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"Competitive Equilibria in Semi-Algebraic Economies"

by

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Competitive Equilibria in Semi-Algebraic Economies*

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Abstract

This paper examines the equilibrium correspondence in Arrow-Debreu exchange economies with semi-algebraic preferences. We show that a generic semi-algebraic exchange economy gives rise to a square system of polynomial equations with finitely many solutions. The competitive equilibria form a subset of the solution set and can be identified by verifying finitely many polynomial inequalities.

We apply methods from computational algebraic geometry to obtain an equivalent polynomial system of equations that essentially reduces the computation of all equilibria to finding all roots of a univariate polynomial. This polynomial can be used to determine an upper bound on the number of equilibria and to approximate all equilibria numerically.

We illustrate our results and computational method with several examples. In particular, we show that in economies with two commodities and two agents with CES utility, the number of competitive equilibria is never larger than three and that multiplicity of equilibria is rare in that it only occurs for a very small fraction of individual endowments and preference parameters.

(**JEL** D50, C63; *Keywords*: Computable general equilibrium, semi-algebraic economy, Groebner bases)

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1 Introduction

This paper examines the equilibrium correspondence in Arrow-Debreu exchange economies with semi-algebraic preferences. We show that for a generic economy all equilibria are among the finitely many solutions of a square system of polynomial equations. We apply methods from computational algebraic geometry to obtain an equivalent polynomial system of equations that essentially reduces the computation of all equilibria to finding all roots of a univariate polynomial. Sturm's Theorem allows us to determine the number of positive real roots of this polynomial, and we can then approximate all equilibria numerically by simple one-dimensional root-finding procedures.

Applied general equilibrium models are ubiquitous in many areas of modern economics, in particular in macroeconomics, public finance or international trade. The usefulness of the predictions of these models and the ability to perform sensitivity analysis are seriously challenged in the presence of multiple equilibria. Unfortunately, it is now well understood in general equilibrium analysis that sufficient assumptions for the global uniqueness of competitive equilibria are too restrictive to be applicable to models used in practice. However, it remains an open problem whether non-uniqueness of competitive equilibria poses a serious challenge to applied equilibrium modeling or whether non-uniqueness is a problem that is unlikely to occur in so-called 'realistically calibrated' models. Given specifications for endowments, technology and preferences, the fact that the known sufficient conditions for uniqueness do not hold obviously does not imply that there must be several competitive equilibria in the model economy. Also, considering that algorithms which are used in practice to solve for equilibrium in applied models are never designed to search for all solutions of the model, there is no proof that there might not always be several equilibria in these models after all. The fundamental problem is that for general preferences one cannot prove that equilibria are unique for a given set of endowments.

In this paper we develop a theoretical foundation for the analysis of multiplicity in general equilibrium models. We examine a standard Arrow-Debreu exchange economy with finitely many agents and goods. The simplicity of this model allows us to best illuminate the mathematical foundations of the analysis. We emphasize, however, that our theoretical results and computational method carry over to models with production technologies or incomplete asset markets. They can also easily be applied to stationary equilibria in infinite-horizon models.

Since we use methods from computational algebraic geometry to characterize the set of all equilibria and to approximate them numerically, the first step of our analysis is to identify an assumption on agents' preferences so that the resulting equilibrium conditions can be written as a system of polynomial equations and inequalities. Our key assumption is that agents' marginal utilities are continuous semi-algebraic functions. We argue that from a practical point of view, this assumption imposes few restrictions on the economic fundamentals and on equilibrium multiplicity. The Tarski-Seidenberg Principle (see e.g.

Bochnak et al. (1998)) implies that it is decidable whether competitive equilibria are unique. In fact, it follows from this principle that for any semi-algebraic class of economies we can algorithmically determine whether there are economies in this class for which multiplicity of equilibria occurs. Unfortunately, the Tarski-Seidenberg procedure is known to be highly intractable and therefore, while it allows us to derive theoretical results, it is useless for calculations in even the smallest exchange economies.

In this paper, we characterize and compute all equilibria of a semi-algebraic economy as solutions to a polynomial system of equations. We show for a generic economy that under our preference assumption all equilibria are among the finitely many solutions of a square system of polynomial equations. A solution to this system of equations is an equilibrium if it also satisfies a finite number of polynomial inequalities. Thus, finding all equilibria requires first to find all solutions to a polynomial system of equations. We solve these equations using Gröbner bases (see e.g. Cox et al. (1997)). In particular, we use a special version of the Shape Lemma (see Sturmfels (2002)) from computational algebraic geometry to prove that for a generic set of endowments the economic equilibria also satisfy an equivalent system of polynomial equations that essentially reduces the computation of equilibria to finding all roots of a univariate polynomial called the univariate representation. The values of all remaining variables are found by simple back substitution from the remaining equations in the new system. Moreover, this 'univariate representation' of the equilibrium equations can be computed as a polynomial in one endogenous variable and in exogenous variables such as endowments and preference parameters.

The important feature of Gröbner bases is that they can be computed exactly and in finitely many steps by Buchberger's algorithm (see Cox et al. (1997)). This algorithm is a cornerstone of computational algebraic geometry. In this paper we use a variation of this algorithm as implemented in the computer algebra system SINGULAR (see Greuel at al. (2005)), available free of charge at www.singular.uni-kl.de. We illustrate our theoretical results using several examples and use the methods to give bounds on the maximal number of equilibria in exchange economies with CES preferences.

We also develop methods to show that within a given class of preferences, equilibrium is unique for 'most' realistic specifications of endowments and preferences, i.e. for some compact set of exogenous parameter values. It is not clear how the idea that multiplicity of equilibria is rare in 'realistically calibrated' economies could possibly be formalized. The first observation is that we must impose joint restriction on preferences and endowments to have any hope to guarantee uniqueness. For any profile of endowments we can construct preferences such that the resulting economy has an arbitrary (odd) number of equilibria. Moreover, Gjerstad (1996) claims that in a pure exchange economy with CES utility functions with elasticities of substitution above 2 (arguably realistically calibrated utility functions), multiplicity of equilibrium is a prevalent problem. The question then becomes whether for 'most' endowments and preference parameters these economies have unique equilibria. Intuitively, in the case of Arrow-Debreu pure exchange economies no-trade equilibria are always unique, and so we may guess that a large departure from Pareto-efficient endowments is necessary to obtain non-uniqueness. Balasko (1979) formalizes the idea that the set of endowments for which there are n equilibria shrinks as n increases. Going beyond this result in the general case seems impossible. Instead, we suggest in this paper to estimate the size of the set of parameters for which non-uniqueness occurs in models with CES utility. Following Kubler (2007), we use a result from Koiran (1995) that gives estimates for the volume of semi-algebraic sets after verifying that finitely many points are contained in the set. We apply this result to CES economies with two agents and two goods and find that multiplicity is extremely rare for standard choices of parameters. Specifically for elasticities of substitution below 10, the volume of the set of individual endowments and preference parameters for which multiplicity can occur is bounded above by about half a percent.

One drawback of using SINGULAR for our computations is that with the current state of technology we can only solve models of moderate size, say of about $10 - 15$ polynomial equations of small or moderate degree. While our paper builds the theoretical foundation for computing all equilibria in general equilibrium models, we currently cannot solve applied models that often have hundreds or thousands of equations. We expect that the development of ever faster computers and more efficient or perhaps even parallelizable algorithms will allow for the computation of Gröbner bases for larger and larger systems. For recent advances see, for example, Faugère (1999).

While there is a large literature on sufficient conditions for uniqueness in general equilibrium models (see e.g. Mas-Colell (1991) for an overview), there have been few attempts to use numerical methods to explore multiplicity in any detail. Datta (2003) applies Gröbner bases to the computation of totally mixed Nash equilibria in normal form games, see also Sturmfels (2002). To the best of our knowledge, there has so far not been an attempt to use these methods to make statements about the number of equilibria in general equilibrium models.

In a seminal contribution, Blume and Zame (1992) show how one can conduct 'genericity analysis' in semi-algebraic exchange economies and prove that equilibria are generally locally unique. For this, they introduce Hardt's Triviality Theorem (see Bochnak et al. (1998)) to economic analysis. In our theoretical analysis we make use of this result and of some of the ideas in Blume and Zame (1992). Brown and Matzkin (1996) use the Tarski-Seidenberg principle to derive testable restrictions on observables in a pure exchange economy. These conditions are necessary and sufficient for the construction of a semi-algebraic exchange economy from a finite data set.

The paper is organized as follows. In Section 2 we define semi-algebraic exchange economies and show that equilibria can be characterized as solutions to polynomial equations. Section 3 uses results from computational algebraic geometry to characterize all solutions to polynomial systems of equations. In Section 4 we provide examples of semi-algebraic economies to illustrate our results and computational method. In Section 5 we examine uniqueness in Arrow-Debreu economies with CES utility functions.

2 Semi-algebraic Exchange Economies

We consider standard finite Arrow-Debreu exchange economies with H individuals, $h \in \mathcal{H} = \{1, 2, \ldots, H\}$, and L commodities, $l = 1, 2, \ldots, L$. Consumption sets are $\mathbb{R}^L_+$, prices are denoted by $p \in \mathbb{R}^L_+$. Each individual h is characterized by endowments, $e^h \in \mathbb{R}^L_{++}$, and a utility function, $u^h : \mathbb{R}^L_+ \to \mathbb{R}$.

A competitive equilibrium consists of prices p and an allocation $(c^1, \ldots, c^H)$ such that

$$c^h \in \arg\max_{c \in \mathbb{R}^L_+} u^h(c) \text{ s.t. } p \cdot (c - e^h) \leq 0, \qquad \text{for all } h \in \mathcal{H},$$

and

$$\sum_{h \in \mathcal{H}} (c^h - e^h) = 0.$$

It will simplify notation considerably to denote the profile of endowments across individuals by $e^{\mathcal{H}} = (e^1, \ldots, e^H) \in \mathbb{R}^{HL}_{++}$, and allocations by $c^{\mathcal{H}} \in \mathbb{R}^{HL}_+$ and to define $\lambda^{\mathcal{H}} = (\lambda^1, \ldots, \lambda^H)$. We assume that for each agent $h \in \mathcal{H}$, u^h is C^1, strictly increasing and strictly concave. We also assume that for each agent h the gradient $\partial_c u^h(c) \gg 0$ is a semi-algebraic function. This assumption will be explained and discussed in detail below.

We define an (interior) Walrasian equilibrium to be a strictly positive solution $\left(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p\right)$ to the following system of equations.

$$\begin{aligned} \partial_c u^h(c^h) - \lambda^h p &= 0, \quad \forall h \in \mathcal{H} && (1) \\ p \cdot (c^h - e^h) &= 0, \quad \forall h \in \mathcal{H} && (2) \\ \sum_{h \in \mathcal{H}} (c^h_l - e^h_l) &= 0, \quad l = 1, \ldots, L-1 && (3) \\ \sum_{l=1}^{L} p_l - 1 &= 0 && (4) \end{aligned}$$

Of course, equations (1) and (2) are the first-order conditions to the agents' utility maximization problem, equations (3) are the market-clearing conditions for all but the last good, and equation (4) is a standard price normalization. An economy is called regular if at all Walrasian equilibria the Jacobian of this system of equations has full rank.

We again emphasize that we only focus on a standard finite Arrow-Debreu exchange economy for ease of exposition. The ideas and results of this paper apply to much more general models.

Our model description contains one non-standard assumption: marginal utilities of all agents are semi-algebraic functions. Under this assumption all Walrasian equilibria are among the finitely many solutions to a polynomial system of equations that encompasses (1)–(4). The next subsection defines polynomials. The subsequent subsection summarizes those properties of semi-algebraic functions that turn out to be valuable for our analysis. We refer the interested reader to the excellent book by Bochnak et al. (1998) for an exhaustive treatment of real algebraic geometry.

2.1 Polynomials

For the description of a polynomial f in the n variables $x_1, x_2, \ldots, x_n$ we first need to define monomials. A monomial in $x_1, x_2, \ldots, x_n$ is a product $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ where all exponents α_i, $i = 1, 2, \ldots, n$, are non-negative integers. It will be convenient to write a monomial as $x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^N$, the set of non-negative integer vectors of dimension n. A polynomial is a linear combination of finitely many monomials with coefficients in a field $\mathbb{K}$. We can write a polynomial f as

$$f(x) = \sum_{\alpha \in S} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{K}, \quad S \subset \mathbb{Z}_+^N \text{ finite.}$$

We denote the collection of all polynomials in the variables $x_1, x_2, \ldots, x_n$ with coefficients in the field $\mathbb{K}$ by $\mathbb{K}[x_1, \ldots, x_n]$, or, when the dimension is clear from the context, by $\mathbb{K}[x]$. The set $\mathbb{K}[x]$ satisfies the properties of a commutative ring and is called a polynomial ring. In this paper we do not need to allow for arbitrary fields of coefficients but instead we can focus on three commonly used fields. These are the field of rational numbers $\mathbb{Q}$, the field of real numbers $\mathbb{R}$, and the field of complex numbers $\mathbb{C}$.

A polynomial $f \in \mathbb{K}[x]$ is irreducible over $\mathbb{K}$ if f is non-constant and is not the product of two non-constant polynomials in $\mathbb{K}[x]$. Every non-constant polynomial $f \in \mathbb{K}[x]$ can be written uniquely (up to constant factors and permutations) as a product of irreducible polynomials over $\mathbb{K}$. Once we collect the irreducible polynomials which only differ by constant multiples of one another, then we can write f in the form $f = f_1^{a_1} \cdot f_2^{a_2} \cdots f_s^{a_s}$, where the polynomials f_i, $i = 1, \ldots, s$, are distinct irreducible polynomials and the exponents satisfy $a_i \geq 1$, $i = 1, \ldots, s$. Being distinct means that for all $i \neq j$ the polynomials f_i and f_j are not constant multiples of each other. The polynomial f is called reduced or square-free if $a_1 = a_2 = \ldots = a_s = 1$.

2.2 Semi-algebraic Sets and Functions

A subset $A \subset \mathbb{R}^n$ is a semi-algebraic subset of $\mathbb{R}^n$ if it can be written as the finite union and intersection of sets of the form $\{x \in \mathbb{R}^n : g(x) > 0\}$ or $\{x \in \mathbb{R}^n : f(x) = 0\}$ where f and g are polynomials in x with coefficients in $\mathbb{R}$, that is, $f, g \in \mathbb{R}[x]$. More valuable for our purposes than this definition is the following lemma. It is a special case of Proposition 2.1.8 in Bochnak et al. (1998) and provides a useful characterization of semi-algebraic sets.

Lemma 1 *Every semi-algebraic subset of* $\mathbb{R}^n$ *can be written as the finite union of semi-algebraic sets of the form*

$$\{x \in \mathbb{R}^n : f_1(x) = \cdots = f_l(x) = 0,\ g_1(x) > 0, \ldots, g_m(x) > 0\}, \tag{5}$$

where $f_1, \ldots, f_l, g_1, \ldots, g_m \in \mathbb{R}[x]$.

Sets of the form (5) are called basic semi-algebraic sets.

Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be two semi-algebraic sets. A function $f : A \to B$ is semi-algebraic if its graph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m :\ x \in A,\ y \in B,\ y = \phi(x)\}$ is a semi-algebraic subset of $\mathbb{R}^{n+m}$. Semi-algebraic functions have many nice properties. For example, if $f : A \to B$ is a semi-algebraic mapping then the image $f(S)$ of a semi-algebraic subset $S \subset A$ is also semi-algebraic. Similarly, the preimage $f^{-1}(T)$ of a semi-algebraic subset $T \subset B$ is also semi-algebraic.

A semi-algebraic set A can be decomposed into a finite union of disjoint semi-algebraic sets $(A_i)_{i=1}^p$ where each A_i is (semi-algebraically) homeomorphic to an open hypercube $(0, 1)^{d_i}$ for some $d_i \geq 0$, with $(0, 1)^0$ being a point (see e.g. Bochnak et al. (1998), Theorem 2.3.6). This decomposition property of semi-algebraic sets naturally motivates the definition of the dimension of such sets. The dimension of the semi-algebraic set A is $\dim(A) = \max(d_1, \ldots, d_p)$. For any two semi-algebraic sets A and B it holds that $\dim(A \times B) = \dim(A) + \dim(B)$.

The following lemma proves invaluable for our approach to finding all equilibria. The proof is constructive and therefore helpful for our computations.

Lemma 2 *Let* $A \subset \mathbb{R}^n$ *be a semi-algebraic set and* $\phi : A \to \mathbb{R}$ *a semi-algebraic function. Then there exists a nonzero polynomial* $f(x, y)$ *in the variables* $x_1, \ldots, x_n, y$ *with* $f \in \mathbb{R}[x, y]$ *such that for every* $x \in A$ *it holds that* $f(x, \phi(x)) = 0$. *More generally, if* $B \subset \mathbb{R}^n$ *is a semi-algebraic set of dimension less than* n *then there exists a nonzero polynomial* $f(x)$ *in the variables* $x_1, \ldots, x_n$ *with* $f \in \mathbb{R}[x]$ *such that for every* $x \in B$ *it holds that* $f(x) = 0$.

Proof. Lemma 1 states that the graph of the semi-algebraic function $\phi : A \to \mathbb{R}$ is the finite union of basic semi-algebraic sets, each of which is of the form

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : f_1(x, y) = \cdots = f_l(x, y) = 0,\ g_1(x, y) > 0, \ldots, g_m(x, y) > 0\}.$$

Note that in each basic semi-algebraic set at least one of the polynomials f_i must be nonzero, since otherwise the set would be open, which in turn would imply that the graph of ϕ contains a nonempty open subset of $\mathbb{R}^{n+1}$. But that would contradict the fact that ϕ is a function. Now consider the product f of all nonzero polynomials f_i across all basic semi-algebraic sets. This product is itself a nonzero polynomial and it satisfies $f(x, \phi(x)) = 0$.

The proof of the second statement is analogous. □

The following lemma concerning semi-algebraic sets and functions proves very useful in our analysis. It is a simple consequence of Hardt's Triviality Theorem, see Bochnak et al. (1998, Theorem 9.3.2) or Basu et al. (2003, Theorem 5.45). For applications of this theorem in economics, see Blume and Zame (1992, 1994).

LEMMA 3 *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set and $f : A \to \mathbb{R}^k$ a continuous semi-algebraic function. Then there is a finite partition of $\mathbb{R}^k$ into semi-algebraic sets $C_1, \ldots, C_m$ such that for each C_i and every $b \in C_i$*

$$\mathit{dim} f^{-1}(b) = \mathit{dim} f^{-1}(C_i) - \mathit{dim}(C_i) \leq \mathit{dim}(A) - \mathit{dim}(C_i),$$

where negative dimension means the set is empty. In fact, the partition can be chosen such that the union of all C_i with $dim C_i < k$ is a closed subset of $\mathbb{R}^k$.

We also need a semi-algebraic version of Sard's theorem to characterize the set of regular economies (see e.g. Bochnak et al. (1998), Theorem 2.9.2).

LEMMA 4 *Let $N \subset \mathbb{R}^n$ be an open semi-algebraic set and $f : N \to \mathbb{R}^n$ be a C^∞ semi-algebraic function. Then the set of $y \in \mathbb{R}^n$ for which there exists an $x \in N$ with $f(x) = y$ and $\det(\partial_x f(x)) = 0$ is a semi-algebraic subset of $\mathbb{R}^n$ of dimension strictly smaller than n.*

2.3 Tarski-Seidenberg Principle

The Tarski-Seidenberg Principle (see e.g. Bochnak et al. Chapter 5) implies that it is 'decidable' whether a given semi-algebraic economy has one or multiple equilibria. Algorithmic quantifier elimination (see Basu et al. 2003) provides an algorithm to do so. In this subsection, we explain how *theoretically* algorithmic quantifier elimination can be used to compute the number of competitive equilibria for any semi-algebraic economy. However, it is practically infeasible to implement this theoretical algorithm even for very small problems. This fact motivates us to reformulate the problem of determining the number of equilibria to solving a system of polynomial equations and to consider algorithms from computational algebraic geometry that find all solutions to polynomial systems of equations.

Given any semi-algebraic set X, with $(x_0, x_1) \in X \subset \mathbb{R}^{l_0} \times \mathbb{R}^{l_1}$, define

$$\Phi = \{x_0 \mid \exists\, x_1\ [(x_0, x_1) \in X]\}.$$

The Tarski-Seidenberg Theorem implies that the set Φ is itself a semi-algebraic set. Several algorithms have been developed to eliminate the quantifiers and write any such set Φ as the finite union of basic semi-algebraic sets of the form (5) as in Lemma 1.

As a first application of the Tarski-Seidenberg Principle in combination with Hardt-triviality, we prove the following result which will be used in our analysis below.

LEMMA 5 *Let $E \subset \mathbb{R}^l$ be an open semi-algebraic set. Suppose that a semi-algebraic function $f : E \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ with $n \geq 1$ and $0 \leq m \leq n$ has finitely many zeros for each $e \in E$. Then for all μ outside a closed lower-dimensional subset $D_0 \subset \Delta^{n-1}$ and all e outside a closed lower-dimensional subset $E_0 \subset E$, there cannot be $(x', y') \neq (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $f(e; x, y) = f(e; x', y') = 0$ and*

$$\sum_{i=1}^{n} \mu_i x_i = \sum_{i=1}^{n} \mu_i x_i', \quad \sum_{i=1}^{m} \mu_i y_i = \sum_{i=1}^{m} \mu_i y_i'.$$

Proof. Consider the set

$$\begin{aligned} A \quad = \quad \Big\{ e \in E, \mu \in \Delta^{n-1} \ : \ & \exists (x,y) \neq (x',y') \quad f(e;x,y) = f(e;x',y') = 0 \quad \text{and} \\ & \sum_{i=1}^{n} \mu_i x_i = \sum_{i=1}^{n} \mu_i x_i', \quad \sum_{i=1}^{m} \mu_i y_i = \sum_{i=1}^{m} \mu_i y_i' \Big\}. \end{aligned}$$

The Tarski-Seidenberg Theorem implies that the set A is semi-algebraic. Under the assumption that $f(e; \cdot, \cdot)$ has only finitely many zeros, the set has at most dimension $l+n-2$. Consider the projection of A onto Δ^{n-1}, $g : A \to \Delta^{n-1}$ with $g(e, \mu) = \mu$. This is a continuous semi-algebraic function and so Lemma 3 ensures that for all μ outside a closed lower-dimensional subset $D_0 \subset \Delta^{n-1}$ the set $g^{-1}(\mu)$ has dimension at most $l+n-2-(n-1)$. Therefore, the dimension of the corresponding set of parameters e must be less than l. Define the set E_0 as its closure. Proposition 2.8.2 in Bochnak et al. (1998) ensures that the closure has the same dimension less than l. □

The Tarski-Seidenberg Theorem implies immediately that in our framework demand functions are semi-algebraic; they can be written as $\{(c, p) \mid \exists\ \lambda\ [\partial_c u(c) - \lambda p = 0 \text{ and } p \cdot (c - e^h) = 0]\}$. Of course, in this case it is trivial to eliminate the quantifier by simply eliminating λ.

More interestingly, the Tarski-Seidenberg Principle implies that for each $\mu \in \Delta^{L-1}$, the set

$$E_\mu = \{(e^{\mathcal{H}}, y) \in \mathbb{R}^{HL}_{++} \times (0,1) \ : \ \exists \ \left(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p\right) \text{ that solve } (1) - (4) \text{ and } y = \sum_{l=1}^{L} \mu_l p_l\}$$

is a semi-algebraic set. Using a variant of Lemma 3 Blume and Zame (1992) prove that for all endowments outside a closed, lower-dimensional semi-algebraic subset of $\mathbb{R}^{HL}_{++}$ the set of equilibrium prices is finite. This fact readily implies that E_μ has dimension HL. Therefore, by Lemma 2, there exist a polynomial $\omega_\mu(e^{\mathcal{H}}, y)$ such that

$$\omega_\mu(e^{\mathcal{H}}, y) = 0 \text{ whenever } (e^{\mathcal{H}}, y) \in E_\mu\ .$$

Observe that for fixed $e^{\mathcal{H}}$ this polynomial is a univariate polynomial in y. Walrasian equilibria for the economy with endowments $e^{\mathcal{H}}$ then correspond to those solutions of $\omega_\mu(e^{\mathcal{H}}, y) = 0$

for which also finitely many additional inequalities are satisfied. The following theorem shows that for all μ outside a closed, lower-dimensional subset of Δ^{L-1} the (finite) number of zeros of this polynomial is an upper bound on the number of Walrasian equilibria for all $(e^{\mathcal{H}})$ outside a lower-dimensional subset of $\mathbb{R}^{HL}_{++}$.

THEOREM 1 *There exists a closed lower-dimensional subset $D_0 \subset \Delta^{L-1}$ and a closed lower-dimensional subset $E_0 \subset \mathbb{R}^{HL}_{++}$ such that for all $\mu \in \Delta^{L-1} \setminus D_0$ and for all $(e^{\mathcal{H}}) \in \mathbb{R}^{HL}_{++} \setminus E_0$ there exists no $y \in \Delta^{L-1}$ satisfying $(e^{\mathcal{H}}, y) \in E_\mu$ for which two distinct Walrasian equilibrium price vectors $p, p' \in \Delta^{L-1}$ satisfy $y = \sum_l \mu_l p_l$ and $y = \sum_l \mu_l p'_l$.*

Proof. Define the semi-algebraic set

$$A = \{(e^{\mathcal{H}}, \mu) \in \mathbb{R}^{HL}_{+} \times \Delta^{L-1} : \exists \text{ W. E. prices } p \neq p' \text{ with } \sum_l \mu_l p_l = \sum_l \mu_l p'_l\}.$$

Blume and Zame (1992) show that for all endowments outside a closed, lower-dimensional semi-algebraic subset $E_0 \subset \mathbb{R}^{HL}_{++}$ the set of equilibrium prices is finite. Hence the set A has dimension at most $HL + L - 2$. Now apply Lemma 5 with $l = HL$, $n = L$, $m = 0$ to obtain the result. $\square$

Somewhat surprisingly the theorem is not true for just any fixed μ. Intuitively, an economy may have many more equilibria than parameters (for example, an economy with 2 agents and 2 goods has 4 endowment parameters but might have hundreds of equilibria) and so by perturbing parameters equilibrium prices cannot be perturbed independently. We illustrate this issue in an example in Section 4 below.

If we knew the polynomial ω_μ we could easily determine the number of Walrasian equilibria for the economy with endowments $(e^{\mathcal{H}})$. Recall that for fixed $e^{\mathcal{H}}$ this polynomial is a univariate polynomial in y. Counting equilibria then reduces to simply counting the number of solutions of the univariate polynomial in y and verifying finitely many polynomial inequalities (for the corresponding equilibrium variables). Solving a univariate polynomial is straightforward and Sturm's Theorem gives us an algorithm to count the number of (positive) solutions of a univariate polynomial. We return to this issue in the next section.

While quantifier elimination provides an algorithm for computing ω_μ, this approach is hopelessly inefficient. Surprisingly it turns out that, using tools from computational algebraic geometry, we can determine the polynomial ω much more efficiently. This insight provides the basis of our strategy to find all Walrasian equilibria. First we need to characterize equilibria by a system of polynomial equations.

2.4 From Equilibrium to Polynomial Equations

The central objective of this paper is to characterize and compute equilibria as solutions to a polynomial system of equations. Recall that interior Walrasian equilibria of our model are defined as solutions to the system of equations (1)–(4). Equations (2)–(4) are simple polynomial equations. But equations (1) are often not polynomial – even under our fundamental assumption that marginal utilities are semi-algebraic functions. This assumption, however, allows us to transform these equations into polynomial expressions. Unfortunately this transformation comes at the price of numerous new technical difficulties.

The marginal utility $\partial_{c_l} u^h : \mathbb{R}^L_+ \to \mathbb{R}$ is assumed to be semi-algebraic. Lemma 2 then ensures the existence of a nonzero polynomial $m^h_l(c, y)$ with $m^h_l \in \mathbb{R}[c, y]$ such that for every $c \in \mathbb{R}^L_+$,

$$m^h_l(c, \partial_{c_l} u^h(c)) = 0. \tag{6}$$

Without loss of generality we can assume the polynomial m^h_l to be square-free. In a slight abuse of notation we define $m^h(c, \partial_c u^h(c)) = (m^h_1(c, \partial_{c_1} u^h(c)), \ldots, m^h_L(c, \partial_{c_L} u^h(c)))$. Our assumptions on $\partial_{c_l} u^h$ yield some properties for the polynomials m^h_l, $l = 1, \ldots, L$, $h \in \mathcal{H}$.

Proposition 1 *Consider square-free nonzero polynomials m^h_l satisfying equation (6) for $l = 1, \ldots, L$, $h \in \mathcal{H}$. Then the following statements hold.*

(1) The dimension of the set $V(m^h_l) = \left\{(c, y) \in \mathbb{R}^L_+ \times \mathbb{R} : m^h_l(c, y) = 0\right\}$ is L.

(2) The set

$$\begin{aligned} S^h_l \;\; = \;\; & \{(c, y) \in \mathbb{R}^L_+ \times \mathbb{R} : m^h_l(c, y) = \partial_{c_1} m^h_l(c, y) = \partial_{c_2} m^h_l(c, y) = \ldots \\ & \qquad \ldots = \partial_{c_L} m^h_l(c, y) = \partial_y m^h_l(c, y) = 0\} \end{aligned}$$

is a closed semi-algebraic subset of $\mathbb{R}^L_+ \times \mathbb{R}$ with dimension of at most $L-1$. The projection of S^h_l on $\mathbb{R}^L_+$ is also a closed semi-algebraic subset with dimension of at most $L-1$.

(3) The set

$$\left\{c \in \mathbb{R}^L_+ : u^h \text{ is not } C^\infty \text{ at } c\right\} \cup \bigcup_{l=1}^{L} \left\{c \in \mathbb{R}^L_+ : \partial_y m^h_l(c, \partial_{c_l} u^h(c)) = 0\right\}$$

is a closed semi-algebraic subset of $\mathbb{R}^L_+$ with a dimension of at most $L-1$. Put differently, at every point of the complement of a closed lower-dimensional semi-algebraic subset of $\mathbb{R}^L_+$ it holds that u^h is C^∞ and $\partial_y m^h_l(c, \partial_{c_l} u^h(c)) \neq 0$ for all $l = 1, \ldots, L$.

(4) The set

$$B^h = \left\{c \in \mathbb{R}^L_+ : u^h \text{ is not } C^\infty \text{ at } c\right\} \cup \left\{c \in \mathbb{R}^L_+ : \det\left(\partial_c m^h(c, \partial_c u^h(c))\right) = 0\right\}$$

is a closed semi-algebraic subset of $\mathbb{R}^L_+$ with a dimension of at most $L-1$.

Proof. Statement (1) follows by construction of m^h_l since the marginal utility function $\partial_{c_l} u^h$ is defined for all $c \in \mathbb{R}^L_+$. Thus, for all $c \in \mathbb{R}^L_+$ there is a $y \in \mathbb{R}$ satisfying $m^h_l(c, y)$. The dimension of $V(m^h_l)$ cannot be $L+1$ since m^h_l is a nonzero polynomial. Statement (2) follows from m^h_l being square-free and the fact that the projection of a semi-algebraic set is itself semi-algebraic.

Marginal utility $\partial_c u^h$ is a semi-algebraic function and thus C^∞ at every point of the complement of a closed semi-algebraic subset of $\mathbb{R}^L_+$ of dimension less than L. The implicit function theorem implies that at a point $\bar{c}$ with $\partial_y m^h_l(\bar{c}, \partial_{c_l} u^h(\bar{c})) \neq 0$ the function $\partial_{c_l} u^h$ is C^∞. The implicit function theorem also implies that at a point $\bar{c}$ with $\partial_y m^h_l(\bar{c}, \partial_{c_l} u^h(\bar{c})) = 0$ the function $\partial_{c_l} u^h$ can be C^∞ only if $\partial_{c_k} m^h_l(c, \partial_{c_l} u^h(c)) = 0$ for all $k = 1, \ldots, L$. Statement (2) implies that this property can hold only in a semi-algebraic set with dimension of at most $L - 1$. The finite union of semi-algebraic sets of dimension less than L is again just that, a semi-algebraic sets with dimension of at most $L - 1$. Thus, Statement (3) holds.

Utility u^h is strictly concave and so $\partial_c u^h$ is strictly decreasing. Statement (3) and the implicit function theorem then imply Statement (4). □

We can now use the implicit representation (6) of marginal utility to transform each individual equation of system (1),

$$\partial_{c_l} u^h(c^h) - \lambda^h p_l = 0, \tag{7}$$

into the polynomial equation

$$m^h_l(c^h, \lambda^h p_l) = 0. \tag{8}$$

Simply by construction any solution to (7) also satisfies (8). Define the polynomial $F \in \mathbb{R}[c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p]$ by

$$F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = \begin{cases} m^h(c^h, \lambda^h p), & h \in \mathcal{H} \\ p \cdot (c^h - e^h), & h \in \mathcal{H} \\ \sum_{h \in \mathcal{H}} (c^h_l - e^h_l), & l = 1, \ldots, L-1 \\ \sum_l p_l - 1 & \end{cases}$$

Instead of focusing on the equilibrium system (1)–(4) our attention now turns to the system of equations $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0$. This system has the original equations (1) replaced by polynomial equations of the form (8) but otherwise continues to include the original equations (2)–(4). Therefore, this system consists only of polynomial equations. We collect the first two sets of polynomial expressions in $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)$ in the 'demand system' and define for each $h \in \mathcal{H}$,

$$D^h(c, \lambda, p) = \begin{pmatrix} m^h(c, \lambda p) \\ p \cdot (c - e^h) \end{pmatrix}.$$

Theorem 2 *All Walrasian equilibria are solutions to the system of polynomial equations*

$$F((c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0. \tag{9}$$

For every endowment vector $e^{\mathcal{H}}$ in the complement of a closed lower-dimensional semi-algebraic subset of $\mathbb{R}^{HL}_{++}$ all Walrasian equilibria also have the property that for each $h \in \mathcal{H}$, the rank of the matrix

$$\left[\partial_{(c,\lambda)} D\left(c^h, \lambda^h, p\right)\right]$$

is $(L+1)$ and thus is full.

To simplify the proof of the theorem we make use of individual demand functions. For this purpose we introduce the following notation. The positive price simplex is $\Delta_{++}^{L-1} = \{p \in \mathbb{R}_{++}^L : \sum_l p_l = 1\}$. Individual demand of agent h at prices p and income τ is $d^h(p,\tau) = \arg\max_{c\in\mathbb{R}_+^L} u^h(c)$ s.t. $p \cdot c = \tau$. Individual demand functions are continuous. As explained in Section 2.3, the Tarski-Seidenberg Principle ensures that the continuous function $d^h : \Delta_{++}^{L-1} \times \mathbb{R}_{++} \to \mathbb{R}^L$ is also semi-algebraic.

Proof. Simply by construction all solutions to (1)–(4) are solutions to system (9).

The individual demand $d^h(p,\tau)$ of agent h is determined by the agent's first-order conditions,

$$\begin{aligned} \partial_c u^h(c^h) - \lambda^h p &= 0, \\ p \cdot c^h - \tau &= 0. \end{aligned}$$

Since $p \in \Delta_{++}^{L-1}$ these equations are equivalent to

$$\begin{aligned} \frac{\partial_c u^h(c^h)}{\sum_l \partial_{c_l} u^h(c^h)} &= p, \\ \sum_l c_l^h \frac{\partial_{c_l} u^h(c^h)}{\sum_{l'} \partial_{c_{l'}} u^h(c^h)} &= \tau. \end{aligned}$$

Observe that the function $G : \mathbb{R}_+^L \to \Delta_{++}^{L-1} \times \mathbb{R}_{++}$ given by the expressions on the left-hand side

$$G(c^h) = \begin{cases} \frac{\partial_c u^h(c^h)}{\sum_l \partial_{c_l} u^h(c^h)} \\ \sum_l c_l^h \frac{\partial_{c_l} u^h(c^h)}{\sum_{l'} \partial_{c_{l'}} u^h(c^h)} \end{cases}$$

is a continuous semi-algebraic function. Consider the set B^h from Statement (4) of Proposition 1. This set has dimension of at most $L-1$ and so the same must be true for the semi-algebraic set

$$G\left(B^h\right) = \left\{(p,\tau) \in \Delta_{++}^{L-1} \times \mathbb{R}_{++} : G(c^h) = (p,\tau) \text{ for some } c^h \in B^h\right\}.$$

Next consider the following function from Blume and Zame (1992),

$$H(p,\tau,e^2,\ldots,e^H) = \begin{cases} d^1(p,\tau) + \sum_{h=2}^H (d^h(p, p\cdot e^h) - e^h) \\ e^2 \\ \vdots \\ e^H \end{cases} \tag{10}$$

for $H : G(B^1) \times \mathbb{R}_{++}^{(H-1)L} \to \mathbb{R}_{++}^{HL}$. Note that the domain of H is a semi-algebraic subset with dimension at most $HL-1$. Lemma 3 then ensures the existence of a finite partition of $\mathbb{R}_{++}^{HL}$ into semi-algebraic subsets $C_1, \ldots, C_m$ such that for all subsets C_i of dimension HL and $e \in C_i$ it holds that $H^{-1}(e)$ is empty.

Thus, only for a closed lower-dimensional subset of endowments it will be true that $c^1 \in B^1$. This argument works for all agents $h \in \mathcal{H}$. The finite union of semi-algebraic subsets of dimension less than HL is again a semi-algebraic subset of dimension less than HL. Therefore, for all endowment vectors $(e^1, \ldots, e^H)$ outside a closed lower-dimensional semi-algebraic subset of $\mathbb{R}^{HL}_{++}$ all Walrasian equilibria have consumption allocations such that $c^h \notin B^h$ for all $h \in \mathcal{H}$. For such consumption allocation the standard argument for showing that

$$\partial_{(c,\lambda)} D^h \left(c^h, \lambda^h, p\right)$$

has full rank now goes through. □

The following corollary to Theorem 2 is a consequence of Sard's Theorem, i.e. Lemma 4 above.

COROLLARY 1 *For every endowment vector* $e^{\mathcal{H}}$ *in the complement of a closed lower-dimensional semi-algebraic subset of* $\mathbb{R}^{HL}_{++}$ *all Walrasian equilibria have the property that the rank of the matrix*

$$\left[\partial_{c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p} F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)\right] \tag{11}$$

is $H(L+1) + L$ *and thus is full.*

Proof. Theorem 2 and its proof imply that there exists a subset of $\Delta^{L-1} \times \mathbb{R}_{++} \times \mathbb{R}^{(H-1)L}_{++}$ such that the function H as defined by Equation (10) is C^∞ on this set and the complement of its image in $\mathbb{R}^{HL}_{++}$ is closed and has dimension less than HL. By Lemma 4 it must therefore be true that there is a semi-algebraic set $\bar{E} \subset \mathbb{R}^{HL}_{++}$ whose complement is lower dimensional and closed such that for each $e^{\mathcal{H}} \in \bar{E}$, if p is a W.E. price we must have that the matrix $\sum_{h \in \mathcal{H}} \partial_p d^h(p, p \cdot e^h)$ has full rank $L - 1$. Since by the implicit function theorem, at these points for each h,

$$\partial_p d^h(p, p \cdot e^h) = -\left(\partial_{c,\lambda} D^h(c^h, \lambda^h, p)\right)^{-1} \partial_p D^h(c^h, \lambda^h, p),$$

the matrix $\sum_{h \in \mathcal{H}} \left(\partial_{c,\lambda} D^h(c^h, \lambda^h, p)\right)^{-1} \partial_p D^h(c^h, \lambda^h, p)$ must have full rank. The matrix $\partial_{c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p} F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)$ must have then have full row rank, since after elementary row operations one obtains an equivalent matrix of the form

$$\begin{pmatrix} I_{H(L+1) \times H(L+1)} & \left[\left(\partial_{c,\lambda} D^h(c^h, \lambda^h, p)\right)^{-1} \partial_p D^h(c^h, \lambda^h, p)\right]_{h=1}^{H} \\ M & 0 \end{pmatrix},$$

where M consists of the derivatives of market clearing with respect to consumptions and $\lambda^{\mathcal{H}}$. □

We illustrate some of the possible complications in the context of an example.

EXAMPLE 1 *Consider the piece-wise continuous function*

$$u'(c) = \begin{cases} \frac{4}{\sqrt{c}} & 0 < c \leq 1, \\ 6 - 2c & 1 < c \leq 2, \\ \frac{4}{c} & 2 < c. \end{cases}$$

The polynomial

$$m(c, y) = (16 - cy^2)(6 - 2c - y)(4 - cy)$$

satisfies $m(c, u'(c)) = 0$ *for all* $c > 0$.

Unfortunately, for all values of c the equation $m(c, y) = 0$ allows positive solutions other than $y = u'(c)$. For example, for $c = 4$ not only $y = u'(4) = 1$ but also $y = 2$ yields $m(4, y) = 0$. Intuitively, the solution $(4, 2)$ is on the "wrong" branch of the function. At $(4, 2)$ the term $(16 - cy^2)$ is zero but the domain for this term is only $(0, 1]$. For each value of $c \in \mathbb{R}_{++}$ there are altogether four (real) solutions to the equation $m(c, y) = 0$.

The system $m(c, y) = \partial_c m(c, y) = \partial_y m(c, y) = 0$ has three solutions, $(1, 4)$, $(2, 2)$, and $(4, -2)$. For each value of $c \in \mathbb{R}_{++}$ the partial derivative term $\partial_y m(c, y)$ is a cubic polynomial in y with at most three real solutions. Observe that u is differentiable in $\mathbb{R}_{++} \setminus \{1, 2\}$. So, the set B of ill-behaved points in the sense of Proposition 1, Statement (4), is finite and thus of dimension $L - 1 = 0$.

This last fact would not be true if the polynomial $m(c, y)$ were not square-free. The polynomial $\tilde{m}(c, y) = (16 - cy^2)(6 - 2c - y)(4 - cy)^2$ has the identical zero set as $m(c, y)$. But note that $\partial_c \tilde{m}(c, y) = \partial_y \tilde{m}(c, y) = 0$ whenever $(4 - cy) = 0$.

The example highlights the fact that the system of polynomial equilibrium equations (9) may have more solutions than the original equilibrium equations (1) – (4). But it still only has finitely many of them. Once one has finitely many candidate Walrasian equilibrium one can find the actual equilibria by verifying finitely many systems of polynomial equalities and inequalities: The Walrasian equilibria lie in a semi-algebraic set that can be written as in Lemma 1. Given individual endowments $e^{\mathcal{H}}$, we are thus interested in the set of competitive equilibria,

$$\mathcal{E} = \{(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) \in \mathbb{R}_{++}^{H(L+1)+L} \text{ that solve (9)} : 1 - 4 \text{ hold}\} \tag{12}$$

2.5 Semi-algebraic Classes of Economies

So far, the analysis was done for a fixed profile of utility functions. However, it is straightforward to extend the method and to consider parameterized classes of utility functions that are semi-algebraic. In particular, if we assume that for each agent h, utility u^h is parameterized by some $\xi \in \Xi \subset \mathbb{R}^M$ and we assume that $\partial_{x_l} u^h(x, \xi)$ is semi-algebraic in both x and ξ, all results carry through and we obtain that there exist non-zero polynomials $m^h(c, y; \xi)$ such that

$$m_l^h(c, \partial_{c_l} u^h(c, \xi), \xi) = 0.$$

Furthermore, for each $\xi \in \Xi$, m_l^h satisfies the properties of Proposition 1 and for each $(\xi^h)_{h\in\mathcal{H}} \in \Xi^H$, Theorem 2 holds true.

2.6 Economic Implications of Semi-algebraic Utility

Before we show how to solve the polynomial system (9) and thereby compute all Walrasian equilibria of our semi-algebraic economy, some words on the relevance and restrictiveness (actually, the lack thereof) of our key assumption are in order. How general is the premise of semi-algebraic marginal utility?

From a practical point of view, it is easy to see that Cobb-Douglas and CES utility functions with rational elasticities of substitution, $\sigma \in \mathbb{Q}$, are semi-algebraic utility functions. Therefore, a large number of interesting applied economic models satisfy our assumption.

From a theoretical point of view, note that if a function is semi-algebraic, so are all its derivatives (the converse is not true, as the example $f(x) = \log(x)$ shows). It follows from Blume and Zame (1992) that semi-algebraic preferences (i.e. the assumption that better sets are semi-algebraic sets) implies semi-algebraic utility.

Also note that by Afriat's theorem (Afriat (1967)), any finite number of observations on Marshallian individual demand that can be rationalized by arbitrary non-satiated preferences can be rationalized by a piecewise linear, hence semi-algebraic function. While Afriat's construction does not yield a semi-algebraic, C^1, and strictly concave function, the construction in Chiappori and Rochet (1987) can be modified to our framework and we obtain the following lemma.

Lemma 6 *Given N observations $(c^n, p^n) \in \mathbb{R}^{2l}_{++}$ with $p^i \neq p^j$ for all $i \neq j = 1, \ldots, N$, the following are equivalent.*

(1) There exists a strictly increasing, strictly concave and continuous utility function u such that

$$c^n = \arg\max_{c\in\mathbb{R}^l_+} u(c) \text{ s.t. } p^n \cdot c \leq p^n \cdot c^n.$$

(2) There exists a strictly increasing, strictly concave, semi-algebraic and C^1 utility function v such that

$$c^n = \arg\max_{c\in\mathbb{R}^l_+} v(c) \text{ s.t. } p^n \cdot c \leq p^n \cdot c^n.$$

To prove the lemma, observe that if statement (1) holds, the observations must satisfy the condition 'SSARP' from Chiappori and Rochet (1987). Given this one can follow their proof closely to show that there exists a C^1 semi-algebraic utility function that rationalizes the data. The only difference to their proof is that in the proof of their Lemma 2, one needs to use a polynomial 'cap'-function which is at least C^1. In particular, the argument in Chiappori and Rochet goes through if one replaces C^∞ everywhere with C^1 and uses the cap-function $\rho(c) = \max(0, 1 - \sum_l c_l^2)^2$. Since the integral of a polynomial function is polynomial, the resulting utility function is piecewise polynomial, i.e. semi-algebraic.

Mas-Colell (1977) shows, in light of the theorems of Sonnenschein, Mantel and Debreu, that for any compact (non-empty) set of positive prices $P \subset \Delta^{L-1}$ there exists an exchange economy with (at least) L households, $((u^h)_{h=1}^L, (e^h)_{h=1}^L)$, with u^h strictly increasing, strictly concave and continuous such that the equilibrium prices of this economy coincide precisely with P.

Given Lemma 6 above, this result directly implies that for any finite set of prices $P \subset \Delta$, there exists an exchange economy $((u^h)_{h=1}^L, (e^h)_{h=1}^L)$, with u^h strictly increasing, strictly concave, semi-algebraic and C^1 such that the set of equilibrium prices of this economy contains P. Therefore, the abstract assumption of semi-algebraic preferences imposes *no* restrictions on multiplicity of equilibria. Mas-Colell (1977) also shows that if the number of equilibria is odd, one can construct a regular economy and that there exist open sets of individual endowments for which the number of equilibria can be an arbitrary odd number.

Finally note that the results we obtain below are robust with respect to perturbations of preferences outside of the semi-algebraic class: If a semi-algebraic utility is C^2, and a regular economy has n equilibria, it follows from Smale (1974) that there is a C^2 Whitney-open neighborhood around the profile of utilities for which the number of equilibria is n.

In summary, our key assumption of semi-algebraic utility offers little if any room for objection. Much applied work in economics assumes semi-algebraic utility. Utility functions derived from demand observations are semi-algebraic. And the assumption does not entail any restrictions on the number of equilibria.

3 Polynomial Equation Solving and Gröbner Bases

We have seen that all Walrasian equilibria of our economic model are among the solutions of a system of polynomial equations. We now turn to the issue of solving such systems. The study of solving polynomial equations requires us to considerably change the mathematical focus of our discussion. So far our analysis relied heavily on fundamental results from the mathematical discipline of 'Real Algebraic Geometry', notably the Tarski-Seidenberg Principle and the Hardt Triviality Theorem. We now move into the discipline of '(Computational) Algebraic Geometry' and use concepts such as Gröbner Bases and Buchberger's Algorithm.

The parameters in our polynomial equations are real or even rational numbers. Much of the study of polynomial equations, however, is done on algebraically closed fields, that is, on fields where each non-constant univariate polynomial has a zero. Neither the field $\mathbb{Q}$ of rational numbers nor the field $\mathbb{R}$ of real numbers is algebraically closed, but the field $\mathbb{C}$ of complex numbers is. Therefore, we need to set our system of equations also into the field $\mathbb{C}$. But, of course, after we have found all (complex) solutions to that system, only the real solutions can be of economic interest.

Given a polynomial system of equations $f(x) = 0$ with $f : \mathbb{C}^n \to \mathbb{C}^n$ that has finitely many complex solutions, there are now a variety of algorithms to numerically approximate all complex zeros of f. Sturmfels' (2002) monograph provides an excellent overview on polynomial equation solving. To make use of some results in this literature we need to introduce some definitions and concepts from algebraic geometry.

3.1 Some Algebraic Geometry

Recall that the set of all polynomials in n variables with coefficients in some field $\mathbb{K}$ forms a ring which we denote by $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$. A subset I of the polynomial ring $\mathbb{K}[x]$ is called an ideal if it is closed under sums, $f + g \in I$ for all $f, g \in I$, and it satisfies the property that $h \cdot f \in I$ for all $f \in I$ and $h \in \mathbb{K}[x]$. For given polynomials $f_1, \ldots, f_k$, the set

$$I = \{\sum_{i=1}^{k} h_i f_i : h_i \in \mathbb{K}[x]\} = \langle f_1, \ldots, f_k \rangle,$$

is an ideal. It is called the ideal generated by $f_1, \ldots, f_k$. This ideal $\langle f_1, \ldots, f_k \rangle$ is the set of all linear combinations of the polynomials $f_1, \ldots, f_k$, where the "coefficients" in each linear combination are themselves polynomials in the polynomial ring $\mathbb{K}[x]$. The Hilbert Basis Theorem states that for any ideal $I \subset \mathbb{K}[x]$ there exist finitely many polynomials that generate I. A set of such polynomials generating the ideal I is called a basis of I.

The notion of ideals is fundamental to solving polynomial equations. The set of common complex zeros of the polynomials $f_1, f_2, \ldots, f_k$, that is, the set

$$V(f_1, f_2, \ldots, f_k) = \{x \in \mathbb{C}^n : f_1(x) = \ldots = f_k(x) = 0\}$$

is called the complex variety defined by $f_1, \ldots, f_k$. The variety does not change if we replace the polynomials $f_1, \ldots, f_k$ by another basis $g_1, \ldots, g_l$ generating the same ideal. That is, the notion of complex variety can be defined for ideals and not just for a set of polynomials. For an ideal $I = \langle f_1, \ldots, f_k \rangle = \langle g_1, \ldots, g_l \rangle$ we can write

$$V(I) = V(f_1, f_2, \ldots, f_k) = V(g_1, g_2, \ldots, g_l).$$

Let us emphasize this point. The set of common zeros of a set of polynomials $f_1, f_2, \ldots, f_k$ is identical to the common set of zeros of all (infinitely many!) polynomials in the ideal $I = \langle f_1, f_2, \ldots, f_k \rangle$. In particular, any other basis of I has the same zero set. If the set $V(I)$ is finite and thus zero-dimensional, we call the ideal I itself zero-dimensional.

At this point of our discussion the reader may already have guessed a promising strategy for solving a system of polynomial equations. Considering that the set of solutions to a system $f_1(x) = \ldots = f_k(x) = 0$ is the same for any basis of the ideal $I = \langle f_1, f_2, \ldots, f_k \rangle$, we ask whether we can find a basis that has "nice" properties and which makes describing the solution set $V(I)$ straightforward. Put differently, our question is: Can we transform the original system $f_1(x) = \ldots = f_k(x) = 0$ into a new system $g_1(x) = \ldots = g_l(x) = 0$ that can be easily solved, particularly if the solution set is zero-dimensional? The mathematical field of 'Algebraic Geometry' answers our question with a resounding "Yes!".

'Gröbner Bases' are such bases that have desirable algorithmic properties for solving polynomial systems of equations. Specifically, the 'reduced Gröbner basis $\mathcal{G}$ in the lexicographic term order' is ideally suited for solving systems of polynomial equations. A proper definition of the relevant notions of Gröbner basis, reduced Gröbner basis, and lexicographic term order is rather tedious. But the main mathematical result that is useful for our purposes is easily understood without many additional mathematical definitions. Therefore we do not give all these definitions here and instead refer the interested reader to the books by Cox et al. (1997) and Sturmfels (2002).

The one remaining term we need to define is that of a radical ideal. The *radical* of an ideal I is defined as $\sqrt{I} = \{f \in \mathbb{K}[x] : \exists m \geq 1 \text{ such that } f^m \in I\}$. The radical $\sqrt{I}$ is itself an ideal and contains I, $I \subset \sqrt{I}$. An ideal I is called radical if $I = \sqrt{I}$. We mention some motivation for this definition in our discussion of the following lemma, the so-called Shape Lemma. For a proof of the Shape Lemma see Becker et al. (1994).

LEMMA 7 (SHAPE LEMMA) *Let I be a zero-dimensional radical ideal in $\mathbb{K}[x_1, \ldots, x_n]$ with $\mathbb{K} \subset \mathbb{C}$ such that all d complex elements of $V(I)$ have distinct values for the last coordinate x_n. Then the reduced Gröbner basis of I (in the lexicographic term order) has the shape*

$$\mathcal{G} = \{x_1 - v_1(x_n), \ldots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$$

where r is a polynomial of degree d and the v_i are polynomials of degree strictly less than d.

The Shape Lemma implies that the zero set $V(f_1, f_2, \ldots, f_n)$ of a system of polynomial equations $f_1(x) = f_2(x) = \ldots = f_n(x) = 0$ is also the solution set to another equivalent system of polynomial equations having a very simple form. The equivalent system consists of one univariate equation $r(x_n) = 0$ in the last variable x_n and $n-1$ equations, each of which depends only on a (different) single variable x_i and the last variable x_n. These equations are linear in their respective x_i, $i = 1, 2, \ldots, n-1$. The Shape Lemma clearly suggests how we can find all solutions to a polynomial system of equations. If the assumptions of the lemma are satisfied, we should first compute the Gröbner basis G. Then we need to find all solutions to a univariate equation in the last variable. The values for all other variables are then trivially given by the remaining equations. Finding all solutions to a complicated system of polynomial equations in many variables thus requires determining the Gröbner basis $\mathcal{G}$ and finding all solutions to a univariate polynomial equation.

Some simple examples shed some light on the assumptions of the Shape Lemma. Consider the system of equations $x_1^2 - x_2 = 0$, $x_2 - 4 = 0$ and its solutions $(2, 4)$ and $(-2, 4)$. Both solutions have the same value for the last coordinate x_2. Clearly, no polynomial of the form $x_1 - v_1(x_2)$ can yield the two possible values -2 and 2 for x_1 when $x_2 = 4$. The linearity in x_1 prohibits this from being possible. Next consider the system $x_1^2 - x_2 + 1 = 0$, $x_2 - 1 = 0$ and its solution $(0, 1)$. Observe that for $x_2 = 1$ the first equation yields $x_1^2 = 0$ and so 0 is a multiple zero of this equation. There cannot be a Gröbner basis linear in x_1 that yields a multiple zero. For polynomial systems with zero-dimensional solution sets, multiple zeros are ruled out by the Shape Lemma's assumption that I is a radical ideal. In a nutshell, a multiple zero requires the ideal to contain a polynomial of the form f_i^m with $m \geq 2$ but not to contain f_i. This cannot happen for a radical ideal. (Note that this simple intuition is only correct for zero-dimensional ideals. In higher dimensions additional tricky issues arise.)

There is a large literature on the computation of a Gröbner basis for arbitrary sets of polynomials. In particular, *Buchberger's algorithm* always produces a Gröbner basis in finitely many steps. We refer the interested reader to the book by Cox et al. (1997).

In this paper we use the computer algebra system SINGULAR to compute Gröbner basis. The implementation uses a variation of Buchberger's original algorithm that improves efficiency considerably (see Greuel et al. (2005) for a description of the algorithm). While this algorithm is well defined independently of the field $\mathbb{K}$, it can be performed *exactly* over $\mathbb{Q}$. That is, the polynomials $r, v_1, v_2, \ldots, v_{n-1}$ can be determined exactly. The coefficients of these polynomials are rational numbers with (often very large) numerators and denominators. Note that the exactness property is important to us from the viewpoint of economic theory. It means that we can prove statements about the maximal number of solutions to a given equilibrium system. Only once we calculate the solutions to the system we need to allow for approximation errors.

For our economic model we do not only want to solve a single system of polynomial equations characterizing an economic equilibrium. Instead, we often think of our economy being parameterized by a set of parameters and so would like to make statements about the equilibrium manifold. Economic parameters lead to polynomial systems with parameters as coefficients. Therefore, we need a specialized version of the Shape Lemma with parametric coefficients. In addition, in our economic models we cannot prohibit multiple equilibria to have identical values for one or several variables. We cannot in general assume that the assumption of the Shape Lemma on the distinct values of the last variable x_n is satisfied. Introducing a new last variable y and a generic linear equation $y = \sum_i u_i x_i$ with random coefficients u_i relating all existing variables to the new variable guarantees that this assumption does hold.

Lemma 8 (Parameterized Shape Lemma)
Let $E \subset \mathbb{K}^m$, $\mathbb{K} \subset \mathbb{C}$, be an open set of parameters, $(x_1, \ldots, x_n) \in \mathbb{C}^n$ a set of variables and let $f_1, \ldots, f_n \in \mathbb{K}[e_1, \ldots, e_m; x_1, \ldots, x_n]$. Assume that for each $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_m) \in E$, the ideal $I(\bar{e}) = \langle f_1(\bar{e}; \cdot), \ldots, f_n(\bar{e}; \cdot)\rangle$ is zero-dimensional and radical in $\mathbb{K}[x]$. Furthermore, assume that there exist $u_1, \ldots, u_n \in \mathbb{K}$, such that for all $\bar{e}$ any solutions $\bar{x} \neq \bar{x}'$ satisfying

$$f_1(\bar{e}; \bar{x}) = \ldots = f_n(\bar{e}; \bar{x}) = 0 = f_1(\bar{e}; \bar{x}') = \ldots = f_n(\bar{e}; \bar{x}')$$

also satisfy $\sum_i u_i \bar{x}_i \neq \sum_i u_i \bar{x}'_i$. Then there exist $r, v_1, \ldots, v_n \in \mathbb{K}[e; y]$ and $\rho_1, \ldots, \rho_n \in \mathbb{K}[e]$

such that for $y = \sum_i u_i x_i$ and for all $\bar{e}$ outside a closed lower-dimensional subset E_0 of E,

$$\begin{aligned} V(I(\bar{e})) &= \{x \in \mathbb{C}^n : f_1(\bar{e}, x) = \ldots = f_n(\bar{e}, x) = 0\} \\ &= \{x \in \mathbb{C}^n : \rho_1(\bar{e}) x_1 = v_1(\bar{e}; y), \ldots, \rho_n(\bar{e}) x_n = v_n(\bar{e}; y) \text{ for } r(\bar{e}; y) = 0\}. \end{aligned}$$

For all $\bar{e} \in E \setminus E_0$ the complex variety $V(I(\bar{e}))$ has an identical number of d elements. The degree of r in y is d, the degrees of $v_1, \ldots, v_n$ in y is at most $d-1$.

If the coefficients of the polynomials $f_1, \ldots, f_n$ are parameters, then Buchberger's algorithm yields a set of polynomials $g_1, \ldots, g_n$ with coefficients that are themselves polynomial functions of the parameters. This set of polynomials forms a Gröbner basis for the ideal $\langle f_1, \ldots, f_n \rangle$ for all values of the parameters outside the union of the solution sets to finitely many polynomial equations. Intuitively, for some parameter values a polynomial in some denominator may be zero. In that case the Gröbner basis would be different since Buchberger's algorithm performed an ill-defined division. (It is possible to compute a Gröbner basis that simultaneously works for all choices of parameters. Such bases are called 'Comprehensive Gröbner Bases', see Suzuki and Sato (2006) and Weispfenning (1992). We do not need this notion for our purposes.)

3.1.1 Sufficient Condition for Shape Lemma

Given a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ one can define partial derivatives with respect to complex numbers in the usual way. Write

$$f = c_0(x_{-j}) + c_1(x_{-j}) x_j + \ldots + c_d(x_{-j}) x_j^d,$$

where the c_i are polynomials in the variables $x_{-j} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Then,

$$\frac{\partial f}{\partial x_j} := c_1(x_{-j}) + \ldots + d c_d(x_{-j}) x_j^{d-1}.$$

Given a system of polynomial equations $f : \mathbb{C}^n \to \mathbb{C}^n$, the Jacobian $\partial_x f(x)$ is defined as usual as the matrix of partial derivatives. A sufficient condition for an ideal $\langle f_1, \ldots, f_n \rangle$ to be radical and zero-dimensional is that $det(\partial_x(f_1(x), \ldots, f_n(x))) \neq 0$ whenever $f_1(x) = \ldots = f_n(x) = 0$.

3.1.2 The Number of Real Zeros

The Shape Lemma reduces the problem of solving a system of polynomial equations essentially to solving a single univariate polynomial equation. This equivalence enables us to employ bounds on the number of zeros of univariate polynomials to derive bounds on the number of solutions to polynomial systems. From an economic perspective we are particularly interested in bounding the number of positive real solutions of our equilibrium systems.

The Fundamental Theorem of Algebra, see e.g. Sturmfels (2002), states that a univariate polynomial, $f(x) = \sum_{i=0}^{d} a_i x^i$, with rational, real or complex coefficients a_i, $i = 0, 1, \ldots, d$, has d zeros, counting multiple roots, in the field $\mathbb{C}$ of complex numbers. That is, the degree d of the polynomial f is an upper bound on the number of complex zeros. More importantly for our economic analysis even better bounds are available for the number or real zeros. For a finite sequence $a_0, \ldots, a_k$ of real numbers the number of sign changes is the number of products $a_i a_{i+l} < 0$, where $a_i \neq 0$ and a_{i+l} is the next non-zero element of the sequence. Zero elements are ignored in the calculation of the number of sign changes. The classical Descartes's Rule of Signs, see Sturmfels (2002), states that the number of positive real zeros of f does not exceed the number of sign changes in the sequence of the coefficients of f. This bound is remarkable because it bounds the number of (positive) real zeros. It is possible that a polynomial system is of very high degree and has many solutions but the Descartes bound on the number of positive real zeros of the representing polynomial f in the Shape Lemma proves that the system has a single real positive solution.

The Descartes bound is not tight and overstates the true number of positive real solutions for many polynomials. Sturm's Theorem, see Sturmfels (2002) or Bochnak et al. (1998), yields an exact bound on the number of positive real solutions of a univariate polynomial. For a univariate polynomial f, the Sturm sequence of $f(x)$ is a sequence of polynomials $f_0, \ldots, f_k$ defined as follows,

$$f_0 = f, \;\; f_1 = f', \;\; f_i = f_{i-1} q_i - f_{i-2} \;\; \text{for } 2 \leq i \leq m$$

where f_i is the negative of the remainder on division of f_{i-2} by f_{i-1}, so q_i is a polynomial and the degree of f_i is less than the degree of f_{i-1}. The sequence stops with the last nonzero remainder f_m. Sturm's Theorem provides an exact root count, see e.g. Bochnak et al. (1998) for a proof.

LEMMA 9 (STURM'S THEOREM) *Let f be a polynomial with Sturm sequence $f_0, \ldots, f_k$ and let $a < b \in \mathbb{R}$ with neither a nor b a root of f. Then the number of roots of f in the interval $[a, b]$ is equal to the number of sign changes of $f_0(a), \ldots, f_k(a)$ minus the number of sign changes of $f_0(b), \ldots, f_k(b)$.*

3.2 Shape Lemma and Competitive Equilibria

We apply the Shape Lemma to the system of polynomial equations (9) derived in Section 2.4. For this purpose we view equation (9) as a system of equations in complex space. To simplify the notation in our application of the Shape Lemma let $M = H(L + 1) + L$ and associate with $x \in \mathbb{C}^M$ the vector $(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)$. We are now concerned with the system of $M + 1$ polynomial equations

$$F(e^{\mathcal{H}}; x) = 0, \tag{13}$$

$$y - \sum_{i=1}^{M} \mu_i x_i = 0, \tag{14}$$

with parameters $\mu = (\mu_1, \ldots, \mu_M) \in \Delta^{M-1}$ and the variables $x \in \mathbb{C}^M$ and $y \in \mathbb{C}$. Recall that equations (13) rely fundamentally on our assumptions on utility functions $(u^h)_{h\in\mathcal{H}}$. The following theorem provides the basis for all further analysis.

THEOREM 3 *There exists a closed lower-dimensional subset $D_0 \subset \Delta^{M-1}$ and a closed subset $E_0 \subset \mathbb{R}^{HL}_{++}$ of Lebesgue measure zero such that for all $\mu \in \Delta^{M-1} \setminus D_0$ and all $e^{\mathcal{H}} \in \mathbb{R}^{HL}_{++} \setminus E_0$ every Walrasian equilibrium x^* of the economy along with the accompanying $y^* = \sum_{i=1}^M \mu_i x_i^*$ is among the finitely many complex common zeros of the polynomials in a set $\mathcal{G}$ of the shape*

$$\mathcal{G} = \left\{\rho_1(e^{\mathcal{H}})x_1 - v_1(e^{\mathcal{H}}; y),\ \ldots\ ,\ \rho_M(e^{\mathcal{H}})x_M - v_M(e^{\mathcal{H}}; y),\ r(e^{\mathcal{H}}; y)\right\}. \tag{15}$$

The non-zero polynomial $r \in \mathbb{R}[e^{\mathcal{H}}; y]$ is not constant in y. Moreover, v_i, $i = 1, \ldots, M$, is a non-zero polynomial in $\mathbb{R}[e^{\mathcal{H}}; y]$ with a degree in y that is less than the degree of r in y. Each ρ_i is a non-zero polynomial in $\mathbb{R}[e^{\mathcal{H}}]$.

Proof. Equations (13) together with the condition

$$1 - t\ det[\partial F(e^{\mathcal{H}}; x)] = 0 \tag{16}$$

generate a zero-dimensional radical ideal in $\mathbb{K}[e^{\mathcal{H}}; x, t]$. The system (13),(16) consists of $M + 1$ equations in the $M + 1$ complex variables $x_1, \ldots, x_n, t$. We can identify a complex number $z \in \mathbb{C}$ with the vector $(Re(z), Im(z)) \in \mathbb{R}^2$ consisting of its real part $Re(z)$ and its imaginary part $Im(z)$. Then we can view the left-hand sides of these equations as a system of semi-algebraic functions $g : \mathbb{R}^{2M+2} \to \mathbb{R}^{2M+2}$. For all $e^{\mathcal{H}} \in \mathbb{R}^{HL}_{++}$ this function has finitely many zeros. Lemma 5 implies that the set of $\mu \in \Delta^{M-1}$ for which there are two distinct solutions $x \neq x' \in \mathbb{R}^{2M}$ with $g(Re(x), Im(x)) = g(Re(x'), Im(x')) = 0$ and $\sum_{i=1}^M \mu_i(Re(x)_i - Re(x')_i) = \sum_{i=1}^M \mu_i(Im(x)_i - Im(x')_i) = 0$ is lower-dimensional and closed. Thus for μ outside this closed lower-dimensional set equation (14) yields different values for y for all solutions to the system (13),(16). Therefore we can now apply Lemma 8 to the entire system (13) – (16). The set of solutions to this system is identical to the set of solutions of a system with the shape $\mathcal{G}$. Finally, by construction all Walrasian equilibria of the economy satisfy equations (13) and (14). Corollary 1 implies that for an open set of endowments of full Lebesgue measure all Walrasian equilibria also satisfy equation (16). $\square$

Note that since the set of 'good' weights μ is semi-algebraic, the fact that its complement is lower dimensional implies that there in fact must exist rational $(\mu_1, \ldots, \mu_M) \in \Delta^{M-1}$ that allow for the Shape-lemma representation. This is important for our computations since SINGULAR only performs exact computations over $\mathbb{Q}$.

Following the discussion in Section 2.5, we can obtain an analogue of Theorem 3 for the case where parameters consist of both profiles of individual endowments and preference parameters. In this case the Shape Lemma representation yields the correct competitive equilibria for a generic set of endowments and preference parameters.

Since there are algorithms to compute Gröbner bases exactly for the case of rational coefficients, the set of polynomials $\mathcal{G}$ can be computed exactly whenever marginal utility can be written as a polynomial with rational coefficients. Once the set $\mathcal{G}$ for an economy (or a class of economies parameterized by endowments or preference parameters) is known, we can use it to determine the number of real zeros of the system and the number of competitive equilibria.

In order to find all equilibria for a given generic semi-algebraic economy, it suffices to find all real solutions to a univariate polynomial equation. Sturm's algorithm provides an exact method to determine the number of solutions to a univariate polynomial in the interval $[0, \infty)$. Therefore, we can determine the exact number of solutions of the univariate polynomial. Using simple bracketing, we can then approximate all solutions numerically, up to arbitrary precision. Given the solutions to the univariate representation, the other solutions can then be computed with arbitrary precision by evaluating polynomials up to arbitrary precision. This is the only point in the procedure where the computation is not exact. Therefore, equilibria in this model are Turing computable (in contrast, see Richter and Wong (1999) who show that without restrictions on preferences Walrasian equilibria are generally not Turing computable).

4 Applications

In this section we apply our theoretical results to some parameterized economies. A simple class of semi-algebraic utility can be obtained by assuming that utility is separable, i.e. $u^h(x) = \sum_{l=1}^{L} v_{hl}(x_l)$ with each v_{hl} being semi-algebraic. In order to illustrate our methods, we examine two special cases. We first consider the case of quadratic utility and then move to the case where utility exhibits constant elasticity of substitution (CES). This latter case is prevalent in economic applications.

4.1 Quadratic Utility

There are two agents and two commodities, utility functions for agent h and good l are

$$v'_{hl}(c) = a_{hl} - b_{hl}c.$$

For the case where utility is symmetric across goods, i.e. $v_{h1} = v_{h2}$, there always exists a unique Walrasian equilibrium. The following polynomial system of equations is solved by any interior Walrasian equilibrium. (We write b_h for $b_{h1} = b_{h2}$ and normalize utility so that

$a_{hl} = 1$ for $h = 1, 2$ and $l = 1, 2$.)

$$\begin{aligned}
1 - b_1 c_1^1 - \lambda^1 p_1 &= 0 \\
1 - b_1 c_2^1 - \lambda^1 p_2 &= 0 \\
1 - b_2 c_1^2 - \lambda^2 p_1 &= 0 \\
1 - b_2 c_2^2 - \lambda^2 p_2 &= 0 \\
p_1(c_1^1 - e_1^1) + p_2(c_2^1 - e_2^1) &= 0 \\
p_1(c_1^2 - e_1^2) + p_2(c_2^2 - e_2^2) &= 0 \\
c_1^1 + c_1^2 - e_1^1 - e_1^2 &= 0 \\
p_1 + p_2 - 1 = 0
\end{aligned}$$

We observe that for a specific value of p_2 the last equation fixes the value of p_1. The remaining equations are then linear in the remaining variables. Therefore, the system cannot have two solutions with the same value for p_2. Thus, if we use p_2 as the last variable then the Shape Lemma holds for this system without an additional linear form. Implementing this system in SINGULAR yields an equivalent system with the shape $\mathcal{G}$ of the Shape Lemma with a last equation in the variable p_2 of the form

$$r(e^{\mathcal{H}}, b_1, b_2; p_2) = C_2 p_2^2 + C_1 p_2$$

with the coefficients

$$\begin{aligned}
C_2 &= b_1 b_2 e_1^1 + b_1 b_2 e_2^1 + b_1 b_2 e_1^2 + b_1 b_2 e_2^2 - 2b_1 - 2b_2, \\
C_1 &= -b_1 b_2 e_2^1 - b_1 b_2 e_2^2 + b_1 + b_2.
\end{aligned}$$

Obviously, the equation $r(\cdot; p_2) = 0$ has two solutions. One solution is $p_2 = 0$, which is not a Walrasian equilibrium. It is easy to check that for economically meaningful values of the parameters b_h and endowments $e^{\mathcal{H}}$ it holds that $C_2 < 0$ and $C_1 > 0$ and so $p_2^* = -C_1/C_2 \in (0, 1)$ is a Walrasian equilibrium price. The remaining equations (which we do not report here) then yield all remaining variable values. Theorem 3 now asserts that for a generic set of parameter values the interior Walrasian equilibrium (if there is one) is unique.

Next we allow utility to differ across agents and goods. For this general case the univariate polynomial r has the form

$$r(e^{\mathcal{H}}, (a_{hl}, b_{hl})_{h=1,2,l=1,2}; p_2) = C_4 p_2^4 + C_3 p_2^3 + C_2 p_2^2 + C_1 p_2,$$

where C_1, C_2, C_3 and C_4 are polynomials in the parameters. All four polynomials contain positive and negative monomials in the parameters and so their respective signs depend on the actual parameter values.

Again $p_2 = 0$ is a solution to this equation which does not correspond to a Walrasian equilibrium. Thus, there can be at most 3 Walrasian equilibria. For many parameter values only exactly one of the solutions to $r = 0$ corresponds to a Walrasian equilibrium. However, it is easy to "reverse-engineer" parameter values to obtain an economy with 3 equilibria. For example, suppose $e^1 = (10, 0)$, $e^2 = (0, 10)$ and

$$v'_{11}(c) = 9 - c, \quad v'_{12}(c) = 29/4 - 7/8c, \quad v'_{21}(c) = 116 - 26c, \quad v'_{22}(c) = 24 - 4c.$$

It is easy to verify that this economy has at 3 equilibria with prices (p_1, p_2) being $(4/5, 1/5)$, $(3/5, 2/5)$ and $(1/2, 1/2)$, respectively. In fact, the representing polynomial from the Gröbner basis for the equilibrium system is $r(p_2) = 50p_2^4 - 55p_2^3 + 19p_2^2 - 2p_2$. By Descartes' bound this system has at most three positive solutions. For them to be equilibrium prices they must lie in $(0, 1)$. We can apply Sturm's theorem and use SINGULAR to compute the number of sign changes of the Sturm sequence at 0 and the number of sign changes of the Sturm sequence at 1. It turns out that there are exactly 3 solutions in $(0, 1)$.

4.2 CES Utility

Suppose that each agent has CES utility, with marginal utility being of the form

$$v'_{hl}(c) = (\alpha_l^h)^{-\sigma_h}(c_l)^{-\sigma_h} \tag{17}$$

This transformation of the standard CES-form may appear unusual at first but considerably simplifies the notation during our analysis. We need to assume that σ_h is a rational number for all $h \in \mathcal{H}$ and set $\sigma_h = \frac{N}{M^h}$ such that the greatest common divisor of the natural numbers N and M^h is equal to one for at least one $h \in \mathcal{H}$.

After transforming agents' first-order conditions into polynomial expressions we obtain the specific form of Equations (9) for our CES-framework.

$$\begin{aligned}
\alpha_{hl}^N (c_l^h)^N (\lambda^h)^{M^h} p_l^{M^h} - 1 &= 0, \quad h \in \mathcal{H}, l = 1, \ldots, L, \\
\sum_{l=1}^{L} p_l (c_l^h - e_l^h) &= 0, \quad h = 1, \ldots, H, \\
\sum_{h=1}^{H} c_l^h - e_l^h &= 0, \quad l = 1, \ldots, L-1, \\
\sum_{l=1}^{L} p_l - 1 &= 0.
\end{aligned}$$

We can greatly reduce running times of SINGULAR if we write the equilibrium equations slightly differently. In particular, we normalize $p_1 = 1$ and eliminate all Lagrange multipliers, $\lambda^h = 1/(\alpha_{h1} c_1^h)^{N/M^h}$. Defining $q_l = p_l^{1/N}$, $l = 2, \ldots, L$, we obtain a similar system of equations, which has the same real positive solutions but often fewer complex and negative real solutions.

$$\begin{aligned} \alpha_{h1} c_1^h - \alpha_{hl} c_l^h q_l^{M^h} &= 0, \quad h \in \mathcal{H}, l = 2, \ldots, L, & (18)\\ c_1^h - e_1^h + \sum_{l=2}^{L} q_l^N (c_l^h - e_l^h) &= 0, \quad h = 1, \ldots, H, & (19)\\ \sum_{h=1}^{H} c_l^h - e_l^h &= 0, \quad l = 1, \ldots, L-1. & (20) \end{aligned}$$

The following theorem states properties of the real solutions to this system of equations. The statement is useful for a choice of ordering for the variables to ensure that the Shape Lemma holds.

THEOREM 4 *All real solution $c^{\mathcal{H}}, q$ to equations (18) – (20) satisfy $c^h \gg 0$ whenever $q \gg 0$. Moreover, if N and M^h are odd for all $h \in \mathcal{H}$, all real solutions satisfy $q \gg 0$.*

Proof. Suppose $c^{\mathcal{H}}, q$ solve (18) – (20), $q \gg 0$ but $c_l^h < 0$ for some h, l. Then Equation (18) implies that $c^h \ll 0$ for this agent h, but then the budget equation (19) cannot hold for this agent.

Now assume N, M^h odd and $q_l < 0$ for at least one l. Define $\bar{\mathcal{H}} = \{h : c_1^h > 0\}$. Market clearing implies that this set is non-empty. Moreover, the budget equations for the agents $h \in \bar{\mathcal{H}}$ imply

$$\sum_{h \in \bar{\mathcal{H}}} \left(c_1^h - e_1^h + \sum_{l=2}^{L} q_l^N (c_l^h - e_l^h) \right) = 0.$$

By definition of $\bar{\mathcal{H}}$, $\sum_{h \notin \bar{\mathcal{H}}} (c_1^h - e_1^h) \leq 0$ and with market clearing $\sum_{h \in \bar{\mathcal{H}}} (c_1^h - e_1^h) \geq 0$. By (18), whenever $q_l < 0$, then $c_l^h < 0$ and therefore $q_l^N (c_l^h - e_l^h) > 0$ for all $h \in \bar{\mathcal{H}}$. Similarly, if $q_l > 0$, then $c_l^h < 0$ for all $h \notin \bar{\mathcal{H}}$. By market clearing $\sum_{h \in \bar{\mathcal{H}}} (c_l^h - e_l^h) \geq 0$ and thus $\sum_{h \in \bar{\mathcal{H}}} q_l^N (c_l^h - e_l^h) \geq 0$. In total, since by assumption there is at least one l with $q_l < 0$,

$$\sum_{h \in \bar{\mathcal{H}}} \left(c_1^h - e_1^h + \sum_{l=2}^{L} q_l^N (c_l^h - e_l^h) \right) > 0,$$

yielding a contradiction. Furthermore, the case $q_l = 0$ for some l is ruled out since this implies that $c_1^h = 0$ for all $h \in \mathcal{H}$, contradicting market clearing. $\square$

4.2.1 Equilibria with CES Utility

We consider economies with $H = 2$ agents and $L = 2$ goods. In the case of two commodities there is only a single relative price and thus only the single price variable q_2 in equations (18) – (20). Note that for a specific value of q_2 the remaining equations are linear in the remaining variables and so cannot have multiple solutions. Therefore, the Shape Lemma holds for the equations (18) – (20) without an extra linear form if we choose q_2 as our last variable. Moreover, Theorem 4 implies that the number of real positive zeros of the univariate representation in q_2 is identical to the number of equilibria in the economy. Without loss of generality we can normalize $\alpha_2^h = 1 - \alpha_1^h \in (0,1)$ for both agents $h = 1, 2$. We denote q_2 simply by q and α_1^h simply by α^h.

For identical and integer-valued $\sigma = \sigma_1 = \sigma_2 \geq 3$ (that is, $M^h = 1$ for all $h \in \mathcal{H}$) the univariate representation in the Shape Lemma is given by

$$\begin{aligned} r(e^{\mathcal{H}}, \alpha^{\mathcal{H}}; q) \quad &= \quad \left(\alpha^1 e_2^2 + \alpha^2 e_2^1 - \alpha^1\alpha^2(e_2^1 + e_2^2)\right) q^{\sigma} - \alpha^1\alpha^2(e_1^1 + e_1^2)\ q^{\sigma-1} + \\ & \quad\quad (1-\alpha^1)(1-\alpha^2)(e_2^1 + e_2^2)\ q + \left(\alpha^1\alpha^2(e_1^1 + e_1^2) - \alpha^1 e_1^1 - \alpha^2 e_1^2\right) \end{aligned}$$

The univariate representation has exactly four terms for $\sigma \geq 3$. Since $0 < \alpha^1, \alpha^2 < 1$ the polynomial r has always exactly three sign changes. Descartes's Rule of Signs implies that there can be at most 3 real positive solutions.

For arbitrary parameters the bound of three equilibria is tight, as the following simple case illustrates Suppose $\sigma_1 = \sigma_2 = 3$, $\alpha_1 = 1/5, \alpha_2 = 4/5$ and $e_2^1 = e_1^2 = 1$. If $e_1^1 = e_2^2 = f > 44$ the economy has three equilibria – with these parameters the univariate representation above becomes

$$r(q) = (f+16)q^3 - (4f+4)q^2 + (4f+4)q - f - 16$$

whose 3 positive real roots for $f > 44$ correspond to 3 Walrasian equilibria.

For the general case of distinct and rational σ_h, as above, let $\sigma_h = N/M^h$ and define $\xi^h = ((1-\alpha_1^h)/\alpha_1^h)^{M^h}$ for $h = 1, 2$.

Depending on the actual values of M^1, M^2 and N, the univariate representation will look differently. We assume that $\sigma_1 \neq \sigma_2$ and so $M_1 \neq M_2$. If in addition $N - \min[M^1, M^2] > \max[M_1, M_2]$ (i.e. the denominators are relatively small) then we can define $K_1 = N + |M^2 - M^1|$, $K_2 = N$, $K_3 = N - \min[M^1, M^2]$, $K_4 = \max[M^1, M^2]$ and $K_5 = |M^2 - M^1|$ and obtain the univariate representing polynomial

$$r(q_2) = -e_2^2\xi_2 q_2^{K_1} - e_2^1\xi_1 q_2^{K_2} + (e_1^2 + e_1^1)q_2^{K_3} - \xi_1\xi_2(e_2^1 + e_2^2)q_2^{K_4} + e_1^1\xi_2 q_2^{K_5} + e_1^2\xi_1.$$

By Descartes' bound, the number of positive real solutions is then again uniformly bounded by three! The result is intuitively appealing: we would not expect the number of Walrasian equilibria to increase if some σ_h changes from 2 to, say, 360/179.

If the above conditions on N and M^1, M^2 do not hold, the results are very similar. A notable special case results if one agent has log-utility, e.g. if $M_1 = N$. In this case the representing polynomial simplifies as follows

$$r(f; q_2) = -e_2^2\xi_2 q_2^{K_1} - e_2^1\xi_1 q_2^{K_2} + (e_1^2 + e_1^1)q_2^{K_3} + e_1^2\xi_1,$$

and Descartes' bound implies that equilibria are unique for all endowments and all ξ_1, ξ_2. This is independent of σ_2, the elasticity of substitution of the second agent.

The fact that in the example, there are always at most 3 equilibria, independently of preference parameters or endowments can only be explained by the fact that we looked at a very special class of preferences, CES utility is both homothetic and separable! Yet, its use in applications in prevalent in public finance, macroeconomics, international trade.

4.2.2 The Need for a Linear Form

Recall that the statement of Theorem 3 relies on the presence of an additional variable q and the linear equation $q = \sum_i \mu_i x_i$ relating the economic variables x_i to the new variable. The additional equation guarantees (for almost all values of the weights μ_i) that all solutions to the equilibrium equations differ in their value of q. This condition is crucial for the Shape Lemma. So far we performed all computations in our applications without this complication. The following examples show that in general we cannot dispense with this equation.

First consider a trivial economy with two goods and a single agent who has CES utility with $\alpha_l = 1$, $l = 1, 2$, and $\sigma = 1/2$. The equilibrium equations are as follows

$$\begin{aligned} 1 - c_1\lambda^2 &= 0 \\ 1 - c_2\lambda^2 p^2 &= 0 \\ c_1 - e_1 + p(c_2 - e_2) &= 0 \\ c_1 - e_1 &= 0. \end{aligned}$$

This system has a total of four solutions. All four have $c_l = e_l$, $l = 1, 2$. The multiplier λ can take the values $\pm\sqrt{1/e_1}$ and the price p the values $\pm\sqrt{e_1/e_2}$. All four combinations of these values yield a solution to the equilibrium equations. In particular, for each value of p there are two distinct values for λ. Clearly, the Shape Lemma cannot hold with p as the last variable. Note that the Shape Lemma also cannot hold with any of the other three variables being the last one. But with the additional linear equation $y = c_1 + c_2 + \lambda + p$ the Shape Lemma holds.

As a second example suppose there are 2 agents and 3 commodities and utility functions are CES with identical $\sigma_1 = \sigma_2 = 2$ and with $\alpha_l^2 = 1/3$ for $l = 1, 2, 3$, $\alpha_1^1 = \alpha_3^1 = 1/4$, $\alpha_2^1 = 1/2$. The Gröbner basis for the Equations (18)-(20) determines q_3 by $(e_3^1 + e_3^2)q_3 - (e_1^1 + e_1^2) = 0$. However, for generic parameter values q_2 is not a linear function of q_3 but is itself determined by a polynomial,

$$C_2\ q_2^2 + C_1\ q + C_0 = 0$$

with C_0, C_1 and C_2 polynomials in the individual endowments. The reason for the failure of the shape lemma in this example is clear. Utility functions are identical with respect to commodities 1 and 3. The relative price of commodity 3 to commodity 1 is therefore just a function of aggregate endowments in these commodities. The price of commodity 2 on the other hand, depends on individual endowments since preferences with respect to this commodity differ across the two agents. If agent 1 is relatively rich, the equilibrium price of the commodity is high, if he is relatively poor the equilibrium price is low. This fact is independent of the price of commodity 3.

4.2.3 More Commodities or Agents

The rather small upper bound on the number of equilibria that we found above is no longer valid once we consider economies with more than two commodities or more than two agents. While we detect still a lot of structure in the equations, we are unable to derive general bounds on the number of equilibria. We encountered two particular difficulties. Currently SINGULAR cannot determine a Gröbner basis in shape form $\mathcal{G}$ for most parameterized economies with four or more goods and agents. And for those economies that SINGULAR can compute the univariate representation Descartes's Rule of Signs no longer gives tight bounds on the number of equilibria. We conjecture that a much more detailed analysis of parameterized economies is necessary and bounds on the number of equilibria will vary considerably by the choice of parameter and their respective values. Such an analysis is beyond the scope of this paper and is subject of further research.

Here we just illustrate some difficulties with another example. Consider an economy with 2 agents but $L > 2$ commodities. Suppose agents have CES utility functions with $\sigma = \sigma_1 = \sigma_2 \in \mathbb{Z}$, $\sigma > 1$. Using the linear form $y = \sum_{l=2}^{L} q_l$ (with q_l as in Section 4.2.1) we obtain the univariate representation

$$r(e^{\mathcal{H}}, \alpha^{\mathcal{H}}; y) = \sum_{s=0}^{\sigma} C_s(e^{\mathcal{H}}, \alpha^{\mathcal{H}}) y^{\sigma},$$

where the C_s-terms are polynomials whose total degree increases with L and which cannot be signed. Independently of L, we can therefore bound the number of equilibria by the elasticity of substitution σ. But since for $\sigma \to \infty$ the number of equilibria remains finite, this bound cannot be tight for sufficiently large σ. We have no computational evidence that the maximal number of equilibria actually does increase with σ.

5 Likelihood of Multiplicity with CES Utility

Although for many specifications of the model with two goods and two agents the upper bound of 3 equilibria is tight, it seems likely that for 'most' specifications of endowments and preferences the economy possesses a unique Walrasian equilibrium.

In this section, we estimate the relative size of the set of endowments and preference parameters for which multiplicity occurs. To illustrate the method, we focus on the simple case with 2 agents and 2 goods and assume that utility functions are of the CES-form (17) with odd M^h and N as in Theorem 4. The free parameters are endowments $e^{\mathcal{H}}$ and preference parameters α_l^h. We take $\alpha_1^h = 1 - \alpha_2^h$ for both agents as well as $(e_1^1, e_2^1, e_1^2, e_2^2) \in [0,1]^4$. The resulting set of exogenous parameters is therefore $[0,1]^6$. For fixed σ_1, σ_2, we are then interested in the volume of the subset of parameters in $[0,1]^6$ for which the resulting economy has multiple equilibria.

In order to do so, we use the method from Koiran (1995) (see also Kubler (2007)) and make probabilistic statements about the size of the set of parameters $(e^h, \alpha^h)_{h\in\mathcal{H}}$ for which Walrasian equilibria in economies with CES-utility functions are unique. We first review Koiran's method.

5.1 Estimating the Size of Sets

Suppose the set of exogenous parameters is $[0,1]^l$. Let $\Phi \subset [0,1]^l$ be the set of parameters for which there exists a unique Walrasian equilibrium. Let G be a grid of points in $[0,1]^l$, $G = \{1/n, \ldots, 1\}^l$. For $x \in [0,1]^l$, let $\Im(x) = 1$ if there is a unique Walrasian equilibrium for the corresponding economy and zero otherwise. We want to estimate the volume of Φ which is given by $\int_{[0,1]^l} \Im(x)dx$. Suppose it is known that the fraction of points in G for which there are multiple equilibria is not larger than some $\gamma \in (0,1)$. Let λ denote a bound on the maximal number of connected components of Φ intersected with any axes-parallel line. Then Koiran (1995) shows that

$$\left| \int_{[0,1]^l} \Im(x)dx - \gamma \right| < \frac{l}{n}\lambda \tag{21}$$

We address the question of how to determine good bounds for λ below. First, we want to focus on the question of how to obtain a good probabilistic estimate for γ. Suppose m random vectors $x^1, \ldots, x^m$ are drawn i.i.d. from $\{1/n, \ldots, 1\}^l$. First suppose that uniqueness of equilibria is found for all draws. By the binomial formula one obtains that the probability of the event that the fraction of points $x \in G$ for which there is multiplicity is greater than δ must be less or equal to $(1-\delta)^m$. If multiplicity is detected for a fraction $\hat{\gamma} > 0$ of the m draws, the argument becomes slightly more complicated and one needs to use Hoeffding's inequality to bound γ.

Suppose the $(X_i)_{i=1}^m$ are Bernoulli random variables with success probability $p \in (0,1)$ and $\hat{p}$ denotes the empirical frequency of success. Denoting $S = n\hat{p}$, Hoeffding' inequality can be written as

$$P\left(p - \hat{p} > t\right) \le \exp(-2mt^2).$$

In the experiments below, we use 200000 draws and want to bound γ from below with precision $t = 0.005$. Hence we obtain that

$$P\left(\gamma - \hat{\gamma} > 0.005\right) \le e^{-10}.$$

Note that while the probability is very small, the bound on γ is only about half of a percent.

5.2 Bounds on the Number of Connected Components

In the framework where all agents have CES utility with odd elasticities of substitution, by Theorem 4, it suffices to examine the number of real zeros of the univariate representation

$\rho(q_L, e^{\mathcal{H}}, \alpha^{\mathcal{H}})$. As explained above, one does not need to introduce an additional variable for the Shape Lemma to hold.

To determine a bound on the number of connected components λ of sets economies with unique equilibria, it suffices to find a bound on the number of critical points of r, i.e. to consider the system of equations

$$r(e^{\mathcal{H}}, \alpha^{\mathcal{H}}; q_L) = 0 \quad (22)$$

$$\frac{\partial r(e^{\mathcal{H}}, \alpha^{\mathcal{H}}; q_L)}{\partial q_L} = 0 \quad (23)$$

Since λ bounds the number of connected components in parameter-space along axes-parallel lines, we need to find an upper bound on the number of zeros of the above system holding all but one of the parameters $(e^{\mathcal{H}}, \alpha^{\mathcal{H}})$ fixed. This results in a system of two equations in two unknowns, the Shape-lemma holds and we can find a bound by Descartes rule. The number of connected components then does not exceed the number of zeros divided by two plus one.

For the case of identical σ, we obtain that independently of σ, the number of connected components λ never exceeds 2. In fact for the variables q and α_l^h and for $\sigma \geq 3$, the first polynomial of the Gröbner basis for (22) and (23) becomes

$$C_4 q^{2\sigma-1} + C_3 q^{\sigma} + C_2 q^{\sigma-1} + C_1 q^{\sigma-2} + C_0 = 0,$$

where $C_0, C_1 > 0$ and $C_3, C_4 > 0$ are polynomials in the other parameters. The number of zeros is at most 2, therefore λ is bounded by 2 (for the endowment variables the expressions look very similar and one obtains the same bound).

For $\sigma_1 \neq \sigma_2$, the number of connected components, λ, varies in the examples below to lie between 7 and 13, which means that for a 6 dimensional space of parameters, one needs n, the number of grid points, to lie between 10000 and 30000 to ensure that the actual volume of Φ does not deviate from α by more than 0.002. So overall, the experiments below will allow us to estimate the size of Φ to be within about a percent of the empirical frequency with very high probability.

5.3 Results

We first consider the case of identical $\sigma < \infty$. For this case, as shown above, the univariate representation is particularly simple. The number of connected components of sets with unique equilibria along axes-parallel lines of parameters can in fact be bounded by 2, independent of σ. We take $m = 100000$ and $n = 10000$. The size of the grid plays a role because larger integers are more difficult to handle for computer algebra systems. With this, the volume of the set of parameters that result in multiple equilibria can be bounded by the relatively frequency plus 0.005 (from Hoeffding) plus $\frac{2\times 6}{10000}$. In Table 1 we consider the case of identical σ and report the estimate for the volume of Φ with probability e^{-10}.

σ	λ	$\hat{\gamma}$	VOL
3	2	0	0.001
5	2	13/100000	0.006
7	2	36/100000	0.006
9	2	71/100000	0.006
25	2	338/100000	0.009
51	2	629/100000	0.01

Table 1: Fraction of Economies with Multiple Equilibria

In the column denoted by 'VOL' we report the resulting bound of the volume of the set of parameters which could yield multiple equilibria.

The table shows that multiplicity is extremely rare in the economies considered. Even for $\sigma = 51$, the empirical frequency of multiplicity is very low. The resulting estimate of the size of the set of parameters for which there could be multiplicity is not larger than one percent.

5.3.1 Heterogeneous σ

Table 2 shows that heterogeneous σ does not increase the possibility of multiple equilibria. In this case, λ increases with σ but our results are still quite precise for large values of σ.

σ^1, σ^2	λ	$\hat{\gamma}$	VOL
3,1	5	0	0.005
5,3	7	3/100000	0.01
25,5	13	87/100000	0.01

Table 2: Fraction of Economies with Multiple Equilibria

In fact it appears that the only thing that matters is how large the average of the two parameters is. For the case of one agent having an elasticity of substitution of 25 and the other of 5, the empirical frequency is comparable to the case of identical σ of not much larger than 9 above. Note that for the case $(3, 1)$, we have actually shown above that there is always a unique equilibrium, which is consistent with the results here.

5.3.2 Leontief Utility

For comparison, it is interesting to consider the limiting case of Leontief utility. As $\sigma \to \infty$, we can write utility as

$$u^1(c_1, c_2) = \max(\alpha_1 c_1, \alpha_2 c_2) \quad \text{and} \quad u^2(c_1, c_2) = \max(\beta_1 c_1, \beta_2 c_2).$$

For this case, we can characterize the set of parameters with multiplicity by simple inequalities. The maximal number of equilibrium prices is three and the conditions for the economy to allow multiplicity for some endowments are $\alpha_2 e_2 > \alpha_1 e_1$ and (note the reverse inequality) $\beta_2 e_2 < \beta_1 e_1$ where e_1, e_2 are the aggregate endowments in the two goods. These two conditions imply that

$$\frac{\alpha_2}{\alpha_1} > \frac{\beta_2}{\beta_1}.$$

Individual endowments of agent 1 that yield multiplicity must satisfy

$$e_1^1 \geq \alpha_2 \frac{\beta_1 e_1 - \beta_2 e_2}{\alpha_2 \beta_1 - \alpha_1 \beta_2} \quad \text{and} \quad e_2^1 \leq \alpha_1 \frac{\beta_1 e_1 - \beta_2 e_2}{\alpha_2 \beta_1 - \alpha_1 \beta_2}.$$

The volume of the set of endowments and preference parameters for which the economy has multiple equilibria is then

$$e_2^1 (e_1 - e_1^1) = \alpha_1 \frac{\beta_1 e_1 - \beta_2 e_2}{\alpha_2 \beta_1 - \alpha_1 \beta_2} \left(e_1 - \alpha_2 \frac{\beta_1 e_1 - \beta_2 e_2}{\alpha_2 \beta_1 - \alpha_1 \beta_2} \right).$$

Hoeffding's inequality implies that with probability less than e^{-10}, there are three equilibria for a fraction of less than 21.9 percent of all parameter values – for all other cases, equilibrium prices are unique.

Note that the number of connected components of sets of parameters which imply multiplicity still does not exceed 2, however, these sets just become much larger. It is surprising that the frequency of multiplicity is so large compared to the case $\sigma = 51$. This example shows that while it is true that there are many economies with homothetic utility with multiple equilibria. However, for 'reasonable' values of σ, multiplicity is extremely unlikely.

6 Conclusion

This paper has laid the theoretical foundation for the analysis of equilibrium multiplicity in general equilibrium models. We have presented a tractable algorithm to bound the number of Walrasian equilibria for exchange economies with semi-algebraic preferences. We have shown how equilibria in these economies can be characterized as particular solutions to square polynomial systems of equations and how methods from algebraic geometry solve these systems effectively. Using the computer algebra system SINGULAR we have illustrated our results and computational method for several economies. We have show that in exchange economies with two agents and two goods where all agents have CES utility, multiplicity of equilibria is rare. Moreover, even if there are multiple equilibria there are never more than three.

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