which bidders may form a ring. As, e.g., Bloch (1996), Bloch and Gomes (2006), Ray and Vohra (1999, 2001), we restrict our attention to a specific bargaining protocol. A complementary approach omits the specification of the bargaining process and focuses on examining properties of the bargaining result, i.e., the admitted partition of the society as in, e.g., Ray and Vohra (1997) and chapter 2. The latter apply a core notion on auctions in order to study the stability of small cartels vs. the grand cartel without referring to the question of how a given cartel emerged.

3	-4	0
0	1	0
0	-3	1

[2]Robinson (1985) considers also common value auctions.

The notion of identity dependent externalities, which plays a major role in this chapter, is taken from the work of, e.g., Jehiel and Moldovanu (1996, 1999), Caillaud and Jehiel (1998), and Jehiel, Moldovanu and Stacchetti (1996, 1999).

1.1.3 The collusion game

Following, e.g., Jehiel and Moldovanu (1996) we consider a market with complete information where players assign positive valuations to a single indivisible good. The valuation of a player is the utility he derives if consuming the good. Additionally, each player exerts identity dependent externalities on the others if consuming the good. In other words, each player gets a certain utility, which (possibly negative) value is determined as a function of the identity of the consumer.

A first price auction is organized in this market.[3] We assume that the winner is obliged to consume the good, and no resale is allowed after the auction ends. Such an assumption is reasonable, for example, in state tenders where the winning firm has to carry out the project in question and cannot resell the execution rights to a third party. (See, e.g., the South-Korean high-speed train case study in Jehiel and Moldovanu (1996).)

The auction is preceded by a negotiation stage in which bidders may form a bidding ring. The bargaining protocol we consider takes the following form. One of the bidders is chosen by a chance move to be the *collusion designer*. He may then address any subset of the others, offering them to form a cartel. He designates one of the members of the proposed cartel, possibly himself, as the representative of the cartel, or the *designated cartel bidder*. Finally, he specifies a contingent transfer scheme which is implemented if the cartel wins the good.[4] If all addressed agents accept the offer then the cartel forms, and all but the designated cartel bidder are committed to place an irrelevant bid in the auction. Otherwise, agents act individually in the auction.

We analyze bidders' behavior in sub-game perfect Nash equilibrium. As we consider an auction game, we restrict our attention to pure bidding strategies.[5] Such a restriction, however, puts into question the existence of an equilibrium bid in the auction. We handle this difficulty in appendix A providing a characterization of first price auction equilibrium bids in pure bidding strategies in the presence of externalities.

1.1.4 Main results

We start by studying, as a benchmark, an auction which takes place in the absence of externalities. Namely, every losing bidder is indifferent, in terms of his final utility, regarding the identity of the winner. Not surprisingly, we find that the primary intuition holds. Bidders always form the grand coalition, represented by the bidder with the highest valuation. As a consequence they win the good for a minimal price, and the seller's surplus as a whole is divided between ring members through transfer payments.

Introducing externalities between bidders changes the outcome dramatically. Externalities lead to a trade-off between reducing competition and compensating some of the participating ring members via transfer payments. For instance, a player may ask for a high transfer payment if he anticipates a considerably low externality if participating in the collusion, compared to his anticipated utility if he declines to participate. Hence, in order to avoid paying high transfers, the collusion designer may find it optimal to exclude players from the considered bidding ring, although risking a tougher competition in the auction. We, therefore, identify externalities as a likely reason for the formation of small bidding rings

[3] Our main result, namely, that direct externalities lead to the formation of small bidding rings, can be verified in second price auctions as well.

[4] Collusion case studies find that as transfer payments between cartel members are easily tracked, cartels tend to participate in several auctions, letting each member be the relevant bidder according to "the phases of the moon" (see, e.g., Porter and Zona (1999), Bajari and Ye (2003)). As we consider a single-auction setup, we use transfer payments as motivation for cartel members to cooperate.

[5] The results we are presenting, namely, the formation of small bidding rings in the presence of direct externalities, hold for the non-restricted case as well, where mixed bidding strategies are allowed.

instead of the grand cartel. We demonstrate a market with externalities, where the collusion designer indeed forms a small cartel in a sub-game perfect Nash equilibrium.

We then move on to discuss the question of the identity of the bidding ring's representative. As mentioned above, in the absence of externalities the bidder with the highest valuation optimally represents the grand cartel in the auction. The intuition is quite clear. The bidder with the highest valuation allows splitting the "largest pie" among cartel members. This intuition does not extend to markets with externalities. We say that a player is the cartel's efficient member if the sum of his valuation and the externalities he exerts on other cartel members is maximal. In order to be able to split the "largest pie" the efficient member should represent the cartel. However, the efficient member may happen to exert terribly low externalities on players outside the cartel. Such a threat translates into aggressive bids in the auction, which in turn reduces the chances of the cartel to win, or alternatively yields a high winning price. As a result the net benefit of the cartel from collusion decreases. The cartel may therefore find it optimal to be represented by an "inefficient" and less threatening member.

Finally, we compare our results with Jehiel and Moldovanu (1996)'s strategic non-participation. They proved that bidders may find it optimal to commit not to participate in the auction just before it takes place. In this way, a bidder eliminates himself from being a potential consumer, hence the externalities he may exert on others become irrelevant. Such a decision changes the market description and may lead to a different winner in the auction. For example, a bidder may find it optimal not to participate if by doing so he anticipates that a bidder he prefers as a consumer will win the good. We show that the collusion designer is strictly better off forming an appropriate cartel rather than choosing not to participate. Thus, strategic non-participation is redundant if bidder collusion is considered.

This chapter takes the following structure: In section 1.2 we present the model of the collusion game. In section 1.3 we analyze bidders' behavior in the collusion game in markets without externalities, and prove that full collusion always emerges in equilibrium. In section 1.4 we introduce externalities and show that partial collusion may arise. In section 1.5 we demonstrate that in the presence of externalities a formed cartel may prefer to be represented in the auction by an inefficient member. Section 1.6 analyzes strategic non-participation in view of the collusion game. In section 1.7 we study possible extensions of our model. We obtain a characterization of equilibrium bids in first price auctions in markets with externalities, and of weakly dominated strategies in this setup in appendices A and B. Appndices C, D and E contain proofs of the main propositions of sections 1.3, 1.4 and 1.5 correspondingly.

1.2 The model

1.2.1 First price auction with externalities

The market consists of a non-strategic seller S, $n \in \mathbb{N}$ potential buyers $B = \{B_1, B_2, \ldots, B_n\}$, and one indivisible good. Each buyer B_i assigns a valuation π_i to the good in case he consumes it. An identity dependent externality $\alpha_{ij} \in \mathbb{R}$, $i \neq j$, is the utility to buyer B_j in case buyer B_i consumes the good. We refer to this setup of valuations and externalities as an $n \times n$ *matrix of externalities.*

We consider a first price auction in this market, which the seller organizes. All participants place simultaneously their non-negative bids. The highest positive bid which was placed, denoted p, wins. The winner, denoted B_w, pays p to the seller, consumes the good, and gets his valuation, π_w. All other agents, $B_j \neq B_w$, get their corresponding externality, α_{wj}. We make the following assumptions:

- The winner must consume the good, and no resale to another agent is allowed.
- If the highest positive bid was placed by several agents (tie), each of them has an equal probability of winning the good.
- If all participants place a zero-bid, the good stays in the possession of the seller, and each agent gets a utility normalized to zero.

We assume that each agent in the market prefers consuming the good rather than either having some other agent consuming it, or leaving it in the possession of the seller, i.e. status-quo. Formally, for all B_i, $\pi_i > \alpha_{ji}$ for all $j \neq i$, and $\pi_i > 0$.[6]

Following, e.g., Jehiel and Moldovanu (1996), we will assume the existence of a smallest money unit in the market, denoted ϵ, in order to avoid problems related to the existence of Nash equilibrium in the first price auction. In particular, bids, valuations and externalities are discrete with respect to this money unit. For instance, if B_i places a bid equals to $b_i = k\epsilon$, and B_j wishes to overbid it, then B_j must bid at least $b_i + \epsilon = (k+1)\epsilon$.

Finally, we assume that the valuations and externalities are generic in the following sense.

Definition 1.2.1. We will refer to a market as *generic* , if the following holds:

- All valuations and externalities are linearly independent with respect to the set of coefficients $\{-1, 0, 1\}$. Namely, for any two sets of coefficients $\{\delta_i\}_{i=1}^n$ and $\{\delta_{ij}\}_{i\neq j}$ which take values in $\{-1, 0, 1\}$, if not all coefficients are null then $\sum_{i=1}^{n} \delta_i \pi_i + \sum_{i \neq j} \delta_{ij} \alpha_{ij} \neq 0$.

- Adding or subtracting up to $(n+2)\epsilon$ to any of the valuations and externalities maintains the linear independence. Namely, for any two sets of coefficients $\{\eta_i\}_{i=1}^n$, and $\{\eta_{ij}\}_{i\neq j}$ such that $-(n+2)\epsilon \leq \eta_i, \eta_{ij} \leq (n+2)\epsilon$, the valuations $\{\pi_i + \eta_i\}_{i=1}^n$, and externalities $\{\alpha_{ij} + \eta_{ij}\}_{i\neq j}$, are independent with respect to the set of coefficients $\{-1, 0, 1\}$.

1.2.2 Collusion

The first price auction is preceded by a negotiation process between the potential bidders, in which they may agree to collude. One of the agents is chosen by a chance move to be the *collusion designer.* He may then address the others, and propose them a take-it-or-leave-it offer to form a cartel. He designates one of the proposed cartel members, who will make a relevant bid while the others place irrelevant bids. Finally, the collusion designer proposes a configuration of transfer payments. We emphasize that the *designated cartel bidder* participates in the auction as an actual player, and no fictitious player which represents the cartel is added to the game.[7] We assume the following:

- The offer is observed by all agents in B.

- By accepting the offer, all addressed agents, but the designated cartel bidder, commit to make an irrelevant bid, namely 0, in the auction.

- If the offer is accepted, all members of the cartel commit to implement the transfer payment scheme if the designated cartel bidder indeed wins the auction.[8]

- Agents' responses are also observed by all agents in B.

Note that the designated cartel bidder, does not commit to a specific bid, and in particular may eventually bid zero in the auction. The probability of every agent to be the collusion designer, is given by a probability vector, denoted σ, which is part of the game data. Right after the addressed agents respond to the offer, the auction takes place. Hence, the formed cartel, its representative in the auction, and the transfer payments agreed upon among the members of the cartel, define the state of the economy at the beginning of the auction.

Definition 1.2.2. A *state* s in the game is the tuple (C, B_l, d) where:

[6] Caillaud and Jehiel (1998) also assume that every agent prefers the status-quo, i.e. no sale, over a sale to another, namely, negative externalities ($\forall i, \alpha_{ji} < 0$ for all $j \neq i$).

[7] See, e.g., Haeringer (2004) who considers a game between "meta-players", where a "meta-player" stands for a coalition.

[8] We assume that if the cartel loses the auction then there is no motivation for further cooperation. In particular, no transfer payments are made in such a case.

- $C \subset B$
- $B_l \in C$
- $d \in \mathbb{R}^n$ such that:
  - For all $1 \leq j \leq n$ there exists an integer m_j such that $d_j = m_j\epsilon$
  - For all $j \notin C$ it holds that $d_j = 0$
  - $\sum_{j \in C} d_j = 0$

The interpretation is that C is a cartel, of which B_l is the representative, and d are transfer payments which the agents receive, if B_l wins the auction. For the simplicity of notations we refer to d as a vector of transfer payments to the members of the society as a whole. Transfer payments correspond to the money unit ϵ. Surely enough, transfer payments outside the cartel are zero. Within the cartel, transfer payments are balanced.

We denote s^0 the initial state of the economy, where there is no cartel, and no commitment to transfer payments. If the offer which the collusion designer makes is declined by any of the addressed agents, the economy stays in the state s^0. In this case, as no cartel is formed, all agents go to the auction as individual bidders, without any commitment to bids nor to transfer payments. Finally, the collusion designer may prefer to leave the economy at the initial state s^0, which we refer to as *negotiation status-quo*.

Definition 1.2.3. Let s be a state. The set of *relevant bidders* in the state s, denoted $B(s)$, is given by:

- If $s = s^0$ then $B(s^0) = B$.
- Otherwise, let $s = (C, B_l, d)$, then $B(s) = \{B_l\} \cup (B \setminus C)$.

Consider B_i as the collusion designer, we say that a *proposal* is a state s such that either $s = s^0$, or $s = (C, B_l, d)$ and $B_i \in C$. The interpretation is that B_i may either choose the negotiation status-quo, or may alternatively propose to move the economy to a state s, by suggesting to form a cartel in which he is a member.

Given the assumptions detailed above, the *collusion game* is the following:

- Stage 1: A collusion designer, B_i, is chosen by a chance move according to σ.
- Stage 2: B_i makes a take-it-or-leave-it offer, to move the economy to a state s.
- Stage 3: If $s = s^0$ the game moves to the next stage. Otherwise, if $s = (C, B_l, d)$, all members of C, but the designer, signal sequentially whether they accept or reject the proposal.[9] At the end of this stage, if all agents accepted the proposal, then the economy moves to the new state s. Otherwise, if at least one agent rejected the proposal, then the economy stays at the primary stage s^0, where all agents are singletons.
- Stage 4: A first price auction takes place, with respect to the established state.

Let $s = (C, B_l, d)$ be the state of the economy when the auction takes place, and let $b(s)$ be a valid bidding vector with respect to the state s. Assume, for the sake of simplicity, that if the highest bid p is positive then it is placed by a single agent $B_w \in B(s)$. The utility function, $u(s, b(s))$, is defined as follows: (The case where there are several winners is treated similarly.)

- If no agent wins the good in the auction, namely $b(s) = \overline{0}$, then the utility of the seller is 0, and every B_j gets $u_j(s, \overline{0}) = 0$

[9] Assume that agents are addressed in ascending order with respect to their indices. As we discuss subgame perfect Nash equilibria (SPNE), the analysis does not depend on the order in which agents are addressed and respond. In particular, the restriction to SPNE rules out equilibria where all agents reject because if another rejects all responses are equivalent. Equivalently, one can consider trembling hand perfect equilibria of the game, in which the agents respond simultaneously.

- Otherwise, the seller's utility is p.
- If $w = l$ then $u_j(s, b(s)) = \begin{cases} \pi_l + d_l - p & \text{if } j = l \\ \alpha_{lj} + d_j & \text{if } j \in C \setminus \{l\} \\ \alpha_{lj} & \text{if } j \notin C \end{cases}$
- If $w \neq l$ then $u_j(s, b(s)) = \begin{cases} \pi_w - p & \text{if } j = w \\ \alpha_{wj} & \text{if } j \neq w \end{cases}$

Finally, we wish to consider the following definition of efficiency, as we study the question of efficiency in equilibrium.

Definition 1.2.4. Let C be a cartel in a generic market. We say that B_i is the *efficient* member of C if $i = \arg\max_{j \in C}(\pi_j + \sum_{l \in C \setminus \{j\}} \alpha_{jl})$. We refer to the *efficient agent* as the efficient member of B.[10]

1.3 The zero-externality case

We start the game analysis by discussing the case where no agent exerts externalities on the others, namely, for all $i \neq j$, $\alpha_{ij} = 0$. We prove that in all Subgame Perfect Nash Equilibria (SPNE), with pure bidding strategies in the auction, full collusion emerges with probability one.

We emphasize that our results hold for the non-restricted case as well, where mixed strategies are allowed. However, the analysis one should follow is somewhat more complicated. For instance, in a generic market with externalities the winner in equilibrium of the first price auction is not uniquely determined (see, e.g., Jehiel and Moldovanu (1996)). Moreover, even in the case without externalities, there are non-generic markets, where both a bid leading to a tie between two agents, and a bid which yields one of them as a single winner, are in equilibrium. We note, however, that considering mixed strategies in the auction simplifies the discussion regarding the existence of SPNE in the collusion game. One may conclude the existence of an equilibrium of the auction game in mixed strategies, as bids are discrete, and continue in backward induction to deduce the existence of an SPNE of the collusion game.

We start by understanding how an agreement made by agents affects the market description at the last stage of the game, when the auction takes place. The relation between an agreement and the market description is explained by transfer payments. For example, suppose that B_1 and B_2 form a cartel, where B_1 is the cartel bidder. In addition, suppose that they agree that if B_1 wins the good then B_2 gets a transfer of x, and B_1 gets a transfer of $-x$. When going to the auction and considering his bid, B_1 needs to take into account that if indeed he wins the good and consumes it, his payoff will not be his original valuation π_1, but his valuation fixed by his transfer payment, namely, $\pi_1 - x$. Therefore, the relevant valuation for B_1 while considering his bid should be updated according to the agreement he is part of.

The matrix of externalities is used both to determine what bids are in equilibrium in the auction, and to conclude agents' utilities when the good is consumed. Given a state, all cartel members but the cartel bidder, are committed to bid 0. Therefore, an equilibrium of a first price auction in this state is a function of the bids of the relevant bidders in this state only. Namely, the cartel bidder and fringe bidders, i.e., agents outside the cartel. Moreover, as bidding 0 cannot lead to winning the good and consuming it, the cartel members, but the cartel bidder, are not potential consumers. Hence, we should reduce the original matrix of valuations and externalities to a matrix composed of the valuations and externalities of relevant bidders only, with respect to the state in question. Hence, we start by erasing the rows and columns of the cartel members but the cartel bidder.

We continue by updating the valuation of the cartel bidder with respect to the agreed transfer payment. Since a valuation is defined as what an agent gets if he wins the auction and consumes the good, and the cartel bidder gets his transfer payment only if he indeed wins, we conclude that with respect to the agreement, the valuation of the cartel bidder is the sum of his original valuation and the transfer

[10]The efficient agent in every cartel C is unique due to genericity.

payment agreed upon. The valuations of the other relevant bidders stay as the original ones, as they are not involved in any agreement, and expect no transfer payments if winning. Finally, all externalities in the reduced matrix also stay as the original ones, as no relevant bidder is to get a transfer payment if another relevant bidder wins. This idea is formalized in the following lemma, which proof follows directly from definition 1.2.2.

Lemma 1.3.1. *Consider a generic zero-externality market.*[11] *Let $s = (C, B_l, d)$ be the state of the economy when the auction takes place. For all $k, j \in B(s)$ denote $X_{kj}(s)$ the payoff to agent B_j if B_k consumes the good in state s. Then the matrix $X_{kj}(s)$ of dimension $|B(s)|$ is well defined and is given by:*

- *If $k = j = l$, $X_{kj}(s) = \pi_l + d_l$.*
- *If $k = j \neq l$, then $X_{kj}(s) = \pi_k$.*
- *If $k \neq j$, then $X_{kj}(s) = 0$.*

Example 1.3.2. Consider a generic 0-externality market with 3 potential buyers. Consider the state $s = (\{B_1, B_2\}, B_1, (-x, x))$. That is, B_1 and B_2 form a cartel, where B_1 is its bidder, and if B_1 eventually wins the good, he commits to pay x to B_2. Then $B(s) = \{B_1, B_3\}$, and $X(s)$ is given by the following matrix:

	1	3
1	$\pi_1 - x$	0
3	0	π_3

∎

Finally, the discussed link between agents behavior in the auction and the agreements they are involved in is used in order to prove the formation of the grand-cartel in equilibrium in the absence of externalities.

The proof of the concluding proposition of this section is brought in appendix C. The intuition, however, is quite clear. Due to the absence of externalities, the agent with the highest valuation, is the efficient one. If the negotiation status-quo is followed, the efficient agent wins the auction and consumes the good. By forming the grand-cartel with the efficient agent as the cartel bidder, the collusion designer can extract the seller's surplus from the efficient agent as a transfer payment, since the latter is going to win the good for the price of ϵ. As the efficient agent wins the good in the negotiation status-quo in the first place, all other agents stay indifferent to the offer to form the grand-cartel with the efficient agent as the cartel bidder. Hence, the collusion designer need not compensate any of them, and the seller's surplus is his net gain. Any alternative offer to form a smaller cartel, preserves competition between potential buyer in the auction, which raises the winning price, and in turn decreases the surplus which the collusion designer can extract.

Proposition 1.3.3. *In a generic market without externalities the set of SPNE points of the collusion game is not empty. Moreover, full collusion, i.e. the grand cartel, emerges with probability 1 in all SPNE points of the game.*

1.4 Formation of small cartels in the presence of externalities

We establish in this section that the presence of externalities may lead agents to form small cartels. In order to understand how an agreement to form a cartel affects the state of the economy we start by an analysis which is similar to the one in the previous section. We show that in every possible state there is a bid which is in equilibrium in a first price auction. In appendix D we prove that the set of SPNE points of the game in the presence of externalities is not empty. (As stated in the previous section, we consider pure bidding strategies in the auction stage, hence, the existence of SPNE is to be proved.) Finally, we

[11] Genericity in zero-externality markets is defined as in definition 1.2.1 with respect to valuations only.

demonstrate that there exists a market with externalities where partial collusion arises in SPNE with positive probability.

Lemma 1.4.1. *Consider a generic market with externalities. Let* $s = (C, B_l, d)$ *be the state of the economy when the auction takes place. For all* $k, j \in B(s)$ *denote* $X_{kj}(s)$ *the payoff to agent* B_j *if* B_k *consumes the good in state* s*. Then* $X_{kj}(s)$ *is well defined and is given by:*

- *If* $k = j = l$*, then* $X_{kj}(s) = \pi_l + d_l$.
- *If* $k = j \neq l$*, then* $X_{kj}(s) = \pi_k$.
- *If* $k \neq j$*, then* $X_{kj}(s) = \alpha_{kj}$.

The proof follows directly from definition 1.2.2.

The following example shows that the presence of externalities, may lead to a situation where the collusion designer is strictly better off forming a cartel smaller than the grand one. The intuition is that externalities may be so low, that some agents will demand a high compensation from the collusion designer in order to join the grand cartel. The designer in this case, is better off leaving those agents outside the cartel he forms, although he is risking a tougher competition in the auction by doing so. In the proof of proposition 1.4.3 below we will use the following example.

Example 1.4.2. Consider the following 4-player market with externalities: $\pi_1 = 8, \pi_2 = \pi_3 = \pi_4 = 1; \alpha_{1j} = -2, \forall j \neq 1; \alpha_{2j} = 0, \forall j \neq 2; \alpha_{3j} = -8, \forall j \neq 3; \alpha_{4j} = -7, \forall j \neq 4.$[12]

8	-2	-2	-2
0	1	0	0
-8	-8	1	-8
-7	-7	-7	1

We claim that in this market, B_2 can gain more as a collusion designer by forming a cartel with B_1 only, rather than forming the grand cartel. The following proposition 1.4.3 proves this claim formally, however, we would like to precede the formal discussion with an intuitive one.

If no cartel is formed and all agents go to the auction as individual bidders, i.e. negotiation status-quo, we consider an equilibrium bid as a result of which B_2 wins the auction paying the seller $p = 8$ for the good. (See corollary A.2, with $m = 1$.)

Considering the formation of the grand cartel, it is but natural, that the collusion designer is best off designating the efficient agent, B_1, as the cartel bidder. That is as the efficient agent maximizes the utility of the grand cartel if he consumes the good. However, if B_2, the collusion designer, wishes to do so, he needs to compensate B_3 and B_4, as the latter gain a lower externality if indeed the efficient agent, B_1, consumes the good, compared to the negotiation status-quo (e.g., $\alpha_{13} = -2 < 0 = \alpha_{23}$).

And indeed, by excluding B_3 and B_4 from the cartel (and by that avoiding the compensations they would claim), and forming a small cartel with B_1 only, B_2 gains a higher utility as detailed in the proof below. Note, that the latter is true although clearly by forming a small cartel, the price payed for the good in the auction rises, compared to an auction in which the grand cartel forms ($p = 3$ as opposed to $p = \epsilon$ respectively). It means that the collusion designer is better off experiencing tougher competition in the auction, than compensating B_3 and B_4. ∎

Proposition 1.4.3. *There exists a generic market with externalities, and an SPNE of the game in this market, in which a cartel smaller than the grand one forms with a positive probability.*

Proof. Consider the 4-player market with externalities in example 1.4.2.[13] We consider the following strategies of the agents:

[12] As already stated, partial collusion may emerge in a 3-player market as well as in a non-symmetric setup. See footnote 1 for further details.

[13] There exist $\epsilon > 0$ and $\delta > 0$, both small enough, such that we can change the valuations and externalities in the neighborhood of δ, in order to achieve genericity of the market. Moreover, for small enough ϵ and δ the analysis in the proof holds for the generic market as well.

- In the state s^0, agents bid $b(s^0) = (8 - 2\epsilon, 8, 8 - \epsilon, 8 - 2\epsilon)$. That is an equilibrium bid of the first price auction in the state s^0, according to corollary A.2 (with $m = 1$).
- For every state s, such that B_l is a single bidder in s, namely $B(s) = \{B_l\}$, B_l bids ϵ if $X(s) \geq \epsilon$, and 0 otherwise. According to proposition A.8 this is an equilibrium bid in this state.
- In the state $s^{12} = (\{B_1, B_2\}, B_1, d^{12})$, where $d_1^{12} = -d_2^{12} = -5$

	1	3	4
1	3	-2	-2
3	-8	1	-8
4	-7	-7	1

we consider the bid $b(s^{12}) = (3, 0, 3 - \epsilon, 3 - 2\epsilon)$, which is in equilibrium by corollary A.2.

- For every other state s, consider some equilibrium bid $b(s)$, which exists as established in appendix D (lemma D.1).
- Finally, with respect to the above described function $b(s)$, for every proposal made by B_i to move to the state $s = (C, B_l, d)$, every $B_j \in C \setminus \{B_i\}$ accepts the proposal if and only if $u_j(s, b(s)) \geq u_j(s^0, b(s^0))$.

As proved in appendix D (corollary D.3), there exists an SPNE of the game which includes these strategies. It is enough to demonstrate that in such an SPNE, the utility that B_2 derives as the collusion designer by proposing a cartel different than the grand one, is strictly greater than the utility he can derive by full collusion.

Consider first the initial state s^0. As stated above we consider the bid $b(s^0) = (8-2\epsilon, 8, 8-\epsilon, 8-2\epsilon)$. As a result of this bid B_2 wins the good, pays $p = 8$ to the seller and consumes. The utility vector of the agents is therefore $u(s^0, b(s^0)) = (\alpha_{21}, \pi_2 - p, \alpha_{23}, \alpha_{24}) = (0, -7, 0, 0)$.

Consider now the offer to form the grand cartel with the efficient agent as its representative, namely, to move to the state $s^{GC} = (B, B_1, d^{GC})$, where $d^{GC} = (-8 + \epsilon, 4 - \epsilon, 2, 2)$. If the offer is accepted then $X(s^{GC}) = \pi_1 + d_1^{GC} = 8 + (-8 + \epsilon) = \epsilon$. Therefore, B_1 bids ϵ and wins the auction. He pays $p = \epsilon$ to the seller, and gets $\pi_1 - p + d_1^{GC} = 8 - \epsilon + (-8 + \epsilon) = 0$. As $u_1(s^0, b(s^0)) = 0$, B_1 accepts the offer. In a similar way, if the offer is accepted, and B_1 wins the good and consumes, B_3 gains $\alpha_{13} + d_3^{GC} = -2 + 2 = 0$. As $u_3(s^0, b(s^0)) = 0$, B_3 accepts the offer. The same holds for B_4. To conclude, B_2 gains by proposing the discussed offer a utility of $\alpha_{12} + d_2^{GC^1} = -2 + 4 - \epsilon = 2 - \epsilon$. Clearly, B_2 cannot gain by proposing a higher transfer to any of the agents. On the other hand, by proposing a lower transfer to either B_3 or B_4, the offer will be rejected and the grand cartel will not form. Offering a lower transfer payment to B_1, will lead to a state where B_1 bids 0 in the auction, the good stays in the possession of the seller, and B_2 gains 0, which is strictly less than what he gains by proposing s^{GC}.

The same analysis can be repeated considering other possible bidders on behalf of the grand cartel, to conclude that by forming the grand cartel B_2 can gain at most $2 - \epsilon$ with respect to the considered strategy profile.

We now proceed to demonstrate an alternative offer to form a cartel with B_1 only, i.e., partial collusion, which is accepted by B_1 according to the considered strategy profile, and yields B_2 a strictly greater utility. Consider the proposal to move to the state $s^{12} = (\{B_1, B_2\}, B_1, d^{12})$, where $d_1^{12} = -d_2^{12} = -5$. $X(s^{12})$ is given by:

	1	3	4
1	3	-2	-2
3	-8	1	-8
4	-7	-7	1

where, $X(s^{12})_{11} = \pi_1 + d_1^{12} = 8 + (-5) = 3$. We consider the following equilibrium bid $b(s^{12}) = (3, 0, 3-\epsilon, 3-2\epsilon)$. As a result of which B_1 wins the auction as a single winner, pays $p = 3$ to the seller, and consumes the good. He therefore gains, $X(s^{12})_{11} - p = 3 - 3 = 0$ which is equal to $u_1(s^0, b(s^0))$. Therefore, B_1 accepts the offer, and B_2 can gain $\alpha_{12} + d_2^{12} = 0 + 3 = 3$. Indeed, that is a greater utility than the one that can be achieved by full collusion, and therefore there is a positive probability that the grand cartel will not form in the considered SPNE. ∎

1.5 Efficiency

In the previous section we demonstrated that partial collusion may arise in SPNE in the presence of externalities. In this section, we further show that the cartel bidder in SPNE in the presence of externalities is not necessarily the cartel's efficient member. This phenomenon is explained by a trade-off between the welfare of the cartel on one hand, which is maximized if the efficient member of the cartel consumes the good, and the price paid for the good on the other hand, which depends on the externalities that the cartel bidder exerts on fringe agents, i.e. agents outside the cartel.

Consider a 3-player market in which the only feasible cartel is $C = \{B_1, B_2\}$, and let B_1 be the collusion designer.[14] The following example presents a setup in which B_2 is the cartel's efficient member, and at the same time introduces a major threat on the fringe agent B_3. A-priori in order to maximize the cartel's profits, it should be represented in the auction by B_2. Doing so, however, would increase dramatically the price that the cartel would need to pay in the auction if winning, as the fringe agent would be bidding aggressively in order to avoid the potential loss he might suffer if B_2 indeed wins. The collusion designer in this case is better off if the cartel is not represented by its efficient member, as the low price the cartel will pay compensates for the loss of potential welfare.

Example 1.5.1. Consider the following 3-player market with externalities: $\pi_1 = \pi_2 = 2, \pi_3 = 1, \alpha_{12} = -1, \alpha_{23} = -10, \alpha_{31} = \alpha_{32} = -2$, and all other externalities are null. Let B_1 be the collusion designer in the beginning of the second stage.

2	-1	0
0	2	-10
-2	-2	1

Consider a proposal to form the cartel $C = \{B_1, B_2\}$. B_2 is the efficient member of this cartel as $\pi_2 + \alpha_{21} > \pi_1 + \alpha_{12}$. Note that B_2 is a great threat to B_3 as $\alpha_{23} = -10$. Therefore, B_3 would be willing to place a high bid in order to prevent B_2 from winning the auction. Hence, if B_1 wishes to send B_2 to the auction as the cartel bidder, he has to commit to a high transfer payment to B_2, to enable him to overcome B_3's expected high bid. We demonstrate that B_1 can gain more by proposing himself as the cartel bidder, rather than proposing B_2.

Consider first the initial state s^0, where agents bid $b(s^0) = (4-2\epsilon, 4-\epsilon, 4)$. (This is an equilibrium bid in this state according to corollary A.2.) As a result of this bid, B_3 wins the good, pays $p = 4$ to the seller, and consumes the good. The utility vector of the agents is therefore $u(s^0, b(s^0)) = (\alpha_{31}, \alpha_{32}, \pi_3 - p) = (-2, -2, -3)$.

Consider now the proposal to move the economy to the state $s^1 = (C, B_1, d^1)$, where $d_1^1 = -d_2^1 = 1$. If B_2 accepts the offer then the economy moves to the state s^1. $X(s^1)$ is given by:

	1	3
1	3	0
3	-2	1

[14] Caillaud and Jehiel (1998) consider such an example where 2 European firms were considering a joint venture vis-a-vis a Japanese firm in the 1992 South-Korean high speed train tender. A joint venture in the context of this example between a European firm and the Japanese one is less likely to be feasible.

where, $X(s^1)_{11} = \pi_1 + d_1^1 = 2 + 1 = 3$. We consider the following equilibrium bid $b(s^1) = (1, 0, 1 - \epsilon)$. (This is an equilibrium bid in this state according to corollary A.2.) As a result of which B_1 wins the auction as a single winner, pays $p = 1$ to the seller, and consumes the good. Therefore, B_2 gains $\alpha_{12} + d_2^1 = -1 + (-1) = -2$ which is equal to what he gains by rejecting the offer (i.e., $u_2(s^0, b(s^0))$). Hence, B_2 may accept the offer in SPNE, and let us consider a strategy profile in which he does. As a result, by proposing to move to s^1, B_1 gains $X(s^1)_{11} - p = 3 - 1 = 2$.

Let us now look at the alternative proposal to move the economy to the state $s^2 = (C, B_2, d^2)$, where the cartel C is represented by its efficient member B_2. If B_2 accepts the offer then the economy moves to the state s^2, where $X(s^2)$ is given by:

	2	3
2	$2 + d_2^2$	-10
3	-2	1

In order for B_2 to win the auction in state s^2 it must hold that $X(s^2)_{22} - X(s^2)_{32} > X(s^2)_{33} - X(s^2)_{23}$ (see corollary A.2), namely $2 + d_2^2 - (-2) > 1 - (-10)$, which is equivalent to $d_2^2 > 7$. We conclude that if B_1 proposes B_2 as the cartel bidder, he ends up with a negative utility (his transfer payment would be negative, and the externality that B_2 exerts on B_1 is null).[15] Hence, B_1 can profit more by going to the auction himself as the cartel bidder, rather than sending the cartel's efficient member. ∎

The following proposition discusses inefficient collusion in the presence of externalities in the general case, namely, where any form of collusion is feasible. The proof appears in appendix E.

Proposition 1.5.2. *There exists a generic market with externalities, and there exists an SPNE of the collusion game in this market, where with positive probability, the cartel bidder is not the cartel's efficient member.*

1.6 Strategic non-participation

Jehiel and Moldovanu (1996) presented the idea of strategic non-participation in auctions. They considered an extended auction game, where in the first stage all agents decide simultaneously whether they wish to participate or not in the auction. At the next stage the auction takes place, and only agents who have decided to participate during the first stage take part in it. At the end of the auction, the winner consumes the good, gains his valuation and pays the proper price to the seller. Every other agent, whether he has participated or not in the auction, gains the corresponding externality according to the identity of the consumer.

They show that agents may be better off committing not to participate in the auction, providing the following intuition to explain this phenomenon. The absence of a specific agent in the auction may remove a potential threat he imposes on others. As a result, the bidding strategy of the participants may change, which in turn may lead to a different winner. This alternative winner may be better off for the non-participating agent, as an alternative consumer of the good, in terms of the externalities which the alternative winner imposes.

Collusion provides agents with the possibility of non-participating, by joining a cartel in which they are not the cartel's representative. The following proposition provides a link between the strategic non-participation presented in Jehiel and Moldovanu (1996) and the collusion game. We prove that in the collusion game, the collusion designer can gain strictly more than what he could have achieved by not participating à la Jehiel and Moldovanu. The intuition is, that by full collusion where the negotiation status-quo winner is the designated cartel bidder, the collusion designer can have all agents accepting to join for an ϵ transfer payment. In this way he is capable of extracting a net transfer approximately equal to the seller's surplus in negotiation status-quo. This net transfer is large enough to overcome any externality, especially the externality he would have gained if not participating à la Jehiel and Moldovanu.

[15] The analysis omits the case of a tie between B_2 and B_3 in the state s^2, and the case where the cartel forms and loses the auction. B_1 cannot gain a positive utility in any of these cases as well.

As we wish to compare with non-participation of the collusion designer only, the following definitions suffice for the proposition we present.

Consider a generic market with externalities consisting of n potential buyers, and let B_i be a potential buyer. We say that the bidding vector $b^{NP_i} \in \mathbb{R}^n$ is an equilibrium bid of the first price auction in the market which corresponds to the non-participation of B_i, if $b_i^{NP_i} = 0$, and if omitting the i'th coordinate of b^{NP_i} yields a bidding vector in $\mathbb{R}^{n-1}$, which is an equilibrium bid in a first price auction held in the market derived from the original market by omitting the i'th row and column.[16]

We denote $u_i(b^{NP_i})$ the utility of the non-participating agent. If B_w is the winner in a first price auction held in a market which corresponds to the non-participation of B_i, then $u_i(b^{NP_i}) = \alpha_{wi}$.

Finally, in order to be consistent with their framework of externalities, we will respect their restriction to non-positive externalities, namely, for all $i \neq j$, $\alpha_{ij} \leq 0$.

Proposition 1.6.1. *Consider a generic market with externalities, where all externalities are non-positive. Let B_i be the collusion designer at the second stage. Then in every SPNE of a sub-game of the collusion game where B_i is the collusion designer, he gains strictly more than what he could have gained by not participating à la Jehiel and Moldovanu.*

Namely, consider an SPNE of the sub-game of the collusion game where B_i is drawn to be the collusion designer, let s be the state in which the auction takes place in this SPNE, and let $b(s)$ be the corresponding bidding vector, then for every b^{NP_i}, a bidding vector which is an equilibrium of the first price auction in the market which corresponds to the non-participation of B_i, it holds that:

$$u_i(s, b(s)) > u_i(b^{NP_i})$$

Proof. Consider an SPNE of the sub-game of the collusion game where B_i is drawn to be the collusion designer. Denote $b(s)$ the function which maps every state s to an equilibrium bid of the first price auction in this SPNE. Finally, let b^{NP_i} be an equilibrium bid of the first price auction in the market which corresponds to the non-participation of B_i. Denote B_w the winner of the auction in state s^0 according to $b(s^0)$, and denote $B_{w'}$ the winner of the auction if B_i is not participating, according to b^{NP_i}. Due to genericity there is no tie in s^0, nor in the market which corresponds to the non-participation of B_i. Moreover, it is readily verified that the no-winner bid, $b = \overline{0}$, is not in equilibrium. Hence, by non-participating B_i gains $\alpha_{w'i}$.

Let us consider first the case where B_i is the winner in s^0, namely $B_w = B_i$. Consider the state where the grand cartel forms with B_i as the cartel bidder, namely $s^{GC^i} = (B, B_i, d^{GC^i})$, where $d_j^{GC^i} = \epsilon$, for all $j \neq i$, and $d_i^{GC^i} = -(n-1)\epsilon$. If all agents accept the offer, then the grand cartel forms, and B_i is a single bidder. Genericity yields $X(s^{GC^i}) = \pi_i + d_i^{GC^i} = \pi_i - (n-1)\epsilon > \epsilon$. By proposition A.8, B_i wins the good in s^{GC^i} for the price of $p = \epsilon$. Therefore, every agents $B_j \neq B_i$ derives a utility of $\alpha_{ij} + d_j^{GC^i} = \alpha_{ij} + \epsilon$. The latter is strictly greater than his utility if he refuses the offer, α_{ij}. As lemma C.7 clearly holds also for a market with externalities, it follows that this offer is accepted in every SPNE of the considered sub-game. Thus, B_i gains in every SPNE at least $\pi_i + d_i^{GC^i} - p = \pi_i - n\epsilon$, which is strictly greater than $\alpha_{w'i}$.

Consider now the case $B_w \neq B_i$. The analysis is similar. Consider the state $s^{GC^w} = (B, B_w, d^{GC^w})$, where $d_j^{GC^w} = \epsilon$, for all $j \neq i, w$, $d_w^{GC^w} = -p(s^0) + 2\epsilon$, and $d_i^{GC^i} = p(s^0) - n\epsilon$. If the offer is rejected, every agent $B_j \neq B_i, B_w$ gains α_{wj}, whereas B_w gets $\pi_w - p(s^0)$. If all agents accept the offer, then the grand cartel forms, and B_w is a single bidder. We consider 2 different cases according to B_w's bid in SPNE at the state s^{GC^w}.

If B_w bids 0 in SPNE at the state s^{GC^w}, then the good stays in the possession of the seller and all agents gain a 0 utility. According to proposition A.8, bidding 0 in SPNE yields $\epsilon \geq \pi_w + d_w^{GC^w} = \pi_w - p(s^0) + 2\epsilon$, otherwise, B_w would profitably win the good for a minimal price of $p = \epsilon$. It therefore holds that $0 > \pi_w - p(s^0)$, hence, from lemma C.7, B_w accepts the considered offer in every such SPNE of the game.

[16]All other agents but B_i participate in the auction as single bidders. Note that a market derived from a generic market by omitting a row and a column is also generic. It follows that the bidding vector $b = \overline{0}$ is not an equilibrium of the auction in the derived market. Moreover, a bidding vector which leads to a tie is not in equilibrium.

As all externalities are negative, for all $j \neq i, w$, $0 < \alpha_{wj}$, and therefore, all other agents accept the considered offer as well in every such SPNE of the game. We conclude that B_i can gain in such SPNE 0, which is strictly greater than $\alpha_{w'i}$.

Alternatively, being the only bidder in the auction, B_w bids ϵ in the state s^{GC^w}, and wins the good for the price of $p = \epsilon$. Hence, every agents $B_j \neq B_i, B_w$ derives a utility of $\alpha_{wj} + d_j^{GC^w} = \alpha_{wj} + \epsilon$, and B_w derives $\pi_w - p + d_w^{GC^w} = \pi_w - p(s^0) + \epsilon$. As all agents gain strictly more by accepting this offer than by rejecting it, it follows from lemma C.7, that this offer is accepted in every SPNE of the game, where B_w bids ϵ in s^{GC^w}. Thus, B_i gains in every such SPNE at least $\alpha_{wi} + d_i^{GC^w} = \alpha_{wi} + p(s^0) - n\epsilon$. By corollary A.2, it holds that $p(s^0) \geq \max_{j \neq w}(\pi_j - \alpha_{wj})$. Therefore, B_i's utility in every such SPNE is at least $\alpha_{wi} + \max_{j \neq w}(\pi_j - \alpha_{wj}) - n\epsilon \geq \alpha_{wi} + \pi_i - \alpha_{wi} - n\epsilon = \pi_i - n\epsilon$, which is strictly greater than $\alpha_{w'i}$.
∎

1.7 Model extension - Contingent transfers on a winner outside the cartel

In this section we consider a possible extension to the collusion game, which corresponds to the following motivation. Consider a market where the collusion designer is interested in the winning of a specific agent, a "preferred consumer", due to a high externality which this "preferred consumer" exerts on the collusion designer, for example. However, forming a cartel represented by this "preferred consumer", as the collusion game suggests, may not be feasible, as the "preferred consumer" may claim a high transfer. In such a market, the collusion designer might profit from the possibility to form a cartel with other agents who will also enjoy high externalities if this "preferred consumer" wins the auction. The designer may then extract some transfer payments from these agents, which are contingent on the winning of the "preferred consumer" who is not part of the formed cartel.

With respect to definition 1.2.2, a state is now extended to be the tuple (C, B_l, B_w, d). The interpretation is that, as before, the cartel C is represented in the auction by its member B_l, who is the only member who may make a positive bid in the auction. However, the transfer payments d are made among the members of the cartel C, if and only if B_w wins the auction. Restricting to $B_w = B_l$ yields the collusion game, discussed so far in this chapter.

When the extended collusion game reaches a state s, where $B_l \neq B_w$, then in the matrix of the updated externalities $X(s)$, it is the term $X(s)_{wl}$ which is updated, to take the value $X(s)_{wl} = \alpha_{wl} + d_l$ (See lemma 1.4.1).

Note, that in a market without externalities, as agents have no reason to prefer one consumer over the other, there is no motivation to form a cartel with contingent transfers on a winner outside the cartel. The analysis in this case is analogous to the one presented in the collusion game, and indeed full collusion arises in SPNE (See proposition 1.3.3). We shall not go further to formalize the extended collusion game, but demonstrate its motivation instead.

Example 1.7.1. Consider the following 4-player market with externalities: $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 1, \alpha_{12} = \alpha_{13} = \alpha_{14} = -1, \alpha_{21} = -5, \alpha_{34} = -4, \alpha_{41} = \alpha_{42} = \alpha_{43} = -6$, and all other externalities are null.

6	-1	-1	-1
0	1	0	0
5	0	1	-5
-1	-6	-6	1

Consider B_2 as the collusion designer. B_2 can gain more in the extended collusion game by forming a cartel with B_1, where the transfer are contingent on the winning of the agent they both prefer as a consumer, B_3. In status-quo, we consider a bid, as a result of which B_2 wins the good for the price of $p = 6$ (See corollary A.2). Hence, the utilities of the agents in negotiation status-quo are $u = (0, -5, 0, 0)$.

Following an analysis similar to the one presented in proposition 1.5.2 leads to the conclusion that in all SPNE points of the collusion game, B_2 cannot gain more than 3 (e.g., forming the grand cartel with the efficient agent, B_1, as the cartel bidder).

Consider now the following alternative proposal that B_2 may make in the extended collusion game. B_2 forms a cartel with B_1, where B_2 is the cartel bidder, and B_1 commits to pay B_2 a transfer of $d_2 = 5$ if B_3, their preferred consumer, wins the auction. Note that B_1 is ready to pay such a transfer due to "free-riding" if B_3 indeed wins. The externality which the latter exerts on B_1 is higher compared to the externality which is exerted on B_1 in the negotiation status-quo (i.e., $\alpha_{31} = 5 > 0 = \alpha_{21}$).

Such an agreement leads to a state s, corresponding to the following matrix of externalities $X(s)$:

	2	3	4
2	1	0	0
3	5	1	-5
4	-6	-6	1

where $X(s)_{32} = \alpha_{32} + d_2 = 0 + 5 = 5$, as B_2 gets a transfer payment from B_1 if indeed B_3 wins the auction and consumes the good. We consider a bid in this state, which results in the winning of B_3 for the price of $p = 6$ (See corollary A.2). Therefore, B_1 makes the transfer agreed upon to B_2, who ends up, after taking into account the corresponding externality which is exerted on him, with a utility of $\alpha_{32} + d_2 = 0 + 5 = 5$. This is more than he can gain in the collusion game as explained before. ∎

As a concluding remark for this section we wish to note that the model might be further on extended, to consider agreements where agents commit to different transfer payments for different possible consumers. As mentioned earlier in this section, since in the absence of externalities agents have no reason to prefer one consumer over the other, one concludes that in such an extended model the behavior of agents in a market without externalities is similar to their behavior in the collusion game. Namely, full collusion will always appear in equilibrium (See proposition 1.3.3). Moreover, as the collusion game is a restricted case of such an extension, and if the grand cartel forms there is only one possible consumer (the cartel bidder), which narrows down the extended model to the collusion game again, one concludes that small cartels may form in equilibrium in the presence of externalities in an extended model as well.

1.8 Conclusion

We considered a first price auction in which the winner exerts direct externalities on losing bidders. We further specified a negotiation protocol according to which agents may form a bidding ring prior to the auction, where all bidders but the cartel representative commit to place an irrelevant bid in the auction.

We showed that in the absence of externalities bidders will form the grand cartel, which in turn eliminates competition in the auction, and allows winning the good for a minimal price. In the presence of externalities inefficiencies may arise. The collusion designer may find it profitable to form a small cartel excluding some bidders while risking tougher competition in the auction. Furthermore, we showed that the formed cartel may be better off designating an inefficient representative if the efficient cartel member is a major threat to fringe bidders, as such a threat may lead to aggressive bids and a (too) high winning price.

Finally, a comparison was made between strategic non-participation (Jehiel and Moldovanu (1996)) and strategic collusion, finding that the collusion designer is strictly better off forming an appropriate bidding ring than not participating in the auction at all.

Appendix

A Appendix: First price auction equilibrium in the presence of externalities

The collusion game we study in this work has, generally speaking, two phases. It starts with a negotiation phase, in which agents are allowed to form a cartel, given some transfer payments among them. In the second phase, a first price auction takes place. The participating bidders are the cartel bidder and players outside the cartel. All members of the cartel other than its representative, are committed to bid 0. As only positive bids may win, a 0-bid in the auction is practically equivalent to non-participating.

Every SPNE point of the collusion game includes a bidding strategy for every possible state. Namely, a bidding strategy which corresponds to a market with externalities in which a certain cartel was formed with a certain representative, and a commitment to transfer payments (See definition 1.2.2). This bidding strategy is, in particular, a set of equilibrium points of the first price auction which takes place in the different states. The following discussion characterizes the pure equilibrium bids in a first price auction, which takes place in a market with externalities, not necessarily generic (Due to transfer payments, see example 1.3.2). The grand cartel may form, and in this case only one bidder actively participates in the auction. Hence, we will discuss also the case of a first price auction with a single bidder. We remind the reader that bids in the auction are discrete, and correspond to a given smallest money unit, denoted ϵ.

We start the analysis with the case of a single winner. In a first price auction, the winner bids in equilibrium just a bit above the second highest bid, formally, ϵ. Therefore, if the winner lowers his bid a bit, he is in tie with the second highest bidders. Thus, we give special attention to the number of second highest bids, in the equilibrium analysis.

We draw the reader's attention to the fact that the following necessary and sufficient condition is met in a generic market. Hence, in a generic market there is always an equilibrium of the first price auction. As corollary A.7 shows, this is not necessarily the case in a non-generic market.

Proposition A.1. *Let $n \geq 3$, and $m < n-1$. The following condition is necessary and sufficient for having an equilibrium bid $b = (b_1, b_2, ..., b_n)$ where B_i wins the auction as a single winner, and $B_{k_1}, B_{k_2}, \ldots, B_{k_m}$ are the second highest bidders. Namely, $b_i > b_{k_1} = b_{k_2} = ... = b_{k_m} > b_j$ for all $j \neq i, k_1, k_2, .., k_m$:*

$$\forall j \neq i \quad \pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq \max\{\pi_j - \alpha_{ij}, 2\epsilon\}$$[17]

Proof. Assume that such an equilibrium point exists. We will demonstrate that the condition holds. Note, that as b is an equilibrium bid, it holds that $b_{k_l} = b_i - \epsilon$, for all $1 \leq l \leq m$. In order for B_i to profit from winning the auction it must hold that $\frac{1}{m+1}(\pi_i - (b_i - \epsilon)) + \frac{1}{m+1}\sum_{l=1}^{m} \alpha_{k_l i} \leq \pi_i - b_i$, which is equivalent to $b_i \leq \pi_i - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} - \frac{1}{m}\epsilon$. In words, B_i is better off bidding b_i at least as bidding $b_{k_l} = b_i - \epsilon$. The condition we bring here regarding a possible tie with B_{k_l} is stronger than the one corresponding to a situation where B_i bids below b_{k_l}. In addition, it is necessary that any agent B_j, where $j \neq i$, cannot profit from bidding b_i which means that $\frac{1}{2}(\pi_j - b_i) + \frac{1}{2}\alpha_{ij} \leq \alpha_{ij}$, which is equivalent to $b_i \geq \pi_j - \alpha_{ij}$. Again, it is a stronger condition than saying that B_j cannot profit from over-bidding B_i's bid. Finally, let $j \neq i, k_1, k_2, ..., k_m$. As $b_i > b_{k_1} = b_{k_2} = ... = b_{k_m} > b_j \geq 0$ it follows that $b_i \geq 2\epsilon$. Combining the three conditions on b_i yields, $\forall j \neq i$, $\pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq \max\{\pi_j - \alpha_{ij}, 2\epsilon\}$, as required.

[17]If $n = 2$ or $m = n-1$, the necessary and sufficient condition is, $\forall j \neq i \quad \pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq \max\{\pi_j - \alpha_{ij}, \epsilon\}$.

Assume now that the condition holds for some $B_i, B_{k_1}, B_{k_2}...B_{k_m}$, all different. Then there exists p such that $\forall j \neq i$, $\pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq p \geq \max\{\pi_j - \alpha_{ij}, 2\epsilon\}$, and p is a valid bid. Consider the following bidding vector: $b_i = p$, $b_{k_1} = b_{k_2} = ... = b_{k_m} = p - \epsilon$, and for all $j \neq i, k_1, k_2, ..., k_m$, $b_j = p - 2\epsilon$. Then B_i wins the good as a single winner, pays p to the seller and gains $u_i = \pi_i - p$. Every other agent $B_j \neq B_i$ gains $u_j = \alpha_{ij}$. Clearly, B_i cannot gain more by raising his bid. If B_i lowers his bid to $p - \epsilon$, he wins with probability $\frac{1}{m+1}$ and gains $\frac{1}{m+1}(\pi_i - (p - \epsilon)) + \frac{1}{m+1}\sum_{l=1}^{m} \alpha_{k_l i}$. It is readily verified that this utility is at most $\pi_i - p$. If he lowers his bid further below $p - \epsilon$, he gains $\frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i}$ which is strictly less than $\pi_i - p$. For any $j \neq i$, it is clear that B_j cannot gain more by any bid lower than $b_i = p$. By bidding p, B_j gains $\frac{1}{2}(\pi_j - p) + \frac{1}{2}\alpha_{ij}$. It is readily verified that due to the condition this term is lower than $u_j = \alpha_{ij}$. The same holds for overbidding B_i. ∎

Corollary A.2. *Let $n \geq 3$, $m < n - 1$, and let $b = (b_1, b_2, ..., b_n)$ be a bidding vector where B_i makes the single highest bid and $B_{k_1}, B_{k_2}, \ldots, B_{k_m}$ make the second highest bid. Namely, $b_i > b_{k_1} = b_{k_2} = ... = b_{k_m} > b_j$ for all $j \neq i, k_1, k_2, .., k_m$. Then b is an equilibrium point of the first price auction if and only if:*

$$\forall j \neq i \quad \pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq b_i \geq \max\{\pi_j - \alpha_{ij}, 2\epsilon\}^{18}$$

$$\forall 1 \leq l \leq m \quad b_{k_l} = b_i - \epsilon$$

The following analysis, characterizes equilibrium bids in which there is a tie between several bidders. The case were all bidders are in tie is handled separately.

Proposition A.3. *Let $2 \leq m < n$. The following are necessary and sufficient conditions for having an equilibrium bid $b = (b_1, b_2, ..., b_n)$ where there are m winners, denoted $B_{i_1}, B_{i_2}, \ldots, B_{i_m}$:*

$$\forall 1 \leq k \neq l \leq m \quad |(\pi_{i_k} - \frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}) - (\pi_{i_l} - \frac{1}{m-1}\sum_{j=1, j\neq l}^{m} \alpha_{i_j i_l})| \leq \frac{m}{m-1}\epsilon$$

$$\forall 1 \leq k \leq m, \forall m < q \leq n \quad \max\{\epsilon, \pi_{i_q} - \frac{1}{m}\sum_{j=1}^{m} \alpha_{i_j i_q}\} \leq \pi_{i_k} - \frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}$$

Proof. Let b be a bid which leads to m winners, denoted $B_{i_1}, B_{i_2}, \ldots, B_{i_m}$, and denote p the winning bid in b. Namely, $p = b_{i_1} = b_{i_2} = ... = b_{i_m} > b_j$ for all $j \neq i_1, i_2, ..., i_m$. Note that as p is a winning bid it holds that $p \geq \epsilon$. Then the utilities of the agents are:

$$u_{i_k} = \begin{cases} \frac{1}{m}[(\pi_{i_k} - p) + \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}] & \text{if } 1 \leq k \leq m \\ \frac{1}{m}\sum_{j=1}^{m} \alpha_{i_j i_k} & \text{if } m < k \leq n \end{cases}$$

Assume that b is in equilibrium. We demonstrate that the conditions hold. Let $1 \leq k \leq m$. As b is in equilibrium, it holds that B_{i_k} cannot profit from neither overbidding nor underbidding p. Namely, $\pi_{i_k} - (p+\epsilon) \leq \frac{1}{m}[(\pi_{i_k} - p) + \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}]$ and $\frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k} \leq \frac{1}{m}[(\pi_{i_k} - p) + \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}]$ respectively. It follows that $\pi_{i_k} - \frac{m}{m-1}\epsilon - \frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k} \leq p \leq \pi_{i_k} - \frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}$, which yields the first condition

[18] If $n = 2$ or $m = n - 1$, the necessary and sufficient conditions are, $\forall j \neq i \quad \pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq b_i \geq \max\{\pi_j - \alpha_{ij}, \epsilon\}$, and $\forall 1 \leq l \leq m \quad b_{k_l} = b_i - \epsilon$.

in the statement. Moreover, for all $m < q \leq n$ it holds that B_{i_q} cannot profit from bidding p as well. Formally, $\frac{1}{m+1}[(\pi_{i_q} - p) + \sum_{j=1}^{m} \alpha_{i_j i_q}] \leq \frac{1}{m} \sum_{j=1}^{m} \alpha_{i_j i_q}$, which is equivalent to $p \geq \pi_{i_q} - \frac{1}{m} \sum_{j=1}^{m} \alpha_{i_j i_q}$. Together with the former conditions on p we get the second condition in the statement. Note, that the latter is stronger than demanding that B_{i_q} cannot profit from overbidding p.

We now demonstrate that the conditions are sufficient. It follows from the conditions that there exists $p \geq \epsilon$ such that,

$$\forall 1 \leq k \neq l \leq m \quad p \leq \pi_{i_k} - \frac{1}{m-1} \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k} \leq p + \frac{m}{m-1}\epsilon$$

$$\forall m < q \leq n \quad \pi_{i_q} - \frac{1}{m} \sum_{j=1}^{m} \alpha_{i_j i_q} \leq p$$

Consider the following bid, for all $1 \leq k \leq m$ $b_{i_k} = p$, and for all $m < q \leq n$ $b_{i_q} = p - \epsilon$. Following the same analysis as above, it is readily verified that b is in equilibrium. ∎

Corollary A.4. *Let $2 \leq m < n$, and let $b = (b_1, b_2, ..., b_n)$ be a bidding vector where $B_{i_1}, B_{i_2}, ..., B_{i_m}$ make the highest bid, denoted p. Namely, $p = b_{i_1} = b_{i_2} = ... = b_{i_m} > b_j$ for all $j \neq i_1, i_2, ..., i_m$. Then b is an equilibrium point of the first price auction if and only if:*

$$\forall 1 \leq k \leq m \quad \pi_{i_k} - \frac{m}{m-1}\epsilon - \frac{1}{m-1} \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k} \leq p \leq \pi_{i_k} - \frac{1}{m-1} \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}$$

$$\forall m < q \leq n \quad p \geq \max\{\epsilon, \pi_{i_q} - \frac{1}{m} \sum_{j=1}^{m} \alpha_{i_j i_q}\}$$

In the special case where all agents are winners, namely $m = n$, we get the following characterization in a similar way.

Proposition A.5. *The following are necessary and sufficient conditions for having an equilibrium bid $b = (b_1, b_2, ..., b_n)$ where all buyers are winners. Namely, $b_1 = b_2 = ... = b_n \geq \epsilon$:*

$$\forall 1 \leq k \neq l \leq n \quad |(\pi_k - \frac{1}{n-1} \sum_{j=1, j\neq k}^{n} \alpha_{jk}) - (\pi_l - \frac{1}{n-1} \sum_{j=1, j\neq l}^{n} \alpha_{jl})| \leq \frac{n}{n-1}\epsilon$$

$$\forall 1 \leq k \leq n \quad \pi_k - \frac{1}{n-1} \sum_{j=1, j\neq k}^{n} \alpha_{jk} \geq \epsilon$$

Corollary A.6. *Let $b = (b_1, b_2, ..., b_n)$ be a bidding vector where all buyers make the same bid, denoted p. Then b is an equilibrium of the first price auction if and only if there exists $p \geq \epsilon$ such that:*

$$\forall 1 \leq k \leq n \quad \pi_k - \frac{n}{n-1}\epsilon - \frac{1}{n-1} \sum_{j=1, j\neq k}^{n} \alpha_{jk} \leq p \leq \pi_k - \frac{1}{n-1} \sum_{j=1, j\neq k}^{n} \alpha_{jk}$$

As a conclusion of the characterization that we have presented so far in this appendix, we get the following corollary, which demonstrates a non-generic market with externalities, in which none of the sufficient conditions is satisfied, namely, there is no equilibrium bid in the first price auction held in this market. Nevertheless, in the analysis of the collusion game, we prove that although many states of the economy correspond to non-generic markets, in every state of the collusion game, there exists an equilibrium bid in pure strategies in the first price auction held in the market corresponding to the state in question.

Corollary A.7. *In the following non-generic market there is no bid which is in equilibrium in the first price auction for a small enough ϵ: $\pi_1 = 1, \pi_2 = 2, \pi_3 = 3, \alpha_{13} = -1, \alpha_{21} = -3, \alpha_{32} = -2$, and all other externalities are null.*

1	0	-1
-3	2	0
0	-2	3

In order to complete the analysis, and as we consider a negotiation process before the auction during which a cartel may form , we need to consider the special case in which the grand cartel forms in the market, which yields a single bidder in the auction.

Proposition A.8. *Consider a market with externalities. If a single agent, B_l, goes to the auction in $X(s) = X(s)_{ll}$*[19] *then:*

- *If $X(s)_{ll} > \epsilon$ then ϵ is the only equilibrium bid for B_l.*
- *If $X(s)_{ll} < \epsilon$ then 0 is the only equilibrium bid for B_l.*
- *If $X(s)_{ll} = \epsilon$ then 0 and ϵ are the only equilibrium bids for B_l.*

Proof. If $X(s)_{ll} > \epsilon$ then by bidding ϵ, B_l wins the auction consumes the good and gets $X(s)_{ll} - \epsilon$. Clearly, he cannot gain more by raising his bid. By lowering his bid to 0, the good stays in the possession of the seller, and B_l gets $0 < X(s)_{ll} - \epsilon$.

If $X(s)_{ll} < \epsilon$ then by bidding 0, the good stays in the possession of the seller, and B_l gets 0. By raising his bid to ϵ, he wins the auction and gains $X(s)_{ll} - \epsilon < 0$.

Finally, if $X(s)_{ll} = \epsilon$ by bidding ϵ he wins and gains $X(s)_{ll} - \epsilon$, which is the same as what he gains if he bids 0. Clearly, by raising the bid he cannot gain more. ∎

B Appendix: Weakly dominated strategies in first price auction

The following lemma characterizes the undominated bidding strategy in a first price auction which takes place in a generic market in the presence of externalities. The non-generic case follows a similar analysis.

Lemma B.1. *Consider a generic market with externalities. Denote $\beta_i = \pi_i - \min \alpha_{ji} - \epsilon$. Then any bid $b_i > \beta_i$ is weakly dominated by β_i, the bid 0 is dominated by the bid ϵ, and every positive bid $0 < b_i \leq \beta_i$ is undominated.*

Proof. Note that due to genericity $\beta_i > 0$. Let b_{-i} be a bid of all players but B_i, and let $b_i > \beta_i$, or equivalently, $b_i \geq \beta_i + \epsilon$. If b_i is not the highest bid in b, then player B_i achieves the same utility by bidding b_i as by bidding β_i. Assume that b_i is the highest bid. If B_i is a single winner in b then he gains a utility of $u_i = \pi_i - b_i \leq \pi_i - (\beta_i + \epsilon) = \min \alpha_{ji}$, which means that by lowering his bid to β_i he can only do better. If B_i is one among m winners, denoted $\{B_i, B_{j_1}, B_{j_2}, \ldots, B_{j_{m-1}}\}$, then his utility is

$$u_i = \frac{1}{m}(\pi_i - b_i) + \frac{1}{m}\sum_{k=1}^{m-1} \alpha_{j_k i} \leq \frac{1}{m}(\pi_i - (\beta_i + \epsilon)) + \frac{1}{m}\sum_{k=1}^{m-1} \alpha_{j_k i} =$$

$$\frac{1}{m}(\min \alpha_{ji}) + \frac{1}{m}\sum_{k=1}^{m-1} \alpha_{j_k i} \leq \frac{1}{m-1}\sum_{k=1}^{m-1} \alpha_{j_k i}$$

the latter being what B_i gains by lowering his bid to β_i.

Consider now a zero bid made by B_i. If all other agents make a zero bid as well, B_i gains 0. By bidding ϵ instead he gains $\pi_i - \epsilon$, which is positive due to genericity. If alternatively the highest bid is positive, then by bidding ϵ, B_i cannot gain less.

[19] See, lemma 1.4.1.

Let $0 < b_i \leq \beta_i$, and denote $k = \arg\min \alpha_{ji}$. Consider the following bid of all the players but B_i. B_k bids $b_k = b_i - \epsilon$, and every B_j such that $j \neq i, k$ bids 0. B_i is a single winner and he derives a utility of $\pi_i - b_i$. It is clear that by raising his bid he gains strictly less. If $b_i > \epsilon$, then by lowering his bid he gains strictly less as $\pi_i - b_i > \alpha_{ki}$, and $b_k = b_i - \epsilon > 0$. Alternatively, $b_i = \epsilon$, and his utility is $\pi_i - \epsilon$ which is positive and therefore strictly greater than the utility he gets if lowering his bid to 0. ∎

The following example demonstrates a market with externalities in which every equilibrium bid involves a weakly dominated strategy of at least one of the agents.

Example B.2. Consider a 5-player market with externalities where: $\pi_1 = 5, \pi_2 = 1, \pi_3 = 6, \pi_4 = 7, \pi_5 = 8, \alpha_{21} = -5, \alpha_{35} = -1, \alpha_{43} = -3, \alpha_{54} = -2$, and all other externalities are null.

5	0	0	0	0
-5	1	0	0	0
0	0	6	0	-1
0	0	-3	7	0
0	0	0	-2	8

For a small enough ϵ, the only equilibrium bid is where B_1 is a single winner, bidding at least 8, and B_2 makes the second highest bid (See appendix A). Hence, B_2 bids at least $8 - \epsilon$, which is a weakly dominated strategy for him. ∎

C Appendix: Proof of full collusion in the absence of externalities

Throughout this appendix, we will assume, thus WLOG, that in a generic market without externalities, $\pi_1 > \pi_2 > ... > \pi_n$. Note, that due to genericity it holds in particular that for all $1 \leq i < n$, $\pi_i - \pi_{i+1} > (n+2)\epsilon$, as well as for all $1 \leq i \leq n$, $\pi_i > (n+2)\epsilon$.

We learn from Lemma 1.3.1, that indeed for any state s, $X(s)$ is a market without externalities, however, it is not necessarily generic (e.g., consider $x = \pi_1 - \pi_3$ in example 1.3.2). Nevertheless, as at most one valuation has changed, the updated matrix is generic, except maybe for a single valuation. We give special attention to the question of genericity in the update matrix, as non-genericity may lead to potential ties in the auction which follows.[20]

Lemma C.1. *Consider a generic 0-externality market, and let $s = (C, B_l, d)$ be a state. If $dim(X(s)) \geq 2$, denote the 2 highest valuations in $X(s)$, $i_1 = \arg\max X(s)_{jj}$, and $i_2 = \arg\max_{j \neq i_1} X(s)_{jj}$. Then one of the followings holds:*

- *$X(s)$ is of dimension one.*
- *$dim(X(s)) \geq 2$, and $X(s)_{i_1 i_1} - X(s)_{i_2 i_2} > 2\epsilon$.*
- *$dim(X(s)) \geq 2$, $X(s)_{i_1 i_1} - X(s)_{i_2 i_2} \leq 2\epsilon$, and either $i_1 = l$ or $i_2 = l$.*

Proof. If the grand cartel forms in s, namely $C = B$, then by lemma 1.3.1 $X(s) = X(s)_{ll} = \pi_l + d_l$, and the first case in the statement holds. Assume then that $C \subsetneq B$. By lemma 1.3.1 for every $k \in B(s) \setminus \{B_l\}$, it holds that $X(s)_{kk} = \pi_k$. As the original market is generic, if $i_1, i_2 \neq l$ then the second case in the statement holds. Otherwise, either $i_1 = l$ or $i_2 = l$ and one of either the second or the third case in the statement holds. ∎

[20] In a non-generic market with externalities, there is not necessarily an equilibrium bid in a first price auction (see, corollary A.7 in appendix A). This is not the case in a non-generic market without externalities, where there is always an equilibrium bid in the auction.

In order to analyze the SPNE points of the collusion game in the absence of externalities, we now move on to discuss agents' equilibrium behavior in the auction given a state s. We distinguish between different scenarios, according to the different possible market types in which the auction takes place, as characterized by lemma C.1. The proofs follow from the analysis of equilibrium bids in a first price auction in a market with externalities which appears in appendix A, and are therefore omitted.

Lemma C.2. *Consider a generic zero externality market, and let $s = (B, B_l, d)$ be a state where the grand cartel forms. If $X(s)_{ll} > \epsilon$ then bidding ϵ is the only equilibrium strategy of the single bidder B_l in a first price auction in this market. If $X(s)_{ll} < \epsilon$ then bidding 0 is the only equilibrium strategy in a first price auction in this market. If $X(s)_{ll} = \epsilon$ then bidding either ϵ or 0 are the only equilibrium strategies in a first price auction in this market.*

Lemma C.3. *Consider a generic zero-externality market, and let s be a state. Assume that there are at least 2 potential buyers in s, namely $|B(s)| \geq 2$. Denote the 2 highest valuations in $X(s)$, $i_1 = \arg\max X(s)_{jj}$, and $i_2 = \arg\max_{j \neq i_1} X(s)_{jj}$. If $X(s)_{i_1 i_1} - X(s)_{i_2 i_2} > 2\epsilon$ then a bidding vector b in a first price auction held in this state, is in equilibrium if and only if B_{i_1} makes the single highest bid, namely, $b_{i_1} > b_j$ for all $j \neq i_1$. In addition, the price $p = b_{i_1}$ that the winner pays for the good, is in the interval $X(s)_{i_1 i_1} > p \geq \max\{\epsilon, X(s)_{i_2 i_2}\}$.*

Lemma C.4. *Consider a generic zero-externality market, and let s be a state. Assume that there are at least 2 potential buyers in s, namely $|B(s)| \geq 2$. Denote the 2 highest valuations in $X(s)$, $i_1 = \arg\max X(s)_{jj}$, and $i_2 = \arg\max_{j \neq i_1} X(s)_{jj}$. Assume that $X(s)_{i_1 i_1} - X(s)_{i_2 i_2} \leq 2\epsilon$ and let b be a bidding vector of a first price auction in this market.*

- *If $X(s)_{i_1 i_1} = X(s)_{i_2 i_2}$ then b is in equilibrium if and only if both B_{i_1} and B_{i_2} make the highest bid p. In addition, the price they pay for the good is in the interval $X(s)_{i_1 i_1} - 2\epsilon \leq p \leq X(s)_{i_1 i_1}$.*

- *If $X(s)_{i_1 i_1} > X(s)_{i_2 i_2}$ then b is in equilibrium if and only if either $B_{i_1 i_1}$ makes the single highest bid p where $X(s)_{i_2 i_2} \leq p < X(s)_{i_1 i_1}$, or both $B_{i_1 i_1}$ and $B_{i_2 i_2}$ make the highest bid p where $X(s)_{i_1 i_1} - 2\epsilon \leq p \leq X(s)_{i_2 i_2}$.*

As a corollary of lemma C.1, lemma C.2, lemma C.3, and lemma C.4 we conclude, that in every state there exists an equilibrium bid of the first price auction if held in this state. This is a step in order to establish the existence of an SPNE point of the collusion game without externalities. Note, however, that, as remarked before, one may conclude the existence of an equilibrium bid in every state of the collusion game without externalities, from the analysis in appendix A only. Nevertheless, lemma C.1, lemma C.2, lemma C.3, and lemma C.4 are necessary to the proof of proposition 1.3.3.

Corollary C.5. *Consider a generic zero-externality market. There exists a function, denoted $b(s)$, which maps every state s to an equilibrium bid of the first price auction in that state.*

The following corollary follows from lemma C.3 and lemma C.4, and will be useful in the proof of proposition 1.3.3, later in this appendix.

Corollary C.6. *Consider a generic zero-externality market, and let $s = (C, B_l, d)$, $C \subsetneq B$. If the representative of the cartel, B_l, wins the auction in s as a single winner or in a tie, then the price $p(s)$ that he pays to the seller satisfies $p(s) > n\epsilon$.*

Proof. Using the notations of lemma C.3 and lemma C.4, if B_l is a single winner, then $X(s)_{ll} > X(s)_{i_2 i_2} > n\epsilon$, where the last inequality holds due to genericity. Again from lemma C.3 and lemma C.4 we learn that $p(s) \geq X(s)_{i_2 i_2}$, which yields $p(s) > n\epsilon$.

If B_l wins in a tie, then from lemma C.4, $p \geq X(s)_{i_1 i_1} - 2\epsilon \geq \max\{X(s)_{i_1 i_1}, X(s)_{i_2 i_2}\} - 2\epsilon > (n+2)\epsilon - 2\epsilon = n\epsilon$. ∎

The next step in the proof of the existence of an SPNE of the collusion game without externalities, is to study which offers agents accept and reject in SPNE. As denoted in corollary C.5, $b(s)$ is an arbitrary mapping of states to equilibrium bids of the first price auction in these states. In addition, given $b(s)$, we denote $p(s)$ the corresponding price that the winner pays to the auctioneer, namely, $p(s) = \max b_i(s)$. The following lemma follows directly from the definition of SPNE.

Lemma C.7. *Consider a generic market without externalities, and let $b(s)$ be an arbitrary selection of equilibrium bid of the first price auction for every state s. Let $s(C, B_l, d) \neq s^0$ be a proposal made by B_i at the second stage of the game. Denote $u^r = u(s^0, b(s^0))$ the vector of utilities if the considered proposal is rejected, and $u^a(s) = u(s, b(s))$ the vector of utilities if it is accepted. Then for all $j \in C \setminus \{B_i\}$:*

- *If $u_j^a(s) > u_j^r$ then B_j accepts the offer in every SPNE of the game.*
- *If $u_j^a(s) < u_j^r$ then B_j rejects the offer in every SPNE of the game.*

proof of proposition 1.3.3. Let $b(s)$ be an equilibrium point function of the first price auction. We demonstrate that for every collusion designer who is selected in the first stage, there exists a proposal to form the grand cartel, which strictly maximizes his utility, and is accepted by all agents. It follows that the grand cartel forms with probability one in every SPNE of the game, as claimed.

If the collusion designer makes an offer that gets rejected, then the economy stays in state s^0. Namely, all agents go to the auction as single individual bidders. It follows from lemma C.3 that in s^0, B_1 wins the good for a price of $\pi_2 \leq p(s^0) < \pi_1$. Hence, with respect to the previous notations,

$$u_j^r = \begin{cases} \pi_1 - p(s^0) & \text{if } j = 1 \\ 0 & \text{if } j \geq 2 \end{cases}$$

We start with the case where B_1, the agent with the highest valuation, is the collusion designer in the beginning of the second stage. Consider the proposal to form the grand cartel represented in the auction by B_1, with ϵ transfer payments to all agents. Namely, B_1 proposes to move to the state $s^{GC^1} = (B, B_1, d^{GC^1})$, where $d_1^{GC^1} = -(n-1)\epsilon$, and for all $j \neq 1$, $d_j^{GC^1} = \epsilon$. It follows from genericity and lemma C.2 that in s^{GC^1} B_1 wins the good for a price of ϵ. As all transfer payments in s^{GC^1} are ϵ,

$$u_j^a(s^{GC^1}) = \begin{cases} \pi_1 - n\epsilon & \text{if } j = 1 \\ \epsilon & \text{if } j \geq 2 \end{cases}$$

Then from lemma C.7 B_1's proposal to move to the state s^{GC^1} is unanimously accepted, and B_1 gains $\pi_1 - n\epsilon$. We need to show that for any alternative proposal B_1 makes, namely to move to the state $s = (C, B_l, d)$ where $C \subsetneq B$, B_1 gains strictly less than $\pi_1 - n\epsilon$. If the alternative offer is rejected by at least one agent, then B_1 gains $u_1^r = \pi_1 - p(s^0) \leq \pi_1 - \pi_2 < \pi_1 - n\epsilon$ due to genericity. If the alternative offer is accepted by all agents, then from lemma C.7, for all $B_j \in C \setminus \{B_1\}$, $u_j^a(s) \geq u_j^r$. We consider 2 different cases.

First, consider the case where the proposed representative is B_1 himself, namely, $B_l = B_1$. Then for all $B_j \in C \setminus \{B_1\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. As transfers are balanced in C it follows that $d_1 \leq 0$. By corollary C.6, the price that B_1 pays in s if he wins the good, as a single or co-winner, satisfies $p(s) > n\epsilon$. Therefore, if B_1 wins the good as a single winner in s he gains $\pi_1 + d_1 - p(s) \leq \pi_1 - p(s) < \pi_1 - n\epsilon$. If B_1 wins the good with a second winner in s he gains $\frac{1}{2}(\pi_1 + d_1 - p(s))$ which is again strictly less than $\pi_1 - n\epsilon$. Finally, if B_1 loses in s he gets $0 < \pi_1 - n\epsilon$ due to genericity.

Consider now the case where $B_l \neq B_1$. As before, for all $B_j \in C \setminus \{B_1\}$, $u_j^a(s) \geq u_j^r$, since we assume that the offer is accepted. Therefore, for all $B_j \in C \setminus \{B_1, B_l\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. If B_l wins in s he gains $\pi_l + d_l - p(s)$, whereas in the case of a tie in s with another agent he gains $\frac{1}{2}(\pi_l + d_l - p(s))$. If he loses in s he gains 0. If B_l loses in s then B_1 gains $0 < \pi_1 - n\epsilon$. If B_l wins in s, then $0 = u_l^r \leq u_l^a(s) \leq \pi_l + d_l$, as $p(s) > 0$. Therefore, $-d_l \leq \pi_l$, which yields $d_1 \leq \pi_l$. Thus, if B_l is a single winner in s, B_1 gains $d_1 \leq \pi_l < \pi_1 - n\epsilon$ due to genericity. In a similar way, in case of a tie in s, B_1 gains strictly less than

$\pi_1 - n\epsilon$. We conclude that if B_1 is drawn in the first stage to be the collusion designer, he can achieve $\pi_1 - n\epsilon$ by forming the grand cartel, and can only gain strictly less by considering any alternative proposal to form a cartel which is not the grand one. Yet, however, as a function of the strategies of the others, if the offer to form the grand cartel with B_1 as its representative, is accepted by some $B_j \neq B_1$ for a 0-transfer payment, then B_1 would indeed offer them a 0-transfer payment, in order to maximize his utility. Clearly, by offering these lower transfer, B_1 can only gain more than $\pi_1 - n\epsilon$.[21]

We repeat the analysis for the case where at the first stage some $B_i \neq B_1$ is drawn to be the collusion designer. Consider the proposal to form the grand cartel represented in the auction by B_1, with a transfer of $-p(s^0) + 2\epsilon$ to B_1, and ϵ transfer payments to all other agents. Namely, $s^{GC^i} = (B, B_1, d^{GC^i})$ where,

$$d_j^{GC^i} = \begin{cases} \epsilon & \text{if } j \neq 1, i \\ -p(s^0) + 2\epsilon & \text{if } j = 1 \\ p(s^0) - n\epsilon & \text{if } j = i \end{cases}$$

Hence, $X(s^{GC^i})_{11} = \pi_1 - p(s^0) + 2\epsilon \geq \pi_1 - \pi_2 + 2\epsilon$. Therefore, genericity yields $X(s^{GC^i})_{11} > \epsilon$. By lemma C.2 we conclude that in s^{GC^i} B_1 wins the good for a price of ϵ. Therefore,

$$u_j^a(s^{GC^i}) = \begin{cases} d_j & \text{if } j \neq 1, i \\ \pi_1 - \epsilon + d_1 & \text{if } j = 1 \\ d_i & \text{if } j = i \end{cases} = \begin{cases} \epsilon & \text{if } j \neq 1, i \\ \pi_1 - p(s^0) + \epsilon & \text{if } j = 1 \\ p(s^0) - n\epsilon & \text{if } j = i \end{cases}$$

Therefore, B_i's proposal s^{GC^i} is unanimously accepted, and B_i gains $p(s^0) - n\epsilon$. We need to prove that for any alternative proposal $s = (C, B_l, d)$, $C \subsetneq B$, that B_i may make, he gains strictly less than $p(s^0) - n\epsilon$. If B_i proposes to stay in s^0, or if the alternative offer is rejected then B_i gains $u_i^r = 0 < p(s^0) - n\epsilon$. It suffices to assume then that the alternative offer is accepted. From lemma C.7, it holds in this case that for all $B_j \in C \setminus \{B_i\}$, $u_j^a(s) \geq u_j^r$. We consider 2 different cases.

First, consider the case where the proposed representative is B_1, namely, $B_l = B_1$. Then for all $B_j \in C \setminus \{B_1, B_i\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. Let us look at the 3 possible outcomes of the auction in s. If B_1 loses then B_i gains $0 < p(s^0) - n\epsilon$. If B_1 wins the auction as a single winner, then $u_1^a(s) = \pi_1 - p(s) + d_1$. As the offer is accepted, it holds that $\pi_1 - p(s^0) = u_1^r \leq u_1^a(s) = \pi_1 - p(s) + d_1$, and therefore, $d_1 \geq p(s) - p(s^0)$. As all other transfer payments are non-negative we conclude that $d_i \leq p(s^0) - p(s)$. So B_i gains in this case $d_i \leq p(s^0) - p(s) < p(s^0) - n\epsilon$, where the last inequality follows from corollary C.6. Finally, if B_1 wins the auction in s in a tie with another agent then $u_1^a(s) = \frac{1}{2}(\pi_1 - p(s) + d_1)$, which now yields $d_1 \geq p(s) - 2p(s^0) + \pi_1$. In such a case B_i gains $\frac{1}{2}d_i \leq -\frac{1}{2}d_1 \leq -\frac{1}{2}(p(s) - 2p(s^0) + \pi_1) = p(s^0) - \frac{1}{2}\pi_1 - \frac{1}{2}p(s) < p(s^o) - n\epsilon$, where the last inequality follows from genericity, and from corollary C.6.

Consider now the case where $B_l \neq B_1$. First, assume that $B_l = B_i$. It holds that for all $B_j \in C \setminus \{B_1, B_i\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. If B_i loses the auction in s then he gets $0 < p(s^0) - n\epsilon$. It is therefore sufficient to assume that, either B_i wins in s as a single winner, or B_i wins in a tie with another agent. If $B_1 \notin C$, then $1 = \arg\max_{j \neq l} X(s)_{jj}$. By lemma C.3 and lemma C.4, it must hold that $\pi_i + d_i = X(s)_{ii} \geq X(s)_{11} - 2\epsilon = \pi_1 - 2\epsilon$. It yields $d_i \geq (\pi_1 - \pi_i) - 2\epsilon > 0$. This stands in a contradiction to the fact that transfers to all agents but B_i are non-negative. It thus suffices to consider the case $B_1 \in C$. We distinguish between two possible outcomes in the auction in s according to the bid $b(s)$. If B_i wins in s as a single winner, then as $i \neq 1$ it holds that $u_1^a(s) = d_1$. Furthermore, as $u_1^r = \pi_1 - p(s^0)$, and the offer to move to state s is accepted in the considered case, we conclude that $d_1 \geq \pi_1 - p(s^0)$. We conclude that B_i gains $d_i \leq -d_1 \leq -(\pi_1 - p(s^0)) < p(s^0) - n\epsilon$. It therefore holds that B_i's utility in this case is strictly less than $p(s^0) - n\epsilon$. Finally, the analysis in the case where B_i wins in a tie in s is similar.

[21]If B_1 is the collusion designer then in the different equilibrium points of the game, he gains a utility of $\pi_1 - (k+1)\epsilon$, where $0 \leq k \leq n-1$ is the number of agents who reject forming the grand cartel with B_1 as its representative for a 0-transfer payment, in the equilibrium in question.

Finally, assume that $B_l \neq B_i$. As before, it holds that for all $B_j \in C \setminus \{B_1, B_i, B_l\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. If B_l loses the auction in s then B_i gets $0 < p(s^0) - n\epsilon$. It is therefore sufficient to assume that, either B_l wins in s as a single winner, or B_l wins in a tie with another agent. If $B_1 \notin C$, then $1 = \arg\max_{j \neq l} X(s)_{jj}$. By lemma C.3 and lemma C.4, it must hold that $\pi_l + d_l = X(s)_{ll} \geq X(s)_{11} - 2\epsilon = \pi_1 - 2\epsilon$. Hence, $d_l \geq (\pi_1 - \pi_l) - 2\epsilon > 0$. As transfer for all the other agents are non-negative, it follows that $d_i < 0$, which means a negative final utility to B_i in this case. It thus suffices to consider the case $B_1 \in C$. We distinguish between two possible outcomes in the auction in s according to the bid $b(s)$. If B_l wins in s as a single winner, then $u_l^a(s) = \pi_l - p(s) + d_l$. As $u_l^r = 0$, and the offer to move to state s is accepted in the considered case, we conclude that $d_l \geq p(s) - \pi_l$. As $l \neq 1$ it holds that $u_1^a(s) = d_1$. Furthermore, as $u_1^r = \pi_1 - p(s^0)$, and the offer to move to state s is accepted in the considered case, we conclude that $d_1 \geq \pi_1 - p(s^0)$. We conclude that B_i gains $d_i \leq -d_l - d_1 \leq -(p(s) - \pi_l) - (\pi_1 - p(s^0)) \leq p(s^0) + (\pi_l - \pi_1) < p(s^0) - n\epsilon$. It therefore holds that B_i's utility in this case is strictly less than $p(s^0) - n\epsilon$. Finally, the analysis in the case where B_l wins in a tie in s is similar.

Thus, as in the previous case, we conclude that if B_i, where $i \neq 1$, is drawn in the first stage to be the collusion designer, he can achieve $p(s^0) - n\epsilon$ by forming the grand cartel, and can only gain strictly less by considering any alternative proposal to form a cartel which is not the grand one. As we remarked in the previous case, as a function of the strategies of the others, if the offer to form the grand cartel with B_1 as its representative, is accepted by some $B_j \neq B_i, B_1$ for a 0-transfer payment, or by B_1 for a transfer of $-p(s^0) + \epsilon$, then B_i would indeed offer these agents a 0-transfer payment in order to maximize his utility. Offering these lower transfer payments can gain B_i only more than $p(s^0) - n\epsilon$ (Up to $p(s^0) - \epsilon$ in the SPNE point which is best to B_i as the collusion designer).

As for every state s the considered bidding strategy $b(s)$ is in equilibrium of a first price auction in this state, and as for every possible collusion designer, there exists an offer to form the grand cartel, which strictly maximizes his utility, and is accepted by all agents in SPNE, the set of SPNE points of the game is not empty, and in every SPNE point of the game the grand cartel forms with probability 1 as claimed. ∎

D Appendix: Existence of SPNE of the collusion game in the presence of externalities

As previously stated in this chapter, we restrict agents to pure strategies as we consider an auction game. Note however, that following this approach prevents us from using Nash's (1951) result regarding the existence of equilibrium point in strategic form games with complete information. Therefore, the existence of SPNE of the collusion game is to be proved explicitly.

In addition to proving the existence of SPNE points of the collusion game in the presence of externalities, we also discuss in the following appendix some features of such SPNE points, which we use in the proof of proposition 1.4.3 to demonstrate partial collusion in the presence of externalities.

Lemma D.1. *Consider a generic market with externalities and let $s = (C, B_l, d)$ be a state, such that $|B(s)| \geq 2$. There exists an equilibrium bid of a first price auction in $X(s)$ in which the good does not stay in the possession of the seller.*

Proof. Denote $(i_1, k_1) = \arg_{(j,m)} \max_{m \neq j} (\pi_j - \alpha_{mj})$. Such a pair is unique due to genericity. If there exists a pair of indices $g \neq h$ such that

$$X(s)_{gg} - X(s)_{hg} > X(s)_{jj} - X(s)_{gj} \quad \forall j \neq g \tag{D1}$$

then by proposition A.1 there is an equilibrium bid in the state s. Otherwise, consider two different cases. Assume first that $l \neq i_1$. By lemma 1.4.1 we learn that the only difference between $X(s)$ and the original matrix $X(s^0)$ may be the term $X(s)_{ll}$. Together with genericity, we conclude that $X(s)_{ll} - X(s)_{i_1 l} \geq X(s)_{i_1 i_1} - X(s)_{k_1 i_1}$, otherwise the pair (i_1, k_1) would be satisfying equation D1. If $l \neq k_1$, then by the definition of (i_1, k_1) it holds that for all $j \neq l$, $X(s)_{i_1i_1} - X(s)_{k_1i_1} = \pi_{i_1} - \alpha_{k_1i_1} > \pi_j - \alpha_{lj} = X(s)_{jj} - X(s)_{lj}$. It follows that for all $j \neq l$, $X(s)_{ll} - X(s)_{i_1l} > X(s)_{jj} - X(s)_{lj}$. The pair (l, i_1) maintains equation D1, in contradiction to the case assumption. Hence $l = k_1$. If $X(s)_{ll} - X(s)_{i_1l} > X(s)_{i_1i_1} - X(s)_{k_1i_1}$, then again as for all $j \neq l$ it holds that $X(s)_{i_1i_1} - X(s)_{k_1i_1} \geq X(s)_{jj} - X(s)_{lj}$, we get a contradiction to the case assumption with the pair (l, i_1). We conclude therefore that $X(s)_{ll} - X(s)_{i_1l} = X(s)_{i_1i_1} - X(s)_{k_1i_1}$, and as $l = k_1$, we get $X(s)_{ll} - X(s)_{i_1l} = X(s)_{i_1i_1} - X(s)_{li_1}$. In addition, by the definition of (i_1, k_1), for all $q \neq i_1, k_1$ it holds that $X(s)_{i_1i_1} - X(s)_{k_1i_1} \geq X(s)_{qq} - \frac{1}{2}(X(s)_{i_1q} + X(s)_{k_1q})$. Therefore, according to proposition A.3, there is an equilibrium bid in s, where B_{i_1} and B_l win the good in tie.

The case $l = i_1$ is handled in a similar way. As we assume that equation D1 does not hold, and as $|B(s)| \geq 2$, there exists $i_3 \neq i_1$ such that $X(s)_{i_3i_3} - X(s)_{i_1i_3} \geq X(s)_{i_1i_1} - X(s)_{k_1i_1}$. Denote $(i_2, k_2) = \arg_{(j,m)} \max_{m \neq j, j \neq i_1} (X(s)_{jj} - X(s)_{mj})$, such a pair is unique due to genericity, lemma 1.4.1, and the case assumption $l = i_1$. It follows that for every $j \neq i_1, i_2$, $X(s)_{i_2i_2} - X(s)_{k_2i_2} > X(s)_{jj} - X(s)_{i_2j}$. Therefore, on one hand it holds that $X(s)_{i_2i_2} - X(s)_{k_2i_2} \leq X(s)_{i_1i_1} - X(s)_{i_2i_1} \leq X(s)_{i_1i_1} - X(s)_{k_1i_1}$, where the first inequality is a result of the assumption that equation D1 does not hold, and the second inequality follows from the definition of (i_1, k_1). On the other hand it holds that, $X(s)_{i_2i_2} - X(s)_{k_2i_2} \geq X(s)_{i_3i_3} - X(s)_{i_1i_3} \geq X(s)_{i_1i_1} - X(s)_{k_1i_1}$, where the first inequality follows from the definition of (i_2, k_2), and the second from the definition of i_3. Thus, we get an equality. Therefore, by genericity, lemma 1.4.1, and as $i_1 \neq i_2, i_3$ we conclude that $k_1 = i_2 = i_3$, and $k_2 = i_1$. It holds therefore, that $X(s)_{i_3i_3} - X(s)_{i_1i_3} = X(s)_{i_1i_1} - X(s)_{i_3i_1}$. Moreover, as in the previous case, by the definition of (i_2, k_2), for all $q \neq i_1, i_2$ it holds that $X(s)_{i_2i_2} - X(s)_{k_2i_2} \geq X(s)_{qq} - \frac{1}{2}(X(s)_{i_1q} + X(s)_{i_2q})$. According to proposition A.3 there is an equilibrium bid in s, where B_{i_1} and B_{i_2} win the good in tie. ∎

Proposition D.2. *The set of SPNE points of the collusion game in the presence of externalities is not empty.*

Proof. Let $b(s)$ be a selection of equilibrium bids of the first price auction, where the good does not stay in the possession of the seller, for every state s, such that $|B(s)| \geq 2$. According to lemma D.1, and lemma A.8, such a selection exists. For every proposal $s = (C, B_l, d)$ made by B_i, let $B_j \in C \setminus \{B_i\}$ accept the offer if and only if $u_j(s, b(s)) \geq u_j(s^0, b(s^0))$. It suffices to demonstrate that for every agent B_i who is selected in the first stage to be the collusion designer, there exists an offer which maximizes his utility. Let B_i be the selected designer. If he makes an offer that gets rejected, or by proposing s^0, he gains $u_i(s^0, b(s^0))$. If he offers to move to a state $s = (C, B_l, d)$, the offer is accepted and B_l does not win the auction, he gains a value in the set $\{\alpha_{ki}\}_{k \neq i} \cup \{0\}$. Both scenarios yield a finite set of potential utilities for B_i.

Therefore, it suffices to demonstrate that the set of offers which may be profitable for B_i (i.e., can gain him more than his worst externality, $\min\{0, \min_{k \neq i} \alpha_{ki}\}$), may be accepted by the addressed agents, and where B_l indeed wins the auction in the state s, is finite. If so, then there exists an offer in this set which maximizes B_i's utility under these assumptions, and therefore there exists an offer which maximizes his utility in general.

Due to the fact that transfer payments are ϵ discrete and balanced, it suffices to demonstrate that for every agent B_j there exists a threshold $\underline{d_j}$, such that $B_j \in C \setminus \{B_i\}$ rejects any offer where $d_j < \underline{d_j}$, and B_i does not propose an offer where $d_i < \underline{d_i}$. We claim that if B_l wins as a single winner then the required threshold is given by $\underline{d_j} = \min_{k \neq j} \alpha_{kj} - \pi_j$. The analysis for the case where B_l wins in a tie is similar. Indeed, if $B_j \in C \setminus \{B_i\}$ accepts an offer to move to a state s where B_l wins as a single winner, he gains at most $\pi_j + d_j$. If $d_j < \underline{d_j} = \min_{k \neq j} \alpha_{kj} - \pi_j$, then he gains strictly less than $\pi_j + \underline{d_j} = \pi_j + \min_{k \neq j} \alpha_{kj} - \pi_j = \min_{k \neq j} \alpha_{kj}$. He can profitably deviate by refusing the offer, and gain at least $\min_{k \neq j} \alpha_{kj}$, as in s^0 the good does not stay in the possession of the seller. In the same manner, if B_i offers to move to a state s where B_l wins as a single winner, he gains at most $\pi_i + d_i$. Following the same calculation, if $d_i < \underline{d_i}$ he gains strictly less than $\min_{k \neq i} \alpha_{ki}$, where by deviating and proposing s^0 he gains at least $\min_{k \neq i} \alpha_{ki}$. ∎

Corollary D.3. *Consider a generic market with externalities, and let $b(s)$ be a function which maps every state to an equilibrium bid of the first price auction in this state. There exists an SPNE of the collusion game in which agents bid in the auction in every state s according to $b(s)$, and for every proposal $s = (C, B_l, d)$ made by B_i, every $B_j \in C \setminus \{B_i\}$ accepts the offer if and only if $u_j(s, b(s)) \geq u_j(s^0, b(s^0))$.*

E Appendix: Proof of non-efficiency in the presence of externalities

Proof of proposition 1.5.2. Consider the following 5-player market with externalities: $\pi_1 = \pi_2 = \pi_5 = 1, \pi_3 = 6, \pi_4 = 2, \alpha_{14} = \alpha_{15} = \alpha_{51} = \alpha_{52} = \alpha_{53} = \alpha_{54} = -1, \alpha_{21} = -4, \alpha_{24} = 2, \alpha_{25} = 1, \alpha_{34} = -10, \alpha_{42} = -20$, and all other externalities are null. Let B_1 be the collusion designer in the beginning of the second stage.

1	0	0	-1	-1
-4	1	0	2	1
0	0	6	-10	0
0	-20	0	2	0
-1	-1	-1	-1	1

This market is clearly non-generic, but we can change the valuations and externalities a little to get a generic market in which the same analysis holds (see also footnote 13). σ is a probability vector, such that B_1 is the collusion designer with a positive probability, $\sigma_1 > 0$. We consider the following strategies of the agents:

- In the state s^0, agents bid $b(s^0) = (5 - \epsilon, 6, 6 - 2\epsilon, 6 - \epsilon, 2 - \epsilon)$. That is an equilibrium bid of the first price auction in the state s^0, according to corollary A.2.
- In the state $s^1 = (\{B_1, B_2, B_3\}, B_1, d^1)$, where $d_1^1 = 5, d_2^1 = -5, d_3^1 = 0$

	1	4	5
1	6	-1	-1
4	0	2	0
5	-1	-1	1

  we consider the bid $b(s^1) = (3, 0, 0, 3 - \epsilon, 2 - \epsilon)$, which is in equilibrium by corollary A.2.
- In every state $s \neq s^0, s^1$, such that $|B(s)| \geq 2$ and there exists a unique pair of indices (i, k) such that
$$(i, k) = \arg_{(j,m)} \max_{m \neq j \in B(s)} (X(s)_{jj} - X(s)_{mj})$$
  we consider an equilibrium bid $b(s)$, where B_i wins the auction, and pays $p(s) = \max_{j \neq i \in B(s)} (X(s)_{jj} - X(s)_{ij})$. Such an equilibrium bid exists according to corollary A.2.
- For every other state s, consider some equilibrium bid $b(s)$, which exists by lemma D.1, and lemma A.8.
- Finally, with respect to the above described function $b(s)$, for every proposal $s = (C, B_l, d)$ made by B_i, every $B_j \in C \setminus \{B_i\}$ accepts the offer if and only if $u_j(s, b(s)) \geq u_j(s^0, b(s^0))$.

According to corollary D.3, there exists an SPNE of the game which respects these strategies. We will demonstrate that the proposal to move to state s^1 maximizes B_1's utility as the collusion designer with respect to the considered strategy profile. As B_3 is the efficient member of the cartel $\{B_1, B_2, B_3\}$ which forms in the state s^1, and B_1 is the cartel bidder in this state, the argument follows.

Consider first the initial state s^0. As stated above we consider the bid $b(s^0) = (5-\epsilon, 6, 6-2\epsilon, 6-\epsilon, 2-\epsilon)$. As a result of this bid, B_2 wins the good, pays $p = 6$ to the seller, and consumes the good. The utility vector of the agents is therefore $u(s^0, b(s^0)) = (\alpha_{21}, \pi_2 - p, \alpha_{23}, \alpha_{24}, \alpha_{25}) = (-4, -5, 0, 2, 1)$.

Consider now the proposal to move the economy to the state s^1, where $s^1 = (\{B_1, B_2, B_3\}, B_1, d^1)$, and $d_1^1 = 5, d_2^1 = -5, d_3^1 = 0$. If B_2 and B_3 accept the offer then the economy moves to the state s^1, where $X(s^1)$ is given by:

	1	4	5
1	6	-1	-1
4	0	2	0
5	-1	-1	1

where, $X(s^1)_{11} = \pi_1 + d_1^1 = 1+5 = 6$. We consider the following equilibrium bid $b(s^1) = (3, 0, 0, 3-\epsilon, 2-\epsilon)$. As a result of which B_1 wins the auction as a single winner, pays $p = 3$ to the seller, and consumes the good. Therefore, B_2 gains $\alpha_{12} + d_2^1 = -5$, and B_3 gains $\alpha_{13} + d_3^1 = 0$, which is equal to what they gain by rejecting the offer. Hence, B_2 and B_3 accept the offer in the considered strategy profile. As a result, by proposing to move to s^1, B_1 gains $X(s^1)_{11} - p = 6 - 3 = 3$. It is clear, that by proposing higher transfers to B_2 or B_3, B_1 gains less. Moreover, lowering any of the discussed transfer payments will yield a rejection. The conclusion is that by forming a cartel with B_2 and B_3, which is represent by B_1, the latter can gain at most 3. It suffices then to demonstrate that every other proposal will gain him strictly less than 3.

Note first that in s^0 he gains -4, which is indeed strictly less than 3. Therefore, we should consider only offers which may be accepted in the discussed strategy profile. Moreover, note that for all $j \neq 1$, $\alpha_{j1} < 3$. It yields, that we should not consider offers where B_1 proposes to move to a state $s = (C, B_l, d)$, if B_l does not win the auction in s according to the considered $b(s)$.

We continue by discussing 5 different case, according to the possible identity of the designated cartel bidder, B_l. We start with the case where $B_l = B_1$. In order to have $B_j \neq B_1$ to join a cartel C with respect to the considered strategy profile, it must hold that by accepting the offer, B_j gains at least as much as he does by rejecting it. As we consider only offers where $B_l = B_1$ wins if the offer is accepted, B_j accepts if and only if $\alpha_{1j} + d_j \geq u_j(s^0, b(s^0))$, where d_j is the proposed transfer. Equivalently, B_j accepts the offer to join a cartel of which B_1 is the representative, if and only if $d_j \geq u_j(s^0, b(s^0)) - \alpha_{1j}$. More specifically, B_2 will accept such an offer if and only if $d_2 \geq -5 - 0 = -5$, B_3 if and only if $d_3 \geq 0 - 0 = 0$, B_4 if and only if $d_4 \geq 2 - (-1) = 3$, and finally B_5 if and only if $d_5 \geq 1 - (-1) = 2$. Since transfers inside the cartel are balanced, $\pi_1 = 1$, and the price paid in the auction in order to win the good is positive, it follows that by forming a cartel C with himself as the cartel bidder, B_1 can gain at most $\pi_1 - p(C, B_1, d) - \sum\limits_{j \in C, j \neq 1} d_j < 1 - \sum\limits_{j \in C, j \neq 1} d_j$. In order to gain at least 3, B_1 should therefore consider one of the following cartels only: $\{B_1, B_2\}$,$\{B_1, B_2, B_4\}$,$\{B_1, B_2, B_3, B_4\}$,$\{B_1, B_2, B_5\}$,$\{B_1, B_2, B_3, B_5\}$. Consider first a proposal to form the cartel $\{B_1, B_2\}$. It corresponds to the updated matrix of externalities $X(s)$:

	1	3	4	5
1	$1 + d_1$	0	-1	-1
3	0	6	-10	0
4	0	0	2	0
5	-1	-1	-1	1

As calculated above, $d_1 = -d_2 \leq 5$, and therefore, with respect to the considered strategy profile, B_4 wins in this state and not B_1. Hence, such a proposal cannot yield B_1 a utility greater than 3. The same analysis can be repeated for proposals to form the cartels $\{B_1, B_2, B_4\}$ and $\{B_1, B_2, B_5\}$ with B_1 as the representative.

Consider now a proposal to form the cartel $\{B_1, B_2, B_3, B_4\}$ with B_1 as its representative. It corresponds to the updated matrix of externalities $X(s)$:

	1	5
1	$1+d_1$	-1
5	-1	1

As calculate above, $d_1 = -d_2 - d_3 - d_4 \leq 5 - 0 - 3 = 2$. If $d_1 > 0$, B_1 wins in this state with respect to the considered strategy profile, and pays $p(s) = 2$. He therefore gains $X(s)_{11} - p(s) = 1 + d_1 - 2 \leq 1 + 2 - 2 = 1$, which is strictly less than 3. If $d_1 = 0$ then there is a tie in this state with respect to the considered strategy profile. By corollary A.6, the price is at least $2 - 2\epsilon$, hence, B_1 gains strictly less than 3. Eventually if $d_1 < 0$, B_2 wins. The same analysis can be repeated for a proposal to form the cartel $\{B_1, B_2, B_3, B_5\}$ with B_1 as the cartel bidder.

Consider now the case where the designated cartel bidder is $B_l = B_2$. As calculated in the previous case, in order to have an agent to join a cartel, B_1 should offer him a transfer payment which will guarantee him a utility at least as high as the utility that he will get if he declines the offer with respect to the considered strategy profile. As B_2 wins the auction and consumes the good in the state s^0, it is clear that no money can be extracted from B_3, B_4 and B_5 in order to form a cartel represented by B_2. As B_2 gains -5 in s^0, and his valuation is $\pi_2 = 1$, B_1 would not be able to extract more than 6 from B_2. Hence, whatever cartel B_1 forms with B_2 as the cartel bidder, if B_2 indeed wins the auction in the new state, B_1 gains $\alpha_{21} + d_1$, which is at most $-4 + 6 = 2$.

Consider the case $B_l = B_3$. Following the same analysis, B_1 cannot extract more than 6 from B_3, and can extract at most 5 from B_2. On the other hand, B_4 will demand at least 12 in order to participate, and B_5 will demand at least 1. It follows that it suffices to consider the following cartels: $\{B_1, B_3\}$,$\{B_1, B_2, B_3\}$,$\{B_1, B_3, B_5\}$,$\{B_1, B_2, B_3, B_5\}$. Consider a proposal to form the cartel $\{B_1, B_3\}$. It corresponds to the updated matrix of externalities $X(s)$:

	2	3	4	5
2	1	0	2	1
3	0	$6+d_3$	-10	0
4	-20	0	2	0
5	-1	-1	-1	1

Note that in order to gain at least 3, as $\alpha_{31} = 0$, it follows that $d_1 \geq 3$, hence, $d_3 = -d_1 \leq -3$. Therefore, with respect to the considered strategy profile, it is B_2 who wins the auction, and not B_3. Hence, such a proposal cannot yield B_1 a utility greater than 3. The same analysis can be repeated for proposals to form the other relevant cartels.

Consider the case $B_l = B_4$. B_1 cannot extract any money from B_4 as his valuation is equal to the externality he gains in s^0. He will need to compensate B_2 for his participation by at least 15, and would not be able to extract any money from B_3 and B_5. As $\alpha_{41} = 0$, whatever cartel B_1 forms with B_4 as its representative, he would not be able to gain a positive utility. The analysis in the last possible case $B_l = B_5$ is similar.

The conclusion is that the proposal to move the economy to the state s^1 is maximizing B_1's utility with respect to the considered strategy profile. Therefore, there exists an SPNE in this market, in which with a positive probability B_1 is the collusion designer, he offers to form a cartel with a representative who is not its efficient member, and this offer is accepted. ∎

Chapter 2

Core-stable rings in auctions with independent private values[1]

Abstract

We propose a semi-cooperative game theoretic approach to check whether a given coalition is stable in a Bayesian game with independent private values. The ex ante expected utilities of coalitions, at an incentive compatible (noncooperative) coalitional equilibrium, describe a (cooperative) partition form game. A coalition is core-stable if the core of a suitable characteristic function, derived from the partition form game, is not empty. As an application, we study collusion in auctions in which the bidders' final utility possibly depends on the winner's identity. We show that such direct externalities offer a possible explanation for cartels' structures (not) observed in practice.

2.1 Introduction

Collusion in auctions is mostly studied as a mechanism design problem for a given ring (see, e.g., Graham and Marshall (1987), Mailath and Zemsky (1991) and McAfee and McMillan (1992) for early references and Marshall and Marx (2007) for a recent one). This framework imposes individual participation constraints to every member of the ring. In second price auctions with independent private values, Mailath and Zemsky (1991) further consider participation constraints for all subrings of any potential ring. In this particular framework, equilibria in weakly dominant strategies considerably limit the strategic externalities that coalitions might incur. Mailath and Zemsky (1991)'s analysis does not apply if equilibria in weakly dominant strategies do not exist, e.g., in the case of common values (see Barbar and Forges (2007)). In this chapter, we keep the assumption of independent private values but except for that, allow for an arbitrary auction game. We ask whether a given coalition is stable, in the sense that no subgroup of players would like to leave it. Such collective participation constraints are traditionally captured by core-like solution concepts. However, two difficulties arise when trying to define the core of an arbitrary auction game, or, more generally, a Bayesian game.

A first difficulty, which already appears under complete information, is that every coalition faces strategic externalities, so that it must make conjectures on the behavior of the players who are outside the coalition. To solve this difficulty, Aumann (1961) introduced the $\alpha-$ characteristic function, which measures the worth of a coalition in a strategic form game as the amount that it can guarantee whatever the complementary coalition does. However, the corresponding core, namely the $\alpha-$core, can be criticized on the grounds that it involves incredible threats from the complementary coalition. As a remedy, Ray (2007) and Ray and Vohra (1997) construct a partition form game (as defined by Lucas and Thrall (1963)) in which, given a partition of the players, coalitions evaluate their worth at a Nash equilibrium of an auxiliary game between the coalitions. We extend Ray (2007)'s coalitional equilibrium to games

[1] Co-written with Françoise Forges.

with incomplete information and construct a partition form game from the noncooperative Bayesian game which models the auction. We then apply a notion of core for partition form games, the core with "cautious expectations" (see Hafalir (2007)). Under complete information, it is included in the $\alpha-$core.

In a coalitional equilibrium of a Bayesian game, it is understood that the strategy of every coalition is a function of its members' private information. The description of the previous paragraph hides a second difficulty, which is specific to incomplete information: every coalition faces incentive constraints. We establish (in proposition 2.2.1 and its corollary) that, in a class of Bayesian games which includes standard auctions (namely, games with independent private values and quasi-linear utilities), this difficulty can be ignored: every coalitional equilibrium can be made incentive compatible. More precisely, coalitional equilibria are "first best" solutions, in which every coalition plays a best reply to the strategies outside the coalition, as if information sharing was not an issue. We construct an incentive compatible revelation mechanism for the coalition, which involves exactly balanced monetary transfers among its members and achieves the "first best" reply of the coalition. The fact that, in a coalitional equilibrium, every coalition maximizes its payoff given the strategies of the other coalitions in the underlying partition is crucial to our construction, in particular, in the formulation of incentive constraints.

In order to associate a partition form (cooperative) game to a (noncooperative) Bayesian game, we assume that coalitions can commit to an incentive compatible mechanism at the ex ante stage, i.e., before their members get their private information. This assumption first requires that an ex ante stage can be identified, which is true in many economic applications, like auctions, in which private information reduces to the value of some parameter, like a valuation or a cost. According to empirical data (see, e.g., Pesendorfer (2000), Porter and Zona (1993)), bidding rings often consist of well-identified groups (e.g., "incumbents", as opposed to "newcomers") whose characteristics do not depend on particular information states. Such bidding rings typically form at an early stage. For instance, local suppliers may be aware that a procurement auction will take place and consider to collude before the precise project specifications are published. At the time they commit to a collusion mechanism, they do not figure out their exact valuations, i.e., the costs incurred by the project. Art auctions are another example: the objects to be sold are often available for examination only a few days before the auction.

The ex ante formation of rings is assumed explicitly in Bajari (2001), Marshall et al. (1994) and Waehrer (1999). In these papers, ring mechanisms are investigated within an a priori given partition of the bidders. The partition itself does not depend on the bidders' private information, which reflects the ex ante formation of the rings. To the best of our knowledge, interim formation of rings has only been investigated to a limited extent, e.g., in Caillaud and Jehiel (1998), Graham and Marshall (1987), Marshall and Marx (2007) , and McAfee and McMillan (1992). These papers focus on an ex ante given bidding ring, the grand coalition for instance, and formulate interim participation constraints for the individual members of the ring. More precisely, every member of the ring can decide to leave the ring once he knows his private information. The precise form of the participation constraints depends on the reaction of ring members when one of them leaves the ring. Mailath and Zemsky (1991) start with the latter model but restrict themselves on ex ante expected payoffs when studying the stability of rings. Being interested in the participation constraints of coalitions, rather than individuals, we assume, to keep the analysis tractable, that coalitions can commit to an incentive compatible mechanism ex ante, as in Forges and Minelli (2001) and Forges et al. (2002). These papers introduce the notion of ex ante incentive compatible core in exchange economies with differential information. In this setup, there are no externalities, i.e., only the second difficulty above arises. The basic solution concept in this chapter is an ex ante incentive compatible core for Bayesian games.[2]

Coming back to auctions, we construct a partition form game, which reflects the ex ante commitments of bidding rings. A coalition is core-stable if all its subcoalitions agree to participate in its collusion mechanism. In this definition, we focus on a single ring and assume, as in Marshall and Marx (2007),

[2]Forges et al. (2002) discuss interim collective participation constraints in the absence of externalities. The latter assumption takes the form of "orthogonal coalitions" in Myerson (1984)'s study of interim binding agreements in Bayesian games and in Myerson (2007)'s definition of an interim incentive compatible core. A. Kalai and E. Kalai (Unpublished results) propose a cooperative-competitive solution to two-person Bayesian games. They consider interim participation but with only two players, incentives only matter for the grand coalition, which does not face externalities.

that the bidders outside the ring do not collude. We first apply the solution concept to standard auctions, without direct externalities. In the case of second price auctions, possibly with asymmetric players, our partition form game reduces to a characteristic function and we prove in proposition 2.3.1 that all rings are core-stable. In particular, strategic externalities have limited effects on collusion. Mailath and Zemsky (1991) already obtain this result. They directly focus on the equilibrium in weakly dominant strategies so that they can deal with every coalition separately, without taking account of possible externalities. They thus face a single mechanism design problem for every coalition and derive a characteristic function.

In first price auctions, we derive a genuine partition form game. As is well-known, asymmetric bidders are difficult to handle in this case (see Krishna (2002)). Several papers, e.g., Bajari (2001), Lebrun (1991, 1999), Marshall et al. (1994), and Waehrer (1999), introduce bidding rings in first price auctions as prototypes of asymmetric bidders. However, in these contributions, rings operate as single entities, which automatically share their information, without relying on any (incentive compatible) mechanism. It follows from our proposition 2.2.1 that this simplifying assumption is fully justified if bidding rings can make inside transfers. Thanks to results of Lebrun (1999) and Waehrer (1999), we establish that the grand coalition is always core-stable in a first price auction (proposition 2.3.2). In the absence of general, analytical solutions for first price auctions with asymmetric bidders, we only check that all coalitions are core-stable in two specific examples, borrowed from Marshall et al. (1994) and McAfee and McMillan (1992).

We finally consider the effect of direct externalities on collusion. We first assume, as in Jehiel and Moldovanu (1996)'s model, that a bidder suffers more if a competitor wins the auction than if the object is not sold at all ("negative externalities"). We check to which extent the grand coalition is (not) core-stable in this case. We then propose a three person first price auction game in which a two bidder ring is not stable. If direct externalities can possibly be positive, we show that the grand coalition is not core-stable and that there exist "small" (i.e., non-singleton) rings which are core-stable. All these examples confirm that direct externalities make cooperative behavior difficult, which was already suggested in Jehiel and Moldovanu (1996), but we give a more precise content to that phenomenon. Indeed, Jehiel and Moldovanu (1996) only show that, under reasonable assumptions, no agreement between (some of) the buyers and/or the seller can be stable. They thus depart from collusion of the bidders in the original auction game.

The chapter is organized as follows. Section 2.2 is devoted to the model and solution concept. In subsection 2.2.1, we define coalitional equilibria in games with incomplete information. In subsection 2.2.2, we address the issue of incentives. Proposition 2.2.1 and its corollary state that every coalitional equilibrium become incentive compatible once appropriate balanced transfers are made in every coalition. In subsection 2.2.3, we propose a notion of core-stability for a bidding ring, which does not necessarily gather all the bidders. In section 2.3, we apply core-stability to auctions. As a benchmark, we consider standard auctions (second price in subsection 2.3.1 and first price in subsection 2.3.2). In subsection 2.3.3, we turn to auctions with direct externalities. Section 2.4 concludes with some suggestions for future research.

2.2 Model and solution concept

2.2.1 From Bayesian games to cooperative games

Let us fix a Bayesian game with independent, private values $\Gamma \equiv \left[N, \{T_i, q_i, A_i, u_i\}_{i\in N}\right]$, namely a set of players N and for every player i, $i \in N$,

- a set of types T_i
- a probability distribution q_i over T_i
- a set of actions A_i
- a utility function $u_i : T_i \times A \to \mathbb{R}$, where $A = \prod_{i\in N} A_i$.

Let P be a coalition structure, namely a partition of N. From Γ and P, we construct an auxiliary Bayesian game $\Gamma(P) \equiv \left[P, \{T_S, q_S, A_S, U_S\}_{S\in P}\right]$, in which the players are the coalitions S, $S \in P$, and

- $T_S = \prod_{i\in S} T_i$, $q_S = \bigotimes_{i\in S} q_i$, $A_S = \prod_{i\in S} A_i$
- $U_S(t_S, (a_K)_{K\in P}) = \sum_{i\in S} u_i(t_i, (a_K)_{K\in P})$, where $t_S = (t_i)_{i\in S}$, $a_K = (a_i)_{i\in K}$

A strategy[3] of S in $\Gamma(P)$ is a mapping $\sigma_S : T_S \to A_S$. Such a definition makes sense if the members of coalition S fully share their information in T_S before jointly deciding on an action profile in A_S. We justify such strategies in the next subsection by showing that they can be derived from coalitions' mechanisms, which allow for appropriate transfers between the coalitions' members. Thanks to these mechanisms, utilities become transferable and incentive compatibility conditions are satisfied (see proposition 2.2.1).

As in Ray (2007) and Ray and Vohra (1997), we define a coalitional equilibrium relative to P as a Nash equilibrium $(\sigma_S)_{S\in P}$ of $\Gamma(P)$. We assume that for every P, there exists a coalitional equilibrium relative to P and in case of multiple equilibria, we fix a mapping σ associating a coalitional equilibrium $\sigma(P)$ with every P.[4] We denote as $v_\sigma(S; P)$ the expected utility of S at $\sigma(P)$, for every $S \in P$, namely

$$v_\sigma(S; P) = E\left[\sum_{i\in S} u_i\left(\widetilde{t_i}, \sigma(P)(\widetilde{t})\right)\right] \tag{2.2.1.1}$$

where r.v.'s are denoted with a $\sim$ and $\sigma(P)(\widetilde{t}) = \left(\sigma(P)_K(\widetilde{t}_K)\right)_{K\in P}$. (2.2.1.1) defines a partition form game, which is constructed from Γ and σ, with $\Gamma(P)$ as an intermediary step.

Let $T = \prod_{i\in N} T_i$. By evaluating (2.2.1.1) at the grand coalition N, we get

$$v_\sigma(N; P) \equiv v(N) = \max_{\tau\in A^T} E\left[\sum_{i\in N} u_i(\widetilde{t_i}, \tau(\widetilde{t}))\right] \quad \text{for every } \sigma \text{ and } P. \tag{2.2.1.2}$$

$v(N)$ is thus the first best Pareto optimal payoff of the grand coalition. Given any coalitional equilibrium mapping σ and any partition P of N, $\sigma(P)$ is a feasible strategy for N (i.e., $\sigma(P)$ can be viewed as an element of A^T). Hence, v_σ is "grand coalition superadditive", or, according to an equivalent terminology, N is efficient in v_σ:

$$v(N) \geq \sum_{S\in P} v_\sigma(S; P) \qquad \text{for every } P \tag{2.2.1.3}$$

2.2.2 Coalitions' mechanisms

Let us fix a coalition S. A mechanism μ_S for S is a pair of mappings $\mu_S = (\tau_S, m_S)$:

$$\begin{aligned} \tau_S &: T_S \to A_S \\ m_S &: T_S \to \left\{z \in \mathbb{R}^S : \sum_{i\in S} z_i \leq 0\right\} \end{aligned}$$

τ_S is S's decision scheme and m_S is a balanced transfer scheme. As usual, the interpretation is that members of S are invited to report their types to a planner who then chooses a profile of actions and transfers as a function of these reports only[5]. According to Marshall and Marx (2007)'s terminology, we

[3]In view of our application to auctions, we focus on pure strategies.

[4]Ray and Vohra (1997) give sufficient conditions for the existence of a coalitional equilibrium but their result is not useful in our applications to auctions. However, many specific results are available in this context (see section 2.3). Ray (2007) argues that the partition form game only makes sense if a unique coalitional equilibrium can be associated with every partition (possibly up to transfers). We rather take the view that in case of multiple equilibria, some "standard of behavior" allows us to select among them. Again, this seems appropriate in the context of auctions.

[5]Proposition 2.2.1 below only necessitates interim transfers, which are defined over (the set of *reports*) T_S. In particular, we do not rely on transfers depending on the actual players' utilities as in A. Kalai and E. Kalai (Unpublished results). In this case, transfers also depend on the players' *types*.

use "bid submission mechanisms", in which the bidders' delegate their decision power to a planner (as opposed to "bid coordination mechanisms", in which the planner just recommends bids to the players).

We assume that utilities over mechanisms are quasi-linear. More precisely, the utility of μ_S for player $i \in S$, given his type t_i, reported types $r_S = (r_j)_{j \in S}$, a "strategy" $\sigma_{N \setminus S} : T_{N \setminus S} \to A_{N \setminus S}$ for the players outside S (e.g., $\sigma_{N \setminus S} = (\sigma_K)_{K \in P, K \neq S}$, for some partition P of N) and types $t_{N \setminus S}$ for the players outside S is

$$u_i(t_i, \tau_S(r_S), \sigma_{N \setminus S}(t_{N \setminus S})) + m_S^i(r_S)$$

As this expression explicitly shows, every member i of S incurs an externality from the strategic choices of the players in $N \setminus S$ but, thanks to the private value assumption, does not face any direct informational externality. We define the incentive compatibility (I.C.) of the mechanism μ_S given a mapping $\sigma_{N \setminus S} : T_{N \setminus S} \to A_{N \setminus S}$. More precisely, μ_S is I.C. given $\sigma_{N \setminus S}$ iff for every $i \in S$, every type t_i and reported type r_i,

$$\begin{aligned} & E\left[u_i(t_i, \tau_S(t_i, \widetilde{t}_{S \setminus i}), \sigma_{N \setminus S}(\widetilde{t}_{N \setminus S})) + m_S^i(t_i, \widetilde{t}_{S \setminus i})\right] \\ \geq \quad & E\left[u_i(t_i, \tau_S(r_i, \widetilde{t}_{S \setminus i}), \sigma_{N \setminus S}(\widetilde{t}_{N \setminus S})) + m_S^i(r_i, \widetilde{t}_{S \setminus i})\right] \end{aligned}$$

This definition makes sense because, in any coalitional equilibrium, coalition S must take account of the behavior of the players in $N \setminus S$ in elaborating its own strategy. In the case of complete information, S just looks for a best reply to $N \setminus S$'s action profile. In the case of incomplete information with private values, S looks for an I.C. best reply to $N \setminus S$'s strategy $\sigma_{N \setminus S}$, without entering the details of $\sigma_{N \setminus S}$ (whether the players lie or not, how they possibly gather into subcoalitions, etc.). The next proposition justifies the coalitions' strategies in the auxiliary Bayesian game; in particular, we show that explicit I.C. conditions are not necessary. The construction, which goes back to Arrow (1979) and d'Aspremont and Gérard-Varet (1979, 1982), has been widely used in economic frameworks which do not involve externalities (see, e.g., Forges et al. (2002)).

Proposition 2.2.1. *Let $S \subseteq N$; let $\sigma_{N \setminus S} : T_{N \setminus S} \to A_{N \setminus S}$ be an arbitrary strategy of $N \setminus S$ and let σ_S be a best response of S to $\sigma_{N \setminus S}$ in $\Gamma(\{S, N \setminus S\})$. There exists a transfer scheme m_S such that*

1. $\sum_{i \in S} m_S^i(r_S) = 0$ *for every* $r_S \in T_S$
2. *The mechanism (σ_S, m_S) is I.C. given $\sigma_{N \setminus S}$.*

Proof: Let us fix S, $\sigma_{N \setminus S}$ and σ_S as in the statement. For every $i \in S, t_i \in T_i, a_S \in A_S$ let us set

$$h_i(t_i, a_S) = E[u_i(t_i, a_S, \sigma_{N \setminus S}(\widetilde{t}_{N \setminus S}))]$$

Since σ_S is a best response to $\sigma_{N \setminus S}$,

$$\sum_{i \in S} h_i(t_i, \sigma_S(t_S)) \geq \sum_{i \in S} h_i(t_i, a_S) \qquad \forall t_S \in T_S, a_S \in A_S \tag{2.2.2.1}$$

Let $\widehat{m}_S^i(r_S) = \sum_{j \in S \setminus i} h_j(r_j, \sigma_S(r_S))$. For every $i \in S$, type $t_i \in T_i$, reported type $r_i \in T_i$ and reported types $r_{S \setminus i} \in \prod_{j \in S \setminus i} T_j$ of the other members of S,

$$\begin{aligned} h_i(t_i, \sigma_S(r_S)) + \widehat{m}_S^i(r_S) &= h_i(t_i, \sigma_S(r_S)) + \sum_{j \in S \setminus i} h_j(r_j, \sigma_S(r_S)) \\ &\leq h_i(t_i, \sigma_S(t_i, r_{S \setminus i})) + \sum_{j \in S \setminus i} h_j(r_j, \sigma_S(t_i, r_{S \setminus i})) \\ &= h_i(t_i, \sigma_S(t_i, r_{S \setminus i})) + \widehat{m}_S^i(t_i, r_{S \setminus i}) \end{aligned} \tag{2.2.2.2}$$

where the inequality is due to (2.2.2.1) w.r.t. the type vector $(t_i, r_{S\backslash i})$.

Hence, the mechanism $(\sigma_S, \widehat{m}_S)$ is I.C. given $\sigma_{N\backslash S}$, but not yet balanced. Let $\overline{m}_S^i(r_i) = E[\widehat{m}_S^i(r_i, \widetilde{t}_{S\backslash i})]$. By taking expectations in (2.2.2.2) we conclude that $(\sigma_S, \overline{m}_S)$ is I.C. given $\sigma_{N\backslash S}$.

Finally, let $m_S^i(r_S) = \overline{m}_S^i(r_i) - \frac{1}{|S|-1} \sum_{j \in S\backslash i} \overline{m}_S^j(r_j)$. Then (σ_S, m_S) is I.C. given $\sigma_{N\backslash S}$ and $\sum_{i\in S} m_S^i(r_S) = 0$ for every $r_S \in T_S$. ■

As a direct consequence of this proposition, we get the following

Corollary *Every coalitional equilibrium can be made incentive compatible: let P be a partition of N and σ be a coalitional equilibrium relative to P; for every $S \in P$, there exists a transfer scheme m_S such that (σ_S, m_S) is I.C. given $(\sigma_K)_{K\in P, K\neq S}$ and*

$$v_\sigma(S; P) = E\left[\sum_{i\in S} \left(u_i(\widetilde{t}_i, \sigma_S(\widetilde{t}_S), (\sigma_K(\widetilde{t}_K))_{K\in P, K\neq S}) + m_S^i(\widetilde{t}_S)\right)\right]$$

The previous result provides a justification for the assumption that is made in Marshall et al. (1994) and Waehrer (1999), according to which types are common knowledge inside every ring of a given partition of the bidders.

2.2.3 Core-stability of a (single) ring

Let us denote as $\mathcal{P}(K)$ the set of all partitions of K, for $K \subseteq N$. Let $R \subseteq N$; from $v_\sigma(S; P)$, we derive the following characteristic function over R

$$w_\sigma^R(S) = \min_{\Pi \in \mathcal{P}(R\backslash S)} v_\sigma(S; \left\{S, \Pi, \{k\}_{k\in N\backslash R}\right\}), \quad S \subseteq R$$

In particular, for the grand coalition N,

$$w_\sigma^N(S) = \min_{\Pi \in \mathcal{P}(N\backslash S)} v_\sigma(S; \{S, \Pi\}) \qquad (2.2.3.1)$$

We say that R is *core-stable* (w.r.t. σ) iff the (standard) core of w_σ^R, $C(w_\sigma^R)$, is not empty. The interpretation is the following:

- The coalitional equilibrium mapping σ is given.
- The ring R considers to form; the players outside R are supposed to act individually. R proposes to every $i \in R$ a share x_i of the total expected payoff $w_\sigma^R(R) = v_\sigma(R; \left\{R, \{k\}_{k\in N\backslash R}\right\})$, to be achieved by means of an I.C. mechanism $\mu_R = (\sigma_R, m_R)$.
- *Every* subcoalition S of R considers non-participation; if S does not participate, the players outside R remain singletons, the players in $R \setminus S$ partition themselves as they wish. Hence S can guarantee the total expected payoff $w_\sigma^R(S)$ to its members.
- If the participation constraint of every $S \subseteq R$ is satisfied, R forms; every player observes his type; R implements μ_R.

Basic properties

- Every singleton $\{k\}$, $k \in N$, is core-stable.

- Recalling (2.2.1.2), for every σ, $w_\sigma^N(N) = v(N)$; by (2.2.1.3) and (2.2.3.1), w_σ^N is grand coalition superadditive (N is efficient in w_σ^N). This property does not necessarily hold for w_σ^R, $R \subsetneq N$ (see example 2.3.4 in section 2.3.3).

- $C(w_\sigma^R)$ corresponds to cautious expectations of the subcoalitions of R. In particular, $C(w_\sigma^N)$ contains the usual variants of the core of the partition form game v_σ (see Hafalir (2007)). For instance, the core with singleton expectations, or $s-$core, of v_σ, denoted as $C_s(v_\sigma)$, is defined as the standard core $C(f_\sigma^s)$ of the characteristic function

$$f_\sigma^s(S) = v_\sigma\left(S; \left\{S, \{j\}_{j\in N\setminus S}\right\}\right) \tag{2.2.3.2}$$

Similarly, the core with merging expectations (see Maskin (2003)), or $m-$core, of v_σ, $C_m(v_\sigma)$, is defined as $C_m(v_\sigma) = C(f_\sigma^m)$, where

$$f_\sigma^m(S) = v_\sigma(S; \{S, N \setminus S\}) \tag{2.2.3.3}$$

It readily follows from the definitions that $C(f_\sigma^s)$ and $C(f_\sigma^m)$ are subsets of $C(w_\sigma^N)$. Unlike w_σ^N, the characteristic functions f_σ^s and f_σ^m are not necessarily grand coalition superadditive (see example 2.3.3).

- Equivalent definition: w_σ^N can be defined in terms of the conjecture of every coalition S on the partition to be formed by the players of $N \setminus S$ if S secedes from the grand coalition N. For every coalition S, let $B(S)$ be a partition of N which contains S as a cell. Given a partition form game v, let $f^B(S) = v(S; B(S))$. The B-core of v is defined as the core of the characteristic function game f^B. The $s-$core and the $m-$core correspond respectively to $B(S) = \{S; \{j\}, j \in N \setminus S\ \}$ and $B(S) = \{S, N \setminus S\}$ for every S. The grand coalition N is then core-stable (w.r.t. σ) if, for some specification of the conjecture $B(S)$ of every coalition S, the $B-$core of v_σ is not empty.

- If Γ is a game with complete information, let

$$v_\alpha(S) = \max_{a_S\in A_S} \min_{a_{N\setminus S}\in A_{N\setminus S}} \left[\sum_{i\in S} u_i(a_S, a_{N\setminus S})\right] \tag{2.2.3.4}$$

In particular, $v_\alpha(N) = v(N)$. The $\alpha-$core of Γ is defined as $C(v_\alpha)$ (see Aumann (1961)). It is easily checked that, for every σ and every $S \subsetneq N$, $w_\sigma^N(S) \geq v_\alpha(S)$. Hence, $C(w_\sigma^N) \subseteq C(v_\alpha)$.[6] The extension of the definition of the $\alpha-$core to incomplete information may be delicate in the presence of incentive constraints. In particular, our previous construction of transfers, which made any coalitional equilibrium incentive compatible (see proposition 2.2.1), cannot be used for the maxmin, since the latter solution concept requires that coalition S considers *any* possible strategy of coalition $N \setminus S$.[7] However, in the framework of standard auctions, the difficulties disappear. Indeed, every coalition S guarantees itself a total expected payoff of 0, whatever the mechanism adopted by $N \setminus S$, by having all its members bidding 0 independently of their types, a strategy that is clearly I.C. for S. Furthermore, S cannot guarantee more than 0, since the members of $N \setminus S$ can all bid the maximal possible amount, which is I.C. for $N \setminus S$. Hence, the $\alpha-$core is well-defined and not empty in standard auctions. But the usual objection against maxmin strategies applies: why should S fear costly overbidding from $N \setminus S$?

[6] Hafalir (2007) focuses on abstract partition form games, which are not necessarily generated by a strategic form game. Hence he does not distinguish the core with cautious expectations from the $\alpha-$core. In our framework, at least under complete information, Aumann (1961)'s original definition of the $\alpha-$core can be used.

[7] Our construction applies to the minmax, i.e., to the $\beta-$characteristic function, in the sense that we can dispense with I.C. constraints in the best replies of the coalition under consideration.

2.3 Applications

In this section, we apply our solution concept to auctions with independent private values. In the first two subsections, we consider standard auctions, that is, without direct externalities. We check the core-stability of coalitions in several specific auction models which have been proposed in the literature. In subsections 2.3.1 and 2.3.2, we illustrate that, in absence of direct externalities, coalitions are core-stable. In subsection 2.3.3, we allow for direct negative externalities. We show that the grand coalition can be made core-stable in this case. However, the $s-$core and the $m-$core of the underlying partition form game can be empty (example 2.3.3) and small coalitions may not be core-stable (example 2.3.4). Finally, if externalities are possibly positive, the $\alpha-$core may be empty (example 2.3.6).

2.3.1 Standard second price auctions

Let player i's type $\widetilde{t}_i$ be a continuous random variable over $\left[\underline{t}_i, \overline{t}_i\right]$, $0 \leq \underline{t}_i \leq \overline{t}_i$, to be interpreted as his valuation for a single object. $A_i = [0, M]$ is the set of possible bids, where $M \geq \max_{i \in N} \overline{t}_i$. Let $a = (a_k)_{k \in N}$ be an $n-$tuple of bids. A second price auction is defined by the following utility functions

$$\begin{aligned} u_i(t_i, a) &= t_i - \max_{j \neq i} a_j \quad \text{if } a_i > \max_{j \neq i} a_j \\ &= \frac{1}{\eta(a)}(t_i - a_i) \quad \text{if } a_i = \max_{j \neq i} a_j \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $\eta(a) = |\{k \in N : a_k = \max_{j \in N} a_j\}|$.

As is well-known, this game has an equilibrium in weakly dominant strategies. More generally, let P be a partition of N. The auxiliary Bayesian game $\Gamma(P)$ has a coalitional equilibrium in weakly dominant strategies described by $\sigma_S^k(t_S) = t_k$ for some $k \in S$ such that $t_k = \max_{j \in S} t_j$ and $\sigma_S^i(t_S) = 0$ for $i \in S$, $i \neq k$, for every $S \in P$ and $t_S = (t_j)_{j \in S}$. It is easily checked that for every P and $S \in P$,

$$v_\sigma(S; P) = v_\sigma(S; \{S, N \setminus S\}) = E\left[\left(\max_{i \in S} \widetilde{t}_i - \max_{j \in N \setminus S} \widetilde{t}_j\right)^+\right] \equiv \varphi(S)$$

where $f^+ = \max\{f, 0\}$. The previous expression shows that, at the equilibrium in weakly dominant strategies, the external effects disappear, so that v_σ reduces to a plain characteristic function. In particular, for every $S \subseteq R \subseteq N$ and every $\Pi \in \mathcal{P}(R \setminus S)$, $v_\sigma(S; \left\{S, \Pi, \{k\}_{k \in N \setminus R}\right\}) = \varphi(S)$ and a ring R is core-stable iff $C(\varphi|_R)$ is not empty, where $\varphi|_R(S) = \varphi(S)$ for every $S \subseteq R$.

Proposition 2.3.1. (Barbar and Forges (2007), Mailath and Zemsky (1991)) *In a standard second price auction, all rings are core-stable.*

Proof: Mailath and Zemsky (1991) establish that φ is balanced. Barbar and Forges (2007) further show that φ is supermodular (convex). If the bidders are symmetric, namely if the types $\widetilde{t}_i$, $i = 1, ..., n$, are i.i.d., an easy direct argument shows that giving the same amount $\frac{\varphi(N)}{|N|}$ to every member of N defines a payoff $n-$tuple in $C(\varphi)$: first, I denoting the indicator function,

$$\varphi(S) \leq E\left[\left(\max_{i \in S} \widetilde{t}_i\right) I\left[\max_{i \in S} \widetilde{t}_i \geq \max_{j \in N \setminus S} \widetilde{t}_j\right]\right] \tag{2.3.1.1}$$

Further, it is easily checked that

$$\begin{aligned} & P\left(\left\{\max_{i \in S} \widetilde{t}_i \leq t\right\} \cap \left\{\max_{i \in S} \widetilde{t}_i \geq \max_{j \in N \setminus S} \widetilde{t}_j\right\}\right) \\ = \; & P\left(\left\{\max_{i \in N} \widetilde{t}_i \leq t\right\} \cap \left\{\max_{i \in S} \widetilde{t}_i \geq \max_{j \in N \setminus S} \widetilde{t}_j\right\}\right) \\ = \; & F^n(t) \frac{|S|}{|N|} \end{aligned}$$

where F is the distribution function of any $\widetilde{t}_i$. It follows then from (2.3.1.1) that $\varphi(S) \leq \varphi(N)\frac{|S|}{|N|}$. ■

2.3.2 Standard first price auctions

In this subsection, we assume that the n initial bidders are symmetric, namely that the valuations $\widetilde{t}_i$, $i = 1, ..., n$ are i.i.d. Let $a = (a_k)_{k \in N}$ be an $n-$tuple of bids. A first price auction is defined by the following utility functions

$$\begin{aligned} u_i(t_i, a) &= t_i - a_i \quad \text{if } a_i > \max_{j \neq i} a_j \\ &= \frac{1}{\eta(a)}(t_i - a_i) \quad \text{if } a_i = \max_{j \neq i} a_j \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $\eta(a)$ is defined as for the second price auction.

Obviously, given a nontrivial partition P of N, the players of the auxiliary Bayesian game $\Gamma(P)$ are not symmetric. By Lebrun (1999), $\Gamma(P)$ has a unique equilibrium, for every partition P. In other words, there exists a unique coalitional equilibrium mapping σ. However, no general analytical solution is available.

Waehrer (1999, proposition 2.3.1) shows that for every partition P and every coalitions R, $S \in P$ such that $|R| \leq |S|$

$$\frac{v_\sigma(S;P)}{|S|} \leq \frac{v_\sigma(R;P)}{|R|} \tag{2.3.2.1}$$

In words, at a first price auction, the per capita expected payoff of a cartel's member is greater in small cartels[8]. Following a rough intuition, this inequality seems to rather reflect the fragility of large coalitions in a first price auction. However, it enables us to draw the reverse conclusion!

Proposition 2.3.2. *In a standard first price auction with symmetric bidders, the grand coalition is core-stable.*

Proof: We will show that the vector payoff allocating the amount $\frac{v(N)}{|N|}$ to every member of N is in the $s-$core of the underlying partition game v_σ. Let $S \subsetneq N$ and $P = \left\{S, \{k\}_{k \in N \setminus S}\right\}$. Recalling the definition of the $s-$core (see (2.2.3.2)), we have to show that

$$\frac{v(N)}{|N|} \geq \frac{v_\sigma(S;P)}{|S|} \tag{2.3.2.2}$$

From (2.3.2.1), we deduce that for every $j \in N \setminus S$,

$$v_\sigma(\{j\};P) \geq \frac{v_\sigma(S;P)}{|S|}$$

while, from the grand coalition superadditivity of v_σ (recall (2.2.1.3)),

$$v(N) \geq v_\sigma(S;P) + \sum_{j \in N \setminus S} v_\sigma(\{j\};P)$$

The latter two inequalities yield (2.3.2.2). ■

The previous reasoning can be applied to establish the stability of a bidding ring $R \subsetneq N$ if v_σ is superadditive on R. Such a property indeed holds in examples proposed by Marshall et al. (1994) and McAfee and McMillan (1992).

[8] Waehrer (1999) also shows that for second price auctions, the inequality goes the other way round.

McAfee and McMillan (1992) assume that $\widetilde{t}_i \in \{0,1\}$, $i = 1, ..., n$. They show (in inequality (13)) that, for every coalition $S \subsetneq N$ and $j \in N \setminus S$,

$$\begin{aligned} & v_\sigma\left(S \cup \{j\}; \left\{S \cup \{j\}, \{k\}_{k \in N \setminus (S \cup \{j\})}\right\}\right) \\ \geq\ & v_\sigma\left(S; \left\{S, \{k\}_{k \in N \setminus S}\right\}\right) + v_\sigma\left(\{j\}; \left\{S, \{k\}_{k \in N \setminus S}\right\}\right) \end{aligned}$$

or, equivalently, recalling our notation f_σ^s (see (2.2.3.2))

$$f_\sigma^s(S \cup \{j\}) \geq f_\sigma^s(S) + v_\sigma\left(\{j\}; \left\{S, \{k\}_{k \in N \setminus S}\right\}\right)$$

One can also check that (2.3.2.1) holds in their framework so that

$$v_\sigma\left(\{j\}; \left\{S, \{k\}_{k \in N \setminus S}\right\}\right) \geq \frac{f_\sigma^s(S)}{|S|}$$

Hence

$$\frac{f_\sigma^s(S \cup \{j\})}{|S| + 1} \geq \frac{f_\sigma^s(S)}{|S|}$$

and, by induction, for every coalitions R, S such that $S \subseteq R$,

$$\frac{f_\sigma^s(R)}{|R|} \geq \frac{f_\sigma^s(S)}{|S|} \tag{2.3.2.3}$$

Since $f_\sigma^s(S) \geq w_\sigma^R(S)$ for $S \subseteq R$, with equality if $S = R$, the latter inequality implies that every ring R is core-stable in McAfee and McMillan (1992)'s example.

Marshall et al. (1994) compute f_σ^s by numerical methods in the case of five initial bidders uniformly distributed over $[0,1]$. Their table III shows that $\frac{f_\sigma^s(S)}{|S|}$ is increasing with the size of S (i.e., (2.3.2.3) holds) so that, in their example too, all rings are core-stable.

2.3.3 First price auction with complete information and direct externalities

Following Jehiel and Moldovanu (1996), we consider first price auctions with complete information, in which every bidder incurs an externality if a competitor acquires the object[9]. The basic game reduces to $\Gamma \equiv \left[N, \{A_i, u_i\}_{i \in N}\right]$, where $A_i = \{0, \epsilon, 2\epsilon, ...\}$ is the set of possible bids, given a smallest money unit $\epsilon > 0$. The utility functions are described by an $n \times n$ matrix $E = [e_{ij}]$; for every i, $e_{ii} \equiv t_i$ is agent i's utility for the object and for every $i \neq j$, e_{ij} is the externality incurred by agent j if agent i gets the object. If all bids are 0, the seller keeps the object; agent i's utility is normalized to 0 in this case. Let $a = (a_k)_{k \in N}$; the utility of player i is

$$\begin{aligned} u_i(a) &= t_i - a_i \quad \text{if } a_i > \left[\max_{j \neq i} a_j\right]^+ \\ &= e_{ji} \quad \text{if } a_j > \left[\max_{k \neq j} a_k\right]^+ \quad \text{for some } j \neq i \\ &= 0 \quad \text{if } a = 0 \end{aligned}$$

To complete this description, we assume that if several players make the highest bid, they all get the object with the same probability.

Recalling (2.2.1.2), we have here

$$v(N) = \max_{a \in A}\left[\sum_{i \in N} u_i(a)\right] = \left[\max_{i \in N}\left\{t_i + \sum_{j \neq i} e_{ij}\right\} - \epsilon\right]^+$$

[9]Given the complete information assumption, all results of this subsection apply to second price auctions as well.

Since Γ is a game with complete information, the $\alpha-$characteristic function v_α is defined by (2.2.3.4).

Core-stability of the grand coalition under negative externalities

Except in example 2.3.6, we assume negative externalities, i.e., $e_{ij} \leq 0$ for every $i \neq j$. In this case, given any strategy profile $(a_i)_{i\in S}$ of S, $N \setminus S$ can inflict a negative payoff on S by bidding over $\max_{i\in S} a_i$; hence $v_\alpha(S) \leq 0$ for $S \subsetneq N$; since $v(N) \geq 0$, the $\alpha-$core $C(v_\alpha)$ is not empty. A similar argument shows that, for a coalitional equilibrium mapping σ proposed in Jehiel and Moldovanu (1996)[10], $w^N_\sigma(N) = v(N) \geq 0$, while for every $S \subsetneq N$, $w^N_\sigma(S) \leq 0$. Hence, for that particular choice of σ, the grand coalition N is core-stable, namely $C(w^N_\sigma) \neq \emptyset$. Jehiel and Moldovanu (1996) establish the emptiness of the α-core of a quite different market game, in which all agreements between the bidders and the seller are possible. Here, we stick to the original format of the first price auction, so that we only allow for collusion between the potential buyers.

At the above coalitional equilibrium mapping σ, *all* conceivable cores (e.g., the $s-$core and the $m-$core, see subsection 2.2.3) are nonempty. The example below illustrates that this property does not necessarily hold for coalitional equilibrium mappings which lead to possibly positive payoffs.[11]

Example 2.3.3. $n = 4$; the matrix of valuations/externalities is

$$E = \begin{pmatrix} t_1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 0 \\ -8 & -8 & 1 & -8 \\ -7 & -7 & -7 & 1 \end{pmatrix}$$

Let us start with $t_1 = 8$. $v(N) = 2 - \epsilon$. Assume first that the bidders act individually. Then the following strategies form an equilibrium: $a_1 = a_4 = 8 - 2\epsilon$, $a_2 = 8$, $a_3 = 8 - \epsilon$. Bidder 2 wins the auction and the payoffs are $(0, -7, 0, 0)$. Hence,

$$v_\sigma(\{i\}; \{\{1\}, \{2\}, \{3\}, \{4\}\}) = 0,\ i = 3, 4 \tag{2.3.3.1}$$

Assume next that the first two bidders collude, while the two others remain singletons. The relevant matrix becomes

$$\begin{pmatrix} 6 & -2 & -2 \\ -16 & 1 & -8 \\ -14 & -7 & 1 \end{pmatrix}$$

The following strategies now form an equilibrium: $a_1 = 3$, $a_2 = 0$, $a_3 = 3 - \epsilon, a_4 = 3 - 2\epsilon$. Coalition $\{1, 2\}$ gets a payoff of 3 so that

$$v_\sigma(\{1, 2\}; \{\{1, 2\}, \{3\}, \{4\}\}) = 3 \tag{2.3.3.2}$$

(2.3.3.1) and (2.3.3.2) imply that the characteristic function f^s_σ is not grand coalition superadditive, hence that the $s-$core $C_s(v_\sigma)$ is empty in that example.

Let us take $t_1 = 4$. We now have $v(N) = 1 - \epsilon$. Let us assume that bidder 3 competes with the cartel $\{1, 2, 4\}$. The matrix is

$$\begin{pmatrix} 1 & 0 \\ -24 & 1 \end{pmatrix}$$

[10] Jehiel and Moldovanu (1996) prove that, under appropriate genericity conditions, the following strategies $(b_j)_{j\in N}$ form an equilibrium in Γ: if $t_i - \min_j e_{ji} \leq 0$ for every $i = 1, ..., n$, then $b_i = 0$ for every i. Otherwise, let (i, k) be a pair of bidders $i \neq k$ such that $t_i - e_{ki}$ is maximal over all $t_j - e_{lj}$, $j \neq l$ (that is, bidder i is willing to pay the highest price for the object, given his valuation and the externalities he might suffer); take $b_i = t_i - e_{ki} - \epsilon$, $b_k = t_i - e_{ki} - 2\epsilon$ and $b_j < b_k$, $j \neq i, k$. At this equilibrium, which typically involves weakly dominated strategies, bidder i's payoff is $e_{ki} + \epsilon \leq 0$ and all other bidders $j \neq i$ get $e_{ij} \leq 0$ (see chapter 1 Appendix A, for a full characterization of equilibria).

[11] The features of the next examples depend crucially on the direct externalities. In a first price auction with complete information and no externalities, there exists a coalitional equilibrium mapping σ in which the outcome (namely, the winner and the price) is as in the equilibrium in undominated strategies of the second price auction. For that σ, the $s-$core and the $m-$core of v_σ are not empty. Furthermore, every bidding ring is core-stable w.r.t. σ.

where the first row corresponds to the utilities in case the cartel obtains the object. The strategies $a_1 = 0$, $a_2 = 1$, $a_3 = 1 - \epsilon$, $a_4 = 0$ form an equilibrium. Hence

$$v_\sigma(\{3\}; \{\{3\}, \{1,2,4\}\}) = 0 \tag{2.3.3.3}$$

and similarly for bidder 4. Let us assume again that the first two bidders collude, but facing the opposite ring $\{3,4\}$. The relevant matrix is now

$$\begin{pmatrix} 2 & -4 \\ -14 & -6 \end{pmatrix}$$

The strategies $a_1 = \epsilon$, $a_2 = a_3 = a_4 = 0$ are in equilibrium, so that

$$v_\sigma(\{1,2\}; \{\{1,2\}, \{3,4\}\}) = 2 - \epsilon \tag{2.3.3.4}$$

(2.3.3.3), the analog of (2.3.3.3) for bidder 4 and (2.3.3.4) imply that the characteristic function f_σ^m is not grand coalition superadditive, hence that the $m-$core $C_m(v_\sigma)$ is empty in that example. ■

Core-stability of a "small" coalition under negative externalities

Jehiel and Moldovanu (1996) report on the case of two European firms who did not cooperate in a procurement auction opposing them to an Asian competitor. They suggest that negative externalities might explain the failure of the natural partners' association but, as explained above, the emptiness of the $\alpha-$core that they consider only shows that no stable agreement can be found between the three potential buyers and the seller. In this particular example, cooperation between the European firms and the Asian one looked unlikely, but the stability of the European coalition could be considered. We illustrate below that, in the presence of externalities, a two firm cartel may not be stable.

Example 2.3.4. $n = 3$; the matrix of valuations/externalities is

$$E = \begin{pmatrix} 5 & -4 & -3 \\ -4 & 6 & -9 \\ -10 & -1 & 3 \end{pmatrix}$$

If a first price auction takes place between the 3 agents, in every equilibrium, agent 1 wins and agent 3 is the second highest bidder; in undominated strategies, $10 \leq p \leq 12$; at the lowest price $p = 10$, the utilities are $(-5, -4, -3)$. Provided that $p < 11$, bidders 1 and 2 get a total utility > -10. If they form a joint venture, in every equilibrium, agent 2 represents $R = \{1,2\}$ at the auction and wins; in undominated strategies, $p = 12$: the price *raises* when agent 1 and agent 2 do not compete. The total utility of $\{1,2\}$ is -10, which is less than the sum of agents 1 and 2's individual payoffs (in our previous notation, $w_\sigma^R(\{1\}) = -5$, $w_\sigma^R(\{2\}) = -4$, $w_\sigma^R(\{1,2\}) = -10$). The interpretation is the following: if agents 1 and 2 get together, they cannot expect more than -10; if agent 3 plays a dominated strategy, they will even get less. If agent 1 breaks the agreement, he does not expect that agents 2 and 3 (like a European firm and the Asian firm above) will collude, but considers a noncooperative equilibrium between the three competitors. At an equilibrium leading to the lowest price, he can expect -5. Similarly, agent 2 can expect -4. ■

Core-stability of the grand coalition under possibly positive externalities

In example 2.3.4, the grand coalition is core-stable. If externalities are negative, the grand coalition can decide not to participate in the auction so as to guarantee 0 to its members, a strategy that is not feasible for small coalitions. More generally, the next proposition, proved in the appendix, states that, if $n \leq 3$, the grand coalition is core-stable w.r.t. every coalitional equilibrium mapping, even if externalities can be positive. Recall that $f^+ = \max\{f, 0\}$.

Proposition 2.3.5. *In every 3-player first price auction with direct externalities such that $t_i > e^+_{ji}$ for every $i, j \neq i$, the grand coalition is core-stable w.r.t. every coalitional mapping σ.*

We conclude this section by illustrating that, if sufficiently many players face possibly positive externalities, the grand coalition may fail to be stable. In the next example, with five players, the $\alpha-$core, $C(v_\alpha)$, is empty. In particular, the grand coalition cannot be stable, whathever the coalitional equilibrium mapping σ and whatever the conjectures B made by coalitions (recall section 2.2.3).

Example 2.3.6. $n = 5$; every agent i has two neighbors ($i-1 \bmod 5$, $i+1 \bmod 5$); $t_i = 3$, $e_{ji} = 2$ if agent j is a neighbor of agent i, $e_{ji} = -2$ otherwise.

One computes that $v(N) = 3 - \epsilon$. By symmetry, if $C(v_\alpha) \neq \emptyset$, the payoff vector in which every agent gets $\frac{3-\epsilon}{5}$ must be in $C(v_\alpha)$. Let us consider a coalition of the form $S = \{i, i+1, i+3\}$ where $+$ is $\bmod 5$, i.e., S contains agent i, a neighbor of agent i and a non-neighbor of agent i. S guarantees $\min\{3-\epsilon, 2\} = 2$ (if agent i bids ϵ and the other members of S bid 0) but cannot guarantee more, since a member of $N \setminus S$ can overbid S; hence, $v_\alpha(S) = 2 > 3\times\ \frac{3-\epsilon}{5}$, contradicting $C(v_\alpha) \neq \emptyset$.[12] Hence the grand coalition is not stable in this example. It can be shown that the same holds for all coalitions of 4 players but that all coalitions of 2 or 3 players are stable, for any coalitional equilibrium mapping in undominated strategies. ■

The previous example relies on positive externalities as far as we have normalized at 0 the agents' utilities in the case where no significant bid is made. In this case, the grand coalition has a non-participation strategy which guarantees a zero payoff, even if externalities are all negative, so that the $\alpha-$core is not empty. Alternatively, let us assume that the object is sold at zero price to a randomly chosen agent when no significant bid is made. This assumption makes sense in a procurement auction for an undesirable task: if nobody makes an offer, an agent is picked at random to perform the job. If this assumption is made in example 2.3.6 and a constant is subtracted to all utilities so that $t_i > 0$ and $e_{ji} < 0$ for all i, j, the $\alpha-$core is still empty, while all externalities are negative. This confirms the role of externalities.

2.4 Concluding remarks

In this chapter, we study collusion in auctions, possibly with direct externalities, by associating a cooperative game to the initial Bayesian game modelling the auction. Such a simple "semi-cooperative" approach, which builds a direct "bridge" between the initial noncooperative game and a cooperative one, enables us to rely on a well-known solution concept, namely, the core, but abstracts from the details of the strategic negotiation between coalitions (see A. Kalai and E. Kalai (Unpublished results) and Ray (2007) for further discussion). Such a synthetic analysis is particularly appropriate under the assumption that coalitions can commit at the ex ante stage. The full power of transfers can then be used to dispense with incentive compatibility constraints inside coalitions and to construct a tractable partition form game, which gives foundations to earlier models, e.g., Marshall et al. (1994) and Waehrer (1999).

As already argued in the introduction, ex ante commitment may be feasible in practice if, when they consider to collude, bidders do not have enough information to assess their valuations precisely. However, in many relevant applications, bidders are already privately informed when they contemplate possible collusion. Rings can then form (or not) as a function of their members' information. In other words, the coalition structure itself can depend on the bidders' types. Such a framework looks quite different from the ones adopted in the auction literature, including this chapter. One may nevertheless ask whether our methodology can be useful to investigate the stability of a *given* coalition at the *interim* stage. Our answer is affirmative as the key notions in section 2.2.2 allow us to define incentive compatible coalitional equilibria, in which incentive constraints can be binding.

To study the interim stability of a coalition, one could start by assuming, as in Vohra (1999)'s incentive compatible coarse core, that, when coalitions decide to form, e.g., when some members of a

[12] Equivalently: $\mathcal{S} = \{\{i, i+1, i+3\}, i = 1, ..., 5\}$ is balanced (with weights $\lambda_S = \frac{1}{3}$) and $\sum_{S\in\mathcal{S}} \lambda_S v_\alpha(S) \geq \frac{10}{3} > 3 - \epsilon$.

coalition consider to leave it, communication is limited to a minimum. Under this assumption, coalitions' mechanisms could still be described exactly as in section 2.2.2. But we could not rely on a straightforward transferable utility game anymore, since blocking at the interim stage should be formulated in terms of the players' conditional expected utilities, given their types. Except for this important difference with ex ante blocking, the players' types being independently distributed, interim (coarse) blocking would essentially affect individual participation constraints. The precise form of these would depend on the players' expectations, as in Hafalir (2007) and this chapter. We thus believe that some of our tools can be developed to study interim coalitional stability, at least in the sense of the coarse core. We expect that coalitions would look for "second best" solutions, with no counterpart to proposition 2.2.1. For instance, the grand coalition could be interim (coarse core-) stable without being ex post efficient. As is well-known, transfers like the ones used in proposition 2.2.1, which make ex post efficiency compatible with incentives, are typically not interim individually rational.[13]

Coming back to the achievements of this chapter, we give a precise content to the idea that "direct externalities make collusion harder". According to the available results, without direct externalities, bidding rings are stable. Examples based on Jehiel and Moldovanu (1996), i.e., with complete information, show that this property no longer holds in the presence of direct externalities. A natural setup to pursue the analysis is the second price auction with externalities proposed by Caillaud and Jehiel (1998), in which the valuations of the initial bidders are independently and identically distributed. They show that if interim individual participation constraints are imposed, the grand coalition may fail from being ex post efficient but do not address the question of its stability, which could be studied at the ex ante stage, as in this chapter, or at the interim one, along the lines suggested above.

Finally, in this chapter, we focused on independent private values. This assumption, which is standard in the auction framework, plays a crucial role in dispensing with explicit incentive compatibility constraints in the definition of coalitional equilibria (i.e., in proposition 2.2.1). In more general models, we expect that coalitional stability would rely on coalitional equilibria with binding incentive compatibility constraints, even at the ex ante stage.

Appendix

A Appendix: proof of proposition 2.3.5

Let us fix an arbitrary coalitional mapping σ, namely, for every partition P of $N = \{1,2,3\}$, a Nash equilibrium $\sigma(P)$ of the auction game in which the players are the coalitions in P. We will show that the core with singleton expectations $C_s(v_\sigma)$ is not empty, i.e., that $C(f^s_\sigma) \neq \emptyset$, where the characteristic function f^s_σ is defined by (2.2.3.2).

Let us assume w.l.o.g. that player 1 is efficient in N, namely that

$$t_1 + e_{12} + e_{13} \geq \max\{t_2 + e_{21} + e_{23}, t_3 + e_{31} + e_{32}\} \tag{A1}$$

Then

$$f^s_\sigma(N) = [t_1 + e_{12} + e_{13} - \epsilon]^+$$

We will consider the modified characteristic function g_σ defined by

$$\begin{aligned} g_\sigma(N) &= t_1 + e_{12} + e_{13} - \epsilon \\ g_\sigma(S) &= f^s_\sigma(S) \quad \text{for every } S \subsetneq N \end{aligned}$$

and show that $C(g_\sigma) \neq \emptyset$. Let us set $x_i = g_\sigma(\{i\})$, $i = 1,2,3$. x_i is player i's payoff at the equilibrium $\tau \equiv \sigma(\{\{1\},\{2\},\{3\}\})$ induced by σ in the 3-person original auction game. Since $t_i > 0$ for every i, the

[13]The property nevertheless holds in standard second price auctions, as shown by Mailath and Zemsky (1991). They prove a slightly stronger result than proposition 2.3.1 above, with individual participation constraints fulfilled at the interim stage (and well-defined, independently of the players' expectations). They stick to ex ante participation for coalitions with more than two players.

seller cannot keep the object at τ. If player i gets the object at a positive price p at τ, $x_i = t_i - p < t_i$; if player $j \neq i$ wins the object at τ, $x_i = e_{ji} < t_i$ by assumption. Hence $x_i < t_i$. Furthermore, $x_2 + x_3 \leq e_{12} + e_{13}$. Indeed, if player 1 wins the object at τ, $x_2 + x_3 = e_{12} + e_{13}$. If, say, player 2 wins the object at τ, the price p must exceed $t_1 - e_{21}$, otherwise player 1 would deviate from τ: $x_2 + x_3 = t_2 - p + e_{23} \leq t_2 - t_1 + e_{21} + e_{23} \leq e_{12} + e_{13}$, where the last inequality follows from (A1).

Let us set[14]

$$y = (t_1 - \epsilon, qx_2 + (1-q)t_2, qx_3 + (1-q)t_3)$$

where q is computed so that

$$y_1 + y_2 + y_3 = g_\sigma(N), \text{ i.e., } y_2 + y_3 = e_{12} + e_{13}$$

namely

$$q = \frac{(t_2 + t_3) - (e_{12} + e_{13})}{(t_2 + t_3) - (x_2 + x_3)}$$

From the properties of x_2 and x_3, q is well-defined and $0 < q \leq 1$. We will show that $y \in C(g_\sigma)$. By construction, y is efficient and individually rational. Let S be a 2-player coalition. $g_\sigma(S)$ is the payoff of S at the equilibrium $\zeta_S \equiv \sigma(\{S, N \setminus S\})$ of the 2-player auction game in which S competes against the singleton $N \setminus S$. It is easily checked that, at every equilibrium of an auction game with 2 players, the most efficient one wins the object (see, e.g., proposition 2 in Jehiel and Moldovanu (1996)). Let $S = \{2,3\}$; by (A1), player 1 wins the object at $\zeta_{\{2,3\}}$, so that $g_\sigma(\{2,3\}) = e_{12} + e_{13} = y_2 + y_3$. Let $S = \{1,2\}$; if player 3 wins the object at $\zeta_{\{1,2\}}$, $g_\sigma(\{1,2\}) = e_{31} + e_{32} \leq t_1 + e_{12} + e_{13} - t_3 \leq y_1 + y_2$, where the first inequality follows from (A1) and the second one from $t_3 \geq y_3 + \epsilon$. If $\{1,2\}$ wins the object at $\zeta_{\{1,2\}}$, let $k = 1$ or 2 be the most efficient player in $\{1,2\}$, i.e., $\max\{t_1 + e_{12}, t_2 + e_{21}\} = t_k + e_{k,k+1}$, where $k+1$ is $\bmod 2$. The price p to be paid by $\{1,2\}$ at $\zeta_{\{1,2\}}$ must exceed $t_3 - e_{k3}$, otherwise player 3 would deviate from $\zeta_{\{1,2\}}$. Hence, $g_\sigma(\{1,2\}) \leq t_k + e_{k,k+1} - t_3 + e_{k3}$ so that $g_\sigma(\{1,2\}) \leq t_1 + e_{12} + e_{13} - t_3$ by (A1); the proof is completed as above. $S = \{1,3\}$ is similar. ■

[14] The idea is that the grand coalition, if it forms, allocates the object to the efficient player 1. Then player 2 and player 3 must share $e_{12} + e_{13}$. Transfers are organized between these two players so that they get at least their individually rational level.

Chapter 3

Core stable bidding rings in independent private value auctions with externalities

Abstract

We consider a second price auction between bidders with independently and identically distributed valuations, where a losing bidder suffers a negative direct externality. Considering ex-ante commitments to form bidding rings we study the question of core stability of the grand coalition, namely: is there a subset of bidders that prefers forming a small bidding ring rather than participating in the grand cartel? We show that in the presence of direct externalities between bidders the grand coalition is not necessarily core stable, as opposed to the zero externality case, where the stability of the grand coalition is a known result. Finally, we study collusion in auctions as a mechanism design problem, insisting on the difficulty to compare ex-ante and interim commitments. In particular, we show that there are situations in which bidders prefer colluding before privately learning their types.

3.1 Introduction

The question of collusion in auctions receives great attention in the auction theory literature. From an empirical point of view, there is clear evidence of collusion in real life auctions and auction-like situations, although strictly prohibited in many countries. From a theoretical point of view, collusion in auctions provides a great challenge as the formation of bidding rings violates the symmetry between bidders, which is a common assumption in auction theory.

We address the question of collusion in auctions in the presence of direct externalities between bidders, focusing on the grand coalition, i.e., a bidding ring that includes all bidders. We examine the core stability of the grand coalition, as early as in the ex-ante stage, namely, before bidders learn their types. Our main goal is to answer the question: Is the grand coalition plausible in the presence of externalities, in the sense that no group of bidders prefers seceding?

3.1.1 A 3-player art auction example

The following example demonstrates the motivation to our work. Consider three art collectors who wish to acquire a valuable piece of art in a future auction. We assume that the object receives a restricted attention in the market, and that the three art collectors are the only possible potential bidders, a fact which is common knowledge. For the time being the auction house publishes a catalog with some details about the art object, however, the object itself is not yet available for close examination by the interested potential bidders or experts on their behalf. As such an examination is necessary in order to determine the true value of such a valuable object (e.g., to determine how well preserved it is), the three collectors can only have a rough estimation regarding their valuations in the planned auction. We, therefore, interpret this stage as the ex-ante stage.

As art collectors extremely vary in their personal preferences (e.g., personal taste), the value that they assign to the good (once it is finally accessible for a close inspection) is considered private.

A losing bidder in this example suffers a negative utility (i.e., direct externality). For instance, if the auctioned object is part of a collection, losing the auction may result in a decrease of value of other works of art that the losing collector already owns from the same collection.[1]

The formation of a bidding ring prior to the auction can be profitable. For instance, full cooperation between the three collectors will eliminate completely the competition in the auction (bid rigging), allowing them to win the object for a low price. Note, however, that in such a case the collector which will eventually own the good will most probably need to compensate the other ring members for their expected loss (i.e., externality). The presence of externalities may therefore interfere with cooperation.

We wish to see to which extent, in the presence of externalities, an ex-ante commitment of the grand coalition is plausible, in the sense that no group of players wishes to secede. In example 3.5.1 we demonstrate a market where a player, anticipating that the two other players will continue cooperating if he secedes, finds it profitable to deviate in the presence of externalities.[2] In such a setup, the grand coalition is said to be "instable".

3.1.2 Related Literature

Empirical evidence of collusion, and in particular of partial collusion, i.e., bidding rings which do not include all bidders, can be found in the work of Porter and Zona (1999) who find proof of bid rigging in school milk procurement in Ohio, as well as in Porter and Zona (1993) who study Long-Island highway construction contracts. Bajari and Ye (2003) study collusion in Midwest seal coats contracts. In particular, these studies provide some interesting links between observed bidders' behavior and theoretical notions. For example, Porter and Zona (1993) describe bidding rings that participate in several auctions, which can be interpreted as an ex-ante commitment, as ring members commit to cooperate in several auctions before learning the considered auctions' details, and in particular, before learning their private types.

With the help of mechanism design tools, the question of collusion in private value auctions without externalities was studied in, e.g., McAfee and McMillan (1992) who consider first price auctions, Mailath and Zemsky (1991), and Graham and Marshall (1987) who study collusion in second price auctions, and Marshall and Marx (2007) and Robinson (1985) who compare between first and second price auctions in order to find which is more vulnerable to bidder collusion.

We follow the model of markets with direct externalities between players, which was studied by Caillaud and Jehiel (1998) and Jehiel, Moldovanu and Stacchetti (1999) in incomplete information setups, as well as by Jehiel and Moldovanu (1996, 1999) and Jehiel, Moldovanu and Stacchetti (1996) in complete information. Caillaud and Jehiel (1998) study collusion in identically distributed independent private value second price auctions with direct externalities, however, they do not consider the seceding of coalitions but of individuals only. The question of partial collusion, i.e., the formation of bidding rings smaller than the grand coalition, in complete information auctions with direct externalities, was studied in chapter 1.

The property of core-stability, to which we refer in this chapter, was presented in chapter 2, where we develop tools to study the plausibility of coalitions in Bayesian games. We propose there an application of core-stability of bidding rings in independent private value (first and second price) auctions. We also consider there examples of instability in the presence of direct externalities with complete information. One of the key notions in this chapter, which naturally plays an important role in this chapter as well,

[1] As opposed to art dealers, art collectors do not buy in order to resell.

[2] For the sake of simplicity we demonstrate the secession of a single player rather than of a group of players. In order to demonstrate the secession of a group of players one needs to consider a larger market which significantly complicates the computations.

is coalitional equilibrium in games with incomplete information, extending the work of Ray and Vohra (1997) and Ray (2007). When discussing core stability we rely on two basic core notions, the core with merging expectations introduced by Maskin (2003), and the core with singleton expectations introduced by Hafalir (2007).

Following chapter 2 we focus on core-stability in the ex-ante stage. The literature which deals with ring formation in the ex-ante stage includes, e.g., Bajari (2001), Waehrer (1999), and Marshall et al. (1994). The question of ex-ante commitment to an incentive compatible (I.C.) mechanism was addressed by, e.g., Forges, Mertens and Vohra (2002) and Forges and Minelli (2001).

3.1.3 Core-stability

As in the model of Caillaud and Jehiel (1998), we consider a second price auction with a reserve price organized between a set of bidders with independent private values which are identically distributed. A losing bidder suffers a deterministic negative externality which is not identity dependent and hence assumed to be of common knowledge.

As we wish to study bidder collusion, given a partition of bidders, we define an auxiliary auction game in which the players are bidding rings, namely, cells in the considered partition.

Consider the existence of a Nash equilibrium in every auxiliary game, with respect to ex-ante expected utilities. In other words, for every bidder partition assume the existence of a *coalitional equilibrium* of a second price auction held between the bidding rings in this partition. Such a mapping of bidder partitions to coalitional equilibria, defines a partition form game (see, e.g., Lucas and Thrall (1963)), in which the value of a coalition (given a partition in which it is a cell) is its ex-ante expected utility in a Nash equilibrium of the corresponding auxiliary game.

In chapter 2 we prove that with appropriate transfer payments between coalition members a coalitional equilibrium can be made I.C. More precisely, the latter results from proposition 2.2.1 where we show that for any coalition S, given a strategy of the others and a best response for S, there exists a mechanism for S, composed of an exactly balanced transfer scheme between the members of S and the best response strategy of S, such that this mechanism is I.C. We therefore assume, thus WLOG, that types are common knowledge inside a coalition. In particular, the type of a bidding ring in the auxiliary game is the highest valuation of its members (net of their externalities).

Consider a mapping which determines the conjecture of every coalition on the behavior (i.e., partitioning) of the others if it decides to deviate and secede from the grand coalition. Namely, to every coalition we assign a partition in which it is a cell. With respect to this mapping the previously described partition form game reduces to a characteristic form game. The grand coalition is said to be *core-stable* if the core of this characteristic form game is not empty. The interpretation is that the grand coalition is plausible as no subset of bidders has an interest to deviate.

In order to examine the question of the stability of the grand coalition in auctions with externalities, we start by proving the existence of a coalitional equilibrium for any given bidder partition. Note that as opposed to the non-collusive auction game in which bidders are symmetric, in an auxiliary auction game the "bidders" (i.e., bidding rings) are asymmetric. Symmetry between bidders is violated in two senses in the auxiliary auction game. First, while bidders' valuations are distributed identically in the non-collusive auction, in the auxiliary auction each bidding ring has a different distribution which depends on its size. That is as the type of a bidding ring is the maximal valuation of its members.

Second, in the non-collusive auction externalities on a losing bidder are not identity dependent. A losing bidder suffers the same externality regardless of his (or the winner's) identity. In an auxiliary auction it is no longer true. The externality of a losing ring is cumulative and is a function of its size, as every member of the ring suffers a personal externality due to a loss. These two aspects of asymmetry introduce significant complications in the equilibrium analysis.

In this model Caillaud and Jehiel (1998) already identified an equilibrium in symmetric auctions with externalities where all bidders act individually (i.e., no bidding rings are allowed). They proved that bidding the difference between one's type and the externality term whenever the type exceeds the reservation price (and making an irrelevant bid otherwise) is in equilibrium. The reservation price therefore serves as a *participation threshold* for bidders. This equilibrium, however, depends crucially on the fact that bidders have identical distribution functions, and does not extend to asymmetric setups, where bidding rings (with nonidentical distribution functions) are considered.

In order to prove the existence of equilibrium in asymmetric collusive auctions (i.e., the auxiliary auction game), where each participating bidding ring has a different distribution function, we borrow the idea of participation thresholds. We prove that for **any** collusion scheme (i.e., bidder partition) there exist in the corresponding (asymmetric) auxiliary auction game participation thresholds which constitute an equilibrium. Specifically, given a strategy of the others, a ring has a threshold type between the reservation price and the sum of the reservation price and the ring's negative externality, such that if the ring's type is higher than its threshold type its best response is to bid the difference between its type and its externality, and whenever its type is lower than the threshold type its best response is to make an irrelevant bid.

We then rely on two specific mappings defining the conjecture of a seceding coalition regarding the partitioning of the others, starting with Maskin's (2003) *merging expectations* (see also, Hafalir (2007)). With merging expectations a seceding coalition expects the others to form the complementary coalition. Such an assumption yields an auxiliary auction game with two (usually asymmetric) bidders. Afterwards we consider the case where due to the secession of a coalition cooperation breaks down completely, as a result of which bidders outside the seceding coalition act individually. Hafalir (2007) refers to such a scenario as *singleton expectations*. It yields an auction game in which a "strong" bidder competes with "weak" bidders (see also, e.g., Maskin and Riley (2000)). We compute explicitly the participation thresholds of the coalitions in these two cases.

Once the equilibrium analysis is completed and the question of stability is to be addressed, for the sake of simplicity, we restrict our attention to an auction with three bidders.[3] As opposed to the case without externalities in which the grand coalition is stable (see, e.g., Mailath and Zemsky (1991)), we demonstrate that in the presence of externalities a bidder with merging expectations may prefer acting individually rather than participating in the all-bidders cartel. We therefore conclude that externalities may lead to the instability of the grand coalition.

As a last part of this chapter we study collusion in auctions with externalities as a mechanism design problem, insisting on the differences between ex-ante and interim collusion. More specifically, we look at collusive ex-post efficient mechanisms which are budget-balanced and incentive compatible and compare interim individual rationality with ex-ante group participation constraints.

As a benchmark we refer to the results of Mailath and Zemsky (1991) in second price auctions without direct externalities. They prove the existence of such collusive mechanisms with ex-ante group participation constraints. In the presence of direct externalities Caillaud and Jehiel (1998) identify a necessary and sufficient condition for the existence of an ex-post efficient mechanism for the grand coalition which is budget-balanced, incentive compatible and satisfies interim individual rationality. Translating our results as described above, (obtained in a model similar to the one of Caillaud and Jehiel (1998)), to the language of mechanism design, the grand coalition being core-stable means the existence of an ex-post efficient mechanism for the grand coalition which is budget-balanced, incentive compatible and satisfies ex-ante group participation constraints.

In order to show that the ex-ante and interim approaches are not logically comparable we give two examples. In the first the grand coalition has an ex-post efficient mechanism, which is budget balanced, incentive compatible and satisfies interim individual participation constraints but not ex-ante group participation constraints. In the second example the grand coalition has an ex-post efficient mechanism which is budget-balanced, incentive compatible and satisfies ex-ante group participation constraints but is not interim individually rational.

We insist that although interim individual rationality is stronger than ex-ante individual rationality, comparing ex-ante and interim commitments is difficult. In chapter 2 we explain this difficulty by saying that the decision of a coalition to block in the interim stage is a function of its conditional expected utility

[3]This is the smallest auction in which the instability of the grand coalition can be demonstrated. Examples exist for the general n-bidder case, $n > 3$. However, the grand coalition is always stable in a 2-bidder auction due to symmetry and the super-additivity of the grand coalition (see, section 3.5).

given the types of its members, and therefore one cannot use a straight forward transfer scheme for the grand coalition in order to ensure that no coalition would block interim (as opposed to ex-ante blocking). Finally, we conclude from this discussion that there are situations in which the grand coalition would try to collude in the ex-ante stage (given that such a stage can be identified), rather than letting the players learn their types before asking them to commit.

This chapter takes the following form: Section 3.2 presents the model. In section 3.3 we prove the existence of coalitional equilibrium in auctions with asymmetric bidders for any given bidder partition. In section 3.4 we compute the participation thresholds of bidding rings with merging and singleton expectation, which are used in section 3.5 to demonstrate the instability of the grand coalition in the presence of externalities. In section 3.6 we study collusion in auctions with externalities as a mechanism design problem, and section 3.7 concludes. In appendix A we bring the proof of coalitional equilibrium with singleton expectations, and appendix B presents simulations of core stability of the grand coalition with singleton expectations.

3.2 Model

3.2.1 Second price auction with externalities

As in Caillaud and Jehiel (1998) we consider a single indivisible object second price auction Γ with a reserve price $R > 0$, held in a market with a set of bidders $N = \{1, 2, ..., n\}$. Bidder i assigns a valuation t_i to the object, which is the utility he derives from the object if winning it. The valuation, or type, of i is private, and is identically and independently distributed with respect to a common continuous density $f > 0$ in $[\underline{t}, \overline{t}]$, with distribution F.

Additionally, symmetric direct external effects are considered. Specifically, a losing bidder suffers a negative externality $e < 0$. The externality term is not identity dependent. Namely, every losing bidder gets a utility equals to e regardless of his or the winner's identity. The externality is, therefore, assumed to be of common knowledge. For the sake of simplicity we assume that the parameters maintain $\underline{t} < R+e$ as well as $R < \overline{t}$. The interpretation is that a highest type bidder is expected to participate in the auction, whereas a lowest type is not.[4]

Let $b \in \mathbb{R}^n_+$ be a bidding vector. If no bidder makes a relevant bid, i.e., $\max_{j\in N} b_j < R$ then the seller keeps the good and all bidders get a utility normalized to zero. Otherwise, the good is allocated to the bidder who makes the highest bid, i, for the second highest relevant price: $p = \max\{R, \max_{j\neq i}\{b_j\}\}$. Considering quasi-linear utilities, i gets $t_i - p$, while every other bidder suffers the externality e. In case of a tie we assume that each of the bidders who placed the highest bid wins with equal probability.

3.2.2 The auxiliary collusion game

We extend now the auction game to consider bidding rings. Note P a partition of N, where the interpretation is that $S \in P$ is a bidding ring. We note t_S the valuation of such a ring, defined by $t_S = \max_{i\in S} t_i + (s-1)e$, where $s = |S|$, as a bidding ring wishes to maximize its profit.

Clearly, when defining the valuation of a coalition in this way we assume full revelation of information between coalition members. This assumption is WLOG as in chapter 2 we show that a given coalitional equilibrium can be made I.C. with appropriate transfers within each coalition.[5] Types are therefore common knowledge inside a coalition. Hence, the distribution of a coalition's type in terms of the original distribution F is:

$$F_S(t) = (F(t-(s-1)e))^s \tag{3.2.2.1}$$

for all $t \in [\underline{t} + (s-1)e, \overline{t} + (s-1)e]$. Additionally, if S does not win the auction it suffers an externality $e_S = se$. Note, that the formation of coalitions clearly violates the symmetry in the market in two different senses. First, valuations of "bidders", i.e., bidding rings, in a collusive auction are not identically

[4]Our results may be recovered without this assumption.

[5]See corollary of proposition 2.2.1.

distributed, as F_S depends on the size of S. Second, the externality of a losing ring depends on its identity as e_S is a function of $|S|$.

Given a partition P we can therefore consider the auxiliary auction game $\Gamma(P)$ where players are coalitions in P. Let $b \in \mathbb{R}_+^{|P|}$ be a bidding vector in $\Gamma(P)$, where the interpretation is that in every coalition the highest valuation ring member makes a relevant bid, while the other members of the coalition make irrelevant bids (below the reservation price) which can be conveniently ignored. Once again, WLOG we may assume full revelation of information within a coalition, hence, such a definition of a bidding vector in $\Gamma(P)$ is justified.

As before, if there is no relevant bid, i.e., $\max_{S\in P} b_S < R$ then the seller keeps the good and all coalitions get a null utility. Otherwise, the good is allocated to the coalition S that made the highest bid, for the second highest relevant price: $p = \max\{R, \max_{T\neq S}\{b_T\}\}$. S derives therefore a utility of $t_S - p$, while every other coalition suffers the (identity dependent) externality e_S. In case of a tie we assume that each of the coalitions that placed the highest bid wins with equal probability.

3.2.3 Core-stability of the grand coalition

Following chapter 2 we address the question of the stability of the grand coalition, applied to the considered second price auction. Roughly speaking, in chapter 2 we say that the grand coalition is core-stable if the core of the underlying cooperative game is non-empty.

As a first step in defining the underlying characteristic function, we consider there for every coalition $S \subset N$, a partition $B(S)$ such that $S \in B(S)$. $B(S)$ is interpreted as the conjecture of S on the partitioning of the rest of the players, $N \setminus S$, if S secedes.

Assume the existence of a coalitional equilibrium mapping σ (see, e.g., Ray (2007)). Namely, for every partition P of N, $\sigma(P)$ is a Nash equilibrium of $\Gamma(P)$. With respect to σ, in chapter 2 we derive a characteristic function, w_σ^B, which assigns to every coalition S its (ex-ante) expected utility in the equilibrium $\sigma(B(S))$ of the auxiliary game $\Gamma(B(S))$.

Finally, as mentioned above, we say there that with respect to the mappings σ and B the grand coalition is core-stable if the core of w_σ^B is non-empty.

In order to examine the stability of the grand coalition in auctions with externalities, we start by proving the existence of a coalitional equilibrium mapping σ (see, section 3.3). We then compute the coalitional equilibria given two specific examples of (symmetric) mappings B (see, section 3.4).[6] Finally, we compute the corresponding characteristic functions w_σ^B (see, sections 3.5).

As the grand coalition may win the auction by offering the reserve price, or alternatively get a zero utility if not participating (no trade), it participates in equilibrium if and only if the highest valuation of its members net of the reserve price is greater than the externalities of the other two members. The value of the grand coalition is therefore:

$$w_\sigma^B(N) = w(N) = E((\max_i \tilde{t}_i + (n-1)e - R)I(\max_i \tilde{t}_i + (n-1)e > R)) \qquad (3.2.3.1)$$

where, $\tilde{t}_i$ is a random variable with distribution F, and I is the indicator function. Obviously, the value of the grand coalition does not depend on the mappings σ and B (see also, e.g., chapter 2). If the core of the underlying cooperative game is not empty, then by symmetry, the allocation in which every agent gets an equal share of the value of the grand coalition is in the core. More precisely, the core is not empty if and only if it contains the following payment vector:

$$\left(\frac{w_\sigma^B(N)}{n}\right)_{i=1}^{n} \qquad (3.2.3.2)$$

Hence, in order to prove the instability of the grand coalition, it suffices to identify a coalition S, such that:

$$\frac{w_\sigma^B(N)}{n} < \frac{w_\sigma^B(S)}{|S|} \qquad (3.2.3.3)$$

[6] For instance, singleton expectations yield a symmetric mapping, as every seceding coalition conjectures that the others will partition themselves in the same way, to singletons. Merging expectations is another example for a symmetric mapping.

Example 3.5.1 demonstrates a case where (3.2.3.3) holds for a singleton (i.e., $|S| = 1$) in a 3-bidder auction, illustrating the instability of the grand coalition in the presence of externalities.[7]

3.3 Coalitional equilibrium

Caillaud and Jehiel (1998) prove that in a symmetric market, if no bidding rings are considered, then for every distribution F there exists an equilibrium in the auction, where a bidder bids his valuation augmented by the externality term if his valuation is greater than the reserve price, and 0 otherwise. We refer to such a strategy as a *participation threshold bidding strategy* with a threshold R:

$$\sigma_i(t_i) = \begin{cases} t_i - e & \text{If } t_i > R \\ 0 & \text{Otherwise} \end{cases} \tag{3.3.0.1}$$

We revise their analysis to establish the existence of equilibrium in threshold strategies in collusive auctions. Fix a partition P of N and a coalition $S \in P$. For all types t_S of S and bids $b \in \mathbb{R}_+^{|P|}$ of the participants in the auxiliary game $\Gamma(P)$, we denote $u_S(t_S; b)$ the utility of S of type t_S in $\Gamma(P)$ when b is played, as defined in section 3.2.2. Conveniently, we denote b_{-S} the bidding vector of all the participants but S. Consider the following lemmas:

Lemma 3.3.1.

$$\forall b_{-S}, \forall t_S < R + se, \forall b_S \geq R \quad u_S(t_S; (0, b_{-S})) \geq u_S(t_S; (b_S, b_{-S}))$$

Lemma 3.3.2.

$$\forall b_{-S}, \forall t_S \geq R + se, \forall b_S \geq R \quad u_S(t_S; (t_S - e_S, b_{-S})) \geq u_S(t_S; (b_S, b_{-S}))$$

Lemma 3.3.3.

$$\forall b_{-S}, \forall t_S > R, \forall b_S < R \quad u_S(t_S; (t_S - e_S, b_{-S})) \geq u_S(t_S; (b_S, b_{-S}))$$

Lemma 3.3.4. $\forall b_{-S}, \forall (t'_S, t_S), t'_S > t_S \geq R + se$, *if* $u_S(t_S; (t_S - e_S, b_{-S})) \geq u_S(t_S; (b_S, b_{-S}))$ *for any* $b_S < R$, *then*

$$u_S(t'_S; (t'_S - e_S, b_{-S})) \geq u_S(t'_S; (b_S, b_{-S})) \quad \texttt{for any } b_S < R$$

The proofs of the lemmas are analogous to Caillaud and Jehiel (1998) and are therefore omitted. Loosely speaking, we conclude from the lemmas that in a collusive auction every bidding ring S has a "dominant" strategy (except for the interval $[R + se, R]$, where a "monotonous" behavior maintains).

Specifically, if the type of S is lower than $R + se$ its "best response" is not to participate regardless of the actions of the others as by winning it gets a utility lower than its externality (lemma 3.3.1). If its type is higher than the reserve price it should participate by bidding its type augmented by its externality (lemmas 3.3.2 and 3.3.3). Finally, if an intermediate type in the interval $[R + se, R]$ prefers participating then any higher type prefers participating as well (lemma 3.3.4). It follows that given any strategy of the others there exist a participation threshold strategy which is a best response. The existence of the following tractable equilibrium in threshold strategies follows.

Proposition 3.3.5. *Let P be a partition of N. Then for all $S \in P$ there exists $t^*_S \in [R + e_S, R]$ such that the following $(\sigma_S)_{S \in P}$ is an equilibrium of $\Gamma(P)$:*

$$\sigma_S(t_S) = \begin{cases} t_S - e_S & \textit{If } t_S > t^*_S \\ 0 & \textit{Otherwise} \end{cases} \tag{3.3.0.2}$$

[7] This is for the sake of simplicity as demonstrating the secession of a group of players requires more bidders which significantly complicates the computations. See also, footnote 2.

We refer to t_S^* as the *participation threshold* of S in the equilibrium σ. An immediate implication of the proposition is that for a given valuation, the bigger a coalition is, the higher it is likely to bid if participating. The intuition is quite clear, as a big coalition suffers a greater externality if losing.

Finally, proposition 3.3.5 provides the coalitional equilibrium mapping σ which will be used to demonstrate the instability of the grand coalition in the presence of externalities (see, example 3.5.1).

3.4 Participation threshold

We wish to compute the coalitional equilibrium (namely, the participation threshold of a coalition) identified in the previous section in two specific cases. We start with the case where the complementary of a seceding coalition forms due to a secession. Then, we go on with the case where cooperation breaks down completely due to the secession of a coalition, namely, all players outside the seceding coalition act individually. The first case corresponds to the core with merging expectation (m-core) introduced by Maskin (2003), while the second case corresponds to the core with singleton expectations (s-core) introduced by Hafalir (2007).

3.4.1 Merging expectations - The two cartels case

Let us start with an analysis of the participation threshold of a coalition which expects its complementary to act cooperatively if seceding, i.e., $\forall S \subset N$, $B(S) = \{S, N \setminus S\}$. Suppose, thus WLOG, that $|S| > |N \setminus S|$.[8] We first claim (proposition 3.4.1) that there exists an equilibrium in threshold strategies where the participation threshold of the smaller coalition is simply the reserve price, while that of the bigger one is in the interval $[R - e_{N\setminus S} + e_S, R]$. We then derive an explicit expression of the latter.

Proposition 3.4.1. *Let $P = \{S, N \setminus S\}$ where $|S| > |N \setminus S|$. Then there exists $t_S^* \in [R - e_{N\setminus S} + e_S, R]$, such that t_S^* and $t_{N\setminus S}^* = R$ constitute an equilibrium of $\Gamma(P)$ in threshold strategies.*

Proof. Suppose first that $N \setminus S$ follows a bidding strategy with a participation threshold equals to R. From the preceding analysis we conclude that there exists a best response strategy for S with a threshold $t_S^* \in [R + e_S, R]$. Let us compute the interim utility of S of type $t_S \geq R + e_S$ if it chooses to participate in the auction and if it chooses not to participate. If S makes an irrelevant bid, then it gets 0 if $N \setminus S$ does not participate as well, and e_S otherwise, which yields:

$$seP(\tilde{t}_{N\setminus S} > R) \tag{3.4.1.1}$$

where, $|S| = s$ and $\tilde{t}_{N\setminus S}$ denotes a random variable with distribution $F_{N\setminus S}$.

If on the other hand S chooses to participate, we distinguish between 3 cases. Either, $N \setminus S$ does not participate in which case S wins the good for the reserve price. Or, $N \setminus S$ participates and wins, in which case S gets its externality. Or, finally, both coalitions participate and S wins, paying the bid of $N \setminus S$. Note that as we consider threshold strategies, if a coalition chooses to participate it bids the difference between its type and its externality (see, (3.3.0.2)). The interim utility of S if participating is therefore:

$$\begin{aligned}(t_S - R)P(\tilde{t}_{N\setminus S} \leq R) + seP(\tilde{t}_{N\setminus S} > R \texttt{ and } \tilde{t}_{N\setminus S} - (n-s)e > t_S - se) \\ + E((t_S - (\tilde{t}_{N\setminus S} - (n-s)e))I(\tilde{t}_{N\setminus S} > R \texttt{ and } \tilde{t}_{N\setminus S} - (n-s)e < t_S - se))\end{aligned} \tag{3.4.1.2}$$

where, $|N \setminus S| = n - s$ and I is the indicator function. As we consider continuous distribution, ties occur with zero probability with respect to threshold strategies.

As the interim utility functions of S if participating, (3.4.1.2), or not participating, (3.4.1.1), are continuous with respect to the type of S, we conclude that in equilibrium, S of threshold type t_S^* is

[8]In case of an equality the collusive auction is symmetric and the equilibrium reduces to the one identified by Caillaud and Jehiel (1998). Namely, both coalitions have a participation threshold equals to the reservation price R (see, section 3.3).

indifferent between participating or not. Namely, replacing t_S with t_S^* in (3.4.1.1) and (3.4.1.2) yields an equality:

$$
\begin{aligned}
seP(\tilde{t}_{N\setminus S} > R) = (t_S^* - R)P(\tilde{t}_{N\setminus S} \le R) + seP(\tilde{t}_{N\setminus S} > \max\{R, t_S^* \\
+ (n-2s)e\}) + E((t_S^* - (\tilde{t}_{N\setminus S} - (n-s)e))I(\tilde{t}_{N\setminus S} > R \\
\texttt{and } \tilde{t}_{N\setminus S} - (n-s)e < t_S^* - se))
\end{aligned} \tag{3.4.1.3}
$$

Suppose by way of contradiction that $t_S^* < R - e_{N\setminus S} + e_S$. Then $R = \max\{R, t_S^* + (n-2s)e\}$ and $I(\tilde{t}_{N\setminus S} > R \texttt{ and } \tilde{t}_{N\setminus S} - (n-s)e < t_S^* - se) = 0, \forall \tilde{t}_{N\setminus S}$. Hence, (3.4.1.3) reduces to:

$$
0 = (t_S^* - R)P(\tilde{t}_{N\setminus S} \le R) \tag{3.4.1.4}
$$

or equivalently, as $f > 0$, $t_S^* = R$, which is a contradiction.

Suppose now that S is following a bidding strategy with a participation threshold $t_S^* \in [R - e_{N\setminus S} + e_S, R]$. There exists a best response strategy for $N \setminus S$ with thresholds $t_{N\setminus S}^* \in [R + e_{N\setminus S}, R]$. Following the same analysis we conclude that in equilibrium:

$$
\begin{aligned}
(n-s)eP(\tilde{t}_S > t_S^*) = (t_{N\setminus S}^* - R)P(\tilde{t}_S \le t_S^*) + (n-s)eP(\tilde{t}_S > \max\{t_S^*, \\
t_{N\setminus S}^* - (n-2s)e\}) + E((t_{N\setminus S}^* - (\tilde{t}_S - se))I(\tilde{t}_S > t_S^* \\
\texttt{and } \tilde{t}_S - se < t_{N\setminus S}^* - (n-s)e))
\end{aligned} \tag{3.4.1.5}
$$

where, $\tilde{t}_S$ is a random variable with distribution F_S. As $t_{N\setminus S}^* \le R \le t_S^* + e_{N\setminus S} - e_S$ it holds that $t_S^* = \max\{t_S^*, t_{N\setminus S}^* - (n-2s)e\}$ and $I(\tilde{t}_S > t_S^* \texttt{ and } \tilde{t}_S - se < t_{N\setminus S}^* - (n-s)e) = 0$, $\forall \tilde{t}_S$. It follows that $t_{N\setminus S}^* = R$, which concludes the proof. ∎

In order to compute the participation threshold of the bigger coalition, we derive from (3.4.1.3):[9]

$$
\begin{aligned}
0 = -se(1 - (F(R-(n-s-1)e))^{n-s}) + (t_S^* - R)(F(R-(n-s-1)e))^{n-s} \\
+ se(1 - (F(t_S^* - (s-1)e))^{n-s}) + \int_R^{t_S^* + (n-2s)e} (t_S^* - t + (n-s)e)(n-s) \\
(F(t-(n-s-1)e))^{n-s-1} f(t-(n-s-1)e)\, dt
\end{aligned} \tag{3.4.1.6}
$$

which, by integration in parts, gives a characterization of t_S^* in the considered case:

$$
0 = -(n-2s)e(F(R-(n-s-1)e))^{n-s} + \int_{R-(n-s-1)e}^{t_S^* - (s-1)e} (F(t))^{n-s}\, dt \tag{3.4.1.7}
$$

Note, that there exists a unique threshold satisfying (3.4.1.7) as the function $h(\tau) = -(n-2s)e(F(R - (n-s-1)e))^{n-s} + \int_{R-(n-s-1)e}^{\tau-(s-1)e} (F(t))^{n-s}\, dt$, $\tau \in [R-(n-2s)e, R]$, is strictly increasing and maintains $h(R-(n-2s)e) \le 0$ and $h(R) \ge 0$.

3.4.2 Singleton expectations - The cartel vs. individuals case

We repeat the previous analysis in a market where due to the secession of a coalition, cooperation breaks down completely. As a result of which, all players outside the seceding coalition act individually, namely, $\forall S \subset N$, $B(S) = \{S, \{i\}_{i \notin S}\}$. We prove in a similar way (see, appendix A) that each individual participates if his type is greater than the reserve price while the coalition participates if its valuation exceeds some $t_S^* \in [R + (s-1)e, R]$, characterized in (3.4.2.1) below.

[9] For the sake of simplicity we assume here $\bar{t} + (n-2)e > R$, otherwise S might have a participation threshold which never allows it to participate in the auction. The analysis can be repeated in the complementary case.

Proposition 3.4.2. *Let $P = \{S, \{i\}_{i \notin S}\}$. Then there exists $t^*_S \in [R + (s-1)e, R]$, such that t^*_S and $t^*_i = R$ for all $i \notin S$ constitute an equilibrium of $\Gamma(P)$ in threshold strategies.*

Finally, as in the previous section we derive the following characterization of t^*_S in the considered case:

$$0 = (s-1)e(F(R))^{n-s} + \int_R^{t^*_S-(s-1)e} (F(t))^{n-s}\, dt \qquad (3.4.2.1)$$

As before, there exists a unique threshold satisfying (3.4.2.1).

3.5 Stability in a 3-player market with externalities

We consider a symmetric market with 3 players, as it is the smallest market in which the instability of the grand coalition can be demonstrated.[10] Mailath and Zemsky (1991) prove that in a standard second price auction, without direct externalities, the grand coalition is core-stable, namely, no subset of bidders has a profitable deviation. We wish to go further and examine the stability of the grand coalition in the presence of externalities.

3.5.1 Coalitions' values

In order to verify (3.2.3.3), namely, to compare the per-capita utility of a seceding coalition with the per-capita utility in the grand coalition, we compute the values of the different coalitions. Let us start with the value of the grand coalition. Consider, therefore, the auxiliary game $\Gamma(\{1, 2, 3\})$. Recalling (3.2.3.1), the value (or, ex-ante utility) of the grand coalition is,

$$v = E((\max_i \tilde{t}_i + 2e - R)I(\max_i \tilde{t}_i + 2e > R)) \qquad (3.5.1.1)$$

Consider now a seceding coalition of two, denoted S. Note, that its value, denoted y, does not depend on the considered mapping B. Consider, therefore, the auxiliary game $\Gamma(\{\{i\}, S\})$. We refer to the threshold equilibrium constructed in proposition 3.4.1, namely, the individual i participates if and only if his valuation is greater than the reserve price, offering his valuation augmented by the externality term, while the coalition S acts similarly with respect to the participation threshold t^* given by (3.4.1.7).[11]

In a similar way to the analysis in the previous section, as a first step in computing y we compute the interim utility function of S. If the type of S is lower than its participation threshold, $t_S \leq t^*$, it gets: e_S if the individual participates (i.e., if $t_i > R$), and 0 otherwise. Its interim utility in this case is therefore,

$$e_S P(\tilde{t}_i > R) \qquad (3.5.1.2)$$

If, alternatively, $t_S > t^*$, i.e., S participates, we distinguish between 3 cases. If the individual does not participate, S wins the good for the reserve price R. If the individual participates (offering his valuation augmented by the externality term) and the coalition overbids him (offering its valuation augmented by its externality) the coalition gets the good paying the offer of the individual. Finally, if the individual overbids the coalition, then the latter gets its externality. Hence, if participating, the coalition gets the following interim utility:

$$(t_S - R)P(\tilde{t}_i \leq R) + E[(t_S - (\tilde{t}_i - e))I(\tilde{t}_i > R \ \texttt{and} \ t_S - e_S > \tilde{t}_i - e)] \\ + e_S P(\tilde{t}_i > R \ \texttt{and} \ t_S - e_S < \tilde{t}_i - e) \qquad (3.5.1.3)$$

[10]Due to symmetry and grand coalition superadditivity (see, e.g., chapter 2) the grand coalition is always stable in a 2-player market with externalities. The instability of the grand coalition in the presence of externalities can be demonstrated also in the general n-player case, $n \geq 3$.

[11]As in the computation of (3.4.1.7) we assume here $\bar{t} + e > R$. See also footnote 9.

By taking expectations on (3.5.1.2) and (3.5.1.3), recalling (3.2.2.1), and by integrating in parts, we get the value (or, ex-ante utility) of a coalition of two:

$$y = 2e(1 - F(R)) + \int_{t^*-e}^{\bar{t}} (F(t) - F(t)^3)\, dt \tag{3.5.1.4}$$

with a (unique) participation threshold t^* (see, (3.4.1.7)) given by:

$$0 = eF(R) + \int_{R}^{t^*-e} F(t)\, dt \tag{3.5.1.5}$$

Finally, we need to compute the value of a seceding single player. We need to distinguish between two cases. With merging expectations the value of a seceding individual is his ex-ante utility with respect to the equilibrium $\sigma(\{\{i\}, S\})$, denoted z. With singleton expectations his value is his ex-ante utility w.r.t $\sigma(\{\{1\}, \{2\}, \{3\}\})$, denoted x.

In a similar way to the computation of the term y above we get:

$$\begin{aligned} z = F(t^* - e)^2(t^* - 2e - R) + e - F(t^* - e)^2 \int_{R}^{t^*-e} F(t)\, dt \\ + \int_{t^*-e}^{\bar{t}} (F(t)^2 - F(t)^3)\, dt \end{aligned} \tag{3.5.1.6}$$

In order to compute x, recall that the participation threshold of any individual in this case is the reserve price R. If $t_1 \leq R$, 1 gets e if any of the others participates and 0 otherwise, which yields his interim utility if not participating:

$$eP(\max\{\tilde{t}_2, \tilde{t}_3\} > R) \tag{3.5.1.7}$$

If $t_1 > R$, the interim utility of 1 is:

$$\begin{aligned} (t_1 - R)P(\max\{\tilde{t}_2, \tilde{t}_3\} \leq R) + E[(t_1 - (\max\{\tilde{t}_2, \tilde{t}_3\} - e))I(\max\{\tilde{t}_2, \tilde{t}_3\} > R \texttt{ and} \\ t_1 - e > \max\{\tilde{t}_2, \tilde{t}_3\} - e)] + eP(\max\{\tilde{t}_2, \tilde{t}_3\} > R \texttt{ and } t_1 - e < \max\{\tilde{t}_2, \tilde{t}_3\} - e) \end{aligned} \tag{3.5.1.8}$$

The distribution of $\max\{\tilde{t}_2, \tilde{t}_3\}$ being F^2, we conclude by taking expectations on the interim utilities and integrating in parts that the ex-ante utility of an individual facing two other individuals is:

$$x = e(1 - F(R)^2) + \int_{R}^{\bar{t}} (F(t)^2 - F(t)^3)\, dt \tag{3.5.1.9}$$

3.5.2 Emptiness of the core

As stated above in order to examine the stability of the grand coalition we need to verify whether a coalition of two players or an individual can gain more by not cooperating. By symmetry, a coalition of two players secedes if $\frac{y}{2} > \frac{v}{3}$. However, if an individual secedes, there are two possible partitions for the remaining two agents. To demonstrate the instability of the grand coalition in the presence of externalities we focus on merging expectations (m-core), namely, a seceding individual expects the others to form the complementary cartel. Therefore, by symmetry, an individual profitably secedes if $z > \frac{v}{3}$.[12]

We conclude that the grand coalition is m-core-stable, (i.e., the m-core is non-empty), if and only if:[13]

$$z \leq \frac{v}{3} \texttt{ and } \frac{y}{2} \leq \frac{v}{3} \tag{3.5.2.1}$$

[12]Note that an individual with singleton expectations will not deviate as by grand coalition superadditivity $x \leq \frac{v}{3}$.

Consider first, as a benchmark, a market without externalities. We know that in such a market offering one's valuation, as long as it is higher than the reserve price, constitutes an equilibrium in weakly dominant strategies. Recalling that the valuation of a coalition is equal to the maximal type of its members, in chapter 2 we prove that the partition function reduces in this case to a characteristic function. In other words, in the absence of externalities the value of a coalition does not depend on the mapping B. In particular, the s-core and the m-core coincide, and $x = z$. Furthermore, we prove there that without externalities all rings are core-stable (see also Mailath and Zemsky (1991) and Barbar and Forges (2007)).[14]

Introducing direct externalities between agents interferes severely with cooperation. While as collusion reduces competition which leads to potential greater profits due to price reduction, agents find it more difficult to collude in the presence of externalities as in large coalitions the cumulating effect of negative externalities decreases dramatically the coalition's net profit.

In the following example an individual facing a cartel of two gains more than if he chooses to join the cartel in order to form the grand coalition. We therefore conclude that the grand coalition is not stable with merging expectations.

Example 3.5.1. Consider a 3-agent symmetric market where the valuation each agent assigns to the good is distributed uniformly in the unit interval, $F \sim U[0, 1]$. A second price auction is held in this market with a reserve price $R = \frac{9}{10}$. The externality on a non-winning agent is $e = -\frac{1}{4}$. Note that as $1 + 2e < R$, if the grand coalition forms it profitably chooses not to participate in the auction. Hence, $v = 0$. By (3.5.1.6) it is readily verified that $z > 0$, hence an individual expecting the others to act cooperatively in case he secedes from the grand coalition, will profitably act independently. The grand coalition is therefore instable with merging expectation. ∎

Note, that considering "merging expectations" in the example is easily justified. Once a player secedes from the grand coalition the remaining two players can either cooperate and get $\frac{y}{2}$ each, or further split up gaining x each. From (3.5.1.4) and (3.5.1.9) we learn that $\frac{y}{2} > x$ in the example, hence, the remaining two players will profitably cooperate.

The individual's behavior in the example can also be interpreted as "free-riding". If he forms a coalition with the two other agents he gets a null payoff. By letting the two others form a coalition he gains a higher payoff, $z > 0$.

Let us note that stability is violated in the presence of externalities also when externalities are close to zero. One should not expect some sort of continuity, in the sense that for a small enough externality term all coalitions remain stable. This discontinuity may be explained, for instance, by the fact that for any negative externality the utility of a coalition depends on the partition of the others. In particular, we can consider an externality term as small as we wish, for a large enough reserve price, the instability with merging expectations demonstrated in the example maintains.

As a concluding remark for this section we recall a property proved by Waehrer (1999): the per capita share in a second price auction increases with the size of the coalition. As the example shows, the property does not extend to markets with direct externalities. Specifically, we obtain in the example $\frac{y}{2} < z$.

3.6 Collusive mechanisms

The question of collusion in auctions is frequently addressed in the literature as a mechanism design problem. Loosely speaking, a collusive mechanism for the grand coalition determines the bidding profile of the players, as well as transfer payments between them in order to share the coalition's gain. When

[13] As for the core with singleton expectations (s-core), it is non-empty if and only if:

$$\frac{y}{2} \leq \frac{v}{3} \tag{3.5.2.2}$$

It follows that the s-core contains the m-core in a 3-player market.

[14] The stability of a coalition of two players may also be recovered from Waehrer (1999). He proves that in the absence of externalities the per capita share in a second price auction increases with the size of the coalition. Specifically, $\frac{y}{2} \geq z = x$.

studying collusion in auctions with the means of mechanism design, we try to establish which properties the collusive mechanism may possess.

Consider a second price auction with independent private value (IPV) without direct externalities between bidders. Mailath and Zemsky (1991) construct a collusive mechanism for the grand coalition (with heterogeneous bidders) which is ex-post efficient, incentive compatible (IC), interim individually rational (IR) and budget-balanced (BB).[15] As is well known, IPV second price auctions have an equilibrium in weakly dominant strategies, where every bidder bids his type, or valuation. Full revelation within bidding rings yields an analogous result in collusive auctions, namely, a ring submits in equilibrium a bid equals to the highest valuation of its members (see, e.g., chapter 2). In particular a ring conveys no strategic externality upon players outside the ring, since the ring's highest type member would make the same bid if the ring did not collude. This allows Mailath and Zemsky (1991) to prove the "coalitional stability" of the grand coalition, namely, the existence of a mechanism for the grand coalition which is ex-post efficient, IC, BB and satisfies ex-ante group participation constraints.[16]

Caillaud and Jehiel (1998) study collusion in auctions with direct externalities where individual bidders decide whether to participate in the grand coalition at the interim stage. They mainly assume that bidders have veto power, namely, the refusal of a bidder to participate in the grand coalition leads to an auction where all bidders act individually.[17] They prove that the following condition is necessary and sufficient for the existence of a mechanism for the grand coalition which is ex-post efficient, BB, IC and interim IR:

$$e + \int_{R+e}^{\bar{t}} F(t)^{n-1}\,dt \le \frac{1}{n}v + \int_{\underline{t}}^{\bar{t}} \left(\int_{\max\{\tau, R-(n-1)e\}}^{\bar{t}} F(t)^{n-1}\,dt \right) f(\tau)\,d\tau \qquad (3.6.0.1)$$

whenever $R \le \bar{t} + (n-1)e$. In the complementary case, namely, when the grand coalition has absolutely no interest in participating in the auction, whatever its valuation is, the necessary and sufficient condition for the existence of a mechanism for the grand coalition which is ex-post efficient, BB, IC and interim IR is:

$$e + \int_{R+e}^{\bar{t}} F(t)^{n-1}\,dt \le 0 \qquad (3.6.0.2)$$

It can be verified that in the case without direct externalities ($e = 0$) the necessary and sufficient condition identified by Caillaud and Jehiel (1998) holds, which in turn revalidates the result of Mailath and Zemsky (1991) regarding the existence of an ex-post efficient mechanism for the grand coalition which is IC, BB and interim IR. Specifically, as $e = 0$ it holds that $R \le \bar{t} + (n-1)e$ and we should therefore look at condition (3.6.0.1). Changing the order of integration in the RHS of (3.6.0.1) replacing $v = E((\max_i \tilde{t}_i + (n-1)e - R)I(\max_i \tilde{t}_i + (n-1)e > R))$ (see, (3.2.3.1)) and setting $e = 0$ yields,

$$\int_{R}^{\bar{t}} F(t)^{n-1}\,dt \le \int_{R}^{\bar{t}} (t-R)F(t)^{n-1}f(t)\,dt + \int_{R}^{\bar{t}} F(t)^{n}\,dt \qquad (3.6.0.3)$$

Integrating in parts the first integral in the RHS of (3.6.0.3) yields the following equivalent inequality:

$$\int_{R}^{\bar{t}} F(t)^{n-1}\,dt \le \int_{R}^{\bar{t}} \frac{1}{n}(1 + (n-1)F(t)^{n})\,dt \qquad (3.6.0.4)$$

[15] Allowing agreements which involve the seller as well, Myerson and Satterthwaite (1983) prove an impossibility theorem regarding the existence of an ex-post efficient, BB, IC, interim IR mechanism. This indicates that in this result, the seller takes an important part with respect to the (in)stability of collusive agreements.

[16] Mailath and Zemsky (1991) prove the existence of such a mechanism using the famous Bondareva-Shapley theorem (see, e.g., Shapley (1967)). In particular, no constructive proof is available, as opposed to the case with interim individual participation constraints, as mentioned above.

[17] Caillaud and Jehiel (1998) state that one can assume veto power to bidders without loss of generality. Since they define the individual rationality level of i as his minmax payoff, it indeed does not depend on the partitioning of $N \setminus \{i\}$. Nevertheless, in our model individual rationality levels are calculated in equilibrium and do depend on the partitioning of the others, as demonstrated by example 3.5.1.

which can be verified by looking at the difference between the integrand in the RHS of (3.6.0.4) and the integrand in the LHS of (3.6.0.4) as a function of the type in the interval $[\underline{t}, \bar{t}]$. This function is non-increasing and is equal to 0 in $\bar{t}$.

With respect to Caillaud and Jehiel (1998)'s condition, consider example 3.5.1. In terms of mechanism design, the coalitional equilibrium considered in the example yields a mechanism for the grand coalition which is ex-post efficient, BB and IC.[18] The fact that a single player (with merging expectations) prefers seceding at the ex-ante stage in this example means that the mechanism fails to be ex-ante IR.

As ex-ante IR is a weaker property than interim IR, such a mechanism would also fail to be interim IR given that an individual has merging expectations, in particular, without veto power. Nevertheless, it can be readily verified that condition (3.6.0.2) does hold in this example, which means that the grand coalition has an ex-post efficient, BB, IC and interim IR mechanism if individuals have veto power. We conclude that the veto power assumption is strong.

To complete the discussion we propose another example, where the grand coalition is core-stable while as the necessary and sufficient condition of Caillaud and Jehiel (1998) does not hold. The interpretation is that there exists an ex-post efficient mechanism for the grand coalition which is BB, IC and satisfies ex-ante group participation constraints (for groups with either merging or singleton expectations, i.e., veto power or not), yet a mechanism for the grand coalition which is ex-post efficient, BB and IC, fails to be interim IR.

Example 3.6.1. Consider a 3-player symmetric market where the valuation that each player assigns to the good is distributed uniformly in the unit interval, $F \sim U[0, 1]$. A second price auction is held in this market with a reserve price $R = \frac{3}{5}$. The externality on a non-winning agent is $e = -\frac{1}{4}$. Note that as $1 + 2e < R$ if the grand coalition forms it profitably chooses not to participate in the auction.[19] It is readily verified that the LHS of (3.6.0.2) is positive, therefore, condition (3.6.0.2) does not hold.

Once again, as $1 + 2e < R$, the grand coalition is not expected to participate in the auction, namely $v = 0$. It can be readily verified that $y, z < 0$ (see, (3.5.1.4) and (3.5.1.6)), hence, the m-core of this game is non-empty (see, (3.5.2.1)), which in turn yields that the s-core is not empty as well (see, footnote 13). Hence the grand coalition is ex-ante core stable with both merging or singleton expectations. ∎

The latter example demonstrates the existence of an ex-post efficient, IC and BB mechanism which satisfies ex-ante group participation constraints while interim IR fails to be maintained. We find this result important for two reasons. First, the context of the mechanism design problem might be so that agents have to commit ex-ante, the interim stage being too late for collusion. Consider, for instance, the example of an art auction previously mentioned (see, e.g., section 3.1.1) where waiting for the art object to be available at the auction house for close examination by experts on behalf of the art collectors (i.e., interim stage) might be too adjacent to the auction, leaving no time for coalition formation negotiations. Another example, Porter and Zona (1999) study collusion in school milk procurements in Ohio where local diaries formed a bidding ring which repeatedly participated in the yearly school milk procurement. Forming a ring with the intention to participate in future auctions explicitly assumes the ability and willing of players to commit ex-ante. That is since the auction details are not yet published while collusion takes place, and therefore the players cannot calculate their cost if winning, and in particular they cannot calculate their valuations or types.

A second reason relates to the point of view of the colluding ring. As proved in chapter 2 a coalition can strive to achieve a "first best" solution at the ex-ante stage, namely, a collusive mechanism which is ex-post efficient, and satisfies group participation constraints, which can be made IC by implementing an appropriate balanced transfer scheme, as introduced, for instance, by Groves (1973). It is well known that a transfer scheme of that kind usually fails from being IR at the interim stage, leaving the coalition with a "second best" solution.

[18]While we follow an ex-ante approach, we achieve ex-ante incentive efficiency in the grand coalition by constructing an ex-post efficient mechanism which is also incentive compatible (see also, e.g., Holmstrom and Myerson (1983), Mas-Colell et al. (1995)).

[19]The same example can be revisited with the parameters $R = \frac{1}{2}$ and $e = -\frac{1}{5}$ for which the grand coalition does participate in the auction.

Moreover, as already explained in chapter 2, defining group participation constraints in the interim stage is difficult. The decision of a coalition to secede in the interim stage is a function of its "type". But what is the "type" of a coalition? How, and to what extent, do coalition members share their information, if at all?[20] Moreover, formulating interim blocking as a function of the conditional expected utility of a coalition given its "type" raises another difficulty. As opposed to the ex-ante blocking case, one can no longer use straight forward transfer payments between the members of the grand coalition in order to avoid all coalitions from blocking. Nevertheless, in example 3.6.1 an individual is blocking in the interim stage. Therefore, even if we could define group participation constraints in the interim stage, the grand coalition would be interim blocked, although, as demonstrated above, it would not be blocked in the ex-ante stage by both individuals and groups. We conclude that in this example the grand coalition would prefer colluding in the ex-ante stage.

3.7 Conclusion

We considered a second price auction in the presence of externalities, studying the question of the core stability of the grand coalition. We derived an auxiliary auction game between coalitions, or bidding rings, from the original non-collusive auction game, defining the valuation of a coalition as the highest valuation of its members net of the externalities of the coalition's members. We then proved the existence of a Nash equilibrium in the auxiliary game for any given partition of the bidders, computing it specifically in two concrete cases. The merging expectations case, where a seceding coalition conjectures that the others will form a complementary coalition. And the singleton expectations case, where a seceding coalition conjectures that the others will act individually.

Given these results, we showed that as opposed to the case without externalities where the grand coalition is always stable, the presence of externalities makes cooperation harder. In particular, we demonstrated an auction with three bidders, where a singleton with merging expectations prefers acting individually than joining the grand coalition, as early as in the ex-ante stage. It follows that the grand coalition is not necessarily stable in the presence of externalities.

Finally, we studied collusion in auctions using mechanism design tools. We showed that ex-ante and interim commitments are not logically dependent, insisting on the difficulty to define interim group participation constraints, and demonstrating an auction where the grand coalition would prefer to collude as early as in the ex-ante stage.

Appendix

A Appendix: Participation threshold of a coalition facing individuals

Proof of proposition 3.4.2. Suppose first that all $i \notin S$ follow a bidding strategy with a participation threshold equals to R. Then S has a best response strategy with a threshold $t_S^* \in [R + e_S, R]$. Let us compute the interim utility of S of type t_S^* if it chooses to participate in the auction and if it chooses not to participate. If S makes an irrelevant bid, then it gets 0 if all $i \notin S$ do not participate as well, and e_S otherwise, which yields:

$$seP(\max_{i \notin S} \tilde{t}_i > R) \tag{A1}$$

where $\tilde{t}_i$ denotes a random variable with distribution F.

If on the other hand S chooses to participate, we distinguish between three cases. Either all $i \notin S$ do not participate, in which case S wins the good for the reserve price. Either there is some $i \notin S$ who

[20]In chapter 2 we propose to look at the incentive compatible coarse-core (see, e.g., Vohra (1999)), where coalitions block on the basis of common knowledge events, i.e., communication between coalition members is reduced to a minimal level.

participates and wins, in which case S gets its externality. Or finally there is some $i \notin S$ who participates, however S wins paying the second highest bid $\max_{i \notin S} \tilde{t}_i - e$. The interim utility of S is therefore:

$$(t_S^* - R)P(\max_{i \notin S} \tilde{t}_i \le R) + seP(\max_{i \notin S} \tilde{t}_i > R \texttt{ and } \max_{i \notin S} \tilde{t}_i - e > t_S^* - se) \\ + E((t_S^* - (\max_{i \notin S} \tilde{t}_i - e))I(\max_{i \notin S} \tilde{t}_i > R \texttt{ and } \max_{i \notin S} \tilde{t}_i - e < t_S^* - se)) \quad \text{(A2)}$$

In equilibrium S of type t_S^* is indifferent between participating or not. Suppose by way of contradiction that $t_S^* < R + (s-1)e$. Then $R = \max\{R, t_S^* - (s-1)e\}$ and $I(\max_{i \notin S} \tilde{t}_i > R \texttt{ and } \max_{i \notin S} \tilde{t}_i - e < t_S^* - se) = 0, \forall (\tilde{t}_i)_{i \notin S}$. Hence,

$$seP(\max_{i \notin S} \tilde{t}_i > R) = (t_S^* - R)P(\max_{i \notin S} \tilde{t}_i \le R) + seP(\max_{i \notin S} \tilde{t}_i > R) \quad \text{(A3)}$$

or equivalently, $t_S^* = R$, which is a contradiction.

Suppose now WLOG that $1 \notin S$ and that S is following a bidding strategy with a participation threshold $t_S^* \in [R + (s-1)e, R]$, while all the individuals but 1 follow a strategy with a threshold R. There exists a best response strategy for 1 with threshold $t_1^* \in [R + e, R]$. Following the same analysis we conclude that in equilibrium:

$$\begin{aligned} eP(\max_{i \notin S \cup \{1\}} \tilde{t}_i > R \texttt{ or } \tilde{t}_S > t_S^*) = (t_1^* - R)&P(\max_{i \notin S \cup \{1\}} \tilde{t}_i \le R \texttt{ and } \tilde{t}_S \le t_S^*) \\ &+ eP(\max_{i \notin S \cup \{1\}} \tilde{t}_i > \max\{R, t_1^*\} \\ &\quad \texttt{or } \tilde{t}_S > \max\{t_S^*, t_1^* + (s-1)e\}) \\ &+ E((t_1^* - \max\{\max_{i \notin S \cup \{1\}} \tilde{t}_i - e, \tilde{t}_S - se\}) \\ &\quad I(\max_{i \notin S \cup \{1\}} \tilde{t}_i \in (R, t_1^*] \\ &\quad \texttt{or } \tilde{t}_S \in (t_S^*, t_1^* + (s-1)e]) \end{aligned} \quad \text{(A4)}$$

It follows that $t_1^* = R$. ∎

B Appendix: Simulations of core stability of the grand coalition with singleton expectations

We consider in this appendix singleton expectations, namely, a deviating coalition expects the others to act individually. We present several simulations of markets with three agents, showing that the grand coalition is core stable with singleton expectations, i.e., the s-core is non-empty. We note, however, that given example 3.5.1 where the grand coalition was proved to be unstable with merging expectation, the question of the stability of the grand coalition in a given market depends strongly on the conjecture of a seceding coalition on the partitioning of the remaining bidders.

We first recall, with respect to the notations in section 3.5, that the s-core is non-empty if and only if $\frac{y}{2} \le \frac{v}{3}$. Namely, the per capita utility in a coalition of two if seceding is lower than the per capita utility in the grand coalition (see, (3.5.2.2)).

We consider symmetric markets where the valuation that each player assigns to the good is distributed uniformly in the unit interval, $F \sim U[0,1]$. According to the parameters of the market, (i.e., the reservation price, R, and the externality term, e), we distinguish between two market types . The first market type maintains $R < 1 + 2e$, namely, a maximal type grand coalition participates in the auction as it can gain a positive utility by bidding the reserve price.

The second type maintains $R > 1+2e$, which means that for any type realization, the grand coalition, if it forms, does not participate in the auction. Nevertheless, we also demand $R < 1 + e$ to avoid a degenerated case, where a coalition of two players does not participate as well.[21]

We executed the following MATLAB simulation using the Symbolic Math Toolbox in order to calculate: v, y, z, x. (Correspondingly, the payoff of: The grand coalition; Coalition of two; Singleton facing a coalition of two; Singleton facing two individuals.)[22]

```
syms t;
v = @(e,R) int((t+2*e-R)*3*t^2,R-2*e,1);
% tstar is the participation threshold of a coalition of two
tstar = @(e,R) e+sqrt(R^2-2*e*R);
y = @(e,R) 2*e*(1-R)+int(t-t^3,tstar(e,R)-e,1);
z = @(e,R) (tstar(e,R)-e)^2*(tstar(e,R)-2*e-R)+e-
            (tstar(e,R)-e)^2*int(t,R,tstar(e,R)-e)+
            int(t^2-t^3,tstar(e,R)-e,1);
x = @(e,R) e*(1-R^2)+int(t^2-t^3,R,1);
```

The following table presents the simulation results in markets of the first type (i.e., the grand coalition may participate):

R	e	v	y	z	x
0.01	-0.005	0.73	0.24	0.078	0.078
0.99	-0.001	$9.5 \cdot 10^{-5}$	$6 \cdot 10^{-5}$	$3.1 \cdot 10^{-5}$	$2.9 \cdot 10^{-5}$
0.33	-0.32	0.001	-0.31	-0.13	-0.21
0.5	-0.2	0.014	-0.124	-0.045	-0.093

It can be readily verified that all simulations maintain $\frac{y}{2} \leq \frac{v}{3}$, namely, the grand coalition is core stable with singleton expectations.

Simulation results of markets of the second type, i.e., the grand coalition does not participate, are presented in the following table:

R	e	v	y	z	x
0.34	-0.33	0	-0.33	-0.13	-0.22
0.99	-0.007	0	$-1.3 \cdot 10^{-4}$	$7.4 \cdot 10^{-6}$	$-9 \cdot 10^{-5}$
0.5	-0.49	0	-0.47	-0.03	-0.31
0.9	-0.25	0	-0.03	0.07	-0.04

Here also, all simulations maintain $\frac{y}{2} \leq \frac{v}{3}$. Note, that the last simulation corresponds to the market from example 3.5.1, where the grand coalition was not core stable with merging expectations.

[21] Simulations ran in the complementary case, $1 + 2e > R > 1 + e$, also find the grand coalition core stable with singleton expectations.

[22] In the second market type, as the grand coalition does not participate, we simply set: v = 0;

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