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"Smooth Monotone Contribution Games"

by

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Smooth Monotone Contribution Games

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Abstract

A monotone game is a multistage game in which no player can lower her action in any period below its previous level. A motivation for the monotone games of this paper is dynamic voluntary contribution to a public project. Each player's utility is a strictly concave function of the public good, and quasilinear in the private good. The main result is a description of the limit points of (subgame perfect) equilibrium paths as the period length shrinks. The limiting set of such profiles is equal to the *undercore* of the underlying static game – the set of profiles that cannot be blocked by a coalition using a smaller profile. A corollary is that the limiting set of achievable profiles does not depend on whether the players can move simultaneously or only in a round-robin fashion. The familiar core is the efficient subset of the undercore; hence, some but not all profiles that are efficient and individually rational can be nearly achieved when the period length is small. As the period length shrinks, any core profile can be achieved in a "twinkling of the eye" – neither real-time gradualism nor inefficiency are necessary.

KEYWORDS: *dynamic games, monotone games, core, public goods, voluntary contribution, gradualism*

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1. Introduction

A monotone game is a multistage game, with ordered stage game actions, in which no player can ever choose an action lower than the one she chose in the previous period. This paper is about a certain class of such games, in a setting with an infinite horizon and discounted payoffs. The goal is to characterize their pure strategy subgame perfect equilibrium outcomes and payoffs, especially when the period length is small.

The irreversibility of a monotone game arises in many settings.[1] An especially prominent one is that of dynamic voluntary contribution, along the lines of Marx and Matthews (2000). Agents in this scenario contribute amounts of a private good each period to a project, which uses the total contribution accumulated to date as capital to produce public goods. Think of a fund drive, or a never-ending sequence of fund drives, to finance university buildings or a charity. Each player's cumulative contribution can only increase over time, thereby generating the monotonicity.

In keeping with this motivation, the stage game payoffs in this paper are those of a neoclassical public goods model. They are quasilinear in private good consumption, and the valuation functions for the total amount contributed are strictly concave and differentiable. Each player's marginal valuation is low enough that she will never want to unilaterally increase her contribution, regardless of the level of past contributions. This prisoners' dilemma feature distinguishes the setting from the literature in which the public good has a threshold provision point, as is discussed below.

An equilibrium path is a nondecreasing sequence of action/contribution profiles. Much of this paper concerns the limits of (pure strategy subgame perfect) equilibrium paths, the "equilibrium limit profiles". A path will spend all but a finite number of periods near its limit, and so the limiting profiles have an important role when discounting is low.

The main result is that as the period length goes to zero, the set of equilibrium limit profiles expands and converges essentially to the "undercore" of the underlying coalitional game.[2] The undercore is defined like the familiar core, except that a contribution profile can only be blocked

[1]Consider, for example, firms irrevocably making entry or standards-adoption decisions over time, as in Gale (1995) or Ochs and Park (2004). Or countries negotiating treaties to progressively lower tariff or pollution levels, as in Lockwood and Zissimos (2005).

[2]Technically, the set of equilibrium limit profiles converges to the "strict undercore", the closure of which is the undercore. See Theorem 3.

with a (component-wise) smaller profile. That is, a profile is "underblocked", by a coalition if there exists a smaller profile that each coalition member prefers, and which prescribes zero contributions for the nonmembers. The undercore is the set of profiles that are not underblocked.

This characterization has several consequences. First, it implies that some efficient profiles can be nearly achieved as equilibrium limit profiles when the period length is small. This is because the undercore contains the core, and any core profile is efficient. (The core is nonempty here, containing, e.g., the Lindahl profile.)

Second, if the number of players is three or more, then generally some profiles that are efficient and individually rational cannot be achieved because they are not in the core. The requirement that an equilibrium limit profile be in the undercore means that the ultimate contributions of the players in an equilibrium cannot be too unbalanced – the total contribution of any coalition is bounded.

Third, as the period length shrinks to zero, the set of equilibrium limit profiles converges to the same set regardless of the move structure of the game. The only assumption made about the move structure is that it satisfies a weak cyclicity property, one that both the simultaneous-move and the round-robin move structures satisfy. Hence, in this limiting sense, whether the players can move simultaneously is irrelevant.

Fourth, as the period length shrinks, any undercore profile can be achieved in a negligible amount of real time. In other words, given any neighborhood of any undercore profile, there exists an equilibrium path that permanently enters the neighborhood in an amount of time that goes to zero with the period length. Any equilibrium limit profile can thus be achieved without significant delay or real-time gradualism if the period length is small enough.

These properties of the set of equilibrium limit profiles carry over to the set of equilibrium payoffs. The payoff generated by any undercore (and hence core) profile is the limit of equilibrium payoffs as the period length vanishes. On the other hand, any efficient payoff that is not a core payoff is not the limit of equilibrium payoffs. Therefore, in contrast to repeated game folk theorems, in general not all feasible individually rational payoffs can be achieved.[3]

[3]The games herein are stochastic games, with the state equal to the stage game action profile. The folk theorem of Dutta (1995) does not apply, however, because its "asymptotic state independence" assumptions (A1) and (A2) are not satisfied.

1.1. Relationship to the Literature

The term "monotone game" is due to Gale (2001). He studies a broad class of them, in a no-discounting setting in which each player's payoff from an equilibrium path is equal to the utility of its limiting profile. (Assumptions are made so that all equilibrium paths converge.) As in this paper, payoffs are assumed to satisfy a positive spillovers property: an increase in one player's action benefits all the others. The key result is that any "strongly minimal positive satiation point" is an equilibrium limit profile. In the terms of this paper, a strongly minimal profile is one that is not weakly underblocked. It is a satiation point if, starting from it, no player would want to unilaterally increase her contribution – in this paper, by assumption, all profiles are satiation points. The sufficiency result of this paper, that any strict undercore profile is an equilibrium limit profile if the discount factor is high enough, is thus a partial extension of Gale's result to a particular class of games with discounting. The necessity result of this paper, that any equilibrium limit profile is in the strict undercore, is not shown in Gale (2001), although it does hold in many no-discounting games (fn. 8 below).

The literature on monotone games with discounting has focused on showing that dynamics can alleviate the coordination/free-rider problem that plagues the corresponding static games. For example, the no-contribution profile is the only equilibrium of the static version of some of the contribution games studied in Marx and Matthews (2000). Nonetheless, the corresponding dynamic games have equilibria in which the players contribute over time, and the limiting profile is either efficient or approximately efficient if the discount factor is low. In these equilibria a player is induced to bear the cost of contributing by the implicit promise that the others will then contribute in the future. Contributions each period must often be small, and the convergence may take many periods or even be asymptotic. This gradualism is required when a large current contribution by one player would increase the incentives of the others to free ride in the future by too much.[4]

The literature on monotone games with discounting has obtained full characterizations of equilibria only for games that have a threshold provision point, which is a contribution level that any player will want to unilaterally achieve, as a dominant strategy, once the total contribution is sufficiently large. This is the case, e.g., in games studied in Bagnoli and Lipman (1989), Admati and Perry (1991), Gale (1995), Compte and Jehiel (2003), and Choi, Gale, and Kariv

[4] Strategic gradualism in related models is the focus of Lockwood and Thomas (2002), Compte and Jehiel (2004), and Lockwood and Zissimos (2005).

(2006). The threshold property implies, by backwards induction, that relatively few equilibria exist, and the set of equilibrium limit profiles is much smaller than the undercore. In contrast, the games of this paper do not have the threshold property, and backwards induction cannot be used to characterize their equilibria.

Monotone games with discounting that lack the threshold property are studied in Marx and Matthews (2000), Lockwood and Thomas (2002), and Pitchford and Snyder (2004). The basic result is that approximately efficient equilibria exist if discounting is close to zero. None of of these papers attempts to characterize all equilibria. The latter two restrict attention to the most efficient equilibria. They also consider only two-person games, in which case the set of undercore payoffs and the set of feasible individually rational payoffs are the same. Lockwood and Thomas (2002) obtain two results that are generalized in this paper. First, they show that if payoffs are differentiable, then the limit of the most efficient symmetric equilibrium path is an inefficient profile. This result is extended to a broader class of games and equilibria in this paper. Second, for the case of "linear kinked payoffs" they show that the most efficient symmetric equilibrium payoff of the simultaneous move game can be attained also in the sequential move game, in the limit as discounting is taken to zero. This foreshadows the result of this paper that any core payoff is the limit of equilibrium payoffs, regardless of the move structure.

1.2. Organization

The class of monotone contribution games studied in this paper is presented in Section 2; Appendix A shows how such a game arises as a model of a fund drive. The underlying coalitional game, i.e., the definitions and characterizations of the undercore and core, are presented in Section 3, and Appendix B contains its longer proofs. The results characterizing the set of equilibrium limit profiles are in Section 4, with the longer proofs in Appendix C. Implications are drawn in Section 5. Section 6 contains a concluding comment on extensions.

2. Monotone Contribution Games

The set of players is $N = \{1, \dots, n\}$, and they interact over periods $t = 1, 2, \dots$. In period t player i chooses $x_i^t \in \mathbb{R}_+$, which is referred to variously as her *action* or her *contribution.* A monotonicity constraint requires a player's action in any period to be no less than it was in the previous period: $x_i^t \geq x_i^{t-1}$.

The order of moves is specified by a *move structure,* which is a sequence $\vec{N} = \{N_t\}_{t=1}^{\infty}$ of subsets of N. The players who are not in N_t cannot raise their actions in period $t : x_i^t = x_i^{t-1}$ for $i \notin N_t$.

An *(action) profile* is denoted $x = (x_1, \ldots, x_n)$. A *feasible path* is a sequence $\vec{x} = \{x^t\}_{t=0}^{\infty}$ of profiles which starts with $x^0 = 0$, and is both monotone and consistent with the move structure: for all $t \geq 1$, [5]

$$x^t \geq x^{t-1}, \text{ and } x_i^t = x_i^{t-1} \text{ for } i \notin N_t.$$

A path $\vec{x}$ gives player i the payoff

$$U_i(\vec{x}, \delta) \equiv (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(x^t), \tag{1}$$

where $u : \mathbb{R}_+^n \to \mathbb{R}^n$ is the stage game payoff function and $\delta \in (0, 1)$ is the players' common discount factor.

Past actions are assumed to be publicly observed (but see Remark 2 below). This completes the description of a monotone game to be denoted as $\Gamma(\delta, \vec{N})$. Its pure strategy subgame perfect equilibria are henceforth referred to simply as "equilibria". Of central interest are the limits of equilibrium paths, *the equilibrium limit profiles.* The move structure and the payoff assumptions are the following.

2.1. Move Structure

Because of the discounting, future rewards to a player will matter only if they are not received too far in the future. Accordingly, the interval between the times at which a player can move should not grow too quickly as the game progresses. To ensure this, the move structure is assumed to satisfy the following "cyclicity" property:

(CY) $m > 0$ exists such that $i \in N_{(nk+i)m}$ for all $i \in N$ and $k \geq 0$.

This property requires player 1 to be able to move at date m, player 2 at date $2m$, and so on until the pattern repeats with player 1 able to move at date $(n + 1)m$. There are no restrictions on who else can move at the dates that are multiples of m, nor on who can move at the other dates. Both the simultaneous move structure defined by $N_t \equiv N$, and the round-robin structure

[5] The convention here regarding vector inequalities is the following: $x \geq x'$ means $x_i \geq x_i'$ for all i; $x > x'$ means $x \neq x'$ and $x \geq x'$; and $x \gg x'$ means $x_i > x_i'$ for all i.

defined by $N_t^R \equiv \{t \bmod n\}$, satisfy (CY) with $m = 1$. It will be clear that (CY) is stronger than required, but it is simple and satisfied by many move structures.

2.2. Payoffs

Payoffs take the following form: for all $x \in \mathbb{R}_+^n$ and $i \in N$,

$$u_i(x) = v_i(X) - x_i,$$

where $X = \sum_{i \in N} x_i$. This allows x_i to be interpreted as the amount of private good that player i contributes to a project that uses the total of the contributions, X, to produce a public good that gives a benefit $v_j(X)$ to each player j. A dynamic scenario behind this interpretation is presented in Appendix A.

Each v_i is continuous, normalized by $v_i(0) = 0$, and increasing. An increase in one player's action therefore benefits all the others, so that the "positive spillovers" property holds:

(PS) $u_i(\cdot)$ increases in x_j for all $i \neq j \in N$.

Each v_i is also assumed to be strictly concave, continuously differentiable, and to satisfy $v_i'(0) \leq 1$. This ensures that a "prisoners' dilemma" property holds:

(PD) $u_i(\cdot)$ decreases in x_i for all $i \in N$.

Hence, in any stage game, a player's dominant strategy is to not raise her action above its previous level. A player will raise her action in a period only if doing so is rewarded in the future by other players raising their actions. Consequently, in no equilibrium is there a final period in which actions are raised, and backwards induction cannot be used to find equilibria.

The final assumption is

$$\sum_{i \in N} v_i'(0) > 1 > \lim_{X \to \infty} \sum_{i \in N} v_i'(X). \tag{2}$$

This ensures that $\sum_{i \in N} v_i(X) - X$, the sum of the players' payoffs when they contribute a total amount X, has a unique and positive maximizer.

3. The Coalitional Game

Underlying the dynamic game is a coalitional game defined by u. In this section its core, undercore, and strict undercore are defined and characterized, as a prerequisite to characterizing equilibrium limit profiles in the next section.

3.1. Core and Undercore

Define a *coalition* to be any nonempty subset of players. A coalition S is said to *block* a profile x using a profile z if $z_{-S} = 0$, and $u_i(z) > u_i(x)$ for all $i \in S$. The *core*, C, is the set of profiles that are not blocked. Any core profile is efficient (Pareto optimal), or else N would block it.[6] It is also individually rational, or else a singleton coalition would block it using the origin.

Blocking *per se* is not relevant for understanding the equilibrium limit profiles of a monotone game. Roughly speaking, it does not matter if a coalition prefers a profile z to a putative limit profile x if $z \nless x$. The coalition members for whom $z_i > x_i$ would need to somehow coordinate upward deviations to obtain z. However, coordination is not required if $z < x$, as then each coalition member has an individual incentive to deviate downwards, or rather, to not raise her action once it reaches z_i. Blocking by a lower profile is thus the relevant concept.

Refer to a profile x as *underblocked* if a coalition blocks it using a profile $z < x$. The *undercore, D*, is then the set of profiles that are not underblocked. Note that the undercore contains the core, since an underblocked profile is blocked. An undercore profile is individually rational, or else it would be underblocked by a singleton coalition using the origin. The origin is itself in the undercore – it is not underblocked because no profile is below it.

The payoff assumptions of this paper imply a useful depiction of the undercore. For any coalition S, refer to

$$f_S(X) \equiv \sum_{i \in S} v_i(X) - X$$

as the *surplus function* of S. It is the sum of the coalition members' payoffs when their total contribution is X, and the non-coalition players contribute nothing. Since f_S is strictly concave, (2) implies it has a unique maximizer – denote it as Y_S. Define the *value* of the coalition to be its maximal surplus, $V(S) \equiv f_S(Y_S)$.

Remark 1. *This V defines a coalitional game with transferable utility. The actual coalitional game here has nontransferable utility, due to the constraint $x \geq 0$. (For example, x is efficient in the transferable utility game if and only if $X = Y_N$, but here it is also efficient if $X > Y_N$ and $x_i = 0$ for some $i \in N$.) The two games have the same core, as is shown below.*

[6] A small argument is needed to prove that inefficient profiles are not in the core. If x is inefficient, z exists such that $u(z) > u(x)$. Choose i such that $u_i(z) > u_i(x)$. By (PS), raising z_i slightly yields a profile $\hat{z}$ satisfying $u(\hat{z}) \gg u(x)$. So x is blocked by N using $\hat{z}$, which proves that x is not in the core.

A profile is underblocked if and only it requires some coalition to make too large a contribution. For any coalition S and profile x, let X_S denote the coalition's total contribution: $X_S \equiv \sum_{i \in S} x_i$. The proof of the following lemma is in Appendix B (as are all the proofs missing from this section).

Lemma 1. *A profile x is underblocked by a coalition S only if*

$$X_S > \max\left(Y_S,\ \sum_{i \in S} v_i(X) - V(S)\right). \tag{3}$$

Conversely, if (3) holds then x is underblocked by a coalition $\hat{S} \subseteq S$.

It is easy to see why (3) holds if S underblocks x. Half of it comes from the fact that S blocks x, and so the sum of the coalition members' payoffs must be less than what they can achieve on their own: $\sum_{i \in S} v_i(X) - X_S < V(S)$. The other half, $X_S > Y_S$, follows from the fact that the blocking profile satisfies $z < x$. Why (3) implies x is underblocked is less straightforward. The underblocking coalition is not S itself if x_i is very small for some of its members, as then they cannot be made better off by any nonnegative $z < x$. The coalition that underblocks x is obtained by deleting these members.

Lemma 1 immediately yields a characterization of the undercore.

Proposition 1. *For any profile x, $x \in D$ if and only if*

$$X_S \le \max\left(Y_S,\ \sum_{i \in S} v_i(X) - V(S)\right) \text{ for all coalitions } S. \tag{4}$$

Given a profile x, the corresponding coalition of contributing players is

$$N(x) \equiv \{i \in N \mid x_i > 0\}.$$

The following corollary shows that if x is an undercore profile, then the total contribution it prescribes is no greater than that which maximizes the surplus of this coalition.

Corollary 1. *Any nonzero $x \in D$ satisfies $X \le Y_{N(x)}$.*

Proof. Let $S = N(x)$. Since $X = X_S$, (4) would imply $V(S) \le f_S(X)$ if $X > Y_S$. This is impossible, since f_S is uniquely maximized by Y_S. ■

The next corollary relates the core to the undercore, and shows that the core is the same as that of the related transferable utility game (see Remark 1).

Corollary 2. *$C = C_a = C_b$, where*

(a) $C_a \equiv \{x \in D \mid X = Y_N\}$, *and*

(b) $C_b \equiv \{x \in \mathbb{R}^n_+ \mid X = Y_N$, *and* $X_S \leq \sum_{i \in S} v_i(Y_N) - V(S)\ \forall$ *coalitions* $S\}$.

Corollaries 1 and 2 together show that the core is equal to the intersection of the northeast surface of the undercore with the simplex defined by $X = Y_N$.

Turning to payoffs, the set of individually rational feasible payoffs is

$$R \equiv \{\hat{u} \in u(\mathbb{R}^n_+) \mid \hat{u} \geq 0\}.$$

(Recall that $u(0) = 0$.) The set of efficient individually rational payoffs is

$$P \equiv \left\{\hat{u} \in R \mid \hat{u} \nless u' \text{ for any } u' \in R\right\}.$$

Clearly, $u(D) \subset R$ and $u(C) \subset P$. Since a profile with $X \leq Y_N$ is efficient if and only if $X = Y_N$, Corollary 2 (a) implies that

$$u(C) = P \cap u(D). \tag{5}$$

Typically, a coalition $S \neq N$ exists such that $v(S) > 0$.[7] The next corollary shows that then some individually rational feasible payoffs are not undercore payoffs, and some efficient individually rational payoffs are not core payoffs.

Corollary 3. *$u(D)$ is a proper subset of R, and $u(C)$ is a proper subset of P, if and only if $V(S) > 0$ for some coalition $S \neq N$.*

3.2. Strict Undercore

A subset of the undercore plays a central role. It's definition relies on extending the underblocking relation by using of weak preferences. Say a coalition S *weakly underblocks* a profile x if $z < x$ exists such that $z_{-S} = 0$, and $u_i(z) \geq u_i(x)$ for all $i \in S$. Thus, an underblocked profile is weakly underblocked, but not conversely. The *strict undercore* is defined by

$$D_s \equiv \{0\} \cup \left\{x \in \mathbb{R}^n_+ \backslash \{0\} \mid X < Y_{N(x)} \text{ and } x \text{ is not weakly underblocked}\right\}.$$

[7] This is not true, however, if $n = 2$, since we have $V(\{i\}) = 0$ for each $i \in N$.

The strict undercore thus consists of the origin together with all nonzero profiles that are not weakly underblocked, and are inefficient for the coalitions they require to contribute. The undercore contains the strict undercore.

Lemma 2 below establishes three properties of the strict undercore. First, it is nonempty because it contains the line segment from the origin to the Lindahl profile defined by $x_i^L \equiv v_i'(Y_N)Y_N$. Second, deleting the origin from it yields a relatively open set. Third, its closure is the undercore – the difference between the undercore and the strict undercore is negligible.

Lemma 2. *(i) For every $\hat{X} \in [0, Y_N)$, D_s contains the profile defined by $x_i \equiv v_i'(Y_N)\hat{X}$. (ii) $D_s \backslash \{0\}$ is relatively open in $\mathbb{R}^n_+$. (iii) $cl\ D_s = D$.*

4. Equilibrium Limit Profiles

Recall that an equilibrium limit profile is the limit of an equilibrium path. (Equilibrium paths will be shown to converge.) Let $E(\delta, \vec{N})$ be the set of equilibrium limit profiles of $\Gamma(\delta, \vec{N})$. The set of all profiles that are equilibrium limit profiles for some discount factor is then

$$E(\vec{N}) \equiv \bigcup_{\delta \in (0,1)} E(\delta, \vec{N}).$$

The main result of this section is that this set is equal to the strict undercore.

4.1. Preliminaries

Given any history, define a player's *passive strategy* in the continuation game to be the one requiring her to not raise her action at any node. Because of (PS), the worst conceivable punishment the other players can impose upon a unilateral deviator is to play their passive strategies thereafter. Because of (PD), the passive strategy profile is an equilibrium of any continuation game. Consequently, any feasible path is an equilibrium path if and only if it is supported by the passive strategies. That is, if $\vec{x}$ is an equilibrium path, then the strategy profile that requires x^t to be played in period t if $(x^1, \ldots, x^{t-1})$ was played in the past, but otherwise requires the previous period's profile to be played, is an equilibrium.

Remark 2. *This argument does not need perfect monitoring. Suppose instead that the players publicly observe only the aggregates, $X^t = \sum_{i \in N} x_i^t$. Any unilateral deviation from a pure strategy profile is then still publicly observed. Any sequential equilibrium path is hence the* *path of a perfect public equilibrium in which any unilateral deviation is punished by playing the passive strategies.*

By (PD), a player's best deviation in period t, given that it triggers the passive equilibrium, is to play x_i^{t-1}. The resulting profile in each period $s \geq t$ is then $\hat{x}^s = (x_i^{t-1}, x_{-i}^t)$, and her continuation payoff is

$$(1-\delta)\sum_{s \geq t} \delta^{s-t} u_i(\hat{x}^s) = u_i(x_i^{t-1}, x_{-i}^t).$$

This deviation payoff cannot exceed the player's equilibrium continuation payoff. Hence, the following condition is necessary and sufficient for a feasible $\vec{x}$ to be an equilibrium path:

$$u_i(x_i^{t-1}, x_{-i}^t) \leq (1-\delta)\sum_{s \geq t} \delta^{s-t} u_i(x^s) \quad \text{for all } t \geq 1, i \in N_t. \tag{6}$$

The following lemma records these observations. In addition, it shows that the inequality in (6) also holds for players $i \notin N_t$. (The rest of its proof is in Appendix C.)

Lemma 3. *Let $\vec{x}$ be feasible for $\vec{N}$. Then condition* (6) *is necessary and sufficient for $\vec{x}$ to be an equilibrium path. Furthermore,* (6) *is equivalent to*

$$u_i(x_i^{t-1}, x_{-i}^t) \leq (1-\delta)\sum_{s \geq t} \delta^{s-t} u_i(x^s) \quad \textit{for all } t \geq 1, i \in N. \tag{7}$$

A consequence of Lemma 3 is that any $\vec{x}$ that leaves the origin and converges in a finite number of periods is not an equilibrium path. To see why, let T be the date at which the path stops, so that $x^{T-1} < x^T = x^s$ for all $s \geq T$. Then, by (PD), a player i for whom $x_i^{T-1} < x_i^T$ would be better off if she did not raise her action at date T. That is, by Lemma 3, $\vec{x}$ is not an equilibrium path because

$$u_i(x_i^{T-1}, x_{-i}^T) > u_i(x^T) = (1-\delta)\sum_{s \geq T} \delta^{s-t} u_i(x^s).$$

An equilibrium path may generate a non-monotonic sequence of stage game payoffs. Nonetheless, no payoff in the sequence is greater than its limit.

Lemma 4. *If $\vec{x}$ is an equilibrium path with limit x, then $u(x^t) \leq u(x)$ for all $t \geq 1$.*

Note that Lemma 4 immediately implies that for any equilibrium path $\vec{x}$ converging to a profile x, the corresponding equilibrium payoff satisfies $U(\vec{x}, \delta) \leq u(x)$.

4.2. Necessity

Any equilibrium limit profile is now shown to be in the strict undercore.

Theorem 1. *Every equilibrium limit profile* x *is* (i) *not weakly underblocked,*[8] *and* (ii) *satisfies* $x = 0$ *or* $X < Y_{N(x)}$. *That is,* $E(\vec{N}) \subseteq D_s$.

The proof of (i) (in Appendix C) proceeds by showing that if a coalition weakly underblocks an equilibrium limit profile using a profile z, then the coalition member who is supposed to be the last to raise her action above z_i can do better by not doing so. The logic of the argument is shown here by using it to prove the convergence of equilibrium paths.

Proof that equilibrium paths converge. Let $\vec{x}$ be a nonconvergent feasible path. It is thus unbounded. This implies $u_i(x^t) \to -\infty$ for some $i \in N$.[9] This player underblocks, using the origin, each profile in the tail of the path: $\tau \geq 1$ exists such that $u_i(x^{\tau-1}) > u_i(x^s)$ for all $s \geq \tau$. This prevents $\vec{x}$ from being an equilibrium path. For, if player i deviates at date τ by staying at $x_i^{\tau-1}$, her continuation payoff will be at least $u_i(x_i^{\tau-1}, x_{-i}^{\tau})$, which weakly exceeds $u_i(x^{\tau-1})$ by (PS). Hence, since $u_i(x^{\tau-1}) > u_i(x^s)$ for all $s \geq \tau$, we have

$$u_i(x_i^{\tau-1}, x_{-i}^{\tau}) > (1-\delta) \sum_{s \geq \tau} \delta^{s-\tau} u_i(x^s).$$

Lemma 3 therefore implies that $\vec{x}$ is not an equilibrium path. ■

The remainder of Theorem 1 follows directly from the following result.

Lemma 5. *For all* $\delta < 1$ *and nonzero* $x \in E(\delta, \vec{N})$,

$$\left(\frac{\delta}{1-\delta}\right)\left(\sum_{i \in N(x)} v_i'(X) - 1\right) \geq 1 - \max_{i \in N(x)} v_i'(X). \tag{8}$$

To see that Lemma 5 proves part (ii) of Theorem 1, note that the right side of (8) is positive because $X > 0$. The left side is therefore positive, and this implies $X < Y_{N(x)}$ by concavity.[10]

[8] Theorem 1 (i) holds for any u satisfying (PD) and (PS). Its proof does not use the assumed quasilinearity, concavity, or perfect substitutability of the players' contributions. It also holds (and is proved more simply) if there is no discounting, i.e., if $\lim_{t\to\infty} u(x^t)$ is the payoff from a path $\vec{x}$.

[9] Recall that $f_N(X)$ is strictly concave and maximized at $Y_N < \infty$. This implies $f_N(X^t) \to -\infty$ as $X^t \to \infty$. Since $f_N(X) = \sum_{i \in N} u_i(x)$, we thus have $u_i(x^t) \to -\infty$ for some $i \in N$.

[10] Lemma 5 also implies that for any coalition S, a nonzero equilibrium limit profile exists in which S is the coalition of contributing players only if $\delta \geq A/(A+B)$, where $A = 1 - \max_{i \in S} v_i'(0)$ and $B = \sum_{i \in S} v_i'(0) - 1$.

The following is a heuristic argument for why (8) must hold. To a first-order approximation, the equilibrium contribution $C^t = X^t - X^{t-1}$ made at date t increases the present value of the surplus of the contributing players in periods $s \geq t+1$ by

$$MB \equiv \left(\frac{\delta}{1-\delta}\right)\left(\sum_{i \in N(x)} v_i'(X^{t-1}) - 1 \right) C^t.$$

A player is willing to raise her contribution only if her share of this benefit exceeds her current net cost of contributing, which is approximately $\left(1 - v_i'(X^{t-1})\right)(x_i^t - x_i^{t-1})$. So MB must exceed the sum, over the contributing players, of these net costs. The lowest this total net cost can be is its value if the entire contribution were to be made by the player who has the smallest net cost per unit of contribution:

$$MC \equiv \left(\min_{i \in N(x)} \left(1 - v_i'(X^{t-1})\right)\right) C^t,$$

Inequality (8) is obtained by setting $MB \geq MC$, deleting the factor C^t, and taking $t \to \infty$.

Remark 3. *Equilibrium limit profiles may be efficient if payoff functions are not differentiable. Suppose each player's marginal valuation v_i' is positive until it drops to zero at an amount X^* that "completes" the project. If X^* is the efficient total contribution and δ is sufficiently large, equilibrium paths may exist for which $X^t \to X^*$. See Marx and Matthews (2000) and Lockwood and Thomas (2002).*

4.3. Sufficiency

Any strict undercore profile is now shown to be an equilibrium limit profile, provided the discount factor is high enough.

Theorem 2. *For any $x \in D_s$, there exists a path $\vec{x}$ converging to x, and a discount factor $\underline{\delta} < 1$, such that $\vec{x}$ is an equilibrium path if $\delta > \underline{\delta}$.*

Theorem 2 relies on the following lemma. Recall that the round-robin move structure is defined by $N_t^R \equiv \{t \bmod n\}$.

Lemma 6. *For any $\delta \in (0,1)$, $E(\delta, \vec{N}^R) \subset E(\delta^{1/m}, \vec{N})$. Hence, $E(\vec{N}^R) \subset E(\vec{N})$.*

This is proved by replacing $N(x)$ in (8) by S, and lowering X to 0. The inequality is maintained because the left (right) side of (8) decreases (increases) with X.

Lemma 6 is proved by converting an equilibrium path of $\Gamma(\delta, \vec{N}^R)$ into a path that is feasible for $\vec{N}$. This is where assumption (CY) is used. The new path is obtained by slowing down the round-robin path: player 1 moves in period m instead of period 1, player 2 moves in period $2m$ instead of period 2, and so on. This results in a postponement of the future reward a player receives for raising her contribution in the current period, but raising the discount factor to $\delta^{1/m}$ increases its present value enough to restore incentives.

Consequently, Theorem 2 needs to be proved only for the round-robin structure. The logic of its proof (in Appendix C) is described here, under the simplifying assumption that the strict undercore profile is strictly positive: $x \gg 0$.

The construction of a path to x begins with the definition of a vector d by

$$d_i \equiv \frac{v_i'(X)}{\sum_{j \in N} v_j'(X)}.$$

Then two profiles are found, $\hat{x} = x - \hat{\theta} d$ and $\bar{x} = x - \bar{\theta} d$, where $0 < \hat{\theta} < \bar{\theta}$. The number $\bar{\theta}$ is chosen so that $\bar{x} \geq 0$. The number $\hat{\theta}$ is chosen small enough that $\hat{x} \in D_s$; this can be done by Lemma 2 (ii), since $x \in D_s \setminus \{0\}$. These constructed profiles satisfy $\bar{x} \ll \hat{x} \ll x$ and $u(\bar{x}) \ll u(\hat{x}) \ll u(x)$. The proof is completed in three steps.

In Step 1, a round-robin path starting at $\bar{x}$ and converging to x is found that is an equilibrium path of the subgame starting at $\bar{x}$, provided δ exceeds some $\delta' < 1$. For each player this path is a geometric sequence with periodic gaps. The amount by which a contribution is raised in any period is small enough that the other players' payoffs are bounded below the target payoff, $u(x)$. This bound shrinks to zero as $t \to \infty$, but slowly enough that for all sufficiently high discount factors and all dates t, a player's continuation utility on the path is close enough to $u_i(x)$ that she is induced to raise her contribution in the current period. This step makes use of $X < Y_N$ and the concavity of each v_i.

Step 2, on the other hand, uses the fact that x, or rather, $\hat{x}$, is not weakly underblocked. Adapting an argument in Gale (2001), a finite, *decreasing* round-robin path from $\bar{x}$ to the origin is constructed, along which the players' payoffs never exceed $u(\hat{x})$. The construction starts with player 1 lowering her contribution from $\bar{x}_1$ either all the way to 0, or to a point at which the resulting profile gives her the same payoff as would $\hat{x}$. This yields the first profile of the sequence. The second profile is obtained next by having player 2 lower her contribution in the same manner. Continuing in round-robin fashion yields a decreasing sequence of profiles that generate payoffs no greater than $u(\hat{x})$. The sequence converges, say to a profile z. Because $\hat{x}$ is in the strict undercore, $z = 0$: otherwise, the coalition $N(z)$ would be nonempty, and its members would be indifferent between z and $\hat{x}$, thereby weakly underblocking $\hat{x}$. Since $u(0) \ll u(\hat{x})$ (as $\hat{x}$ is not weakly underblocked by a singleton coalition), the convergence occurs in a finite number of steps: once the sequence is close enough to the origin, a player's contribution cannot be lowered enough to make her indifferent between the resulting profile and $\hat{x}$.

Step 3 puts together the paths obtained in Steps 1 and 2 to yield a path $\vec{z}$ that converges to x and is feasible for $\vec{N}^R$. At any date for which $z^t \geq \bar{x}$, Step 1 insures that the remainder of the path is an equilibrium path of the continuation game if $\delta > \delta'$. At any date t for which $z^t < \bar{x}$, $u(z^t)$ is bounded strictly below $u(x)$, since $u(z^t) \leq u(\hat{x}) \ll u(x)$ by Step 2. This implies that (7) holds for all δ greater than some $\delta_t < 1$. Hence, $\vec{z}$ is an equilibrium path of $\Gamma(\delta, \vec{N}^R)$ for all δ greater than δ' and each of the finite number of δ_t's.

5. Implications

Theorems 1 and 2 together show that the set of equilibrium limit profiles expands with δ and converges to the strict undercore:

Theorem 3. *$E(\delta, \vec{N}) \subseteq E(\delta', \vec{N})$ for all $\delta < \delta'$, and*

$$E(\vec{N}) = \lim_{\delta \to 1} E(\delta, \vec{N}) = D_s.$$

The set of equilibrium limit profiles is observationally indistinguishable from its closure. By Lemma 2, taking closures in Theorem 3 shows that the closure of the set of equilibrium limit profiles is the undercore:

$$cl\, E(\vec{N}) = D. \tag{9}$$

This has implications for the necessity of gradualism, the nature of equilibrium payoffs, and the role of the move structure.

5.1. Gradualism

Since an equilibrium path converging to a nonzero profile does so only asymptotically, equilibrium contributions must be raised gradually. This accords with gradualism results in, e.g., Marx and Matthews (2000), Lockwood and Thomas (2002), and Compte and Jehiel (2004).

Real-time gradualism, however, is not necessary if the period length is short. To see why, let Δ be the period length, and set $\delta = e^{-r\Delta}$. By Theorems 2 and 3, for any equilibrium limit profile x, a fixed path $\vec{x}$ converging to it exists that is an equilibrium path if Δ is small. Given a neighborhood of x, let T be the finite number of periods it takes for $\vec{x}$ to permanently enter the neighborhood. The amount of time the path takes to reach the neighborhood is then $T\Delta$, which goes to zero as $\Delta \to 0$. Every equilibrium limit profile, or rather, undercore profile, can thus be reached instantaneously in the limit as the period length goes to zero.

5.2. Equilibrium Payoffs

Let $W(\delta, \vec{N}) \subset \mathbb{R}^n$ denote the set of equilibrium payoffs of $\Gamma(\delta, \vec{N})$. The set of limits of equilibrium payoffs is then

$$W(\vec{N}) \equiv \text{closure}\left(\bigcup_{\delta\in(0,1)} W(\delta, \vec{N})\right).$$

The payoff generated by an equilibrium path $\vec{x}$ is a weighted average of the stage game payoffs $u(x^t)$, and hence not determined solely by the corresponding equilibrium limit profile x. However, raising δ shifts weight to the tail of the path, and so $U(\vec{x}, \delta) \to u(x)$ as $\delta \to 1$. Theorems 2 and 3 therefore imply that the payoff generated by any undercore profile is the limit of equilibrium payoffs:

$$u(D) \subset W(\vec{N}). \tag{10}$$

Since $C \subset D$, an implication of (10) is that core payoffs are limits of equilibrium payoffs. The core payoffs are the only efficient payoffs for which this is true:

Corollary 4. *If a limit of equilibrium payoffs is efficient, it is a core payoff.*

Proof. Theorem 1 and Lemma 4 imply that any payoff in $W(\delta, \vec{N})$ is weakly dominated by a strict undercore payoff. Thus, letting $\hat{u} \in W(\vec{N})$, there exists $x \in D$ such that $\hat{u} \leq u(x)$. Assuming $\hat{u}$ is efficient, this implies $\hat{u} = u(x)$. Hence, $\hat{u} \in u(D)$. Recalling from (5) that any efficient undercore payoff is a core payoff, we now have $\hat{u} \in u(C)$. ■

Corollary 4, together with (5) and (10), implies that the set of efficient payoffs that are limits of equilibrium payoffs is equal to the set of core payoffs:

$$P \cap W(\vec{N}) = u(C). \tag{11}$$

Recall from Corollary 3 that if $V(S) > 0$ for some $S \neq N$, then not all efficient individually rational payoffs are core payoffs. In this case efficient individually rational payoffs exist that are not limits of equilibrium payoffs: $P \backslash W(\vec{N}) \neq \varnothing$.

5.3. Move Structure Relevance

A consequence of Theorem 3 is that all move structures give rise to the same equilibrium limit profiles: $E(\vec{N})$ does not depend on $\vec{N}$. In this sense the move structure is irrelevant. Note, however, that the lowest discount factor for which a given strict undercore profile is an equilibrium limit profile may depend on $\vec{N}$. The round-robin structure typically requires a higher discount factor than does the simultaneous structure to achieve a given profile.

Turning to payoffs, (11) shows that the set of efficient payoffs in in the limiting equilibrium payoff set $W(\vec{N})$ is also independent of $\vec{N}$. Any efficiency advantage that the simultaneous structure has over the round-robin structure disappears in the limit as the period length decreases to zero.

By (10), $W(\vec{N})$ contains the (large) set $u(D)$ of undercore payoffs, which does not depend on $\vec{N}$. Whether the remainder, $W(\vec{N})\backslash u(D)$, is also independent of $\vec{N}$ is left for future work to determine.[11]

6. Concluding Comment

A topic for the future is the extent to which the results hold for more general payoffs. For example, consider the polar opposite of this paper's valuation functions, the discrete one defined, given some $X^* > 0$, by

$$v_i(X) = \begin{cases} 0 & \text{for } X < X^* \\ V_i & \text{for } X \geq X^*. \end{cases}$$

A path $\vec{x}$ then generates a payoff of

$$\begin{aligned} U_i(\vec{x}, \delta) &= (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\left[v_i(X^t) - x_i^t\right] \\ &= \delta^{T(\vec{x})-1} V_i - \sum_{t=1}^{\infty} \delta^{t-1}\left(x_i^t - x_i^{t-1}\right), \end{aligned}$$

where $T(\vec{x})$ is the first date at which $X^t \geq X^*$, if such a date exists, and othewise $T(\vec{x}) = \infty$. The interpretation is that player i contributes a private good amount $x_i^t - x_i^{t-1}$ in period t, bearing the cost immediately, and the project generates the benefits V_i once the total cumulative contribution X^t reaches X^* (see Appendix A).

Compte and Jehiel (2003) study this game, assuming there are two players; the move structure is alternating, with player 1 moving first; the values differ, $V_2 < V_1$; free riding is an

[11] A conjecture is that $W(\vec{N})\backslash u(D) = \varnothing$ for all $\vec{N}$.

issue, $V_i < X^*$; and efficiency requires completion, $V_1 + V_2 > X^*$. Their result is that for any $\delta \in (\underline{\delta}, 1)$, where $\underline{\delta} = (K - V_1)/V_2$, the equilibrium path is unique: $x^1 = 0$, $x^2 = (0, X^* - V_1)$, and $x^t = (V_1, X^* - V_1)$ for all $t \geq 3$.

The equilibrium limit profile, $(V_1, X^* - V_1)$, is not in the strict undercore, as it is both efficient and weakly underblocked (by player 1). But it is in the core and undercore, and so the necessity result of Theorem 1 fails in a relatively minor way. Theorem 2, on the other hand, fails more strikingly. The profiles that are not weakly underblocked consist of the origin, which is the only strict undercore profile, and the continuum of profiles for which $x_1 + x_2 = X^*$ and $x_i < V_i$. None of these are equilibrium limit profiles.

This example exhibits the threshold property discussed in the introduction. It may thus be true that results like those of this paper hold whenever the threshold property is absent.

Appendix A. Fund Drive Scenarios

The monotonicity restriction, and the time-separable payoff function shown in (1), are taken in the text as defining features of the games of interest. However, they arise from natural primitive assumptions in some scenarios. Such a scenario is described in this appendix, amplifying on the model of a fund drive in Marx and Matthews (2000).

As noted in the introduction, fund drives, or rather, never-ending sequences of fund drives, are used to finance many public goods, like new university buildings or public television shows. The contributions collected in these drives become the capital used to produce future benefits. Participants can contribute any number of times, and are often informed of the total amounts contributed to date. These features are consistent with a monotone contribution game (see Remark 2). However, modeling a fund drive as a monotone contribution game requires plausible assumptions to be made that yield monotonicity and a time-separable payoff function.

The key to obtaining monotonicity is to let x_i^t denote the *cumulative* contribution that player i has made by date t. Thus, $x_i^t - x_i^{t-1}$ is her incremental contribution in period t. The monotonicity constraint, $x_i^t \geq x_i^{t-1}$, is now the result of assuming contributions must be nonnegative.

Two assumptions imply that fund drive payoffs can be written as in (1). The first is that a player's utility in a period is quasilinear in her incremental contribution that period. The second is that the capital used by the project to produce public goods does not decay, so that the capital available in period t to produce public goods is X^t, the total of all contributions made to date. Given these assumptions, let $b_i(X^t)$ denote the rate at which player i benefits from public good flow in period t. Let her discount rate be r, and the period length be Δ. Her payoff from a sequence of contributions is then

$$\begin{aligned} U &= \sum_{t=0}^{\infty} \left[\int_{t\Delta}^{(t+1)\Delta} b_i(X^t) e^{-r\tau} d\tau - (x_i^t - x_i^{t-1}) e^{-rt\Delta} \right] \\ &= \sum_{t=0}^{\infty} \delta^t \left[(1-\delta) r^{-1} b_i(X^t) - \left(x_i^t - x_i^{t-1} \right) \right], \end{aligned} \tag{12}$$

where $\delta = e^{-r\Delta}$. This payoff is not written as a time-separable sum. However, its linearity in the x_i^t terms allows it to be rewritten:[12]

$$U = (1-\delta) \sum_{t=0}^{\infty} \delta^t \left[r^{-1} b_i(X^t) - x_i^t \right]. \tag{13}$$

[12] Setting $x_i^{-1} = 0$ yields the identity $\sum_{t=0}^{\infty} \delta^t (x_i^t - x_i^{t-1}) = (1-\delta) \sum_{t=0}^{\infty} \delta^t x_i^t$.

Setting $v_i(X) = r^{-1}b_i(X)$ and $u_i(x) = v_i(X) - x_i$, this payoff is as shown in (1).

Observe that this $v_i(X)$ depends on the discount rate r. Taking $\delta \to 1$ corresponds to $\Delta \to 0$, holding r fixed. Thus, if the motivation for studying the monotone game is a fund drive scenario as described here, it is important to interpret $\delta \to 1$ as the period length shrinking, not the discount rate. Taking $r \to 0$ would be of little interest: it would cause $r^{-1}b_i(X) \to \infty$, so that the discounted present value of public good benefits would overwhelm the bounded (by Y_N) cost of contributing. The motivating free-rider problem would trivially vanish.

As a final observation, note that the transformation of (12) into (13) requires quasilinearity. For example, consider the assumption of Admati and Perry (1991) that a player's cost of making a contribution is $w_i(x_i^t - x_i^{t-1})$, where w_i is strictly convex. In this case the player's payoff,

$$U = \sum_{t=0}^{\infty} \delta^t \left[(1-\delta)v_i(X^t) - w_i(x_i^t - x_i^{t-1})\right],$$

cannot be written as a time-separable sum, and so a monotone game is not obtained. This is clear economically. In a monotone game, a (non-equilibrium) path in which the players leap immediately to an efficient profile in the first period and stay there forever is efficient. But in a fund drive with strictly convex cost functions w_i, dynamic efficiency requires contributions to be made incrementally.

Appendix B. Proofs of Undercore and Core Results

Proof of Lemma 1. Suppose S underblocks x. Since $f(X_S) = \sum_{i \in S} v_i(X_S) - X_S$ and $X_S \leq X$,

$$f(X_S) \leq \sum_{i \in S} v_i(X) - X_S. \tag{14}$$

Let $z < x$ be the profile S uses to underblock x. Then $z_{-S} = 0$ and, for all $i \in N$, $v_i(X) - x_i < v_i(Z) - z_i$. Summing these inequalities yields

$$\sum_{i \in S} v_i(X) - X_S < f(Z) \leq V(S). \tag{15}$$

Hence, $X_S > \sum_{i \in S} v_i(X) - V(S)$. From (14) and (15) we obtain $f(X_S) < f(Z)$. This and $Z \leq X_S$ imply $X_S > Y_S$, since f_S is concave and maximized at Y_S. We now have the desired inequality (3):

$$X_S > \max\left(Y_S,\ \sum_{i \in S} v_i(X) - V(S)\right).$$

Now suppose x and S satisfy (3). Then $v(X) \geq v(X_S) \gg v(Y_S)$. Furthermore,

$$\Delta \equiv \frac{V(S) - \left[\sum_{i \in S} v_i(X) - X_S\right]}{|S|} > 0.$$

Define $z \in \mathbb{R}^n$ by $z_{-S} = 0$ and, for $i \in S$,

$$z_i \equiv x_i - \Delta - v_i(X) + v_i(Y_S).$$

Then $z_i < x_i$ for all $i \in S$. Summing z_i over S yields $Z = Y_S$. Hence,

$$\hat{S} \equiv \{i \in S \,|\, z_i \geq 0\} \neq \varnothing.$$

Define $\hat{z} \in \mathbb{R}^n_+$ by $\hat{z}_i \equiv \max(0, z_i)$. Then $\hat{z}_{-\hat{S}} = 0$ and $\hat{z} < x$. Because $\hat{Z} \geq Z = Y_S$, for each $i \in \hat{S}$ we have

$$v_i(\hat{Z}) - z_i \geq v_i(Y_S) - z_i = v_i(X) - x_i + \Delta.$$

Since $\Delta > 0$, this proves that $\hat{S}$ underblocks x. ■

Proof of Corollary 2. We prove $C \subseteq C_a \subseteq C_b \subseteq C$, in this order.

We know $C \subseteq D$, and hence $x \in C$ only if $X \leq Y_N$, by Corollary 1. We also know core profiles are efficient. A profile for which $X \leq Y_N$ is efficient only if $X = Y_N$. Thus, $C \subseteq C_a$.

Now let $x \in C_a$. Let S be any coalition. Since $Y_N \geq Y_S$,

$$\begin{aligned} Y_S &\leq \sum_{i \in S} v_i(Y_N) - \sum_{i \in S} v_i(Y_S) + Y_S \\ &= \sum_{i \in S} v_i(Y_N) - V(S). \end{aligned}$$

Thus, since $X = Y_N$, Proposition 1 implies $X_S \leq \sum_{i \in S} v_i(Y_N) - V(S)$. This proves $C_a \subseteq C_b$.

Now let $x \in C_b$, and let z be any profile satisfying $z_{-S} = 0$ for some coalition S. Note that $\sum_{i \in S} u_i(z) = f_S(Z) \leq V(S)$. Since $x \in C_b$, we have

$$V(S) \leq \sum_{i \in S} v_i(Y_N) - X_S = \sum_{i \in S} u_i(x).$$

This proves that x is unblocked, and hence $C_b \subseteq C$. ■

Proof of Corollary 3 ($\Rightarrow$). We prove this direction by assuming $V(S) = 0$ for all coalitions $S \neq N$, and showing that then $R \subseteq u(D)$. This suffices, since it implies $R = u(D)$, and so $P = P \cap u(D) = u(C)$.

Let $\hat{u} \in R$ and $w \equiv \sum_{i \in N} \hat{u}_i$. Then $w \geq 0 = f_N(0)$. Since $\hat{u} = u(\hat{x})$ for some $\hat{x} \in \mathbb{R}^n_+$, $w = f_N(\hat{x}) \leq V(N)$. The intermediate value theorem thus implies that $f_N(X) = w$ for some $X \in [0, Y_N]$. Define $x \in \mathbb{R}^n$ by $x_i \equiv v_i(X) - \hat{u}_i$. To complete the proof, we show that $x \in D$. This will prove $\hat{u} \in u(D)$, and hence $R \subseteq u(D)$.

We first show $x \in \mathbb{R}^n_+$. Let $S = \{i \in N \mid x_i \geq 0\}$. Because $X \geq 0$, S is not empty. Suppose $S \neq N$. Then $X_S = X - X_{N \setminus S} > X$, which implies

$$\begin{aligned}\sum_{i \in S} \hat{u}_i &= \sum_{i \in S} v_i(X) - X_S \\ &< \sum_{i \in S} v_i(X) - X \leq V(S).\end{aligned}$$

By assumption, $V(S) = 0$, and hence $\hat{u}_i < 0$ for some $i \in S$. This contradiction of $\hat{u} \in R$ proves $S = N$, and so $x \in \mathbb{R}^n_+$.

For any $S' \neq N$ we have $\sum_{i \in S'} \hat{u}_i \geq 0 = V(S')$, and so

$$X_{S'} = \sum_{i \in S'} v_i(X) - \sum_{i \in S'} \hat{u}_i \leq \sum_{i \in S'} v_i(X) - V(S').$$

For $S' = N$, we have $X_{S'} \leq Y_N = Y_{S'}$. Proposition 1 now implies $x \in D$. ■

Proof of Corollary 3 ($\Leftarrow$). We prove this direction assuming $V(S) > 0$ for some $S \neq N$, and showing that then $u(C)$ is a proper subset of P. This also proves $u(D)$ is a proper subset of R, since $u(D) = R$ would imply the contradiction $u(C) = P \cap u(D) = P$.

Since $S \neq N$, $Y_S \neq Y_N$. Thus $\sum_{i \in S} v_i(Y_N) - V(S) < Y_N$, since Y_S uniquely maximizes f_S. We can thus choose an amount X_S for S to contribute such that

$$\sum_{i \in S} v_i(Y_N) - V(S) < X_S < \min\left(Y_N, \sum_{i \in S} v_i(Y_N)\right). \tag{16}$$

The second inequality implies that $x_S \in \mathbb{R}^{|S|}_+$ exists such that $\sum_{i \in S} x_i = X_S$ and, for each $i \in S$, $x_i < v_i(Y_N)$. Observe that

$$\begin{aligned}\sum_{i \notin S} v_i(Y_N) &= Y_N + V(N) - \sum_{i \in S} v_i(Y_N) \\ &> Y_N + V(S) - \sum_{i \in S} v_i(Y_N) \\ &> Y_N - X_S,\end{aligned}$$

where the first inequality follows from $V(N) > V(S)$, and the second follows from the first inequality in (16). Thus, $x_{-S} \in \mathbb{R}^{|N \setminus S|}_+$ exists such that $\sum_{i \notin S} x_i = Y_N - X_S$, and $x_i < v_i(Y_N)$

for each $i \notin S$. We have thus found a profile, $x = (x_S, x_{-S})$, that is individually rational and, since $X = Y_N$, efficient. This proves $u(x) \in P$. By the first inequality in (16), S blocks x, and so $x \notin C$. Since S also blocks any $\hat{x}$ for which $u(\hat{x}) = u(x)$, we have $u(x) \notin u(C)$. This proves $u(C)$ is a proper subset of P. ■

The following lemma, an analog to Proposition 1, characterizes the set of profiles that are not weakly underblocked. It is used below to prove Lemma 2.

Lemma A1. *Any $x \in \mathbb{R}^n_+$ is not weakly underblocked if and only if, for all coalitions S,*

$$X \leq Y_S \text{ or } X_S < \sum_{i \in S} v_i(X) - V(S). \tag{17}$$

Proof. (The proof is like that of Lemma 1.) Suppose x is not weakly underblocked, but (17) does not hold for a coalition S. Hence,

$$X > Y_S \text{ and } X_S \geq \sum_{i \in S} v_i(X) - V(S).$$

Then $v(X) \gg v(Y_S)$ and $\Delta \equiv \left(V(S) - \sum_{i \in S} v_i(X) + X_S \right) / |S| \geq 0$. Define $z \in \mathbb{R}^n$ by $z_{-S} = 0$ and, for $i \in S$,

$$z_i \equiv x_i - \Delta - v_i(X) + v_i(Y_S).$$

Then $z_i < x_i$ for all $i \in S$. Summing z_i over S yields $Z = Y_S \geq 0$. Hence,

$$\hat{S} \equiv \{ i \in S \,|\, z_i \geq 0 \} \neq \varnothing.$$

Define $\hat{z} \in \mathbb{R}^n_+$ by $\hat{z}_i \equiv \max(0, z_i)$. Then $\hat{z}_{-\hat{S}} = 0$ and $\hat{z} < x$. Because $\hat{Z} \geq Z = Y_S$, for each $i \in \hat{S}$ we have

$$v_i(\hat{Z}) - z_i \geq v_i(Y_S) - z_i = v_i(X) - x_i + \Delta.$$

Since $\Delta \geq 0$, we conclude that $\hat{S}$ weakly underblocks x using $\hat{z}$. Therefore, if x is not weakly underblocked, (17) holds for all coalitions S.

Now suppose x is weakly underblocked, say by a coalition S using $z < x$. Then $z_{-S} = 0$ and, for all $i \in N$, $v_i(X) - x_i \leq v_i(Z) - z_i$. Summing these inequalities yields

$$\sum_{i \in S} v_i(X) - X_S \leq f(Z). \tag{18}$$

Hence, since $f(Z) \leq V(S)$,

$$X_S \geq \sum_{i \in S} v_i(X) - V(S). \tag{19}$$

We have $Z \leq X_S$. If $Z = X_S$, then (18) would imply $\sum_{i\in S} v_i(X) \leq \sum_{i\in S} v_i(Z)$, and so $X \leq Z$, contrary to $z < x$. hence, $Z < X_S$. Since $f(X_S) \leq \sum_{i\in S} v_i(X) - X_S$, (18) implies $f(X_S) \leq f(Z)$. This and $Z < X_S$ imply $X_S > Y_S$, since f_S is strictly concave and maximized at Y_S. This and (19) show that (17) does not hold for S. Thus, x is not weakly underblocked if (17) holds for all coalitions. ■

Proof of Lemma 2 (i). We have $X = \hat{X}$, since $\sum_{i\in N} v_i'(Y_N) = 1$. Since $N(x) = N$, we also have $X < Y_{N(x)}$. Thus, by Lemma A1, we show $x \in D_s$ by showing that $\sum_{i\in S} v_i(X) - X_S > V(S)$ for any coalition S for which $X > Y_S$. Letting S be such a coalition, the proof is completed thusly:[13]

$$\begin{aligned}
\sum_{i\in S} v_i(X) - X_S &> \sum_{i\in S} \left[v_i(Y_S) + v_i'(X)(X - Y_S)\right] - X_S \\
&> \sum_{i\in S} \left[v_i(Y_S) + v_i'(Y_N)(X - Y_S)\right] - X_S \\
&= \sum_{i\in S} v_i(Y_S) - Y_S \sum_{i\in S} v_i'(Y_N) \\
&\geq \sum_{i\in S} v_i(Y_S) - Y_S \\
&= V(S). \blacksquare
\end{aligned}$$

Proof of Lemma 2 (ii)**.** Let $x \in D_s \backslash \{0\}$. Let $B \equiv \left\{x' \in \mathbb{R}^n_+ \mid \left\|x' - x\right\| < \varepsilon\right\}$, where $\varepsilon > 0$ is so small that all $x' \in B$ satisfy $x_i' > 0$ if $x_i > 0$ for $i \in N$, and for any S,

$$X' \lesseqgtr Y_S \text{ if } X \lesseqgtr Y_S \tag{20}$$

and

$$X_S' \lesseqgtr \sum_{i\in S} v_i(X') - V(S) \quad \text{if} \quad X_S \lesseqgtr \sum_{i\in S} v_i(X) - V(S). \tag{21}$$

Letting $x' \in B$, we must prove $x' \in D_s \backslash \{0\}$. We have $x' \neq 0$, since $x \neq 0$. Since $x \in D_s \backslash \{0\}$, we have $X < Y_{N(x)}$, and so (20) implies $X' < Y_{N(x)}$. We also have $N(x) \subseteq N(x')$, and so $Y_{N(x)} \leq Y_{N(x')}$. Thus, $X' < Y_{N(x')}$. It remains only to show that x' is not weakly blocked. Letting S be a coalition, we verify that X' and S satisfy (17). They do if $X' \leq Y_S$, so suppose $X' > Y_S$. We must prove

$$X_S' < \sum_{i\in S} v_i(X') - V(S). \tag{22}$$

[13] This chain follows from the strict concavity of each v_i; $X \in (Y_S, Y_N)$; $X_S = \sum_{i\in S} v_i'(Y_N)X$; and $\sum_{i\in S} v_i'(Y_N) \leq 1$.

Since $X' > Y_S$, (20) implies $X \geq Y_S$. If $X > Y_S$, then (17) holding for X and S implies $X_S < \sum_{i \in S} v_i(X) - V(S)$. This and (21) yield (22). On the other hand, if $X = Y_S$, then $S \neq N(x)$ because $X < Y_{N(x)}$. Thus, $X_S < X$, and so

$$X_S < X = Y_S = \sum_{i \in S} v_i(Y_S) - V(S).$$

From this and (21), we again obtain (22). ■

Proof of Lemma 2 (iii)**.** Proposition 1 implies D is closed, and so $cl\ D_s \subseteq D$. To show the reverse, let $x \in D$. Since $x \in D_s$ if $x = 0$, we can assume $x \neq 0$. Choose $\hat{X} \in (0, X)$ such that $\hat{X} > Y_S$ for all coalitions S for which $X > Y_S$. Then define $\hat{x}$ by $\hat{x}_i = v_i'(Y_N)\hat{X}$. By part (i), $\hat{x} \in D_s$. Define $x^\lambda \equiv \lambda x + (1-\lambda)\hat{x}$. Suppose $X^\lambda > Y_S$ for some coalition S and $\lambda \in (0, 1)$. This implies $X > Y_S$ and $\hat{X} > Y_S$. Because $X > Y_S$,

$$\sum_{i \in S} v_i(X) - V(S) > \sum_{i \in S} v_i(Y_S) - V(S) = Y_S.$$

Hence, applying Proposition 1 to $x \in D$ yields

$$X_S \leq \sum_{i \in S} v_i(X) - V(S).$$

Because $\hat{X} > Y_S$, applying Lemma A1 to $\hat{x} \in D_s$ yields

$$\hat{X}_S < \sum_{i \in S} v_i(\hat{X}) - V(S).$$

Thus, since $X_S^\lambda = \lambda X_S + (1-\lambda)\hat{X}_S$, the concavity of each v_i implies

$$X_S^\lambda < \sum_{i \in S} v_i(X^\lambda) - V(S).$$

Lemma A1 now implies x^λ is not weakly underblocked. We also have $X^\lambda < Y_{N(x^\lambda)}$, since the strict positivity of $\hat{x}$ implies $N(x^\lambda) = N$, and $X^\lambda < X \leq Y_{N(x)} \leq Y_N$. Therefore, $x^\lambda \in D_s$. Taking $\lambda \to 1$ proves x is a limit point of D_s. ■

Appendix C. Proofs of Equilibrium Limit Profile Results

This appendix contains the proofs of results in Section 4.

Proof of Lemma 3. Condition (6) was shown in the text to be necessary and sufficient for $\vec{x}$ to be an equilibrium path of $\Gamma(\delta, \vec{N})$. Since (6) obviously holds if (7) does, we must now show the reverse. Assuming $\vec{x}$ satisfies (6), and fixing $t \geq 1$ and $i \in N$, we must prove

$$(1-\delta)\sum_{s\geq t}\delta^{s-t}u_i(x^s) \geq u_i(x_i^{t-1}, x_{-i}^t). \tag{23}$$

We can assume $i \notin N_t$, else (23) follows directly from (6). Let τ be the first date greater than t such that $i \in N_\tau$. Since $\vec{x}$ is feasible for $\vec{N}$, we have $x_i^s = x_i^{t-1}$ for $s = t, \dots, \tau - 1$. Thus,

$$\begin{aligned}
(1-\delta)\sum_{s\geq t}\delta^{s-t}u_i(x^s) &= (1-\delta)\sum_{s=t}^{\tau-1}\delta^{s-t}u_i(x_i^{t-1}, x_{-i}^s) + \delta^{\tau-t}(1-\delta)\sum_{s\geq\tau}\delta^{s-\tau}u_i(x^s) \\
&\geq (1-\delta)\sum_{s=t}^{\tau-1}\delta^{s-t}u_i(x_i^{t-1}, x_{-i}^t) + \delta^{\tau-t}u_i(x_i^{\tau-1}, x_{-i}^\tau) \\
&= (1-\delta^{\tau-t})u_i(x_i^{t-1}, x_{-i}^t) + \delta^{\tau-t}u_i(x_i^{t-1}, x_{-i}^\tau) \\
&\geq (1-\delta^{\tau-t})u_i(x_i^{t-1}, x_{-i}^t) + \delta^{\tau-t}u_i(x_i^{t-1}, x_{-i}^t) \\
&= u_i(x_i^{t-1}, x_{-i}^t).
\end{aligned}$$

The first of these inequalities follows from (PS), $x_{-i}^s \geq x_{-i}^t$, and (6) applied to date τ; the second follows from (PS) and $x_{-i}^\tau \geq x_{-i}^t$. This proves (23). ■

Proof of Lemma 4. Let $t \geq 0$. Since $\vec{x}$ satisfies (7), and $u_i(x^t) \leq u_i(x_i^t, x_{-i}^{t+1})$ by (PS), we have

$$u_i(x^t) \leq (1-\delta)\sum_{s\geq t+1}\delta^{s-t-1}u_i(x^s).$$

The right side of this inequality is a convex combination of $\{u_i(x^s)\}_{s>t}$. Hence, there exists $s > t$ such that $u_i(x^t) \leq u_i(x^s)$. Repeating the argument recursively yields a subsequence $\{s_k\}_{k=1}^\infty$ of dates such that $s_k \to \infty$ as $k \to \infty$, and $u_i(x^t) \leq u_i(x^{s_k}) \leq u_i(x^{s_{k+1}})$. Since $\{u_i(x^{s_k})\}$ converges to $u_i(x)$, we conclude that $u(x^t) \leq u_i(x)$. ■

Proof of Theorem 1 (i). Let $\delta \in (0, 1)$ and $x \in E(\delta, \vec{N})$. Assume x is weakly underblocked. Then a coalition S and profile $z < x$ exist such that $z_{-S} = 0$, and

$$u_i(z) \geq u_i(x) \text{ for all } i \in S. \tag{24}$$

If $z_i = x_i$ for some $i \in S$, then $z < x$ would imply $z_{-i} < x_{-i}$, and so (PS) would yield $u_i(z) = u_i(x_i, z_{-i}) < u_i(x)$. As this is impossible by (24), we have $z_S \ll x_S$.

Let $\vec{x}$ be an equilibrium path of $\Gamma(\delta, \vec{N})$ that converges to x. Define a set of dates,

$$\mathcal{T} \equiv \{t \geq 1 \mid x_i^t \geq z_i \text{ for all } i \in S\}.$$

Since $z_S \ll x_S$, $\mathcal{T} \neq \varnothing$. Let τ be the smallest date in $\mathcal{T}$. Then $j \in S$ exists such that $x_j^{\tau-1} < z_j \leq x_j^{\tau}$. Since $\tau \in \mathcal{T}$ and $z_{-S} = 0$, $z_{-j} \leq x_{-j}^{\tau}$. Hence, by (PS),

$$u_j(z_j, x_{-j}^{\tau}) \geq u_j(z). \tag{25}$$

Because $x_j^{\tau-1} < z_j$, (PD) implies

$$u_j(x_j^{\tau-1}, x_{-j}^{\tau}) > u_j(z_j, x_{-j}^{\tau}). \tag{26}$$

From (24) − (26) we obtain

$$u_j(x_j^{\tau-1}, x_{-j}^{\tau}) > u_j(x). \tag{27}$$

On the other hand, Lemmas 3 and 4 imply

$$\begin{aligned} u_j(x_j^{\tau-1}, x_{-j}^{\tau}) &\leq (1-\delta)\sum_{s\geq\tau}\delta^{s-\tau}u_j(x^s) \\ &\leq (1-\delta)\sum_{s\geq\tau}\delta^{s-\tau}u_j(x) = u_j(x). \end{aligned}$$

This contradiction of (27) proves x is not weakly blocked. ■

Proof of Lemma 5. Let $\vec{x}$ be an equilibrium path of $\Gamma(\delta, \vec{N})$ that converges to x. Fix $i \in N$ and $t \geq 1$. From Lemma 3 we have

$$\sum_{s\geq t}\delta^{s-t}\left[u_i(x^s) - u_i(x_i^{t-1}, x_{-i}^t)\right] = \sum_{s\geq t}\delta^{s-t}u_i(x^s) - (1-\delta)^{-1}u_i(x_i^{t-1}, x_{-i}^t) \geq 0.$$

Using the assumed form of u_i, and letting $\tilde{X}^t \equiv X^t - x_i^t + x_i^{t-1}$, we obtain

$$\sum_{s\geq t}\delta^{s-t}\left[v_i(X^s) - v_i(\tilde{X}^t) - (x_i^s - x_i^{t-1})\right] \geq 0. \tag{28}$$

Because v_i is concave and differentiable,

$$v_i'(\tilde{X}^t)(X^s - \tilde{X}^t) \geq v_i(X^s) - v_i(\tilde{X}^t).$$

For $s \geq t$, the left side of this inequality is not lowered by replacing $v_i'(\tilde{X}^t)$ by $v_i'(X^{t-1})$, since $X^{t-1} \leq \tilde{X}^t$ and $X^s - \tilde{X}^t \geq 0$. Hence, for $s \geq t$,

$$v_i'(X^{t-1})(X^s - \tilde{X}^t) \geq v_i(X^s) - v_i(\tilde{X}^t).$$

This and (28) yield

$$\sum_{s\geq t}\delta^{s-t}\left[v_i'(X^{t-1})(X^s - \tilde{X}^t) - (x_i^s - x_i^{t-1})\right] \geq 0.$$

Summing over $i \in N(x)$ and replacing $\tilde{X}^t$ by $X^t - x_i^t + x_i^{t-1}$ gives

$$\sum_{s \geq t} \delta^{s-t} \left\{ (X^s - X^t) \sum_{i \in N(x)} v_i'(X^{t-1}) + \sum_{i \in N(x)} \left[v_i'(X^{t-1})(x_i^t - x_i^{t-1}) - (x_i^s - x_i^{t-1}) \right] \right\} \geq 0.$$

Using $\sum_{i \in N(x)} (x_i^s - x_i^t) = X^s - X^t$, this can be written as

$$\sum_{s \geq t} \delta^{s-t} \left\{ (X^s - X^t) B + \sum_{i \in N(x)} \left[v_i'(X^{t-1}) - 1 \right] (x_i^t - x_i^{t-1}) \right\} \geq 0,$$

where $B \equiv \sum_{i \in N(x)} v_i'(X^{t-1}) - 1$. This rearranges, upon multiplying by $1 - \delta$, to

$$B \left[(1 - \delta) \sum_{s \geq t} \delta^{s-t} (X^s - X^t) \right] + \sum_{i \in N(x)} \left[v_i'(X^{t-1}) - 1 \right] (x_i^t - x_i^{t-1}) \geq 0.$$

Using the identity $(1 - \delta) \sum_{s \geq t} \delta^{s-t} (X^s - X^t) = \delta \sum_{s \geq t} \delta^{s-t} (X^{s+1} - X^s)$, we obtain

$$\delta B \left[\sum_{s \geq t} \delta^{s-t} (X^{s+1} - X^s) \right] + \sum_{i \in N(x)} \left[v_i'(X^{t-1}) - 1 \right] (x_i^t - x_i^{t-1}) \geq 0. \tag{29}$$

Since $X > 0$, we have $v_i'(X) < 1$ for all i. Choose a number b satisfying

$$\max_{i \in N(x)} v_i'(X) < b < 1.$$

Let T be a date such that $b > v_i'(X^{t-1})$ for any $t \geq T$ and $i \in N(x)$. Hence, considering (29) for $t \geq T$, we can replace each $v_i'(X^{t-1})$ in its last term by b to get

$$\delta B \left(\sum_{s \geq t} \delta^{s-t} (X^{s+1} - X^s) \right) + (b - 1)(X^t - X^{t-1}) \geq 0. \tag{30}$$

Because $\vec{x}$ does not converge in a finite number of periods, $X^{t-1} < X$. Since x is not weakly underblocked by Theorem 1 (i), we have $x \in D$. This implies $X \leq Y_{N(x)}$, by Corollary 1. Hence, $X^{t-1} < Y_{N(x)}$, and the concavity of each v_i implies $B > 0$. Inequality (30) is thus preserved when divided by B. Doing so, and using

$$\max_{s \geq t} (X^{s+1} - X^s) \geq (1 - \delta) \sum_{s \geq t} \delta^{s-t} (X^{s+1} - X^s)$$

and the definition of B, yields

$$\max_{s \geq t} (X^{s+1} - X^s) \geq (X^t - X^{t-1}) Q_t, \tag{31}$$

where

$$Q_t \equiv \left(\frac{1 - \delta}{\delta} \right) \left[\frac{1 - b}{\sum_{i \in N(x)} v_i'(X^{t-1}) - 1} \right].$$

Since Q_t is nondecreasing in t, $Q_t \geq 1$ would imply $Q_s \geq 1$ for all $s \geq t$. But then a recursive application of (31) would prove that $\{X^t\}$ diverges, contrary to $X^t \to X$. Hence, $Q_t < 1$ for all large t, and taking the limit yields

$$\left(\frac{1-\delta}{\delta}\right)\left[\frac{1-b}{\sum_{i\in N(x)} v_i'(X)-1}\right] \leq 1.$$

From this, (8) is obtained by taking $b \to \max_{i\in N(x)} v_i'(X)$. ■

Proof of Lemma 6. Let $x \in E(\delta, \vec{N}^R)$, and let $\vec{x}$ be an equilibrium path of $\Gamma(\delta, \vec{N}^R)$ converging to x. Let m be the parameter given in (CY). Define a path $\vec{z}$ by letting the players move as in $\vec{x}$, but only at dates that are multiples of m. That is, let $z^t = 0$ for $t = 0, \ldots, m-1$, and for $t \geq m$ let $z^t = x^{nk+i}$, where k and i are the unique integers satisfying $k \geq 0$, $i \in N$, and

$$(nk+i)m \leq t < (nk+i+1)m.$$

In $\vec{z}$ player i moves only at dates $(nk+i)m$, since in $\vec{x}$ she moves only at dates $nk+i$. The path $\vec{z}$ is feasible for $\vec{N}$, since $i \in N_{(nk+i)m}$.

Let $\hat{\delta} = \delta^{1/m}$. We show $\vec{z}$ is an equilibrium path of $\Gamma(\hat{\delta}, \vec{N})$ by showing that it and $\hat{\delta}$ satisfy (7) for any given $i \in N$ and $t \geq 1$. This is true trivially, by (PS), if $z_i^s = z_i^{t-1}$ for all $s \geq t$. So we can assume $\tau \geq t$ exists such that $z_i^{t-1} = z_i^{\tau-1} < z_i^\tau$. It is a multiple of m, say $\tau = pm$. Furthermore, $z^\tau = x^p$ and $z^{\tau-1} = z^{t-1} = x^{p-1}$. Observe that

$$\begin{aligned}(1-\hat{\delta})\sum_{s\geq t}\hat{\delta}^{s-t}u_i(z^s) &= (1-\hat{\delta})\sum_{s=t}^{\tau-1}\hat{\delta}^{s-t}u_i(z_i^{t-1}, z_{-i}^s) + \hat{\delta}^{\tau-t}(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s) \\ &\geq (1-\hat{\delta}^{\tau-t})u_i(z_i^{t-1}, z_{-i}^t) + \hat{\delta}^{\tau-t}(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s),\end{aligned}$$

where the inequality follows from $u_i(z_i^{t-1}, z_{-i}^s) \geq u_i(z_i^{t-1}, z_{-i}^t)$ for each $s = t, \ldots, \tau-1$. (The overall inequality holds trivially if $\tau = t$.) Hence, (7) holds if

$$(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s) \geq u_i(z_i^{t-1}, z_{-i}^t), \tag{32}$$

which we now show. The definitions of $\vec{z}$ and $\hat{\delta}$ imply

$$\begin{aligned}(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s) &= (1-\hat{\delta})\sum_{k=0}^{\infty}\sum_{s=\tau+km}^{\tau+(k+1)m-1}\hat{\delta}^{s-\tau}u_i(z^s) \\ &= (1-\hat{\delta})\sum_{k=0}^{\infty}\hat{\delta}^{km}u_i(x^{p+k})\sum_{s=\tau+km}^{\tau+(k+1)m-1}\hat{\delta}^{s-\tau-km} \\ &= (1-\hat{\delta}^m)\sum_{k=0}^{\infty}\hat{\delta}^{km}u_i(x^{p+k}) \\ &= (1-\delta)\sum_{k=0}^{\infty}\delta^k u_i(x^{p+k}).\end{aligned}$$

Because $\vec{x}$ satisfies (7) at date p, we have

$$\begin{aligned}(1-\delta)\sum_{k=0}^{\infty}\delta^k u_i(x^{p+k}) &= (1-\delta)\sum_{s\geq p}\delta^{s-p}u_i(x^s)\\ &\geq u_i(x_i^{p-1},x_{-i}^p)\\ &= u_i(z_i^{t-1},z_{-i}^{\tau}).\end{aligned}$$

The two previous displays, and $u_i(z_i^{t-1},z_{-i}^{\tau}) \geq u_i(z_i^{t-1},z_{-i}^{t})$, imply (32). ■

The proof of Theorem 2 uses the following lemma.

Lemma A2. *A feasible $\vec{x}$ converging to a profile x is an equilibrium path if and only if the path $\vec{z}$ defined by $z^t = (x_i^t)_{i\in N(x)}$ for all $t \geq 0$ is an equilibrium path of the game obtained by deleting the players $i \notin N(x)$.*

Proof. If $\vec{x}$ is an equilibrium path, it satisfies (7). This implies $\vec{z}$ satisfies (7), with N replaced by $N(x)$. Lemma 3 thus implies $\vec{z}$ is an equilibrium path when the set of players is $N(x)$.

Conversely, suppose $\vec{z}$ is an equilibrium path when the set of players is $N(x)$. Then (7) holds for $i \in N(x)$. For $i \notin N(x)$, we have, for any $s \geq t$,

$$u_i(x_i^{t-1},x_{-i}^t) = u_i(0,x_{-i}^t) \leq u_i(0,x_{-i}^s) = u_i(x^s),$$

and hence $u_i(x_i^{t-1},x_{-i}^t) \leq (1-\delta)\sum_{s\geq t}\delta^{s-t}u_i(x^s)$. This shows that (7) holds for all $i \in N$, proving by Lemma 3 that $\vec{x}$ is an equilibrium path for N. ■

Proof of Theorem 2. By Lemma 6, it suffices to prove the result for $\vec{N} = \vec{N}^R$. Since the origin is always an equilibrium limit profile, we may assume $x \neq 0$. We construct a feasible path for $\vec{N}^R$ that converges to x, and which is an equilibrium path for large δ. By Lemma A2, we may assume $N(x) = N$, i.e., $x \gg 0$.

Define a vector $d \in \mathbb{R}_+^n$ by

$$d_i \equiv \frac{v_i'(X)}{\sum_{j\in N} v_j'(X)} \text{ for all } i \in N.$$

Note that $0 < d_i < v_i'(X)$ for all $i \in N$, since $X < Y_N$ implies $\sum_{j\in N} v_j'(X) > 1$. Let $\bar{x} \equiv x - \bar{\theta}d$,where $\bar{\theta} > 0$ is small enough that $\bar{x} \geq 0$. Define another profile $\hat{x} \equiv x - \hat{\theta}d$,where $\hat{\theta} \in (0,\bar{\theta})$ is so small that $\hat{x} \in D_s$. (Lemma 2 (ii) implies this can be done, since $x \in D_s\backslash\{0\}$.)

Since

$$\left.\frac{\partial}{\partial\theta}u_i(x-\theta d)\right|_{\theta=0} = \left.\frac{\partial}{\partial\theta}\left[v_i(X-\theta)-x_i+\theta d_i\right]\right|_{\theta=0}$$
$$= v_i'(X)-d_i$$
$$< 0,$$

we have $u(\bar{x}) \ll u(\hat{x}) \ll u(x)$, as well as $0 \le \bar{x} \ll \hat{x} \ll x$.

Define a sequence $\{x^t\}_{k=0}^{\infty}$ to be a *round-robin path* if for each $t > 0$ and $i = t \pmod n$, $x_{-i}^t = x_{-i}^{t-1}$. The rest of the proof consists of three steps.

Step 1. *There exists a nondecreasing round-robin path* $\{x^t\}_{t=0}^{\infty}$, *and a discount factor* $\delta' < 1$, *such that* $x^0 = \bar{x}$, $x^t \to x$, *and the following holds for all* $t > 0$, $i = t \pmod n$, *and* $\delta \ge \delta'$:

$$u_i(x_i^{t-1}, x_{-i}^t) \le (1-\delta)\sum_{s\ge t}\delta^{s-t}u_i(x^s). \tag{33}$$

Proof of Step 1. Since $d_i < v_i'(X)$ for all $i \in N$, we can find positive numbers a and ε such that

$$\frac{(1+\varepsilon)d_i}{v_i'(X)} < a < 1 \tag{34}$$

for all $i \in N$. Define $\{x^t\}_{t=0}^{\infty}$ by $x^0 \equiv \bar{x}$ and, for $t > 0$,

$$x_i^t \equiv \begin{cases} a x_i^{t-1} + (1-a)x_i & \text{if } i = t \pmod n \\ x_i^{t-1} & \text{otherwise.} \end{cases} \tag{35}$$

This $\{x^t\}_{t=0}^{\infty}$ is a round-robin path that starts at $\bar{x}$ and converges to x. Fix $t > 0$, and let $i = t \pmod n$. Let $q \ge 0$ be the integer for which $t = i + qn$. At the end of period $t-1$, players $j = 1, \ldots, i-1$ have raised their actions $q+1$ times, and players $j = i, \ldots, n$ have raised theirs just q times. Hence, since $x - \bar{x} = \bar{\theta}d$,

$$x_j^{t-1} = \begin{cases} x_j - \bar{\theta}a^{q+1}d_j & \text{for } 1 \le j < i \\ x_j - \bar{\theta}a^q d_j & \text{for } i \le j \le n. \end{cases} \tag{36}$$

This implies

$$X^{t-1} = X - \bar{\theta}a^q\left[a\sum_{j=1}^{i-1}d_j + \sum_{j=i}^{n}d_j\right]. \tag{37}$$

Similarly, for any $k \ge 1$,

$$x_j^{t+(k-1)n} = \begin{cases} x_j - \bar{\theta}a^{q+k}d_j & \text{for } 1 \le j \le i \\ x_j - \bar{\theta}a^{q+k-1}d_j & \text{for } i < j \le n \end{cases} \tag{38}$$

and

$$X^{t+(k-1)n} = X - \bar{\theta} a^{q+k-1} \left[a \sum_{j=1}^{i} d_j + \sum_{j=i+1}^{n} d_j \right]. \tag{39}$$

Turning to the desired inequality (33), note that it is equivalent to

$$A \equiv \sum_{s \geq t} \delta^{s-t} \left[u_i(x^s) - u_i(x_i^{t-1}, x_{-i}^t) \right] \geq 0.$$

Observe that $A = \sum_{k=1}^{\infty} \delta^{(k-1)n} A_k$, where

$$A_k \equiv \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[u_i(x^s) - u_i(x_i^{t-1}, x_{-i}^t) \right].$$

Each A_k is a sum over n consecutive dates, and player i moves only at the first of them, $t+(k-1)n$. Hence, for each of these dates s, $x_i^s = x_i^{t+(k-1)n}$. This implies that

$$\begin{aligned}
A_k &= \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[v_i(X^s) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right] \\
&\geq \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[v_i(X^{t+(k-1)n}) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right] \\
&= \left(\frac{1-\delta^n}{1-\delta} \right) \left[v_i(X^{t+(k-1)n}) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right],
\end{aligned}$$

where the inequality follows from $X^s \geq X^{t+(k-1)n}$ for $s \geq t+(k-1)n$. Using now the concavity of v_i and $X^{t-1} < X^{t+(k-1)n} < X$, we obtain

$$A_k \geq \left(\frac{1-\delta^n}{1-\delta} \right) \left[v_i'(X) \left(X^{t+(k-1)n} - X^{t-1} \right) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right].$$

This expression can be bounded from below. From (37) and (39) we have

$$\begin{aligned}
X^{t+(k-1)n} - X^{t-1} &= \bar{\theta} a^q \left[a \sum_{j=1}^{i-1} d_j + \sum_{j=i}^{n} d_j \right] - \bar{\theta} a^{q+k-1} \left[a \sum_{j=1}^{i} d_j + \sum_{j=i+1}^{n} d_j \right] \\
&= \bar{\theta} a^q \left[a(1-a^{k-1}) \sum_{j=1}^{i-1} d_j + (1-a^k) d_i + (1-a^{k-1}) \sum_{j=i+1}^{n} d_j \right].
\end{aligned}$$

From this, $1-a^k > a(1-a^{k-1})$, and $1-a^{k-1} > a(1-a^{k-1})$, we obtain

$$\begin{aligned}
X^{t+(k-1)n} - X^{t-1} &\geq \bar{\theta} a^q \left[a(1-a^{k-1}) \sum_{j=1}^{i-1} d_j + a(1-a^{k-1}) d_i + a(1-a^{k-1}) \sum_{j=i+1}^{n} d_j \right] \\
&= \bar{\theta} a^{q+1} (1-a^{k-1}) \sum_{j=1}^{n} d_j \\
&= \bar{\theta} a^{q+1} (1-a^{k-1}).
\end{aligned}$$

From (36) and (38), $x_i^{t+(k-1)n} - x_i^{t-1} = \bar{\theta} a^q \left(1 - a^k\right) d_i$. Consequently,

$$A_k \geq \bar{\theta} a^q \left(\frac{1-\delta^n}{1-\delta}\right) \left[v_i'(X) a(1 - a^{k-1}) - \left(1 - a^k\right) d_i\right].$$

This and (34) imply

$$A_k \geq \bar{\theta} a^q d_i \left(\frac{1-\delta^n}{1-\delta}\right) \left[\varepsilon - a^{k-1}(1 + \varepsilon - a)\right].$$

Therefore,

$$\begin{aligned} A &\geq \bar{\theta} a^q d_i \left(\frac{1-\delta^n}{1-\delta}\right) \sum_{k=1}^{\infty} \delta^{(k-1)n} \left[\varepsilon - a^{k-1}(1+\varepsilon-a)\right] \\ &= \bar{\theta} a^q d_i \left(\frac{1-\delta^n}{1-\delta}\right) \left\{\varepsilon \sum_{k=1}^{\infty} (\delta^n)^{k-1} - (1+\varepsilon-a) \sum_{k=1}^{\infty} (a\delta^n)^{k-1}\right\} \\ &= \left(\frac{\bar{\theta} a^q d_i}{1-\delta}\right) \left\{\varepsilon - \left(\frac{1-\delta^n}{1-a\delta^n}\right)(1+\varepsilon-a)\right\}. \end{aligned}$$

Thus, $A \geq 0$ for $\delta \geq \delta' \equiv (1+\varepsilon)^{-1/n}$. As δ' does not depend on t, Step 1 is proved. ■

Step 2. *There exists a finite, nonincreasing round-robin path* $\{x^k\}_{k=0}^K$ *such that* $x^0 = \bar{x}$, $x^K = 0$, *and* $u(x^k) \leq u(\hat{x})$ *for each* $k = 0, \ldots, K$.

Proof of Step 2. Let $x^0 \equiv \bar{x}$. To define x^1, let $x_{-1}^1 = x_{-1}^0$. Let $x_1^1 = 0$ if $u_1(0, x_{-1}^0) \leq u_1(\hat{x})$. Otherwise, let x_1^1 be the $\tilde{x}_1$ for which $u_1(\tilde{x}_1, x_{-1}^0) = u_1(\hat{x})$; this equation has a unique solution, and it is in the interval $(0, x_1^0)$, since $u_1(\cdot, x_{-1}^0)$ is monotonic and $u_1(x^0) < u_1(\hat{x}) < u_1(0, x_{-1}^0)$. Note that $0 \leq x^1 \leq x^0$, $u_1(x^1) \leq u_1(\hat{x})$, and by (PS), $u_j(x^1) < u_j(\hat{x})$ for $j \neq i$.

Now suppose that for some $k \geq 1$, profiles $x^0, \ldots, x^k$ have been defined, and they satisfy $0 \leq x^k \leq x^{k-1}$ and $u(x^k) \leq u(\hat{x})$. Let $i = k + 1 \pmod{n}$. Define $x_{-i}^{k+1} \equiv x_{-i}^k$. Let $x_i^{k+1} = 0$ if $u_i(0, x_{-1}^k) \leq u_i(\hat{x})$. Otherwise, let x_i^{k+1} be the unique $\tilde{x}_i \in \left(0, x_i^k\right]$ for which $u_i(\tilde{x}_i, x_{-i}^k) = u_i(\hat{x})$. By (PS), we have $u(x^{k+1}) \leq u(\hat{x})$.

This defines a nonincreasing and bounded round-robin path $\{x^k\}_{k=0}^{\infty}$. Let z be its limit. We have $z \leq x^k$ for all $k > 0$, and $u(z) \leq u(\hat{x})$.

Assume $z > 0$. In addition, assume $u_i(z) < u_i(\hat{x})$ for some $i \in N(z)$. By continuity, $\tilde{x}_i \in (0, z_i)$ exists such that $u_i(\tilde{x}_i, z_{-i}) < u_i(\hat{x})$. Since $x^k \to z$, there exists k' such that $u_i(\tilde{x}_i, x_{-i}^k) < u_i(\hat{x})$ for all $k > k'$. But then, the construction of the path implies that for any $k > k'$ such that $i = k + 1 \pmod{n}$, $x_i^{k+1} < \tilde{x}_i < z_i$. This contradicts $z_i \leq x_i^{k+1}$. Thus, $u_i(z) = u_i(\hat{x})$ for all $i \in N(z)$. Since $z < \hat{x}$, this shows that $N(z)$ weakly underblocks $\hat{x}$. This contradicts $\hat{x} \in D_s$. We conclude that $z = 0$.

As $\hat{x} \in D_s$, no singleton weakly underblocks $\hat{x}$. Since $0 \ll \hat{x}$, this implies $u(0) \ll u(\hat{x})$. Thus, K' exists such that $u_i(0, x^k_{-i}) < u(\hat{x})$ for all $k \geq K'$ and $i \in N$. Therefore, by the construction of the path, $K \leq K' + n$ exists such that $x^K = 0$. ■

Step 3. *There exists* $\underline{\delta} < 1$ *such that for all* $\delta \in (\underline{\delta}, 1)$, $x \in E(\delta, \vec{N}^R)$.

Proof of Step 3. Reverse the round-robin path obtained in Step 2, and add enough copies of 0 to its beginning and $\bar{x}$ to its end to obtain a finite, nondecreasing round-robin path, $\{z^t\}_{t=0}^T$, from $z^0 = 0$ to $z^T = \bar{x}$, that has player 1 moving first ($z^1_{-1} = 0$), and player n moving last ($z^{T-1}_{-n} = \bar{x}_{-n}$). To the end of $\{z^t\}_{t=0}^T$ add the round-robin path obtained in Step 1: $z^{T+s} = x^s$ for all integers $s \geq 0$. This yields a path $\vec{z} = \{z^t\}_{t=0}^{\infty}$ that is feasible for $\vec{N}^R$ and converges to x.

Let $t \geq 1$ and $i \in N^R_t$, so that $i = t \pmod n$. If $t > T$, then by Step 1,

$$u_i(z_i^{t-1}, z_{-i}^t) \leq (1 - \delta) \sum_{s \geq t} \delta^{s-t} u_i(z^s) \tag{40}$$

for $\delta > \delta'$. If $t \leq T$, then by Step 2,

$$u_i(z_i^{t-1}, z_{-i}^t) \leq u_i(z^t) \leq u_i(\hat{x}) < u_i(x).$$

Therefore, since $(1 - \delta) \sum_{s \geq t} \delta^{s-t} u_i(z^s) \to u_i(x)$ as $\delta \to 1$, there exists $\delta_t < 1$ such that (40) holds for $\delta > \delta_t$. We thus have (40) for all $t \geq 1$, $i \in N^R_t$, and $\delta > \underline{\delta} \equiv \max(\delta', \delta_1, \ldots, \delta_T)$. Lemma 3 now implies $x \in E(\delta, \vec{N}^R)$ if $\delta \in (\underline{\delta}, 1)$. ■

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