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"Informational Smallness and Private Monitoring in Repeated Games"
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by

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# Informational Smallness and Private Monitoring in Repeated Games*

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**Abstract**

We consider repeated games with private monitoring that are "close" to repeated games with public/perfect monitoring. A private monitoring information structure is close to a public monitoring information structure when private signals can generate approximately the same distribution of the public signal once they are aggregated into a public signal by some public coordination device. A player's informational size associated with the public coordination device is the key to inducing truth-telling in nearby private monitoring games when communication is possible. A player is informationally small given a public coordination device if she believes that her signal is likely to have a small impact on the public signal generated by the public coordinating device. We show that a *uniformly strict* equilibrium with public monitoring is robust in a certain sense: it remains an equilibrium in nearby private monitoring repeated games when the associated public coordination device, which makes private monitoring close to public monitoring, keeps every player informationally small at the same time.

We also prove a new folk theorem for repeated games with private monitoring and communication by exploiting the connection between public monitoring games and private monitoring games via public coordination devices.

Keywords: Communication, Folk theorem, Informational size, Perfect monitoring, Private monitoring, Public monitoring, Repeated games, Robustness

JEL Classifications: C72, C73, D82

## 1 Introduction

Cooperation within groups is an important and commonly observed social phenomenon, but the way in which cooperation arises is one of the least understood questions

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in economics. The theory of repeated games has improved our understanding by showing how coordinated threats to punish can prevent deviations from cooperative behavior, but much of the work in repeated games rests on very restrictive assumption that all players share the same public information either perfectly or imperfectly. For the case in which each player can observe all other players' actions directly (perfect monitoring), Aumann and Shapley [5] and Rubinstein [26] proved a folk theorem without discounting, and Fudenberg and Maskin [12] proved a folk theorem with discounting. For the case in which each player observes a noisy public signal (imperfect public monitoring), Abreu, Pearce and Stacchetti [1] characterized the set of pure strategy sequential equilibrium payoffs and Fudenberg, Levine and Maskin [13] proved a folk theorem.

But a theory that rests on the assumption that there is common knowledge of a sufficient statistic about all past behavior is, at best, incomplete. Such a theory is of little help in understanding behavior in groups in which there are idiosyncratic errors in individuals' observations of outcomes.[1] For many problems, it is more realistic to consider players who possess only partial information about the environment and, most importantly, players may not know the information possessed by other players. In such problems, players may communicate their partial information to other players in order to build a "consensus" regarding the current situation, which can be used to coordinate their future behavior. In this view, repeated games with public information can be thought of as a reduced form of a more complex interaction involving information sharing.

This point of view leads us to examine the robustness of equilibria with public monitoring when monitoring is private, but "close" to public monitoring. For example, one can think of a situation in which information contained in the public signal is dispersed among the players in the form of noisy private signals. If the amount of information contained in each player's private signal is negligible, then it is tempting to consider the game with such private signals and the underlying game with public signals as being "close." In this paper, we examine whether an equilibrium with public monitoring remains an equilibrium with respect to a public signal generated from private monitoring and communication, and whether (and how) players can be induced to reveal their private information.

To make these ideas precise, consider a public monitoring game $(G, \pi)$ and a private monitoring game $(G', p)$, where $G$ and $G'$ are normal form games with public monitoring and private monitoring respectively. In $(G, \pi)$, each action profile $a$ generates a public signal $y$ from a set $Y$ with probability $\pi(y|a)$. In $(G', p)$, each action profile $a$ generates a private signal profile $s = (s_1, .., s_n)$ with probability $p(s|a)$. In our analysis of the private monitoring game $(G', p)$, we will augment the model with a "public coordination device" $\phi$ that chooses a public coordinating signal (possibly randomly) from $Y$ based on the reported profile of private signals. In this expanded

[1] For example, team production in which each individual observes the outcome with error lies outside this framework.

game, players choose an action profile $a$, observe their private signals $(s_1, .., s_n)$, and publicly announce the (not necessarily honest) profile $(s'_1, .., s'_n)$. A public coordinating signal $y \in Y$ is then selected with probability $\phi(y|s'_1, .., s'_n)$. If the players report their private signals truthfully, then the probability that the realized public coordinating signal is $y$ given $a$ and $\phi$ is equal to $p^{\phi}(y|a) = \sum_{s \in S} \phi(y|s)p(s|a)$. We say that $(G, \pi)$ and $(G', p)$ are close when $G$ and $G'$ are close in terms of payoffs and there exists a public coordinating device $\phi$ such that $\pi(y|a) \approx p^{\phi}(y|a)$. We call the private monitoring repeated game augmented by such public coordinating devices (which may change over time) a communication extension of the repeated game associated with $(G', p)$. We then ask the following question: for a given perfect public equilibrium $\alpha^*$ of the repeated game with imperfect public monitoring associated with $(G, \pi)$ and given the repeated game associated with any nearby private monitoring game $(G', p)$, can we find a communication extension and an equilibrium in which players (i) truthfully reveal their signals along the equilibrium path and (ii) choose their actions as a function of the history of public coordinating signals according to $\alpha^*$?

If revelation constraints can be ignored and players are assumed to announce their private signals truthfully, then it is straightforward to show that $\alpha^*$ is an equilibrium with respect the public signal generated by $\phi$ as the game reduces to a repeated game of public monitoring $\left(G', p^{\phi}\right)$. Since $G'$ is close to $G$ and $p^{\phi}$ is close to $\pi$, it is easy to show that $\alpha^*$ remains an equilibrium of the repeated game of public monitoring game $\left(G', p^{\phi}\right)$ if it is a *uniformly strict* equilibrium of the original repeated game of $(G, \pi)$. Hence our analysis is mainly concerned with revelation constraints. The revelation of private information can be problematic, as can be seen in a simple trigger strategy equilibrium to support collusion. For a trigger strategy equilibrium to work, it is essential that every player reports "bad" outcomes honestly. However it is clear that players will not want to reveal any private information that may trigger mutual punishment.

We find that the following two concepts are the key to deal with the revelation constraints: *informational size* and *distributional variability.* Roughly speaking, player $i$ is informationally small if for each action profile $a$, her private information is unlikely to have a large effect on the distribution of the public coordinating signal $p^{\phi}(\cdot|a)$. Consequently, small informational size will imply that she will have little incentive to misreport her private signal in order to manipulate the other players' behavior to her advantage. Players are naturally informationally small in numerous settings. Suppose, for example, that there are many players whose signals are noisy observations of an underlying (but unobserved) common signal, and that these noisy observations are conditionally i.i.d.. If $\phi$ maps each signal profile into the posterior distribution of the unobserved signal, then each player is informationally small by the law of large numbers. Alternatively, with the same function $\phi$, agents receiving conditionally i.i.d. signals of the unobserved signal would be informationally small if their signals are very precise, even if the number of players is small (but at least three). Distributional variability is an index that measures a correlation between a player's private signal and the public coordinating signal which she would expect when she reports her signal truthfully. If this index is large, that means that a player's conditional belief about the public coordinating signal varies widely with respect to her private information. The larger this index is, the easier it is to detect and punish a lie. With these concepts, our result can be stated as follows: a uniformly strict equilibrium is robust when, for some public coordination device, (1) $(G', p)$ is close to $(G, \pi)$ in the sense described above and (2) each player's informational size is small relative to her distributional variability.

The way to induce honest reporting is roughly as follows. If $\phi$ is employed in every period, then $p^{\phi}(y|a)$ is always close to $\pi(y|a)$, but players may have an incentive to send false reports. To address this, we employ different public coordinating devices at different public histories $\left\{\phi_{h^t}|h^t \in H\right\}$, where each $\phi_{h^t}$ is a perturbation of $\phi$. When every player's informational size is small relative to her distributional variability, we can construct $\left\{\phi_{h^t}|h^t \in H\right\}$ so that every revelation constraint is satisfied on the equilibrium path (i.e. after they played the equilibrium action in the same period), while keeping each perturbation small so that the incentive constraints regarding the equilibrium actions of $\alpha^*$ are not disturbed.

The second main result of this paper is a new folk theorem for repeated games with private monitoring and communication. For the robustness result, we start with a public monitoring game, then consider nearby private monitoring games to check the robustness of equilibria for repeated games with public monitoring. For the folk theorem, we take the opposite path: we start with a private monitoring game, then generate public monitoring games via public coordinating devices. For repeated games with imperfect public monitoring, there is a well known technique to prove a folk theorem by Fudenberg, Levine and Maskin [13]. We exploit a connection between private monitoring games and public monitoring games via public coordination devices to import their technique into the domain of private monitoring games and extend it to incorporate revelation constraints. Again it is important for our result that every player is informationally small.

There are also a couple of technical contributions in this paper. First, we can prove a *uniformly strict* folk theorem. That is, we prove a folk theorem by using uniformly strict equilibria where every player would lose at least a certain amount of payoffs by deviating from the equilibrium action at any history. As a special case, this result implies a uniformly strict folk theorem for some class of repeated games with imperfect public monitoring. Another technical contribution of the paper, which might be of independent interest, is to prove the theorem corresponding to Theorem 4.1 in [13] without relying on their smoothness condition, which is commonly used to prove a folk theorem in the literature.

The model is described in Section 2 and the concepts of informational size and distributional variability are introduced in Section 3. Section 4 states and proves our robustness result. In Section 5, we prove a new folk theorem for repeated games with private monitoring and communication. Section 6 discusses the related literature. Some proofs are provided in the appendix (Section 7).

# 2 Preliminaries

## 2.1 Repeated Games with Public Monitoring

The set of players is $N = \{1, ..., n\}$. Player $i$ chooses an action from a finite set $A_i$. An action profile is denoted by $a = (a_1, ..., a_n) \in \Pi_i A_i := A$. Actions are not publicly observable, but the players observe a public signal from a finite set $Y$. The probability that $y \in Y$ is realized given $a \in A$ is denoted $\pi(y|a)$. We do not assume full support. That is, the set $\{y \in Y | \pi(y|a) > 0\}$ can depend on $a \in A$. This allows for perfect monitoring ($Y = A$ and $\pi(y|a) = 1$ if $y = a$) as a special case. Player $i'$s stage game payoff is $u_i(a_i, y)$ and player $i$'s expected stage game payoff is $g_i(a) = \sum_y u_i(a_i, y)\pi(y|a)$. Consequently, players do not obtain any additional information regarding the actions of other players from realized payoffs. This stage game is denoted by $(G, \pi)$, where $G = (N, A, g)$. We normalize payoffs so that each player's *pure strategy* minmax payoff is 0. Note that the mixed minmax payoff may be smaller than the pure strategy minmax payoff. The set of feasible payoff profiles is $V = co\{g(a) | a \in A\}$ and $V^* = \{v \in V | v \gg \mathbf{0}\}$ is the set of feasible, strictly individually rational payoff profiles.

A private history for player $i$ at stage $t$ is denoted $h_i^t = \left(a_i^0, ..., a_i^{t-1}\right) \in H_i^t = A_i^t$ while a public history is denoted $h^t = \left(y^0, ..., y^{t-1}\right) \in H^t = Y^t$ with $H_i^0 = H^0 := \{\emptyset\}$. A pure strategy for player $i$ is a sequence $\alpha_i = \left\{\alpha_i^t\right\}_{t=0}^{\infty}$, where $\sigma_i^t$ is a mapping from $H_i^t \times H^t$ to $A_i$. The set of pure strategies for player $i$ is denoted $\Sigma_i$. We restrict ourselves to pure strategies throughout this paper, so we will simply use the term strategy to refer to pure strategies when no confusion can arise. A strategy profile is denoted $\alpha = \{\alpha_i\}_{i \in N} \in \Sigma := \times_i \Sigma_i$. A pure strategy profile induces a probability measure on $A^\infty$. Player $i'$s discounted expected payoff given $\alpha$ and $\delta \in (0, 1)$ is $w_i^{\alpha,\delta} = (1 - \delta) \sum_{t=0}^{\infty} \delta^t E\left[g_i\left(\tilde{a}^t\right) | \alpha\right]$.[2] We denote this repeated game associated with $(G, \pi)$ by $G_\pi^\infty(\delta)$.

A strategy is *public* if it only depends on $H^t$. A profile of public strategies is a *perfect public equilibrium* (PPE) if, after every public history, the continuation (public) strategy profile constitutes a Nash equilibrium (Fudenberg, Levine, and Maskin [13]). Note that a perfect public equilibrium is a subgame perfect equilibrium when the stage game is one of perfect monitoring. Since we focus on perfect public equilibrium, we will omit the dependence of strategies on private histories and write $\alpha_i^t\left(h^t\right)$ instead of $\alpha_i^t\left(h_i^t, h^t\right)$.[3]

Given $\alpha, \delta$ and history $h^{t+1} = \left(h^t, y\right) \in H^{t+1} = H^t \times Y$, let $w_i^{\alpha,\delta}\left(h^t, y\right)$ denote

[2] When $x$ is a generic outcome of some random variable, we often use $\widetilde{x}$ to denote this random variable.

[3] Thus we ignore *private strategies* (Kandori and Obara [17]).

player $i$'s continuation payoff from period $t+1$. We define $\eta-$*uniformly strict perfect public equilibrium* ($\eta-$USPPE) as follows.

**Definition 1** *A pure strategy perfect public equilibrium $\alpha \in \Sigma$ for $G_{\pi}^{\infty}(\delta)$ is $\eta-$uniformly strict if*

$$
\begin{aligned}
& (1-\delta) g_i\left(\alpha^t\left(h^t\right)\right)+\delta \sum_{y \in Y} \pi\left(y|\alpha^t\left(h^t\right)\right) w_i^{\alpha, \delta}\left(h^t, y\right)-\eta \\
\geq \quad & (1-\delta) g_i\left(a_i^{\prime}, \alpha_{-i}^t\left(h^t\right)\right)+\delta \sum_{y \in Y} \pi\left(y|a_i^{\prime}, \alpha_{-i}^t\left(h^t\right)\right) w_i^{\alpha, \delta}\left(h^t, y\right)
\end{aligned}
$$

*for all $h^t \in H^t, t \geq 0$, $a_i^{\prime} \neq \alpha_i^t\left(h^t\right)$, and $i \in N$.*

This means that, at any public history, any player would lose at least $\eta$ by any unilateral one-shot deviation. This is stronger than requiring all incentive constraints to hold strictly. However, a strict PPE is $\eta-$uniformly strict for some $\eta>0$ if it can be represented by a finite state automaton.

## 2.2 Repeated Game with Private Monitoring and Its Public Communication Extension

Fix a stage game $(G, \pi)$ with public monitoring. Consider a private monitoring game with the same set of players and the same action sets as $G$. Player $i$ observes a private signal $s_i$ from a finite set $S_i$ instead of the public signal. A private signal profile is denoted $s=(s_1, .., s_n) \in \Pi_i S_i := S$. Player $i'$s stage game payoff is $v_i\left(a_i, s_i\right)$ and player $i$'s expected stage game payoff is $g_i^{\prime}(a)=\sum_s v_i\left(a_i, s_i\right) p(s|a)$ where the conditional distribution on $S$ given $a$ is denoted $p(\cdot|a)$. We assume that the marginal distributions have full support, that is, $p(s_i|a) := \sum_{s_{-i}} p\left(s_i, s_{-i}|a\right)>0$ for all $s_i \in S_i$, $a \in A$ and $i \in N$. Let $p\left(s_{-i}|a, s_i\right) := \frac{p(s_i, s_{-i}|a)}{p(s_i|a)}$ denote the conditional probability of $s_{-i} \in S_{-i}$ given $(a, s_i)$. We denote this private monitoring stage game by $(G^{\prime}, p)$, where $G^{\prime}=(N, A, g^{\prime})$. Let $V(G^{\prime})$ and $V^*(G^{\prime})$ be the feasible payoff set and the set of individually rational and feasible payoffs for $G^{\prime}$. Discounted average payoffs are defined as in the public monitoring case. Let $G_p^{\prime\infty}(\delta)$ be the corresponding repeated game with private monitoring given $\delta \in (0,1)$.

Players communicate directly each period. At the end of each period, players publicly announce a profile $s \in S$. Then, a public coordinating device $\phi: S \rightarrow \triangle(Y)$ generates public signal $y \in Y$ with probability $\phi(y|s)$. A convex combination of two public coordination devices $\phi$ and $\phi^{\prime}$ is denoted by $(1-\lambda) \phi+\lambda \phi^{\prime}$, which is defined by

$$
\left((1-\lambda) \phi+\lambda \phi^{\prime}\right)(y|s) := (1-\lambda) \phi(y|s)+\lambda \phi^{\prime}(y|s) .
$$

The distribution of the signal generated by $\phi$ given $a$ with honest reporting is denoted by

$$
p^\phi(y|a) := \sum_{s \in S} \phi(y|s) p(s|a) .
$$

We denote expectations with respect to $p^{\phi}(\cdot|a)$ by $E^{\phi}[\cdot|a]$. Player $i$ may not report her signal truthfully. Player $i$'s reporting rule is a function $\rho : S_i \to S_i$. Let $R_i$ be the set of all reporting rules for player $i$ and $\tau_i \in R_i$ be the truth-telling rule defined by $\tau_i(s_i) = s_i$ for all $s_i \in S_i$. When player $i$ uses a reporting rule $\rho_i \in R_i$, we will abuse notation and define

$$p^{\phi}(y|a, \rho_i) := \sum_{s \in S} \phi(y|\rho_i(s_i), s_{-i}) p(s|a)$$

as the distribution of the generated public signal given action profile $a$ when $i$ uses the reporting rule $\rho_i$ and the other players report their private signals truthfully. We denote expectation with respect to $p^{\phi}(\cdot|a, \rho_i)$ by $E^{\phi}[\cdot|a, \rho_i]$. Assuming honest reporting by players $j \neq i$, player $i$'s conditional belief regarding the realization of the public coordinating signal given $(a, s_i)$ and report $s_i'$ is given by

$$p^{\phi}\left(y|a, s_i, s_i'\right) := \sum_{s_{-i}} \phi(y|s_i', s_{-i}) p(s_{-i}|a, s_i).$$

We often use $p^{\phi}(y|a, s_i)$ for $p^{\phi}(y|a, s_i, s_i)$ to economize on notation.

This formulation of communication is very special. A more general communication structure would allow for a mediator who receives and sends confidential private information from and to the players[4]. There are two reasons for not introducing a mediator in this paper. First, a mediator plays no role in our robustness result (Section 4). We ask when a perfect public equilibrium for $G_{\pi}^{\infty}(\delta)$ remains a perfect public equilibrium when players are engaged in a "close" private monitoring game $(G', p)$ augmented with communication. As part of our notion of "closeness," we require that private signals in $(G', p)$ can be aggregated so as to generate a public coordinating signal whose distribution is close to $\pi$. Since public strategies can only depend on this public coordinating signal by definition, there is no role for confidential announcements or confidential recommendations. Second, the lack of a mediator only strengthens our folk theorem result (Section 5).

In the repeated game $(G', p)$ augmented with communication as described above, a public history in period $t$ consists of a sequence of realized public coordinating signals $h^t \in Y^t$ and a sequence of public announcements $h_R^t \in S^t$. We allow different coordinating devices to be employed at different $h^t \in Y^t$. Given a private monitoring game $(G', p)$, a *public communication device* for $(G', p)$ is a collection $\Phi = \{\phi_{h^t} : h^t \in Y^t, t \geq 0,\}$ where each $\phi_{h^t} : S \to \Delta(Y)$ is a public coordination device. Given a private monitoring game $(G', p)$, a discount factor $\delta$, and a public communication device $\Phi$, let $G_p'^{\infty}(\delta, \Phi)$ denote the *public communication extension* of the repeated game with private monitoring $G_p'^{\infty}(\delta)$.

In $G_p'^{\infty}(\delta, \Phi)$, play proceeds in the following way. At the beginning of period $t$, player $i$ chooses an action contingent on $\left(h_i^t, h^t, h_R^t\right)$, where $h_i^t \in A_i^t \times S_i^t$ is a

[4]See Forges [10] and Myerson [22] for mediated communication in dynamic games.

sequence of her own private actions and private signals. If the resulting action profile is $a$, then players receive private signals according to the distribution $p(\cdot|a)$. Let $s$ denote the realized signal profile. Then player $i$ makes a public announcement $s_i'$ contingent on $(h_i^t, h^t, h_R^t, a_i, s_i)$. Of course, $s_i'$ may differ from $s_i$. Let $s' \in S$ denote the profile of announcements. Then a public coordinating signal is chosen according to the probability measure $\phi_{h^t}(\cdot|s')$. If $s'$ is announced and $y$ is realized in period $t$, then $h_R^{t+1}$ and $h^{t+1}$ in period $t+1$ are defined as follows: $h_R^{t+1} = (h_R^t, s')$ and $h^{t+1} = (h^t, y)$.

To describe a strategy in $G_p'^{\infty}(\delta, \Phi)$, let $H^t = Y^t$ denote the set of histories of realized public coordinating signals in period $t$, $H_R^t = S^t$ denote the set of public reporting histories, and $H_i^t = A_i^t \times S_i^t$ denote the set of private histories for player $i$ in period $t$. Player $i'$s (pure) strategy consists of two components, an "action strategy" $\alpha_i^t : H_i^t \times H^t \times H_R^t \longrightarrow A_i$ and a "reporting strategy" $\rho_i^t : H_i^t \times H^t \times H_R^t \times A_i \longrightarrow R_i$. Let $\alpha_i = (\alpha_i^0, \alpha_i^1, ...), \rho_i = (\rho_i^0, \rho_i^1, ...), \alpha = \{\alpha_i\}_{i \in N}, \rho = \{\rho_i\}_{i \in N}$ and let $\sigma = (\alpha, \rho)$. As in the repeated game without communication, pure strategies induce probability measures on $A^{\infty}$. Player $i$' s discounted expected payoff in $G_p'^{\infty}(\delta, \Phi)$ is $w_i^{\sigma,\delta}(\Phi) = (1-\delta)\sum_{t=0}^{\infty} \delta^t E\left[g_i'\left(\tilde{a}^t\right)|\sigma, \Phi\right]$. We usually drop $\Phi$ when it is clear from the context which public communication device is used.

A strategy $\sigma_i = (\alpha_i, \rho_i)$ for player $i$ is *truthful* if player $i$ reports her private signal truthfully whenever she played according to $\alpha_i$ in the same period, i.e., $\rho_i^t\left(s_i|h_i^t, h^t, h_R^t, \alpha_i^t\left(h_i^t, h^t, h_R^t\right)\right) = s_i$ for every $\left(h_i^t, h^t, h_R^t\right)$ and $s_i$. Note that we allow players to lie immediately after a deviation in action. That is, we do not require that $\rho_i^t\left(s_i|h_i^t, h^t, h_R^t, a_i\right) = s_i$ if $a_i \neq \alpha_i^t\left(h_i^t, h^t, h_R^t\right)$. A strategy $\sigma_i = (\alpha_i, \rho_i)$ is public if $\alpha_i^t$ only depends on $h^t = \left(y^0, ..., y^{t-1}\right) \in H^t$ and $\rho_i^t$ depends only on $\left(h^t, a_i^t\right)$. Since we focus on this class of strategies in the public communication extension, we will write $\alpha_i^t\left(h^t\right)$ instead of $\alpha_i^t\left(h_i^t, h^t, h_R^t\right)$ and $\rho_i^t\left(h^t, \alpha_i^t\left(h^t\right)\right)$ instead of $\rho_i^t\left(h_i^t, h^t, h_R^t, \alpha_i^t\left(h_i^t, h^t, h_R^t\right)\right)$. Notice that there is a natural one-to-one relationship between public strategies in $G_{\pi}^{\infty}(\delta)$ and the action strategy components of public strategies in $G_p'^{\infty}(\delta, \Phi)$. Note also that we can ignore incentive constraints across different $\left(h_i^t, h_R^t\right)$ in $G_p'^{\infty}(\delta, \Phi)$ when every player uses a public strategy, as we can ignore incentive constraints across different $h_i^t$ with public strategies for $G_{\pi}^{\infty}(\delta)$.

We extend the standard definition of perfect public equilibrium to the public communication extension as follows: a strategy profile $\sigma$ for the public communication extension is a *perfect public equilibrium with communication* (which we will refer to as PPE from now on) if $\sigma$ is a profile of *truthful* public strategies and the continuation (public) strategy profile constitutes a Nash equilibrium at the beginning of every period. A strategy profile $\sigma$ is $\eta-$uniformly strict perfect public equilibrium with comminication if $\sigma$ is a perfect public equilibrium and any player would lose at least $\eta$ in term of discounted average payoff at any moment when she deviates from the *equilibrium action.*

## 3 Informational Size and Incentive Compatibility

### 3.1 Distance between Stage Games

We focus on public monitoring games and private monitoring games that are close to each other. When we say "close", it means that a public monitoring game $(G, \pi)$ and a private monitoring game $(G', p)$ are close in two respects. First, $g$ and $g'$ are close. Second, there exists a public coordination device $\phi$ that can generate a public signal distribution close to $\pi$.

**Definition 2** *Let $(G', p)$ be a private monitoring game and $(G, \pi)$ be a public monitoring game. Given any public coordinating device $\phi$, we say that $p^{\phi}$ is an $\varepsilon-$*approximation *of $\pi$ if $\max_a \left\|\pi(\cdot|a) - p^{\phi}(\cdot|a)\right\| \leq \varepsilon$.*[5] *$(G', p, \phi)$ is an $\varepsilon-$*approximation *of $(G, \pi)$ if $p^{\phi}$ is an $\varepsilon-$approximation of $\pi$ and $\max_{i,a} |g_i(a) - g'_i(a)| \leq \varepsilon$.*

The following is a canonical example of $\varepsilon-$approximation.

**Example 1** ***Conditionally Independent Signals***

*Let $\widetilde{y}$ denote a random variable that can take the values $\widetilde{y} = 0$ or $\widetilde{y} = 1$ with equal probability. There are $n$ players ($n$ odd). They do not observe the realization of $\tilde{y}$ but each observes a noisy private signal correlated with $\tilde{y}$. Specifically, if $\tilde{y} = y$, then player $i$ observes a private signal $s_i \in S_i = \{0, 1\}$, which agrees with $y$ with probability $\beta \in \left(\frac{1}{2}, 1\right]$ and differs from $y$ with probability $1 - \beta$. These private signals are conditionally independent. Suppose that all players report their private signals $(s_1, .., s_n)$ simultaneously and truthfully and $\phi(1|s_1, .., s_n) = 1$ if the majority of the players announce $1$ while $\phi(0|s_1, .., s_n) = 0$ if the majority of the players announce $0$.*[6] *Clearly the distribution on $\{0, 1\}$ generated this way is a good approximation of the original distribution of $\widetilde{y}$ when $\beta$ is sufficiently close to $1$ or when the number of players is sufficiently large.*

Since our notion of approximation is relatively loose, it may happen that two seemingly different monitoring structures are close to each other. For example, $Y$ and $S_i$ can be be different sets. As an another example, consider the following monitoring structure.

**Example 2** ***Perfectly Complementary Information***

*Suppose that $Y = \{0, 1\}$ and $S_i = \{0, 1\}$ and let $\pi$ be the distribution of a hidden signal on $Y$. There are six players. The distribution $p$ of private signals on $S$ satisfies the following. When the true value of the hidden signal is $1$, the private signal profile is such that three players receive signal $0$ and three players receive signal $1$, and each*

[5] $\|\cdot\|$ is Euclidean norm.

[6] The generated public signal is the maximum likelihood estimate of the true realization of $\widetilde{y}$ in this example.

*such profile of signals is equally likely. When the true value of the hidden signal is* $0$*, the private signal profile is either* $(1,1,1,1,1,1)$ *or* $(0,0,0,0,0,0)$*, each with probability* $\frac{1}{2}$*. Consider a public coordination mechanism* $\phi$ *such that* $\phi(0|s)=0$ *if at least five players announce the same signal and* $\phi(0|s)=1$ *otherwise. Then* $p^{\phi}$ *is a* $0-$*approximation of* $\pi$*.*[7]

In this example, each player is equally likely to observe 0 or 1 for any realization of the underlying signal. Hence her signal alone provides no information about the true value of the underlying signal, yet the aggregated private signals completely reveals the true underlying signal.

In this example (or Example 1 with $\beta = 1$ and $n \geq 3$), one player's information does not affect the generated public signal at all. Following Postlewaite and Schmeidler [25], we say that a pair $(p,\phi)$ is nonexclusive when $p^{\phi}(y|a,\rho_i) = p^{\phi}(y|a)$ for any $a$, $y$, $\rho_i$ and $i$.

### 3.2 Informational Size, Distributional Variability, and One-Shot Revelation Game

We turn to the issue of truthful revelation of private information in this subsection. Although our main interest is in repeated games, it is useful to consider the following simple *one-shot information revelation game* first. Fix any private monitoring game $(G',p)$. For any public coordination device $\phi$, any profile of payoff functions $w: Y \to \mathbb{R}^n$, and any $a \in A$, the one-shot information revelation game $(G',p,\phi,w,a)$ is defined as follows. Player $i$ observes a private signal $s_i$, which is distributed according to $p(s|a)$. Players report $s'$, then a public coordinating signal $y$ is generated with probability $\phi(y|s')$. Finally, player $i$ receives payoff $w_i(y)$ if the realized value of the public signal is $y$. In the context of repeated games, this payoff will be interpreted as player $i$'s continuation payoff. Consequently, $(G',p,\phi,w,a)$ defines a game of incomplete information in which a strategy for player $i$ is a function $\rho_i : S_i \to S_i$ and truthful reporting is an equilibrium if for each $i$,

$$\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i(y)\phi\left(y|s_i,s_{-i}\right)p\left(s_{-i}|a,s_i\right) \geq \sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i(y)\phi\left(y|s_i',s_{-i}\right)p\left(s_{-i}|a,s_i\right)$$

for each $s_i, s_i' \in S_i$.

When do players have incentive to report their private signals truthfully in this game? To fix ideas, consider the extreme case in which $(p,\phi)$ is nonexclusive. Then no player has an incentive to lie because what she reports is irrelevant and does not affect the generated public signal at all. Hence truth-telling can be implemented

[7] We need six players for this example. Suppose that there are only five players. Either $(1,1,1,1,1)$ or $(0,0,0,0,0)$ is observed with probability $\frac{1}{2}$ given $y=0$, and every signal profile at least two 0 and 1 is observed with equal probability given $y=1$. In this case, when $(1,1,1,1,0)$ is observed, we cannot tell whether one of the first four players is lying given $y=1$ or the last player is lying given $y=0$.

in a one-shot revelation game for any specification of the payoff function and any action when $(p, \phi)$ is nonexclusive.

We wish to generalize this simple observation. In general, it should become "easier" to induce truth-telling as each player's influence on the public coordinating signal becomes "smaller." The following index measures the size of this influence for each player.

**Definition 3** *(Informational Size)*

*Player $i'$s informational size $v_i^{\phi}(s_i, a)$ given $\phi$ and $(s_i, a) \in S_i \times A$ is the smallest $\epsilon$ satisfying*

$$\max_{s_i' \in S_i} \Pr(\left\| \phi\left(\cdot|s_i, \widetilde{s}_{-i}\right) - \phi\left(\cdot|s_i', \widetilde{s}_{-i}\right)\right\| > \epsilon | s_i, a) \leq \epsilon.$$

This means that, conditional on $(s_i, a)$, player $i$ believes that the probability of her being able to manipulate the public signal distribution by more than $v_i^{\phi}(s_i, a)$ is at most $v_i^{\phi}(s_i, a)$. For example, each player's informational size is 0 in Example 2. We say a player is *informationally small* when her informational size is small.

Note that there may be a trade-off between keeping each player's informational size small and approximating a particular public signal distribution. Suppose that only player 1 is perfectly informed regarding the realization of $\tilde{y}$ in Example 1, i.e. $\beta_1 = 1$ and $\beta_i = \beta \in (1/2, 1)$ for all $i \neq 1$. If the goal is to replicate the true value of $y$ from private signals, then $\phi$ should assign probability one to the element of $Y = \{0, 1\}$ that coincides with player $1'$s private signal. However, this makes player 1 informationally large. In such a case, it may be preferable to aggregate the private signals of all players, including player 1, using the majority rule. Although this majority rule $\phi$ is informationally inferior to the former $\phi$, it makes every player informationally small and $p^{\phi}(\cdot|a)$ still remains a good approximation of $\pi(\cdot|a)$.

Of course, a player's informational size alone is not enough to induce honest reporting. Since players may still have incentive to misreport their signals, however small it is, we need to introduce some scheme to punish dishonest reporting. So we consider the following mechanism design problem: given that $a \in A$ is played, find a public coordination device $\phi'$ that generates approximately the same distribution as $p^{\phi}(\cdot|a)$ and makes truthful reporting a Bayesian Nash equilibrium for the one-shot revelation game $\left(G', p, \phi', w, a\right)$. For this purpose, we construct a certain scoring rule that relies on a player's distributional variability.

**Definition 4** *(Distributional Variability of player $i$ given $\phi$)*

$$\Lambda_i^{\phi}(s_i, a) = \min_{s_i' \neq s_i} \left\| \frac{p^{\phi}(\cdot|a, s_i)}{\|p^{\phi}(\cdot|a, s_i)\|} - \frac{p^{\phi}(\cdot|a, s_i')}{\|p^{\phi}(\cdot|a, s_i')\|} \right\|^2$$

This measures the extent to which player $i$'s conditional (normalized) beliefs regarding the public coordinating signal are different given different private signals (assuming honest reporting by others). This is close to 1 in Example 1 and equal to 0 in Example 2. We use this variation of player $i$'s beliefs to induce her to report her private signals truthfully.[8],[9]

Intuitively, it must be easier to induce players to report their private signals truthfully when the first index is smaller and the second index is larger. It turns out that what is important for truthful revelation is the ratio of these two indexes.

**Definition 5** *The measure $p^{\phi}$ is $\gamma-$regular for $\phi$ if $v_i^{\phi}(s_i, a) \leq \gamma \Lambda_i^{\phi}(s_i, a)$ for all $s_i \in S_i$, $a \in A$, and $i \in N$.*

For example, $p^{\phi}$ is $0-$regular if $(p, \phi)$ is nonexclusive. We can now prove the following theorem.

**Theorem 1** *For any private monitoring game $(G', p)$ and any $\lambda \in (0,1)$, there exists a $\gamma > 0$ such that the following holds: if $p^{\phi}$ is $\gamma-$regular for some $\phi$, then for any $a \in A$ and any payoff function $w : Y \rightarrow \mathbb{R}^n$, there exists a public coordination device $\phi'_{a,w} : S \rightarrow \triangle(Y)$ such that truthful reporting is a Bayesian Nash equilibrium for the one-shot information revelation game $\left(G', p, (1-\lambda)\phi + \lambda \phi'_{a,w}, w, a\right)$.*

**Proof.** See Appendix B. ■

This theorem means that honest reporting can be induced for any one-shot revelation game by perturbing $\phi$ slightly. The smaller $\gamma$ is, the smaller the size of the required perturbation. Note that $\gamma$ depends on $\lambda$ but is independent of the payoff function and the underlying action. These properties will be important when this result is applied to repeated games. It is natural that $\gamma$ is independent of $a$, since $\gamma-$regularity requires a certain property across all actions. Observe that the choice of $\gamma$ is also independent of $w$. We construct a punishment by perturbing $\phi$

[8] Suppose that player $i$ flips a coin after observing $s_i$ and let her new private signal be $(s_i, \omega) \in S_i \times \{face, down\}$. This change of information structure should not affect our result. However player $i$'s distributional variability is 0 given either $(s_i, face)$ or $(s_i, down)$. To deal with this problem, we could have introduced a more elaborate definition of distributional variability. Let $x_i$ be a mapping from $S_i$ to player $i$'s message space $M_i$. Then define distributional variability for each $x_i$ as follows:

$$\Lambda_i^{\phi, x_i}(s_i, a) = \min_{\left\{s_i' : x_i\left(s_i'\right) \neq x_i(s_i)\right\}} \left\| \frac{p^{\phi}(\cdot|a, s_i)}{\|p^{\phi}(\cdot|a, s_i)\|} - \frac{p^{\phi}(\cdot|a, s_i')}{\|p^{\phi}(\cdot|a, s_i')\|} \right\|^2$$

Then we can keep the distributional variability invariant with respect to such extrinsic private information by using $x_i$, which maps $(s_i, face)$ and $(s_i, down)$ into the same $m_i$.

[9] Our distributional variability is similar, but different from the condition with the same name in McLean and Postlewaite [21]. Our condition measures the distance between a player's conditional belief regarding the aggregated public signal, whereas their condition measures the distance between a player's conditional belief about the other players's private signals, i.e. $\left\| \frac{p(\cdot|s_i)}{\|p(\cdot|s_i)\|} - \frac{p\left(\cdot|s_i'\right)}{\left\|p\left(\cdot|s_i'\right)\right\|} \right\|^2$ (there is no action in [21]).

slightly so that the distribution of the generated signal remains similar, but truth-telling is a BNE. Hence, when $w$ is large, the temptation to deviate may be high, but the size of punishments is large in the same proportion.

This theorem is similar to Theorem 1 in McLean and Postlewaite[21]. However, notice that there is a difference between this result and the result in [21]. In [21], each player's preference is given by $u_i(x,\theta)$, where $\theta$ is an unobservable payoff relevant parameter and $x$ is an allocation, so it is important that the true $\theta$ is recovered almost surely ("*negligible uncertainty*") to implement any allocation function $x : \theta \mapsto x(\theta)$. We do not need that the true public signal $y$ is recovered.

When is $p^{\phi}$ likely to be $\gamma-$regular? Consider a more general version of example with conditionally independent signals.

**Definition 6** *A private monitoring game $(G',p)$ is called a $\beta-$perturbation of the public monitoring game $(G,\pi)$ if $S_i = Y$ for all $i$, $v_i(a_i,y) = u_i(a_i,y)$ for all $(a,y)$ and $i$, and there exists $q_i(\cdot|y) \in \triangle(S_i)$ for all $y$ and $i$ such that $p(s|a) = \sum_{y\in Y}\prod_i q_i(s_i|y)\pi(y|a)$ and $q_i(y|y) \geq \beta$ for any $y$ and $i$.*

Suppose that $(G',p)$ is a $\beta-$perturbation of $(G,\pi)$. Let $\phi_M$ be the "majority rule", which is a public coordination device that chooses $y$ reported by the largest number of players (with some tie-breaking rule). Then

$$\begin{aligned} p^{\phi_M}(y|a) &= \sum_{s\in S}\phi_M(y|s)p(s|a) \\ &= \sum_{s\in S}\phi_M(y|s)\sum_{y\in Y}\left[\prod_{i=1}^{n} q_i(s_i|y)\right]\pi(y|a) \end{aligned}$$

and $p^{\phi_M}$ can generate almost the same signal distribution as $\pi$ as $\beta \to 1$. It follows that $(G',p)$ is an $\varepsilon-$approximation of $(G,\pi)$ for any given $\varepsilon$ if $\beta$ is close enough to zero. Furthermore, each player's informational size converges to 0 as long as $n \geq 3$ and distributional variability converges to a positive constant as $\beta \to 1$. The following proposition summarizes this observation.

**Proposition 1** *Fix a public monitoring game $(G,\pi)$ with $n \geq 3$. For any $\varepsilon > 0$ and $\gamma > 0$, there exists $\beta \in (0,1)$ and a public coordination device $\phi$ such that, for every $\beta$-perturbation $(G',p)$ of $(G,\pi)$, $(G',p,\phi)$ is an $\varepsilon-$approximation of $(G,\pi)$ and $p^{\phi}$ is $\gamma-$regular.*

# 4 Robustness of PPE

Consider any $(G,\pi)$. Suppose that $(G',p)$ is an $\varepsilon-$approximation of $(G,\pi)$. In this section, we ask the following question: can a PPE $\alpha^*$ of $G_{\pi}^{\infty}(\delta)$ be a part of a PPE of $G_p'^{\infty}(\delta,\Phi)$ with the help of some public communication device $\Phi$?

The answer is again simple in some extreme cases. Let $\phi$ be a public coordination device for which $p^{\phi}$ is an $\varepsilon-$approximation of $\pi$. Suppose that $(p,\phi)$ is nonexclusive and let $\Phi$ be the public communication device where $\phi$ is used after every public history, i.e., $\phi_{h^t}=\phi$ for all $t$ and all $h^t$. In this case, we can disregard revelation constraints completely. Then essentially we just have another repeated game with public monitoring where the stage game payoffs and the public signal distribution are slightly perturbed. Therefore every $\eta-$uniformly strict PPE of $(G,\pi)$ is an $\eta'-$uniformly strict PPE of $G_p'^{\infty}(\delta,\Phi)$ for some positive $\eta'$ as long as $\varepsilon$ is small enough. This follows from the observation (which is formally demonstrated in the proof of Theorem 2) that continuation payoffs in $G_p'^{\infty}(\delta,\Phi)$ converge to continuation payoffs in $G_{\pi}^{\infty}(\delta)$ uniformly across all public histories and all public strategies as $\varepsilon\to 0$.

Our robustness result generalizes this observation by relaxing nonexclusivity. It says that every $\eta-$uniformly strict PPE of $G_{\pi}^{\infty}(\delta)$ is an $\eta'-$uniformly strict PPE of $G_p'^{\infty}(\delta,\Phi)$ for some public communication device $\Phi$ and some positive $\eta'$ if, for some public coordinating device $\phi$, $(G',p,\phi)$ is a very good approximation of $(G,\pi)$ and each player's informational size is relatively small given $p^{\phi}$ (i.e. $p^{\phi}$ is $\gamma$-regular for enough small $\gamma$).

**Definition 7** *An $\eta-$ uniformly strict perfect public equilibrium $\alpha^*$ of $G_{\pi}^{\infty}(\delta)$ is strictly robust with respect to $(G',p)$ if there exists a public communication device $\Phi^*$ and a reporting strategy $\rho^*$ such that $\sigma^*=(\alpha^*,\rho^*)$ is an $\eta'-$uniformly strict truthful PPE of $G_p'^{\infty}(\delta,\Phi^*)$ for some $\eta'>0$.*

**Theorem 2** *Fix $\delta\in(0,1)$ and a public monitoring game $(G,\pi)$. For any $\eta>0$, there exist $\gamma,\varepsilon>0$ such that every $\eta-$uniformly strict PPE of $G_{\pi}^{\infty}(\delta)$ is strictly robust with respect to any $(G',p)$ for which there exists $\phi$ such that $(G',p,\phi)$ is an $\varepsilon$-approximation of $(G,\pi)$ and $p^{\phi}$ is $\gamma$-regular.*

**Proof.** See Appendix C. ■

Note that $\gamma$ and $\varepsilon$ only depend on $\eta$, but not on a particular equilibrium strategy. To get an idea behind our proof, suppose that private signals are converted into a public coordinating signal by $\phi$ in every period, i.e., let $\Phi$ be the public communication device where $\phi_{h^t}=\phi$ for all $t$ and all $h^t$. Furthermore, suppose that players truthfully reveal their signals so that information revelation constraints are not present. If $\alpha^*$ is an $\eta-$uniformly strict PPE of $G_{\pi}^{\infty}(\delta)$, then one-shot deviations from $\alpha^*$ are not profitable in $G_p'^{\infty}(\delta,\Phi)$ as long as $\varepsilon$ is small because, as we mentioned, continuation payoffs in $G_p'^{\infty}(\delta,\Phi)$ are close to the continuation payoffs in $G_{\pi}^{\infty}(\delta)$ uniformly across all public histories and all public strategies. Hence, the action strategy $\alpha^*$ defines an equilibrium in $G_p'^{\infty}(\delta,\Phi)$ if players report their private signals truthfully. This step is based only on the idea of $\varepsilon$-approximation.

How can truthful reporting be induced? First focus on the revelation constraint in the first period on the equilibrium path (i.e. after the first period equilibrium action is played). We use both informational smallness and distributional variability in this step. Note that this continuation game can be regard as a one-shot revelation game where the action profile is given by the equilibrium action profile $a^* = \alpha^*\left(h^0\right)$ and $w$ is given by the continuation payoffs from the second period on. Here we can apply Theorem 1 to perturb $\phi$ slightly to induce honest reporting: for any $\lambda \in (0,1)$, there exists a $\gamma > 0$ such that, if $p^{\phi}$ is $\gamma-$regular, then truthful reporting is optimal for every player in a one-shot revelation game $\left(G', p, (1-\lambda)\phi + \lambda\phi'_{a^*,w}, w, a^*\right)$ for some $\phi'_{a^*,w}$. Of course, we need to make sure to keep perturbation small so that the strict incentive to play $a^*$ is preserved. So we choose small enough $\lambda$ (with small enough $\gamma$ needed for Theorem 1) and $\varepsilon$ so that $p^{(1-\lambda)\phi+\lambda\phi'_{a^*,w}}$ is a good approximation of $p^{\phi}$, hence a good approximation of $\pi$, independent of the specification of $\phi'_{a^*,w}$ if players report their signals truthfully. Then the strict incentive to play $a^*$ is preserved in $G'^{\infty}_{p}(\delta, \Phi)$ given truth-telling.

We also need to consider a joint deviation in action and reporting within the first period. This is where informational smallness plays another role. Observe that each player cannot manipulate $\phi$ much when $\varepsilon$ is small. Hence, $p^{(1-\lambda)\phi+\lambda\phi'_{a^*,w}}$ is a good approximation of $\pi$ whether a player lies or not as long as she is informationally small. Therefore it remains unprofitable to deviate from $a^*$ in the first period when $(1-\lambda)\phi + \lambda\phi'_{a^*,w}$ is used instead of $\phi$ when $(\lambda, \varepsilon, \gamma)$ are kept small.

Now let us take into account the revelation constraints in all periods. This is the most delicate step in the proof. To illustrate the problem, focus on some public history $h^t$. We may proceed as we did before in the first period: we use the same $\lambda$. We can choose the same $\varepsilon$ because of the uniform convergence result stated above. Then we choose $\gamma$ and construct $\phi'_{\alpha^*(h^t),w(h^t,\cdot)}$ so that playing $\alpha^*\left(h^t\right)$ and truthful reporting is optimal given the continuation equilibrium payoff function $w\left(h^t,\cdot\right)$. Furthermore, we can use the same $\gamma$ as before across all public histories because $\gamma$ can be chosen independent of the equilibrium action or continuation payoffs thanks to Theorem 1. However, observe that $\phi'_{\alpha^*(h^t),w(h^t,\cdot)}$ depends on the continuation payoffs, and the continuation payoffs depend on the choice of $\phi'_{\alpha^*(h^s),w(h^s,\cdot)}$ at every continuation history $h^s$ with $s > t$. Consequently, we cannot pick a public coordination device for history $h^t$ that is independent of all public coordination devices in the future.

We apply a fixed point theorem to address this problem as follows. First assign an arbitrary public coordination device to every public history, i.e. we pick some set of public coordination devices $\left\{\phi'_{h^t}, h^t \in H\right\}$. Consider the public communication device $\Phi' = \left\{(1-\lambda)\phi + \lambda\phi'_{h^t}, h^t \in H\right\}$, where $\lambda$ is chosen so that the incentive constraints in actions will be satisfied independent of the choice of $\left\{\phi'_{h^t}, h^t \in H\right\}$ as long as the players are reporting truthfully and $\varepsilon$ is chosen small enough. Now compute the continuation payoffs $w'\left(h^t,\cdot\right)$ at every public history $h^t$ given this $\Phi$. Next we choose small $\gamma$ so that Theorem 1 can be applied. Again Theorem 1 is playing a critical role to allow us to choose the same $\gamma$ at every $h^t \in H$. Then we can construct $\left\{\phi'_{\alpha^*(h^t),w'(h^t,\cdot)}, h^t \in H\right\}$so that the revelation constraints are satisfied given such continuation payoffs $w'\left(h^t,\cdot\right)$ and $\left\{(1-\lambda)\,\phi + \lambda\phi'_{\alpha^*(h^t),w'(h^t,\cdot)}, h^t \in H\right\}$. Indeed, we can construct a (compact) set of such public coordination devices at every public history. This defines a well-behaved correspondence from the space of sequences of public coordination devices $\{\phi'_{h^t} : h^t \in H^t, t \geq 0\}$ to itself. Since $\{\phi'_{h^t} : h^t \in H^t, t \geq 0\}$ is an infinite dimensional space, we can apply Fan-Glicksberg fixed point theorem to obtain a fixed point $\{\phi^*_{h^t} : h^t \in H^t, t \geq 0\}$. Then all the revelation constraints are satisfied on the equilibrium path with respect to the continuation payoffs that are based on correct public coordination devices $\{\phi^*_{h^t} : h^t \in H^t, t \geq 0\}$.

What exactly does this result say about the robustness of PPE in $G^\infty_\pi(\delta)$? To make the idea precise, let $(G,\pi)$ be a game with public monitoring and define the *$\varepsilon$-perturbation class* $\Delta_\varepsilon(G,\pi)$ of $(G,\pi)$ as follows:

$$\Delta_\varepsilon(G,\pi) = \{(G',p)|\exists\phi \text{ s.t. } \left(G',p,\phi\right) \text{ is an } \varepsilon\text{-approximation of } (G,\pi)\}$$

As a consequence of Proposition 1, it follows that $\Delta_\varepsilon(G,\pi) \neq \varnothing$ for all $\varepsilon$ if $n \geq 3$. We will say that a PPE $\alpha^*$ of $G^\infty_\pi(\delta)$ is $\varepsilon-$robust if for every $(G',p,\phi) \in \Delta_\varepsilon(G,\pi)$, there exists a public communication device $\Phi^*$ and a truthful reporting strategy $\rho^*$ such that $\sigma^* = (\alpha^*,\rho^*)$ is a PPE of $G'^\infty_p(\delta,\Phi^*)$. In terms of $\varepsilon$-robustness, a stronger continuity result would be stated as follows: for every $\eta > 0$, there exists an $\varepsilon > 0$ such that every $\eta$-uniformly strict PPE of $G^\infty_\pi(\delta)$ is $\varepsilon$-robust. The statement of our Theorem 2 is close to, but not quite the same as, this stronger notion of robustness. To show that every $\eta$-uniformly strict PPE of $G^\infty_\pi(\delta)$ remains a part of PPE of $G'^\infty_p(\delta,\Phi)$ for some public communication device $\Phi$, we require that some coordination mechanism $\phi$ by which $(G,\pi)$ is approximated also satisfies an additional condition: the informational size must be enough small relative to the distributional variability given $\phi$, i.e. $p^\phi$ is $\gamma-$regular.

A simpler robustness result can be obtained with respect to $\beta-$perturbations of public monitoring games. We record the result whose proof is an immediate consequence of Proposition 1 and Theorem 2.

**Corollary 1** *Fix a public monitoring game $(G,\pi)$ with $n \geq 3$ and $\delta \in (0,1)$. For every $\eta > 0$, there exists $\beta > 0$ and $\eta' \in (0,\eta)$ such that, every $\eta-$uniformly strict perfect public equilibrium of $G^\infty_\pi(\delta)$ is strictly robust with respect to any $\beta-$perturbation of $(G,\pi)$.*

Finally we like to mention another interpretation of our theorem. The existence of a nontrivial equilibrium in repeated games with private monitoring is a difficult problem. An ostensibly easier problem is that of the existence of correlated or communication equilibrium. Our theorem says that, under certain circumstances, we can construct a certain type of communication equilibrium for a repeated game with private monitoring if it is "close" to a game with public monitoring in a certain sense.

# 5 Folk Theorem

The previous section focuses on the robustness of PPE in public monitoring games and touches upon private monitoring games mainly as their approximations. In this section, our main target is the repeated game with private monitoring itself; we prove a new folk theorem for repeated games with private monitoring and communication when players are informationally small. We exploit a connection between public monitoring games and private monitoring games and adapt some standard techniques for a public-monitoring folk theorem to the domain of private monitoring games.

Our folk theorem asserts the following. Suppose that for some public coordination device $\phi$ for $(G', p)$ the associated $p^{\phi}$ satisfies a certain condition that guarantees a folk theorem in the repeated game with public monitoring game $(G', p^{\phi})$. Then there exists a $\gamma > 0$ such that a folk theorem is also obtained for a communication extension of the repeated game with private monitoring game $G_p'^{\infty}(\delta, \Phi)$ for some public communication device $\Phi$ when $p^{\phi}$ is $\gamma$-regular. Furthermore, our folk theorem is a uniformly strict folk theorem, i.e., a folk theorem with $\eta-$uniformly strict PPE for some $\eta > 0$.

To state the theorem more precisely, we need to clarify the "certain condition" to which we have alluded in the previous paragraph. Given any public signal distribution $\pi$, let $T_i^{\pi}(a) \subset R^{|Y|}$ be defined as

$$T_i^{\pi}(a) = co\left\{\pi\left(\cdot|a_i', a_{-i}\right) - \pi\left(\cdot|a\right) : a_i' \neq a_i\right\}$$

and let $\widehat{T}_i^{\pi}(a) = co\left\{T_i^{\pi}(a) \cup \{0\}\right\}$.[10] The set $T_i^{\pi}(a)$ consists of those distributional changes that player $i$ can induce by choosing a strategy different from $a_i$ when the remaining players choose $a_{-i}$. The set $\widehat{T}_i^{\pi}(a)$ consists of all feasible distributional changes that player $i$ can induce. We say that a public signal distribution $\pi$ satisfies *distinguishability* at $a \in A$ if for each pair of distinct players $i$ and $j$, the following conditions are satisfied:

$$0 \notin T_i^{\pi}(a) \cup T_j^{\pi}(a) \quad (1)$$

$$\widehat{T}_i^{\pi}(a) \cap \widehat{T}_j^{\pi}(a) = \{0\} \quad (2)$$

$$\left(-\widehat{T}_i^{\pi}(a)\right) \cap \widehat{T}_j^{\pi}(a) = \{0\}. \quad (3)$$

We say that $\pi$ satisfies distinguishability if it satisfies distinguishability at every $a \in A$. (1) means that a unilateral deviation by player $i$ or player $j$ must be

[10] $coX$ denote the convex hull of $X$ in $\mathbb{R}^n$.

statistically detectable. (2) and (3) are conditions regarding the distinguishability of player $i$'s deviation and player $j$'s deviation. It is known that these conditions are sufficient for a folk theorem for repeated games with public monitoring.[11]

Now we can state our folk theorem. Let $E(\delta, \Phi, \eta) \subset \mathbb{R}^n$ be the set of $\eta-$uniformly strict PPE payoff profiles of $G_p'^{\infty}(\delta, \Phi)$ given $\delta$ and $\Phi$.

**Theorem 3** *Fix any private monitoring game $(G', p)$. Suppose that $intV^*(G') \neq \varnothing$ and there exists $\phi$ such that $p^{\phi}$ is distinguishable. Then there exists a $\gamma > 0$ such that, if $p^{\phi}$ is $\gamma-$regular, then the following holds: for each $v \in intV^*(G')$, there exists an $\eta > 0$ and a $\underline{\delta} \in (0,1)$ such that, for each $\delta \in (\underline{\delta}, 1)$, there exists a public communication device $\Phi$ and a $(1-\delta)\eta-$uniformly strict truthful PPE of $G_p'^{\infty}(\delta, \Phi)$ with payoff $v$.*

**Proof.** See Appendix D. ■

Note that $\gamma$ depends only on the underlying stage game $(G', p)$ but not on $v$. On the other hand, $\eta$ depends on $v$, while $\Phi$ depends on both $v$ and $\delta$.[12]

**Remark.**

- The assumption $intV^*(G') \neq \varnothing$ requires that $V^*(G')$ is full dimensional. When $V^*(G')$ is not full-dimensional, we may strengthen the distinguishability condition to prove the same result. To prove this theorem, for each $a \in A$ and $q \in \mathbb{R}^n$ such that $\|q\| = 1$ and $|q_i| < 1$ for all $i$, we construct $x : Y \to \mathbb{R}^n$ that satisfies $E^{\phi}[x_i|a] > E^{\phi}[x_i|a_i', a_{-i}]$ for all $a_i' \neq a_i$ and all $i$. If $V^*(G')$ is not full dimensional, the range of $x$ needs to be the affine space that contains $V^*(G')$, instead of $\mathbb{R}^n$. This additional restriction can be addressed by strengthening the distinguishability condition. The bottom line is that every proof goes through if we restrict our attention to the affine space that contains $V^*(G')$.[13] See Fudenberg, Levine and Takahashi [14] for the characterization of the limit equilibrium payoff when $V^*(G')$ is not full dimensional.

- Is a folk theorem obvious given our robustness result? Take any private monitoring game $(G', p)$ for which there exists $\phi$ such that $p^{\phi}$ satisfies distinguishability. Why not prove a folk theorem with $\eta-$uniformly strict PPE for some $\eta > 0$ for the public monitoring repeated game with $(G', p^{\phi})$ (which is not

[11] These conditions guarantee that the incentive constraints of player $i$ and $j$ are satisfied simultaneously by using appropriate transfers (=continuation payoffs) even when their transfers are required to lie on any hyperplane. They are parallel to (A2) and (A3) in Kandori and Matsushima [16].

[12] However, $\eta$ can be chosen independent of $v$ for generic stage games, namely when the solution for $\max_{a \in A} g_i(a)$ and $\min_{a \in A} g_i(a)$ is unique for every $i$.

[13] In particular, we need to state Lemma 9 with respect affine spaces and in terms of relative interior.

difficult to do) and apply our robustness result? However, this approach is not satisfactory because we need to tailor the informational size to each target equilibrium payoff profile and given discount factor to do so, i.e. $\gamma$ depends on both $v$ and $\delta$. The strength of the above folk theorem is that we can find a *fixed* size of informational smallness for which the folk theorem is obtained, rather than including $\gamma$ as a parameter that depends on each payoff profile in the statement of the folk theorem.

## 5.1 Overview of Proof

We prove our folk theorem in several steps. Some proofs are provided in the appendix.

**Self Decomposability with Private Monitoring and Public Coordinating Device**

In the following, a private monitoring game $(G', p)$ is fixed. Rather than analyzing the repeated game directly, we begin by decomposing discounted average payoffs of a repeated game into current payoffs and continuation payoffs, and then analyze a collection of one-shot revelation games.

For a public monitoring game $(G, \pi)$, an action profile $a \in A$ is said to be *enforceable* with respect to $W \subset \mathbb{R}^n$ and $\delta \in (0,1)$ if there exists a function $w : Y \to W$ such that

$$(1-\delta) g_i(a) + \delta E[w_i(\cdot)|a] \geq (1-\delta) g_i(a'_i, a_{-i}) + \delta E[w_i(\cdot)|a'_i, a_{-i}]$$
$$\text{for all } a'_i \neq a_i \text{ and } i \in N.$$

If $v = (1-\delta) g(a) + \delta E[w(\cdot)|a]$ for some enforceable action $a$ and $w : Y \to W$, then we say that $v$ is decomposable with respect to $W$ and $\delta$. Let $B(\delta, W)$ be the set of payoff profiles that are decomposable with respect to $W$ and $\delta$. It is a known result that, if any bounded set $W'$ is self decomposable i.e. $W' \subset B(\delta, W')$, then every payoff in $B(\delta, W')$ (hence in $W'$) can be supported by a PPE (Abreu, Pearce and Stacchetti [1]).

We now extend these ideas to the private monitoring game $(G', p)$ with public coordination devices. Recall that, given action profile $a \in A$, $E^{\phi}[\cdot|a]$ denotes expectation with respect to $p^{\phi}(\cdot|a)$, $E^{\phi}[\cdot|a, \rho_i]$ denotes expectation with respect to $p^{\phi}(\cdot|a, \rho_i)$ and $\tau_i : S_i \to S_i$ denotes the honest reporting rule for player $i$ defined by $\tau_i(s_i) = s_i$ for all $s_i \in S_i$.

**Definition 8** *An action profile $a \in A$ is $\eta-$*enforceable *with respect to $W \subset \mathbb{R}^n$ and $\delta \in (0,1)$ if there exists a public coordinating device $\phi : S \to \Delta(Y)$ and $w : Y \to W$ such that for all $i \in N$,*

*(i)* $(1-\delta) g'_i(a) + \delta E^{\phi}[w_i(\cdot)|a] - \eta \geq (1-\delta) g'_i(a'_i, a_{-i}) + \delta E^{\phi}[w_i(\cdot)|(a'_i, a_{-i}), \rho_i]$ *for all $a'_i \neq a_i$, $\rho_i : S_i \to S_i$*

*(ii)* $(1-\delta) g'_i(a) + \delta E^{\phi}[w_i(\cdot)|a] \geq (1-\delta) g'_i(a) + \delta E^{\phi}[w_i(\cdot)|a, \rho_i]$ *for all $\rho_i \neq \tau_i$.*

The inequality (i) means that a player would lose more than $\eta$ when deviating from $a$. Inequality (ii) means that dishonest reporting is not profitable after $a$ is played. If $a \in A$ is $\eta-$enforceable with respect to $W$ and $\delta$ with some $v$and $w$ and $v = (1-\delta)\, g_i'(a) + \delta E^{\phi}[w_i(\cdot)\,|a]$, then we say that the triple $(a, \phi, w)$ $\eta-$*enforces* $v$ with respect to $W$ and $\delta$. We say that $v$ is $\eta-$*decomposable* with respect to $W$ and $\delta$ when there exists a triple $(a, \phi, w)$ that $\eta-$enforces $v$ with respect to $W$ and $\delta$.

Next define the set of $\eta-$decomposable payoffs with respect to $W$ and $\delta$ as follows.

$$B(\delta, W, \eta) := \{v \in \mathbb{R}^n | v \text{ is } \eta - \text{decomposable with respect to } W \text{ and } \delta\}.$$

We say that $W$ is $\eta-$*self decomposable* with respect to $\delta \in (0,1)$ if $W \subset B(\delta, W, \eta)$.

It is easy to see that a "uniformly strict" version of Theorem 1 in Abreu, Pearce, and Stacchetti [1] holds here when $\eta > 0$: if $W$ is $\eta-$self decomposable with respect to $\delta$, then every $v \in W$ can be supported by a $\eta-$uniformly strict PPE of $G_p'^{\infty}(\delta, \Phi)$ for some public communication device $\Phi$. Note that each payoff profile may need to be supported by using a different public coordinating device. Hence different public coordinating devices need to be used at different public histories. Since the following lemma is a straightforward implication of the result in Abreu, Pearce and Stacchetti [1], its proof is omitted.

**Lemma 1** *If $W \subset \mathbb{R}^n$ is bounded and $\eta-$self decomposable with respect to $\delta \in (0,1)$, then for any $v \in W$, there exists $\Phi$ such that $v \in E(\delta, \Phi, \eta)$.*

### Local Decomposability is Enough

Fudenberg, Levine, and Maskin [13] showed that local self decomposability is sufficient for self decomposability of any convex, compact set for large $\delta$. Here we prove the corresponding lemma in our setting. First, we prove a lemma that establishes a certain monotonicity property of $B$. The Lemma implies that, if $W$ is $\eta-$self decomposable with respect to $\delta \in (0,1)$, then $W$ is $\frac{1-\delta'}{1-\delta}\eta-$self decomposable for every $\delta' \in (\delta, 1)$.

**Lemma 2** *If $W \subseteq \mathbb{R}^n$ is convex and $C \subseteq B(\delta, W, \eta) \cap W$, then $C \subseteq B\left(\delta', W, \frac{1-\delta'}{1-\delta}\eta\right)$ for every $\delta' \in (\delta, 1)$.*

**Proof.** Suppose that $v \in C$. Since $v \in B(\delta, W, \eta)$, $v$ is $\eta-$decomposable with respect to $W$ and $\delta$, there exists a triple $(a, \phi, w)$ that $\eta-$enforces $v$. For any $\delta' > \delta$, define $w^{\delta'} : Y \to W$ as the following convex combination of $v$ and $w$:

$$w^{\delta'}(y) = \frac{\delta' - \delta}{\delta'(1-\delta)} v + \frac{\delta(1-\delta')}{\delta'(1-\delta)} w(y).$$

Clearly, $w^{\delta'}(y) \in W$ for each $y \in Y$ since $W$ is convex. Furthermore, we can show that, for every $\delta' \in (\delta, 1)$, the triple $\left(a, \phi, w^{\delta'}\right)$ $\frac{1-\delta'}{1-\delta}\eta-$*enforces* $v$ with respect to $W$ and $\delta'$. To see this, note that, for every $\delta' \in (\delta, 1)$, $a \in A$, and $i \in N$,

$$\begin{aligned}
& \left(1-\delta'\right) g_i'(a) + \delta' \sum_{y,s} w_i^{\delta'}(y)\phi(y|s)\,p(s|a) \\
= \; & \left(1-\delta'\right) g_i'(a) + \frac{\delta\left(1-\delta'\right)}{1-\delta} \sum_{y,s} w_i(y)\phi(y|s)\,p(s|a) + \frac{\delta'-\delta}{1-\delta} v_i \\
= \; & \frac{1-\delta'}{1-\delta}\left\{(1-\delta)g_i'(a) + \delta \sum_{y,s} w_i(y)\phi(y|s)\,p(s|a)\right\} + \frac{\delta'-\delta}{1-\delta} v_i.
\end{aligned}$$

Consequently,

$$v = \left(1-\delta'\right) g'(a) + \delta' \sum_{y,s} w^{\delta'}(y)\phi(y|s)\,p(s|a)$$

and conditions (i) and (ii) of Definition 8 hold for $w^{\delta'}$ when $\eta$ is replaced with $\frac{1-\delta'}{1-\delta}\eta$. Therefore, the triple $\left(a, \phi, w^{\delta'}\right)$ $\frac{1-\delta'}{1-\delta}\eta-$enforces $v$ with respect to $W$ and $\delta$ implying that $v$ is $\frac{1-\delta'}{1-\delta}\eta-$self decomposable with respect to $W$ and $\delta'$.Therefore $C \subseteq B\left(\delta', W, \frac{1-\delta'}{1-\delta}\eta\right)$ for every $\delta' \in (\delta, 1)$. ■

Now we show that local self decomposability implies self decomposability. A set $W \subseteq \mathbb{R}^n$ is *locally strictly self-decomposable* if, for any $v \in W$, there exists $\eta > 0$, $\delta \in (0,1)$ and an open set $U$ containing $v$ such that $U \cap W \subset B(\delta, W, \eta)$.

**Lemma 3** *If $W \subset \mathbb{R}^n$ is compact, convex, and locally strictly self decomposable, then there exists $\eta > 0$ and $\underline{\delta} \in (0,1)$ such that $W$ is $(1-\delta)\eta-$self decomposable with respect to $\delta$ for any $\delta \in (\underline{\delta}, 1)$.*

**Proof.** Choose $v \in W$. Since $W$ is locally strictly self decomposable, there exists $\delta_v \in (0,1)$, $\eta_v > 0$, and an open ball $U_v$ around $v$ such that

$$U_v \cap W \subseteq B(\delta_v, W, \eta_v).$$

Since $W$ is compact, there exists a finite subcollection $\{U_{v_k}\}_{k=1}^K$ that covers $W$. Define $\underline{\delta} = \max_{k=1,\dots,K}\{\delta_{v_k}\}$ and $\eta = \min_{k=1,\dots,K}\left\{\eta_{v_k}\right\}$. Then

$$U_{v_k} \cap W \subseteq B\left(\delta_{v_k}, W, \eta_{v_k}\right) \subseteq B\left(\delta_{v_k}, W, \eta\right).$$

Lemma 2 and the convexity of $W$ imply that

$$U_{v_k} \cap W \subseteq B\left(\delta, W, \frac{(1-\delta)}{1-\delta_{v_k}}\eta\right) \subseteq B(\delta, W, (1-\delta)\eta)$$

for any $\delta \in (\underline{\delta}, 1)$ and for $k = 1, ..., K$. Consequently,

$$W = \cup_{k=1}^{K} (U_{v_k} \cap W) \subseteq B(\delta, W, (1-\delta)\eta).$$

■

**Proving Local Decomposability**

Given Lemma 1 and Lemma 3, the proof of Theorem 3 will be complete if, for every individually rational and feasible payoff profile $v \in intV^*(G')$, we can find a compact, convex, locally self decomposable set that contains it. We call a set in $\mathbb{R}^n$ *smooth* if it is closed and convex with an interior point in $\mathbb{R}^n$ and there exists the unique tangent hyperplane at every boundary point.[14] Since any such $v$ can be contained in some smooth set in $intV^*(G')$, we are done if we can show that every smooth set in $intV^*(G')$ is locally self decomposable. Hence the following lemma completes the proof of Theorem 3.

**Lemma 4** *Fix a private monitoring game $(G', p)$. Suppose that there exists $\phi$ such that $p^{\phi}$ is distinguishable. Then there exists a $\gamma > 0$ such that, if $p^{\phi}$ is $\gamma-$regular, then every smooth set $W \subseteq intV^*(G)$ is locally strictly self decomposable.*

**Proof.** See appendix D. ■

To prove this, we follow the argument of Fudenberg, Levine and Maskin [13] in Theorem 4.1. Suppose that we can induce players to report their private signals truthfully. Then the stage game is essentially a public monitoring game $(G', p^{\phi})$. In this case, we can show that almost every boundary point $v$ on a smooth set $W \subseteq intV^*(G)$ is decomposable with respect to the hyperplane that is parallel to the tangent hyperplane at $v$ if $p^{\phi}$ satisfies distinguishability. Since we need to induce truthful reporting at the same time, we need to strengthen this requirement and show that every such boundary point $v$ is *strictly decomposable.* We then perturb $\phi$ and continuation payoffs slightly as in the previous section so that these boundary points remain strictly decomposable and every player has an incentive to report honestly. This can be done when every player is informationally small.

A few comments are in order. First, it may be of some technical interest that we prove this step using some smoothness condition that is weaker than the one in [13], which is commonly invoked to prove a folk theorem in the literature. Second, we choose $\gamma$ independent of the target payoff profiles as we emphasized. Third, it may not possible to obtain strict decomposability when the continuation payoffs lie on the tangent hyperplane that is not "regular" (i.e. it is "vertical" or "horizontal"; all the coefficients except one are 0) because the continuation payoffs are constant for

[14] This notion of smoothness is slightly more general than the one in [13] in the sense that the surface does not need to be twice continuously differentiable.

some player. In this case, we obtain strict decomposability by choosing continuation payoffs from a half space bounded away from the target payoff profile. Finally, our result clearly implies that a uniformly strict folk theorem is obtained for repeated games with imperfect public monitoring when distinguishability is satisfied, because there is no incentive constraint regarding the revelation of private information in this case.

## 6 Related Literature and Discussion

There is a large literature on repeated games with private monitoring and communication. Most papers in the literature focus on a folk theorem rather than robustness. Our approach is similar to Ben-Porath and Kahneman [6]. They prove a folk theorem when a player's action is perfectly observed by at least two other players. For each individually rational and feasible payoff profile, they fix a strategy to support it with perfect monitoring, then augment it with a reporting strategy to support the same payoff profile with direct communication. Their strategies employ draconian punishments when a player's announcement is inconsistent with others' announcements ("shoot the deviator"). Our paper differs from their paper in many respects. Firstly, our paper uses not only perfect monitoring but also imperfect public monitoring as a benchmark. Secondly, private signals can be noisy in our paper.

Compte [8] and Kandori and Matsushima [16] consider communication in repeated games with private monitoring and provide sufficient conditions on the private monitoring structure to obtain a folk theorem. Our sufficient conditions are different from their conditions. Compte [8] assumes that players' private signals are independent conditional on action profiles.[15] Example 1 does not satisfy this condition, but we can prove a folk theorem in such environments by combining Proposition 1 and Theorem 3. Kandori and Matsushima [16] assume that, among others, a deviation by one player and a deviation by another player can be statistically distinguished based on the private signals of the remaining players. Note that this condition is similar to, but different from our condition (2) and (3). Their condition and our condition impose the same restriction on the set of probability measures, but they impose it on the marginal distributions of private signals for each subset of $n-2$ players, whereas we impose it on the public signal distribution that is approximated by the private signal distribution via some public coordination device.

Mailath and Morris [18] study robustness of PPE when a public monitoring structure is perturbed slightly, but without any communication. One of the assumptions they need for robustness is that private monitoring is *almost public*. Private monitoring is almost public if $S_i = Y$ for all $i$ and $\left|\Pr\left(s = (y, ..., y)\,|a\right) - \pi\left(y|a\right)\right|$

[15] Obara [23] extends Compte [8]'s result to the case where private signals are correlated.

is small for all $a$ and $y$. In a subsequent work, Mailath and Morris [20] introduced a weaker notion of approximation called $\varepsilon$-*closeness*, which does not assume $S_i = Y$. Their definition of $\varepsilon$-closeness (see Definition 2 in [20]) implies that $\max_{a,y} |\Pr(\forall i,\ f_i(s_i) = y|a) - \pi(y|a)| \leq \varepsilon$ for all $a$ and $y$ given some $f_i : S_i \to Y \times \{\emptyset\}, i \in N$, which map private signals to a public signal. While we allow communication to prove the robustness of PPE, our notion of closeness is weaker. In Example 2, $p^\phi$ is a $0-$approximation of $\pi$ and is $0-$regular, but it is not close to $\pi$ in their sense. Furthermore, we can show that, when $p$ is $\varepsilon-$close to $\pi$ in their sense, there exists a public coordination device $\phi$ such that $p^\phi$ is $\varepsilon(|Y|+1)-$approximation of $\pi$.[16] They show that a certain class of PPE without bounded recall is not robust to small perturbations of public monitoring. Since there exist uniformly strict PPE within this class that are robust in our sense with respect to more general perturbations, our result suggests that communication is essential for the robustness of a certain class of PPE.

Fudenberg and Levine [11] prove a folk theorem for repeated games with private monitoring and communication when private monitoring is *almost perfect messaging*. Our folk theorem allows for more general perturbations because almost perfect messaging is similar to the above weak version of $\varepsilon$-closeness in Mailath and Morris [18]. On the other hand, their result can be applied to two player games, whereas we need at least three players to guarantee every player is informationally small.

Aoyagi [4] proves a Nash-threat folk theorem in a setting similar to Ben-Porath and Kahneman [6], but with noisy private monitoring. In his paper, each player is monitored by a subset of players. Private signals are noisy and reflect the action

[16] To see this, let

$$S(y) := \{s | f_i(s_i) = y \text{ for all } i\}$$

and define

$$\begin{aligned} \phi(y|s) &= 1 \text{ if } s \in S(y) \text{ for some } y \in Y \\ &= \frac{1}{|Y|} \text{ for all } y \in Y \text{ otherwise.} \end{aligned}$$

Since

$$p^\phi(y|a) = \sum_s \phi(y|s)p(s|a) = \sum_{s \in S(y)} p(s|a) + \frac{1}{|Y|} \sum_{s \notin \cup_y S(y)} p(s|a).$$

it follows that

$$\begin{aligned} \left\| \pi(\cdot|a) - p^\phi(\cdot|a) \right\| &\leq \sum_y \left| \pi(y|a) - p^\phi(y|a) \right| \\ &\leq \sum_y |\pi(y|a) - \sum_{s \in S(y)} p(s|a)| + \frac{1}{|Y|} \sum_y \sum_{s \notin S(y)} p(s|a)| \\ &\leq \varepsilon|Y| + \frac{1}{|Y|}\varepsilon|Y| \\ &= \varepsilon(|Y|+1). \end{aligned}$$

of the monitored player very accurately when they are jointly evaluated. That is, private monitoring is jointly almost perfect. In his paper, players have access to a more general communication device than ours, namely, *mediated communication*.

Anderlini and Lagunoff [3] consider dynastic repeated games with communication where short-lived players care about their offsprings. As in our paper, players may have an incentive to conceal bad information so that future generations do not suffer from mutual punishments. Their model is based on perfect monitoring and their focus is on characterizing the equilibrium payoff set rather than establishing the robustness of equilibria or proving a folk theorem.

There is an extensive literature dealing with repeated games with private monitoring and without communication, starting with Sekiguchi [27]. Many folk theorems have been obtained in this domain by imposing strong assumptions on the private monitoring structure. Bhaskar and Obara [7], Ely and Välimäki [9], Piccioni [24] prove a folk theorem for a repeated prisoner's dilemma game with private almost-perfect monitoring. Matsushima [19] proves a folk theorem for a repeated prisoner's dilemma game with conditionally independent private monitoring. Mailath and Morris [18] prove a folk theorem for general repeated games with almost-perfect and almost-public private monitoring. Hörner and Olszewski [15] prove a folk theorem for general repeated games with private almost-perfect monitoring.

# 7 Appendix

### A. Preliminary Lemma

Here we prove several useful lemmas. First we derive a few upper bounds on player $i$'s ability to manipulate the distribution of a public coordinating signal.

**Lemma 5** $\left\|p^{\phi}(\cdot|a,s_i) - p^{\phi}(\cdot|a,s_i,s_i')\right\| \leq \left(1+\sqrt{2}\right) v_i^{\phi}(s_i,a)$ *for all* $s_i', s_i \in S_i$, $a \in A$ *and* $i \in N$.

**Proof.**

$$\begin{aligned}\left\|p^{\phi}(\cdot|a,s_i) - p^{\phi}\left(\cdot|a,s_i,s_i'\right)\right\| &\leq E\left[\left\|\phi(\cdot|s) - \phi(\cdot|s_i', s_{-i})\right\| |a, s_i\right] \text{ (Jensen's inequality)} \\ &\leq \left(1 - v_i^{\phi}(s_i,a)\right) v_i^{\phi}(s_i,a) + v_i^{\phi}(s_i,a) \cdot \max_{c,d \in \triangle(Y)} \|c-d\| \\ &\leq \left(1+\sqrt{2}\right) v_i^{\phi}(s_i,a)\end{aligned}$$

■

**Lemma 6** $\left\|p^{\phi}(\cdot|a) - p^{\phi}(\cdot|a,\rho_i)\right\| \leq \left(1+\sqrt{2}\right) v_i^{\phi}(s_i,a)$ *for all* $\rho_i$, $a \in A$ *and* $i \in N$.

**Proof.**

$$\begin{aligned}\left\|p^{\phi}(\cdot|a) - p^{\phi}(\cdot|a,\rho_i)\right\| &\leq \sum_{s_i \in S_i} p(s_i|a) \left\|p^{\phi}(\cdot|a,s_i) - p^{\phi}(\cdot|a,s_i,\rho_i(s_i))\right\| \\ &\leq \left(1+\sqrt{2}\right) v_i^{\phi}(s_i,a)\end{aligned}$$

■

The next lemma provides an upper bound on player $i$'s distributional variability.

**Lemma 7**

$$\begin{aligned}\Lambda_i^{\phi}(s_i,a) &\leq 2\left(1 - \frac{p^{\phi}(\cdot|a,s_i) \cdot p^{\phi}(\cdot|a,s_i')}{\|p^{\phi}(\cdot|a,s_i)\| \, \|p^{\phi}(\cdot|a,s_i')\|}\right) \\ \textit{for all } s_i,\ s_i' &\neq s_i,\ a \in A \textit{ and } i \in N.\end{aligned}$$

**Proof.**

$$\begin{aligned}\Lambda_i^{\phi}(s_i,a) &\leq \left\|\frac{p^{\phi}(\cdot|a,s_i)}{\|p^{\phi}(\cdot|a,s_i)\|} - \frac{p^{\phi}(\cdot|a,s_i')}{\|p^{\phi}(\cdot|a,s_i')\|}\right\|^2 \\ &= 2\left(1 - \frac{p^{\phi}(\cdot|a,s_i) \cdot p^{\phi}(\cdot|a,s_i')}{\|p^{\phi}(\cdot|a,s_i)\| \, \|p^{\phi}(\cdot|a,s_i')\|}\right)\end{aligned}$$

■

**B. Proof of Theorem** 1

**Proof.** Fix a private monitoring game $(G', p)$ and $\lambda \in (0,1)$. Pick any payoff function $w : Y \to \mathbb{R}^n$ and $a \in A$. Without loss of generality, we will assume that $\min_{y\in Y} w_i(y) = 0$ for all $i \in N$. First we define a public coordination device $\phi'_{a,w}$. To begin, define the following function $\psi_i : A \times S \to [0,1]$ for each $i \in N$

$$\psi_i(a,s) := \sum_{y\in Y} \frac{p^\phi(y|a,s_i)}{\|p^\phi(\cdot|a,s_i)\|} \cdot \phi(y|s)$$

Next, for any pair of probability distributions $\overline{\mu}_i, \underline{\mu}_i \in \triangle(Y)$, let

$$\phi'_{i,a,\overline{\mu}_i,\underline{\mu}_i}(y|s) := \overline{\mu}_i(y)\,\psi_i(a,s) + \underline{\mu}_i(y)(1-\psi_i(a,s))$$

and define $\phi'_{a,\overline{\mu},\underline{\mu}} := \frac{\sum_{i=1}^n \phi'_{i,a,\overline{\mu}_i,\underline{\mu}_i}}{n}$ as the average of $\phi'_{i,a,\overline{\mu}_i,\underline{\mu}_i}, i = 1,...,n$, where $(\overline{\mu},\underline{\mu}) = \left(\overline{\mu}_1,\underline{\mu}_1,...,\overline{\mu}_n,\underline{\mu}_n\right)$.

Next let $\overline{\mu}_{i,w}$ and $\underline{\mu}_{i,w}$ be any pair of probability distributions on $Y$ that satisfy

$$\overline{\mu}_{i,w} \in \arg\max_{q\in\Delta(Y)} \sum_{y\in Y} q(y) w_i(y)$$

and

$$\underline{\mu}_{i,w} \in \arg\min_{q\in\Delta(Y)} \sum_{y\in Y} q(y) w_i(y).$$

That is $\overline{\mu}_{i,w}$ is a distribution on $Y$ that maximizes player $i$'s expected value of $w_i$ and $\underline{\mu}_{i,w}$ is a distribution that minimizes player $i$'s expected value of $w_i$. Finally, define $\phi'_{a,w} := \phi'_{a,\overline{\mu}_w,\underline{\mu}_w}$ and let

$$\phi^\lambda_{a,w} := (1-\lambda)\,\phi + \lambda\phi'_{a,w}.$$

We prove the following claim. Note that this completes the proof of Theorem 1 because $\gamma$ is chosen independent of $w$ and $a$.

**Claim**: Suppose that

$$0 < \gamma < \frac{1}{\left((1-\lambda)\sqrt{|Y|}+\lambda\right)\left(1+\sqrt{2}\right)} \frac{\lambda}{2\sqrt{|Y|}n}.$$

If $p^\phi$ is $\gamma-$regular, then truthful reporting is a Bayesian Nash equilibrium in the one-shot information revelation game $\left(G', p, \phi^\lambda_{a,w}, w, a\right)$.

**Proof of Claim**: We will prove that, if $\gamma$ satisfies the condition of the claim and if $p^{\phi}$ is $\gamma-$regular, then

$$\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i\left(y\right)\left\{\phi^{\lambda}_{a,w}(y|s_i,s_{-i})-\phi^{\lambda}_{a,w}(y|s'_i,s_{-i})\right\}p\left(s_{-i}|a,s_i\right)\geq 0$$

for each $i\in N$ and each $s_i,s'_i\in S_i$. Let $\overline{w}_i=\max_{y\in Y}w_i(y)$ and note $\overline{w}_i\geq 0$ since $\min_{y\in Y}w_i(y)=0$.

We prove this claim in four steps. Fix player $i$ and suppose that player $i$'s true signal is $s_i$, but she dishonestly reports $s'_i$.

**Step 1:** In this step, we show that

$$\begin{aligned}&\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i\left(y\right)\left\{\phi'_{a,\overline{\mu}_{i,w},\underline{\mu}_{i,w}}(y|s_i,s_{-i})-\phi'_{a,\overline{\mu}_{i,w},\underline{\mu}_{i,w}}(y|s'_i,s_{-i})\right\}p\left(s_{-i}|a,s_i\right)\\ \geq\;&\overline{w}_i\left(\frac{\Lambda^{\phi}_i\left(s_i,a\right)}{2\sqrt{|Y|}}-\left(1+\sqrt{2}\right)v^{\phi}_i\left(s_i,a\right)\right)\end{aligned}$$

To see this, note that

$$\begin{aligned}&\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i\left(y\right)\left\{\phi'_{a,\overline{\mu}_{i,w},\underline{\mu}_{i,w}}(y|s_i,s_{-i})-\phi'_{a,\overline{\mu}_{i,w},\underline{\mu}_{i,w}}(y|s'_i,s_{-i})\right\}p\left(s_{-i}|a,s_i\right)\\ =\;&\overline{w}_i\sum_{s_{-i}\in S_{-i}}\left(\psi_i\left(a,s\right)-\psi_i\left(a,\left(s'_i,s_{-i}\right)\right)\right)p\left(s_{-i}|a,s_i\right)\\ =\;&\overline{w}_i\left[\sum_{y\in Y}\frac{p^{\phi}\left(y|a,s_i\right)}{\|p^{\phi}\left(\cdot|a,s_i\right)\|}p^{\phi}\left(y|a,s_i\right)-\sum_{y\in Y}\frac{p^{\phi}\left(y|a,s'_i\right)}{\|p^{\phi}\left(\cdot|a,s'_i\right)\|}p^{\phi}\left(y|a,s_i,s'_i\right)\right]\\ =\;&\overline{w}_i\sum_{y\in Y}\left(\frac{p^{\phi}\left(y|a,s_i\right)}{\|p^{\phi}\left(\cdot|a,s_i\right)\|}-\frac{p^{\phi}\left(y|a,s'_i\right)}{\|p^{\phi}\left(\cdot|a,s'_i\right)\|}\right)p^{\phi}\left(y|a,s_i\right)\\ &-\overline{w}_i\sum_{y\in Y}\frac{p^{\phi}\left(y|a,s'_i\right)}{\|p^{\phi}\left(\cdot|a,s'_i\right)\|}\left(p^{\phi}\left(y|a,s_i\right)-p^{\phi}\left(y|a,s_i,s'_i\right)\right)\\ =\;&\overline{w}_i\left\|p^{\phi}\left(\cdot|a,s_i\right)\right\|\left(1-\frac{p^{\phi}\left(\cdot|a,s_i\right)\cdot p^{\phi}\left(\cdot|a,s'_i\right)}{\|p^{\phi}\left(\cdot|a,s_i\right)\|\,\|p^{\phi}\left(\cdot|a,s'_i\right)\|}\right)\\ &-\overline{w}_i\sum_{y\in Y}\frac{p^{\phi}\left(y|a,s'_i\right)}{\|p^{\phi}\left(\cdot|a,s'_i\right)\|}\left(p^{\phi}\left(y|a,s_i\right)-p^{\phi}\left(y|a,s_i,s'_i\right)\right)\\ \geq\;&\overline{w}_i\frac{\Lambda^{\phi}_i\left(s_i,a\right)}{2\sqrt{|Y|}}-\overline{w}_i\left(1+\sqrt{2}\right)v^{\phi}_i\left(s_i,a\right)\end{aligned}$$

where the final inequality follows from Lemmas 5 and 7 and the Cauchy-Schwartz inequality.

**Step 2:** We claim that

$$\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i(y)\left\{\phi\left(y|s_i', s_{-i}\right)-\phi(y|s)\right\}p(s_{-i}|a,s_i)\leq \overline{w}_i\sqrt{|Y|-1}\left(1+\sqrt{2}\right)v_i^{\phi}(s_i,a).$$

To see this, note that

$$\begin{aligned}
&\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i(y)\left\{\phi\left(y|s_i', s_{-i}\right)-\phi(y|s)\right\}p(s_{-i}|a,s_i)\\
=\;&\sum_{y\in Y} w_i(y)\left(p^{\phi}\left(y|a,s_i,s_i'\right)-p^{\phi}(y|a,s_i)\right)\\
\leq\;&\|w_i(\cdot)\|\left\|p^{\phi}\left(\cdot|a,s_i,s_i'\right)-p^{\phi}(\cdot|a,s_i)\right\|\\
\leq\;&\overline{w}_i\sqrt{|Y|}\left(1+\sqrt{2}\right)v_i^{\phi}(s_i,a)
\end{aligned}$$

where the final inequality follows from Lemma 5.

**Step 3:** We claim that, if $j\neq i$, then

$$\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i(y)\left\{\phi'_{a,\overline{\mu}_{j,w},\underline{\mu}_{j,w}}\left(y|s_i', s_{-i}\right)-\phi'_{a,\overline{\mu}_{j,w},\underline{\mu}_{j,w}}(y|s)\right\}p(s_{-i}|a,s_i)\leq \overline{w}_i\left(1+\sqrt{2}\right)v_i^{\phi}(s_i,a).$$

To see this, note that

$$\begin{aligned}
&\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i(y)\left\{\phi'_{a,\overline{\mu}_{j,w},\underline{\mu}_{j,w}}\left(y|s_i', s_{-i}\right)-\phi'_{a,\overline{\mu}_{j,w},\underline{\mu}_{j,w}}(y|s)\right\}p(s_{-i}|a,s_i)\\
=\;&\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i(y)\left(\overline{\mu}_j(y)-\underline{\mu}_j(y)\right)\left(\psi_j\left(a,(s_i',s_{-i})\right)-\psi_j(a,s)\right)p(s_{-i}|a,s_i)\\
\leq\;&\left|\sum_{y\in Y} w_i(y)\left(\overline{\mu}_j(y)-\underline{\mu}_j(y)\right)\right|\left|\sum_{s_{-i}\in S_{-i}}\left(\psi_j\left(a,(s_i',s_{-i})\right)-\psi_j(a,s)\right)p(s_{-i}|a,s_i)\right|\\
\leq\;&\overline{w}_i\left|\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y}\frac{p^{\phi}(y|s_j)}{\|p^{\phi}(\cdot|s_j)\|}\left(\phi\left(y|s_i',s_{-i}\right)-\phi(y|s)\right)p(s_{-i}|a,s_i)\right|\\
=\;&\overline{w}_i\left|\sum_{y\in Y}\frac{p^{\phi}(y|s_j)}{\|p^{\phi}(\cdot|s_j)\|}\left(p^{\phi}\left(y|a,s_i,s_i'\right)-p^{\phi}(y|a,s_i)\right)\right|\\
\leq\;&\overline{w}_i\left\|p^{\phi}\left(\cdot|a,s_i,s_i'\right)-p^{\phi}(\cdot|a,s_i)\right\|\\
\leq\;&\overline{w}_i\left(1+\sqrt{2}\right)v_i^{\phi}(s_i,a)
\end{aligned}$$

where the final inequality follows from Lemma 5.

**Step 4:** Combining Steps 1-3, it follows that

$$
\begin{aligned}
&\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i\left(y\right)\left\{\phi^{\lambda}_{a,w}(y|s_i,s_{-i})-\phi^{\lambda}_{a,w}(y|s_i',s_{-i})\right\}p\left(s_{-i}|a,s_i\right)\\
\geq\;& \frac{\lambda}{n}\overline{w}_i\left(\frac{\Lambda_i^{\phi}\left(s_i,a\right)}{2\sqrt{|Y|}}-\left(1+\sqrt{2}\right)v_i^{\phi}\left(s_i,a\right)\right)-(1-\lambda)\overline{w}_i\sqrt{|Y|}\left(1+\sqrt{2}\right)v_i^{\phi}\left(s_i,a\right)\\
&-\frac{\lambda(n-1)}{n}\overline{w}_i\left(1+\sqrt{2}\right)v_i^{\phi}\left(s_i,a\right)\\
=\;& \frac{\lambda}{n}\overline{w}_i\left(\frac{\Lambda_i^{\phi}\left(s_i,a\right)}{2\sqrt{|Y|}}\right)-\left(1+\sqrt{2}\right)\overline{w}_i v_i^{\phi}\left(s_i,a\right)\left[(1-\lambda)\sqrt{|Y|}+\lambda\right].
\end{aligned}
$$

Finally, note that

$$
\frac{\lambda}{n}\overline{w}_i\left(\frac{\Lambda_i^{\phi}\left(s_i,a\right)}{2\sqrt{|Y|}}\right)-\left(1+\sqrt{2}\right)\overline{w}_i v_i^{\phi}\left(s_i,a\right)\left[(1-\lambda)\sqrt{|Y|}+\lambda\right]\geq 0
$$

if

$$
v_i^{\phi}\left(s_i,a\right)\leq\frac{1}{\left((1-\lambda)\sqrt{|Y|}+\lambda\right)\left(1+\sqrt{2}\right)}\frac{\lambda}{2\sqrt{|Y|}n}\Lambda_i^{\phi}\left(s_i,a\right).
$$

It follows immediately that

$$
\sum_{s_{-i}\in S_{-i}}\sum_{y\in Y} w_i\left(y\right)\left\{\phi^{\lambda}_{a,w}(y|s_i,s_{-i})-\phi^{\lambda}_{a,w}(y|s_i',s_{-i})\right\}p\left(s_{-i}|a,s_i\right)\geq 0
$$

if $p^{\phi}$ is $\gamma-$regular for any $\gamma$ satisfying

$$
0<\gamma<\frac{1}{\left((1-\lambda)\sqrt{|Y|}+\lambda\right)\left(1+\sqrt{2}\right)}\frac{\lambda}{2\sqrt{|Y|}n}. \tag{4}
$$

■

**C. Proof of Theorem 2**

**Proof.** Let $(G,\pi)$ and $(G',p)$ be a public monitoring game and a private monitoring game respectively. Define $\kappa$ as follows

$$
\kappa:=\max_{i\in N,a\in A}|g_i\left(a\right)|.
$$

The discount factor $\delta$ is fixed throughout the proof.

**Step 1: Continuation payoffs are close with truthful reporting when stage games are uniformly close.**

**Claim**: Suppose that $\alpha \in \Sigma$ is a public strategy in $G_{\pi}^{\infty}(\delta)$, $\Phi = \{\phi_{h^t} : h^t \in Y^t, t \geq 0\}$ is a public communication device, $\sigma = (\alpha, \rho)$ is a truthful strategy in $G'^{\infty}(\delta, \Phi)$, and $(G', p, \phi_{h^t})$ is an $\varepsilon-$approximation of $(G, \pi)$ for every $\phi_{h^t} \in \Phi$. We claim that

$$\sup_{i,h^t} \left| w_i^{\alpha}\left(h^t\right) - w_i^{\sigma}\left(h^t\right) \right| \leq \left(1 + \frac{\delta\sqrt{|Y|}\kappa}{1-\delta}\right)\varepsilon$$

where (abusing notation)

$$w_i^{\alpha}\left(h^t\right) = (1-\delta) g_i\left(\alpha(h^t)\right) + \delta \sum_y w_i^{\alpha}(h^t, y)\pi\left(y|\alpha(h^t)\right)$$

denotes player $i$'s continuation payoffs after $h^t$ in $G_{\pi}^{\infty}(\delta)$ given $\alpha$ and

$$\begin{aligned} w_i^{\sigma}\left(h^t\right) &= (1-\delta) g_i'\left(\alpha(h^t)\right) + \delta \sum_y \sum_s w_i^{\sigma}(h^t, y)\phi_{h^t}(y|s)\, p\left(s|\alpha(h^t)\right) \\ &= (1-\delta) g_i'\left(\alpha(h^t)\right) + \delta \sum_y w_i^{\sigma}(h^t, y) p^{\phi_{h^t}}(y|\alpha(h^t)) \end{aligned}$$

denotes player $i$'s continuation payoffs after $h^t$ in $G'^{\infty}(\delta, \Phi)$ given $\sigma$.

**Proof**: Suppose that $\alpha \in \Sigma$ is a public strategy in $G_{\pi}^{\infty}(\delta)$, $\Phi$ is a public communication device and $\sigma = (\alpha, \rho)$ is a truthful strategy in $G'^{\infty}(\delta, \Phi)$. Choose any $\varepsilon > 0$ and suppose that $(G', p, \phi_{h^t})$ is an $\varepsilon-$approximation of $(G, \pi)$ for each $\phi_{h^t} \in \Phi$. Let $B = \sup_{i,h^t} \left| w_i^{\alpha}\left(h^t\right) - w_i^{\sigma}\left(h^t\right) \right| < \infty$. For each public history $h^t$, we obtain

$$\begin{aligned} & \left| w_i^{\alpha}\left(h^t\right) - w_i^{\sigma}\left(h^t\right) \right| \\ \leqq\ & (1-\delta)\varepsilon + \delta \left| \sum_{y \in Y} w_i^{\alpha}\left(h^t, y\right)\pi\left(y|\alpha\left(h^t\right)\right) - \sum_{y \in Y} w_i^{\sigma}\left(h^t, y\right) p^{\phi_{h^t}}\left(y|\alpha\left(h^t\right)\right) \right| \\ \leqq\ & (1-\delta)\varepsilon + \delta \left| \sum_{y \in Y} w_i^{\alpha}\left(h^t, y\right) \left\{ \pi\left(y|\alpha\left(h^t\right)\right) - p^{\phi_{h^t}}\left(y|\alpha\left(h^t\right)\right) \right\} \right| \\ & + \delta \left| \sum_{y \in Y} \left\{ w_i^{\alpha}\left(h^t, y\right) - w_i^{\sigma}\left(h^t, y\right) \right\} p^{\phi_{h^t}}\left(y|\alpha\left(h^t\right)\right) \right| \\ \leqq\ & (1-\delta)\varepsilon + \delta \left\| w_i^{\alpha}\left(h^t, \cdot\right) \right\| \left\| \pi\left(\cdot|\alpha\left(h^t\right)\right) - p^{\phi_{h^t}}\left(\cdot|\alpha\left(h^t\right)\right) \right\| + \delta B \\ \leqq\ & (1-\delta)\varepsilon + \delta\sqrt{|Y|}\kappa\varepsilon + \delta B. \end{aligned}$$

Computing the supremum of the left hand side, we obtain $B \leq (1-\delta)\varepsilon + \delta\sqrt{|Y|}\kappa\varepsilon + \delta B$ from which it follows that

$$B \leq \left(1 + \frac{\delta\sqrt{|Y|}\kappa}{1-\delta}\right)\varepsilon$$

and the proof of the claim is complete.

**Step 2: Constructing the public communication device**

**Claim:** Choose any $\lambda \in (0,1)$. If $\gamma$ satisfies (4) and if $\phi$ is a public coordinating device for which $p^{\phi}$ is $\gamma-$regular, then for any pure strategy PPE $\alpha^*$ of $G_{\pi}^{\infty}(\delta)$, there exists a collection of public coordinating devices $\{\phi_{h^t} : h^t \in Y^t, t \geq 0\}$, a public communication device $\Phi^{\lambda} = \{(1-\lambda)\phi + \lambda\phi_{h^t} : h^t \in Y^t, t \geq 0\}$ and a truthful strategy profile $\sigma^* = ((\alpha_1^*, \rho_1^*), .., (\alpha_n^*, \rho_n^*))$ for $G_p'^{\infty}(\delta, \Phi^{\lambda})$ such that, for each public history $h^t$, truthful reporting is a Bayesian Nash equilibrium in the one-shot information revelation game

$$\left(G', p, (1-\lambda)\phi + \lambda\phi_{h^t}, w(h^t, \cdot), \alpha^*(h^t)\right)$$

where $\{w(h^t) : h^t \in Y^t, t \geq 0\}$ is the collection of continuation payoffs in $G_p'^{\infty}(\delta, \Phi^{\lambda})$ generated by the truthful strategy profile $\sigma^*$.

**Proof:** As in the proof of Theorem 1, let

$$\psi_i(a,s) := \sum_{y \in Y} \frac{p^{\phi}(y|a, s_i)}{\|p^{\phi}(\cdot|a, s_i)\|} \cdot \phi(y|s)$$

and for any pair of probability distributions $\overline{\mu}_i, \underline{\mu}_i \in \triangle(Y)$, let

$$\phi'_{i,a,\overline{\mu}_i,\underline{\mu}_i}(y|s) := \overline{\mu}_i(y)\,\psi_i(a,s) + \underline{\mu}_i(y)(1 - \psi_i(a,s))$$

and $\phi'_{a,\overline{\mu},\underline{\mu}} := \frac{\sum_{i=1}^n \phi'_{i,a,\overline{\mu}_i,\underline{\mu}_i}}{n}$.

Let $K = (\triangle(Y) \times \triangle(Y))^n$ and let $M$ denote the product of countably many copies of $K$ indexed by the elements of $H := \cup_{t \geq 0} Y^t$. Suppose that $\alpha^*$ is a PPE in $G_{\pi}^{\infty}(\delta)$. For each

$$\mu = \left((\mu(h^t)\right)_{h^t \in H} = \Big((\overline{\mu}_1(h^t), \underline{\mu}_1(h^t)), .., (\overline{\mu}_n(h^t), \underline{\mu}_n(h^t))\Big)_{h^t \in H} \in M$$

where

$$\mu(h^t) = (\overline{\mu}_1(h^t), \underline{\mu}_1(h^t)), .., (\overline{\mu}_n(h^t), \underline{\mu}_n(h^t)) \in R^{2n|Y|},$$

define a public communication device as follows:

$$\Phi(\mu, \lambda) = \{(1-\lambda)\phi + \lambda\phi_{\alpha^*(h^t), \overline{\mu}(h^t), \underline{\mu}(h^t)} : h^t \in Y^t, t \geq 0\}.$$

For each $i$, choose a reporting strategy $\rho_i^*$ so that $\sigma_i^* = (\alpha_i^*, \rho_i^*)$ is truthful. Then the strategy profile $\sigma^*$ and the public communication device $\Phi(\mu, \lambda)$ define continuation payoffs in the game $G_p'^{\infty}(\delta, \Phi(\mu, \lambda))$ at every public history. For each $\mu \in M$, let $w_i^{\sigma^*}(h^t; \mu, \lambda)$ denote player $i$'s continuation payoff in $G_p'^{\infty}(\delta, \Phi(\mu, \lambda))$ at public history $h^t$. Next, define for each $\mu \in M$ a subset $\Gamma(\mu) \subset M$ as follows: $\mu' \in \Gamma(\mu)$ if and only if

$$\overline{\mu}_i'(h^t) \in \arg\max_{q \in \Delta(Y)} \sum_{y \in Y} q(y) w_i^{\sigma^*}\left((h^t, y); \mu, \lambda\right)$$

and

$$\underline{\mu}_i'\left(h^t\right) \in \arg\min_{q \in \Delta(Y)} \sum_{y \in Y} q(y) w_i^{\sigma^*}\left((h^t, y); \mu, \lambda\right)$$

for all $h^t \in Y^t$ and $i \in N$. We will now show that the correspondence $\Gamma : M \to M$ has a fixed point by applying the Fan-Glicksberg fixed point theorem.

First, let $X$ denote the Cartesian product of countably many copies of $R^{2n|Y|}$ indexed by $H := \cup_{t \geq 0} Y^t$. Since $R^{2n|Y|}$ is a locally convex topological vector space, it follows that $X$ is a locally convex topological vector space with respect to the product topology. (Theorem 5.1 and Lemma 5.54 in Aliprantis and Border [2]). Since $K \subseteq R^{2n|Y|}$ is nonempty, convex and compact, we conclude that $M$ is a nonempty, convex, compact subset of $X$ in the product topology.

Clearly $\Gamma$ is nonempty, convex valued, and compact valued. So we need only verify that $\Gamma$ is upper hemicontinuous. Upper hemicontinuity will follow from Berge's Theorem if we can establish that $\mu \mapsto w_i^{\sigma^*}\left((h^t, y); \mu, \lambda\right)$ is a continuous real valued function on $M$ for each $t$ and $h^t \in Y^t$ since the product of upper hemicontinuous correspondences from $M$ to $K$ is upper hemicontinuous with respect to the product topology on $X$ (to prove the last assertion, modify the argument of Theorem 16.28 in [2]). To see that $\mu \mapsto w_i^{\sigma^*}\left((h^t, y); \mu, \lambda\right)$ is continuous, first define

$$\phi_{h^t}^{\mu} = (1 - \lambda)\, \phi + \lambda \phi_{\alpha^*(h^t), \overline{\mu}(h^t), \underline{\mu}(h^t)}$$

and let $d_1$ denote the $\ell_1$ metric on $R^{2n|Y|}$. Note that

$$\begin{aligned}
& |\phi_{h^t}^{\mu}(y|s) - \phi_{h^t}^{\mu'}(y|s)| \\
\leq\ & \frac{\lambda}{n} \sum_{i=1}^{n} \left( |\overline{\mu}_i\left(h^t\right)(y) - \overline{\mu}_i'\left(h^t\right)(y)| \psi_i\left(\alpha^*(h^t), s\right) + |\underline{\mu}_i\left(h^t\right)(y) - \underline{\mu}_i'\left(h^t\right)(y)| [1 - \psi_i\left(\alpha^*(h^t), s\right)] \right) \\
\leq\ & \frac{\lambda}{n} \sum_{i=1}^{n} \left( |\overline{\mu}_i\left(h^t\right)(y) - \overline{\mu}_i'\left(h^t\right)(y)| + |\underline{\mu}_i\left(h^t\right)(y) - \underline{\mu}_i'\left(h^t\right)(y)| \right)
\end{aligned}$$

so that

$$\begin{aligned}
|p^{\phi_{h^t}^{\mu}}(y|\alpha^*(h^t)) - p^{\phi_{h^t}^{\mu'}}(y|\alpha^*(h^t))| &= \sum_{s} |\phi_{h^t}^{\mu}(y|s) - \phi_{h^t}^{\mu'}(y|s)| p(s|\alpha^*(h^t)) \\
&\leq \sum_{y \in Y} |\phi_{h^t}^{\mu}(y|s) - \phi_{h^t}^{\mu'}(y|s)| \\
&\leq \frac{\lambda}{n} d_1(\mu\left(h^t\right), \mu'\left(h^t\right)).
\end{aligned}$$

Therefore,

$$\left\| p^{\phi^{\mu}_{h^t}}(\cdot|\alpha^*(h^t)) - p^{\phi^{\mu'}_{h^t}}(\cdot|\alpha^*(h^t)) \right\| \leq \frac{\lambda}{n}\sqrt{|Y|}d_1(\mu\left(h^t\right), \mu'\left(h^t\right)).$$

Next, note that the product topology on $X$ is metrizable (recall that $X$ is a countable product) with well known metric $\hat{d}$ as defined, for example, in Theorem 3.24 of [2]. For this metric $\hat{d}$, the following condition holds: for each $\xi > 0$ and each $t$, there exists a $\zeta > 0$ (depending on $\xi$ and $t$) such that for each $s \leq t$ and each $h^s \in Y^s$,

$$\hat{d}(\mu, \mu') < \zeta \Rightarrow d_1(\mu\left(h^s\right), \mu'\left(h^s\right)) < \xi.$$

That is, we can make $\mu\left(h^s\right)$ and $\mu'\left(h^s\right)$ as close with respect to $d_1$ as we wish for all $i \in N$ and for any $h^s$ with $0 \leq s \leq t$ by making $\mu$ and $\mu'$ close enough with respect to $\hat{d}$. Since

$$\sup_{i,h^t,\mu} |w_i^{\sigma^*}\left(h^t; \mu, \lambda\right)| < \infty$$

and we use a discounted average payoff criterion, we conclude that $\mu \mapsto w_i^{\sigma^*}\left((h^t, y); \mu, \lambda\right)$ is a continuous real valued function on $M$ for each $h^t \in H$. Therefore, we conclude that $\Gamma : M \to M$ is nonempty valued, convex valued, compact valued and upper hemicontinuous. Applying the Fan-Glicksberg fixed point theorem, there exists $\mu^* \in M$ such that $\mu^* \in \Gamma(\mu^*)$.

Since

$$\overline{\mu}_i^*\left(h^t\right) \in \arg\max_{q \in \Delta(Y)} \sum_{y \in Y} q(y) w_i^{\sigma^*}\left((h^t, y); \mu^*, \lambda\right)$$

and

$$\underline{\mu}_i^*\left(h^t\right) \in \arg\min_{q \in \Delta(Y)} \sum_{y \in Y} q(y) w_i^{\sigma^*}\left((h^t, y); \mu^*, \lambda\right).$$

for every $h_t \in H$ and $i \in N$, we can apply the claim in the proof of Theorem 1: if

$$0 < \gamma < \frac{1}{\left((1-\lambda)\sqrt{|Y|} + \lambda\right)\left(1 + \sqrt{2}\right)} \frac{\lambda}{2\sqrt{|Y|}n}$$

and if $p^{\phi}$ is $\gamma-$regular, then truthful reporting is a Bayesian Nash equilibrium in the one-shot information revelation game

$$\left(G', p, (1-\lambda)\phi + \lambda\phi_{\alpha^*(h^t), \overline{\mu}^*(h^t), \underline{\mu}^*(h^t)}, w^{\sigma^*}\left((h^t, \cdot); \mu^*, \lambda)\right), \alpha^*(h^t)\right)$$

for every $h_t \in H$.

Defining $\phi_{h^t} := \phi_{\alpha^*(h^t), \overline{\mu}^*(h^t), \underline{\mu}^*(h^t)}$ and

$$\Phi^{\lambda} := \{(1-\lambda)\phi + \lambda\phi_{\alpha^*(h^t), \overline{\mu}^*(h^t), \underline{\mu}^*(h^t)} : h^t \in Y^t, t \geq 0\}$$

completes the proof of the claim.

**Step 3: Checking all one-period deviations.**

Fix any $\eta > 0$. In this step, we find $\varepsilon > 0$ and $\gamma > 0$ that have the property stated in the theorem: if $(G', p, \phi)$ is $\varepsilon-$approximation of $(G, \pi)$ and $p^{\phi}$ is $\gamma-$regular for $\phi$, then for any $\eta-$uniformly strict PPE $\alpha^*$ of $G_{\pi}^{\infty}(\delta)$, there exists a public communication device $\Phi$ and a truthful reporting strategy $\rho^*$ such that $\sigma^* = (\alpha^*, \rho^*)$ is a PPE of $G_p'^{\infty}(\delta, \Phi)$.

Choose $\lambda \in (0,1), \varepsilon > 0$, and $\gamma > 0$ such that (4) and the following strict inequality are satisfied

$$2\sqrt{Y}\kappa\left(\epsilon + \left(1+\sqrt{2}\right)\gamma\right) + 2\left(1 + \frac{\delta\sqrt{|Y|}\kappa}{1-\delta}\right)\varepsilon + 4\lambda\kappa < \frac{\eta}{3} \tag{5}$$

Suppose that $(G', p, \phi)$ is an $\varepsilon-$approximation of $(G, \pi)$ and $p^{\phi}$ is $\gamma-$regular for $\phi$. Furthermore, suppose that $\alpha^*$ is an $\eta-$uniformly strict PPE of $G_{\pi}^{\infty}(\delta)$. Using (4) and applying step 2, there exists a collection $\{\phi_{h^t} : h^t \in Y^t, t \geq 0\}$ and a public communication device $\Phi^{\lambda} = \{\phi_{h^t}^{\lambda} : h^t \in Y^t, t \geq 0\}$, where $\phi_{h^t}^{\lambda} = (1-\lambda)\phi + \lambda\phi_{h^t}$, such that truthful reporting is optimal in $G_p'^{\infty}(\delta, \Phi^{\lambda})$ given $\alpha^*$. Next, define a reporting strategy $\rho^*$ as follows:[17] for $h^t \in H$, $a_i \in A_i$, $s_i \in S_i$,

$$\rho^*\left(s_i | h^t, a_i\right) = \begin{cases} s_i \text{ if } a_i = \alpha_i^*(h^t) \\ \text{any optimal report if } a_i \neq \alpha_i^*(h^t) \end{cases}.$$

We will show that $\sigma^* = (\alpha^*, \rho^*)$ is a truthful PPE of the public communication extension $G_p'^{\infty}(\delta, \Phi^{\lambda})$. It is clearly truthful by definition. To verify sequential rationality constraints, we apply the principle of optimality and check one-period deviations at every public history $h^t \in H$ at the beginning of each period. We must check two types of deviations: those which involve a deviation at the action stage and those that do not. By construction of $\mu^*$, honest reporting is optimal when the equilibrium action is played within the same period, i.e. when $\alpha^*(h^t)$ is played given public history $h^t$. Consequently, the second type of deviation is not profitable. To complete the argument, we must show that, for any $h^t \in H$ and any $i \in N$, player $i$ cannot profitably deviate by first choosing an action $a_i$ different from $a_i^*\left(h^t\right)$ and then choosing any report $\rho$ including the optimal one $\rho^*$.

Abusing notation, let

$$E\left[w_i^{\alpha^*}(h^t, \cdot) | a_i, \alpha_{-i}^*\left(h^t\right)\right] = \sum_y w_i^{\alpha^*}(h^t, y)\pi\left(y | a_i, \alpha_{-i}^*\left(h^t\right)\right).$$

Since $\alpha^*$ is $\eta-$uniformly strict, at every $h^t$ and for every $i \in N$, the following inequality must be satisfied for every $a_i \neq \alpha_i^*\left(h^t\right)$

$$\begin{aligned} & (1-\delta)\left(g_i\left(a_i, \alpha_{-i}^*\left(h^t\right)\right) - g_i\left(\alpha^*(h^t)\right)\right) + \eta \\ \leq \quad & \delta\left\{E\left[w_i^{\alpha^*}\left(h^t, \cdot\right) | \alpha^*\left(h^t\right)\right] - E\left[w_i^{\alpha^*}(h^t, \cdot) | a_i, \alpha_{-i}^*\left(h^t\right)\right]\right\}. \end{aligned} \tag{6}$$

[17] We do not derive the optimal reporting strategy off the equilibrium path explicitly, as it is not needed for our proof.

We compare the left hand side and the right hand of this inequality with the corresponding terms in $G_p'^{\infty}\left(\delta, \Phi^{\lambda}\right)$. In particular, we will show that

$$g_i'\left(a_i, \alpha_{-i}^*\left(h^t\right)\right) - g_i'\left(\alpha^*(h^t)\right) \approx g_i\left(a_i, \alpha_{-i}^*\left(h^t\right)\right) - g_i\left(\alpha^*(h^t)\right)$$

for any $h^t$ and $a_i$ and that

$$\begin{aligned} & E\left[w_i^{\alpha^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)\right] - E\left[w_i^{\alpha^*}(h^t, \cdot)|a_i, \alpha_{-i}^*\left(h^t\right)\right] \\ \approx \;& E^{\phi_{h^t}^{\lambda}}\left[w_i^{\sigma^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)\right] - E^{\phi_{h^t}^{\lambda}}\left[w_i^{\sigma^*}(h^t, \cdot)|a_i, \alpha_{-i}^*\left(h^t\right), \rho_i\right] \end{aligned}$$

for any $h^t$, $a_i \neq \alpha_i^*(h^t)$, and $\rho_i : S_i \to S_i$.

To begin, note that (4) implies that $2\varepsilon < \frac{\eta}{3}$. Since $(G', p, \phi)$ is $\varepsilon-$approximation of $(G, \pi)$, it follows that

$$\begin{aligned} g_i'\left(a_i, \alpha_{-i}^*\left(h^t\right)\right) - g_i'\left(\alpha^*(h^t)\right) &\leq g_i\left(a_i, \alpha_{-i}^*\left(h^t\right)\right) - g_i\left(\alpha^*(h^t)\right) + 2\varepsilon \\ &< g_i\left(a_i, \alpha_{-i}^*\left(h^t\right)\right) - g_i\left(\alpha^*(h^t)\right) + \frac{\eta}{3}. \end{aligned} \tag{7}$$

Next, we show that

$$\begin{aligned} & E^{\phi_{h^t}^{\lambda}}[w_i^{\sigma^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)] - E^{\phi_{h^t}^{\lambda}}[w_i^{\sigma^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right), \rho_i] \\ > \;& E[w_i^{\alpha^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)] - E[w_i^{\alpha^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right)] - \frac{\eta}{3} \end{aligned}$$

To see this, note that player $i$'s expected loss at $h^t$ in $G_p'^{\infty}(\delta, \Phi^{\lambda})$ satisfies, for any $a_i \neq \alpha_i^*\left(h^t\right)$ and $\rho_i$,

$$\begin{aligned} & E^{\phi_{h^t}^{\lambda}}[w_i^{\sigma^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)] - E^{\phi_{h^t}^{\lambda}}[w_i^{\sigma^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right), \rho_i] \\ \geq \;& (1-\lambda)\, E^{\phi}[w_i^{\sigma^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)] - (1-\lambda)\, E^{\phi}[w_i^{\sigma^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right), \rho_i] - 2\lambda\kappa \\ \geq \;& E^{\phi}[w_i^{\sigma^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)] - E^{\phi}[w_i^{\sigma^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right), \rho_i] - 4\lambda\kappa \\ \geq \;& E^{\phi}[w_i^{\alpha^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)] - E^{\phi}[w_i^{\alpha^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right), \rho_i] \\ & -2\left(1 + \frac{\delta\sqrt{|Y|}\kappa}{1-\delta}\right)\varepsilon - 4\lambda\kappa \text{ (By Step 1)} \\ = \;& \left\{E^{\phi}[w_i^{\alpha^*}\left(h^t, \cdot\right)|\alpha^*\left(h^t\right)] - E^{\phi}[w_i^{\alpha^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right)]\right\} \\ & + \left\{E^{\phi}[w_i^{\alpha^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right)] - E^{\phi}[w_i^{\alpha^*}\left(h^t, \cdot\right)|a_i, \alpha_{-i}^*\left(h^t\right), \rho_i]\right\} \\ & -2\left(1 + \frac{\delta\sqrt{|Y|}\kappa}{1-\delta}\right)\varepsilon - 4\lambda\kappa \end{aligned}$$

The first term of this expression can be bounded below as follows:

$$
\begin{aligned}
& E^{\phi}[w_i^{\alpha^*}\left(h^t,\cdot\right)|\alpha^*\left(h^t\right)] - E^{\phi}[w_i^{\alpha^*}\left(h^t,\cdot\right)|a_i,\alpha^*_{-i}\left(h^t\right)] \\
= & \sum_{y\in Y} w_i^{\alpha^*}\left(h^t,y\right)\left(p^{\phi}(y|\alpha^*\left(h^t\right)) - \pi\left(y|\alpha^*\left(h^t\right)\right)\right) \\
& + \sum_{y\in Y} w_i^{\alpha^*}\left(h^t,y\right)\left(\pi\left(y|\alpha^*\left(h^t\right)\right) - \pi\left(y|a_i,\alpha^*_{-i}\left(h^t\right)\right)\right) \\
& + \sum_{y\in Y} w_i^{\alpha^*}\left(h^t,y\right)\left(\pi\left(y|a_i,\alpha^*_{-i}\left(h^t\right)\right) - p^{\phi}\left(y|a_i,\alpha^*_{-i}\left(h^t\right)\right)\right) \\
\geq & E[w_i^{\alpha^*}\left(h^t,\cdot\right)|\alpha^*\left(h^t\right)] - E[w_i^{\alpha^*}\left(h^t,\cdot\right)|a_i,\alpha^*_{-i}\left(h^t\right)] \\
& - \left\|w_i^{\alpha^*}\left(h^t,\cdot\right)\right\|\left\|p^{\phi}(\cdot|\alpha^*\left(h^t\right)) - \pi\left(\cdot|\alpha^*\left(h^t\right)\right)\right\| \\
& - \left\|w_i^{a^*}\left(h^t,\cdot\right)\right\|\left\|\pi\left(\cdot|a_i,\alpha^*_{-i}\left(h^t\right)\right) - p^{\phi}(\cdot|a_i,\alpha^*_{-i}\left(h^t\right))\right\| \\
\geq & E[w_i^{\alpha^*}\left(h^t,\cdot\right)|\alpha^*\left(h^t\right)] - E[w_i^{\alpha^*}\left(h^t,\cdot\right)|a_i,\alpha^*_{-i}\left(h^t\right)] - 2\epsilon\sqrt{Y}\kappa
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
& E^{\phi}[w_i^{\alpha^*}\left(h^t,\cdot\right)|a_i,\alpha^*_{-i}\left(h^t\right)] - E^{\phi}[w_i^{\alpha^*}\left(h^t,\cdot\right)|a_i,\alpha^*_{-i}\left(h^t\right),\rho_i] \\
= & \sum_{y\in Y} w_i^{\alpha^*}\left(h^t,y\right)\left(p^{\phi}(y|a_i,\alpha^*_{-i}\left(h^t\right)) - p^{\phi}\left(y|a_i,\alpha^*_{-i}\left(h^t\right),\rho_i\right)\right) \\
\geq & -\left\|w_i^{\alpha^*}\left(h^t,\cdot\right)\right\|\left\|p^{\phi}(\cdot|a_i,\alpha^*_{-i}\left(h^t\right)) - p^{\phi}(\cdot|a_i,\alpha^*_{-i}\left(h^t\right),\rho_i)\right\| \\
\geq & -\left(1+\sqrt{2}\right)\sqrt{Y}\kappa v_i^{\phi}\left(s_i,a\right) \text{ (by Lemma 6)} \\
\geq & -2\left(1+\sqrt{2}\right)\sqrt{Y}\kappa\gamma \text{ (by } \gamma\text{-regularity and Lemma 7).}
\end{aligned}
$$

Therefore player $i$'s expected loss at $h^t$ in $G_p'^{\infty}(\delta,\Phi^{\lambda})$ is bounded from below by

$$
\begin{aligned}
& E[w_i^{\alpha^*}\left(h^t,\cdot\right)|\alpha^*\left(h^t\right)] - E[w_i^{\alpha^*}\left(h^t,\cdot\right)|a_i,\alpha^*_{-i}\left(h^t\right)] \\
& -2\sqrt{Y}\kappa\left(\epsilon+\left(1+\sqrt{2}\right)\gamma\right) - 2\left(1+\frac{\delta\sqrt{|Y|}\kappa}{1-\delta}\right)\varepsilon - 4\lambda\kappa \\
> & E[w_i^{\alpha^*}\left(h^t,\cdot\right)|\alpha^*\left(h^t\right)] - E[w_i^{\alpha^*}\left(h^t,\cdot\right)|a_i,\alpha^*_{-i}\left(h^t\right)] - \frac{\eta}{3} \text{ (by (5)).}
\end{aligned} \tag{8}
$$

Combining (6), (7) and (8), we conclude that

$$
\begin{aligned}
& (1-\delta)\left(g_i'\left(a_i,\alpha^*_{-i}\left(h^t\right)\right) - g_i'\left(\alpha^*(h^t)\right)\right) + \frac{\eta}{3} \\
< & \delta\left(E^{\phi^{\lambda}_{h^t}}[w_i^{\sigma^*}\left(h^t,\cdot\right)|\alpha^*\left(h^t\right)] - E^{\phi^{\lambda}_{h^t}}[w_i^{\sigma^*}\left(h^t,\cdot\right)|a_i,\alpha^*_{-i}\left(h^t\right),\rho_i]\right)
\end{aligned}
$$

for every $\rho_i$, every $h^t$ and every $i \in N$. Let $\eta' = \frac{\eta}{3}$. Then we can conclude that the suggested strategy $\sigma^* = (\alpha^*, \rho^*)$ is a $\eta'-$uniformly strict truthful PPE of $G_p'^{\infty}(\delta,\Phi^{\lambda})$. This completes the proof of the theorem. ■

**D. Proof of Theorem 3**

Let $Q = \{q \in \mathbb{R}^n | \, \|q\| = 1\}$ and $e^i = (0, 0, .., 1, ..., 0)^\top \in Q$ with the $i$th coordinate equal to 1. First we prove two lemmata to prove Lemma 4.

**Lemma 8** *Suppose that $p^\phi$ is distinguishable for some public coordinating device $\phi$. Then there exists $\gamma > 0$ such that, if $p^\phi$ is $\gamma-$regular, then for any $q \in Q$ and $a \in A$, there exists $\xi : Y \to \mathbb{R}^n$ and another public coordinating device $\phi'$ that satisfy the following conditions:*

*(i)*

$$E^{\phi'}\left[\xi_j | a\right] = 0 \text{ for } j = 1, ..., n \tag{9}$$

*(ii) if $0 \leqq |q_i| < 1$ for each $i \in N$, then*

$$E^{\phi'}[\xi_j|a] > E^{\phi'}[\xi_j|a'_j, a_{-j}, \rho_j] \text{ for all } \left(a'_j, \rho_j\right) \text{ with } a'_j \neq a_j \text{ and for all } j \in N \tag{10}$$

$$E^{\phi'}[\xi_j|a] \geq E^{\phi}[\xi_j|a, \rho_j] \text{ for all } \rho_j \text{ and for all } j \in N \tag{11}$$

$$q \cdot w\left(y\right) = 0 \text{ for all } y \in Y \tag{12}$$

*(iii) if $|q_i| = 1$ for some $i \in N$ (hence $q_j = 0$ for every $j \neq i$), then*

$$E^{\phi'}[\xi_j|a] > E^{\phi'}[\xi_j|a'_j, a_{-j}, \rho_j] \text{ for all } \left(a'_j, \rho_j\right) \text{ with } a'_j \neq a_j \text{ and for all } j \in N \tag{13}$$

$$E^{\phi'}[\xi_j|a] \geq E^{\phi}[\xi_j|a, \rho_j] \text{ for all } \rho_j \text{ and for all } j \in N \tag{14}$$

**Proof. Step 1:** For each $a \in A$ and each pair $(i, j)$ with $i \neq j$, there exist functions $x_a^{i,j,+}, x_a^{i,j,-} : Y \to \mathbb{R}$ satisfying the following conditions

$$\begin{aligned} E^\phi[z x_a^{i,j,z}|a] &> E^\phi[z x_a^{i,j,z}|a'_i, a_{-i}] \text{ for all } a'_i \neq a_i \text{ for } z = +, - \\ E^\phi[x_a^{i,j,z}|a] &> E^\phi[x_a^{i,j,z}|a'_j, a_{-j}] \text{ for all } a'_j \neq a_j \text{ for } z = +, - \end{aligned}$$

and

$$||x_a^{i,j,+}|| = 1 = ||x_a^{i,j,-}||.$$

Such functions $x_a^{i,j,+}$ and $x_a^{i,j,-}$ exist as a consequence of (1)-(3) and an application of the separating hyperplane theorem.

**Step 2:** We first consider the case of (ii). Take any $q \in Q$ such that $|q_j| < 1$ for any $j$. This $q$ is fixed throughout step 1-4. Let $i$ be a player such that $|q_i| \geq |q_j|$ for all $j$. If $q_i < 0$, then define $x^{(a,q)} : Y \to \mathbb{R}^n$ as follows: for each $y \in Y$,

$$\begin{aligned} x_j^{(a,q)}\left(y\right) &: = x_a^{i,j,+}(y) \text{ if } q_j \geq 0 \text{ and } j \neq i \\ x_j^{(a,q)}\left(y\right) &: = x_a^{i,j,-}(y) \text{ if } q_j < 0 \text{ and } j \neq i \\ x_i^{(a,q)}\left(y\right) &: = -\sum_{j \neq i} \frac{q_j}{q_i} x_j^{(a,q)}\left(y\right). \end{aligned}$$

If $q_i > 0$, then define $x^{(a,q)} : Y \to \mathbb{R}^n$ as follows: for each $y \in Y$,

$$\begin{aligned} x_j^{(a,q)}(y) &: = x_a^{i,j,-}(y) \text{ if } q_j \geq 0 \text{ and } j \neq i \\ x_j^{(a,q)}(y) &: = x_a^{i,j,+}(y) \text{ if } q_j < 0 \text{ and } j \neq i \\ x_i^{(a,q)}(y) &: = -\sum_{j \neq i} \frac{q_j}{q_i} x_j^{(a,q)}(y). \end{aligned}$$

From these definitions, it follows that $q \cdot x^{(a,q)}(y) = 0$ for all $y \in Y$ so that condition (12) is satisfied.

**Step 3**: For each $s \in S$ and $a \in A$, let

$$\psi_i(a,s) := \sum_{y \in Y} \frac{p^{\phi}(y|a,s_i)}{\|p^{\phi}(\cdot|a,s_i)\|} \cdot \phi(y|s)$$

as in the proof of Theorem 1. Define $\phi'_{a,x^{(a,q)}} : S \to \Delta(Y)$ as

$$\phi'_{a,x^{(a,q)}} = \frac{\sum_{j=1}^{n} \phi'_{j,a,x^{(a,q)}}}{n}$$

where

$$\phi'_{j,a,x^{(a,q)}}(s) = \psi_j(a,s) \cdot \overline{\mu}_j + \left(1 - \psi_j(a,s)\right) \underline{\mu}_j$$

and $\overline{\mu}_j$ ($\underline{\mu}_j$) is a probability measure on $Y$ that assigns probability zero to any $y$ not a member of $\arg\max_{y' \in Y} x_j^{(a,q)}(y')$ ($\arg\min_{y' \in Y} x_j^{(a,q)}(y')$). Finally, let

$$\phi^{\lambda}_{a,x^{(a,q)}} := (1-\lambda)\phi + \lambda \phi'_{a,x^{(a,q)}}$$

for some $\lambda \in (0,1)$.

Let

$$\begin{aligned} \eta_1 &= \min\{E^{\phi}[z x_a^{i,j,z}|a] - E^{\phi}[z x_a^{i,j,z}|a'_i, a_{-i}] : \text{for all } i,j,a,a'_i \neq a_i \text{ and } z = +,-\}, \\ \eta_2 &= \min\{E^{\phi}[x_a^{i,j,z}|a] - E^{\phi}[x_a^{i,j,z}|a'_j, a_{-j}] \text{ for all } i,j,a,\ a'_j \neq a_j \text{ and } z = +,-\} \end{aligned}$$

and define

$$\eta := \min\{\eta_1, \eta_2\}.$$

Note that $\eta > 0$ and it is defined independent of $a$ or $q$.

**Step 4**: In this step, we prove that condition (10) hold for $x^{(a,q)} : Y \to \mathbb{R}^n$ if $p^{\phi}$ is $\gamma-$regular and the following condition is satisfied for $\gamma$ and $\lambda$.

$$\eta - 2\left(1+\sqrt{2}\right)\gamma - 4\lambda > 0 \ (*)$$

We need to show the following for all $j$ :

$$E^{\phi^{\lambda}_{a,x}}[x_j^{(a,q)}|a] > E^{\phi^{\lambda}_{a,x}}[x_j^{(a,q)}|a'_j, a_{-j}, \rho'_j] \text{ for all } \left(a'_j, \rho'_j\right) \text{ with } a'_j \neq a_j$$

To accomplish this, suppose that $q_i < 0$ and let $x = x^{(a,q)}$ for notational ease. The case with $q_i > 0$ is similar, thus omitted. First consider $j \neq i$ for which $q_j \geq 0$ so that $x_j := x_a^{i,j,+}$. For any $a_j' \neq a_j$ and $\rho_j$,

$$\begin{aligned}
& E^{\phi^{\lambda}_{a,x}}[x_j|a] - E^{\phi^{\lambda}_{a,x}}[x_j|a_j', a_{-j}, \rho_j] \\
= \;& (1-\lambda)\left[E^{\phi}[x_a^{i,j,+}|a] - E^{\phi}[x_a^{i,j,+}|a_j', a_{-j}, \rho_j]\right] \\
& +\lambda\left[E^{\phi'_{a,q}}[x_a^{i,j,+}|a] - E^{\phi'_{a,q}}[x_a^{i,j,+}|a_j', a_{-j}, \rho_j]\right] \\
\geq \;& E^{\phi}[x_a^{i,j,+}|a] - E^{\phi}[x_a^{i,j,+}|a_j', a_{-j}, \rho_j] - 4\lambda \text{ (because } \left\|x_a^{i,j,+}\right\| = 1) \\
= \;& E^{\phi}[x_a^{i,j,+}|a] - E^{\phi}[x_a^{i,j,+}|a_j', a_{-j}] \\
& +E^{\phi}[x_a^{i,j,+}|a_j', a_{-j}] - E^{\phi}[x_a^{i,j,+}|a_j', a_{-j}, \rho_j] - 4\lambda \\
\geq \;& \eta - \left\|x_a^{i,j,+}\right\| \left\|p^{\phi}(\cdot|a_j', a_{-j}) - p^{\phi}(\cdot|a_j', a_{-j}, \rho_j)\right\| - 4\lambda \\
\geq \;& \eta - \left(1+\sqrt{2}\right) v_i^{\phi}(s_i, a) - 4\lambda \\
\geq \;& \eta - 2\left(1+\sqrt{2}\right)\gamma - 4\lambda \\
> \;& 0
\end{aligned}$$

We can use the exactly same proof for player $j \neq i$ with $q_j < 0$ (so that $x_j := x_a^{i,j,-}$) to show that

$$\begin{aligned}
E^{\phi^{\lambda}_{a,x}}[x_j|a] - E^{\phi^{\lambda}_{a,x}}[x_j|a_j', a_{-j}, \rho_j] \;&\geq\; \eta - 2\left(1+\sqrt{2}\right)\gamma - 4\lambda \\
&>\; 0
\end{aligned}$$

for any $a_j' \neq a_j$ and $\rho_j$.

Finally for player $i$, the same proof implies that

$$\begin{aligned}
E^{\phi^{\lambda}_{a,x}}[zx_a^{i,j,z}|a] - E^{\phi^{\lambda}_{a,x}}[zx_a^{i,j,z}|a_i', a_{-i}, \rho_i] \;&\geq\; \eta - 2\left(1+\sqrt{2}\right)\gamma - 4\lambda \\
&>\; 0
\end{aligned}$$

for any $j, z = +, -$, and $a_i'$. Hence, whenever $q_j \neq 0$, we obtain

$$E^{\phi^{\lambda}_{a,x}}[q_j x_j^{(a,q)}(y)\,|a] - E^{\phi^{\lambda}_{a,x}}[q_j x_j^{(a,q)}(y)\,|a_i', a_{-i}, \rho_i] > 0.$$

Observe that $q_j \neq 0$ for some $j \neq i$ and $q_i < 0$ by assumption. Therefore it follows that

$$\begin{aligned}
& E^{\phi^{\lambda}_{a,x}}[x_i^{(a,q)}(y)\,|a] - E^{\phi^{\lambda}_{a,x}}[x_i^{(a,q)}(y)\,|a_i', a_{-i}, \rho_i] \\
= \;& E^{\phi^{\lambda}_{a,x}}[-\sum_{j\neq i}\frac{q_j}{q_i}x_j^{(a,q)}(y)\,|a] - E^{\phi^{\lambda}_{a,x}}[-\sum_{j\neq i}\frac{q_j}{q_i}x_j^{(a,q)}(y)\,|a_i', a_{-i}, \rho_i] \\
> \;& 0.
\end{aligned}$$

**Step 5**: In this step, we prove that condition (11) holds for $x^{(a,q)}: Y \to \mathbb{R}^n$ if $p^{\phi}$ is $\gamma-$regular and the following condition is satisfied for $\gamma$ and $\lambda$ :

$$0 < \gamma < \frac{1}{\left((1-\lambda)\sqrt{|Y|}+\lambda\right)\left(1+\sqrt{2}\right)} \frac{\lambda}{2\sqrt{|Y|}n} \ (**).$$

If $(**)$ is satisfied, it follows directly from Theorem 1 that truthful reporting is a Bayesian Nash equilibrium in the one-shot information revelation game $\left(G', p, \phi^{\lambda}_{a,x}, x, a\right)$ for any $x$ and $a$. Hence we obtain

$$E^{\phi^{\lambda}_{a,x}}[x_j|a] \geq E^{\phi^{\lambda}_{a,x}}[x_j|a, \rho'_j] \text{ for all } \rho'_j$$

for any $a \in A$.

**Step 6**: Next consider the case of (iii). Take any $q \in Q$ such that $|q_i| = 1$ for any $i$. Then it immediately follows from Step 1 and Step 3-5 that we can construct $x^{(a,q)}: Y \to \mathbb{R}^n$ that satisfy (13) and (14) in this case. This is because condition (12), which requires payoff profiles to be on a certain hyperplane, is not imposed this time.

**Step 7**: Finally, choose $\lambda$ and $\gamma$ small enough so that $(*)$ and $(**)$ are satisfied in each case. Observe that we can choose $\lambda$ and $\gamma$ independent of $a$ and $q$. For each $a \in A$ and $q \in Q$, define $\phi' := \phi^{\lambda}_{a,x}$ and $\xi := x^{(a,q)} - E^{\phi'}[x^{(a,q)}|a]$. Then $\xi$ and $\phi'$ satisfy (9) in addition to (10)-(12) in the case of (ii) and (13)-(14) in the case of (iii). Therefore the lemma is proved. ■

We need one more lemma to prove Lemma 4.

**Lemma 9** *Let $M \subset \mathbb{R}^n$ be a closed and convex set with an interior point in $\mathbb{R}^n$. Suppose that each boundary point $v \in M$ is associated with the unique supporting hyperplane and the unique normal vector $\lambda_v (\neq 0) \in \mathbb{R}^n$ such that $\lambda_v \cdot v \geq \lambda_v \cdot x$ for all $x \in M$. Then for any point $y \in \mathbb{R}^n$ such that $\lambda_v \cdot v > \lambda_v \cdot y$, there exists $\alpha^* \in (0,1)$ such that $(1-\alpha)v + \alpha y$ is in the interior of $M$ for any $\alpha \in (0, \alpha^*)$.*

**Proof.** Suppose that this is not the case, i.e. there does not exist such $\alpha^* > 0$. Let $W = \{x \in \mathbb{R}^n | \exists \alpha \in [0,1], x = (1-\alpha)v + \alpha y\}$.

We first show $W \cap intM = \varnothing$. First $v$ is not an interior point of $M$ by definition. If $(1-\alpha')v + \alpha' y$ is an interior point for any $\alpha' \in (0,1]$. Then $(1-\alpha)v + \alpha y$ is an interior point of $M$ for every $\alpha \in (0, \alpha')$ as it is a strictly positive combination of $v \in M$ and $(1-\alpha')v + \alpha' y \in intM$. This is a contradiction. Hence $W \cap intM = \varnothing$.

Since $W \cap intM = \varnothing$, we can apply the separating hyperplane theorem for each $x_\alpha = (1-\alpha)v + \alpha y \in W$, obtaining $\lambda_\alpha (\neq 0) \in \mathbb{R}^n$ such that (a) $\lambda_\alpha \cdot x_\alpha \geq \lambda_\alpha \cdot x$ for all $x \in M$. Normalize them so that $\|\lambda_\alpha\| = 1$. Since $\lambda_\alpha \cdot x_\alpha \geq \lambda_\alpha \cdot v$, it also follows that (b) $\lambda_\alpha \cdot y \geq \lambda_\alpha \cdot v$ for every $\alpha > 0$ by the definition of $x_\alpha = (1-\alpha)v + \alpha y$.

Take a sequence of $\lambda_{\alpha_n}$ such that $\alpha_n > 0$ converges to 0 and $\lambda_{\alpha_n}$ converges to some $\lambda^* (\neq 0) \in \mathbb{R}^n$. Then $\lambda^* \cdot v \geq \lambda^* \cdot x$ for all $x \in M$ (from (a)) and $\lambda^* \cdot y \geq \lambda^* \cdot v$ (from (b)) by continuity.

Finally $\lambda^* \neq \lambda_v$ follows from $\lambda_v \cdot v > \lambda_v \cdot y$. Hence $\lambda^*$ and $\lambda_v$ are different normal vectors that separate $v$ from $M$. This is a contradiction. ■

We now prove Lemma 4, thus completing the proof of Theorem 3.

.

**Proof of Lemma 4**

**Proof.** Choose $\gamma > 0$ satisfying the conditions of Lemma 8 and let $W \subset intV^*(G)$ be a smooth set.We will show that, for any $v \in W$, there exists $\eta > 0$, $\delta \in (0,1)$ and an open set $U$ containing $v$ such that $U \cap W \subset B(\delta, W, \eta)$.

**Step 1**: Suppose that $v$ is a boundary point of $W$. Let $q^* \in Q$ be the vector of utility weights such that $v = \arg\max_{v' \in W} q^* \cdot v'$ and $a^* = \arg\max_{a \in A} q^* \cdot g(a)$. We first show that $v$ is strictly enforceable for some $w'_\delta : Y \to \mathbb{R}^n$ such that $q^* \cdot v > q^* \cdot w'_\delta(y)$ for any $y$.

Let $\xi : Y \to \mathbb{R}^n$ and $\phi'$ be the payoff function and public coordinating device as defined in the conditions of Lemma 8 given $q$ and $a^*$. Note that, by (ii), we can find $c > 0$ and $\eta' > 0$ such that

$$\begin{array}{rcl} g_j(a^*) + cE^{\phi'}[\xi_i | a^*] - \eta' & > & g_j\left(a_j, a^*_{-j}\right) + cE^{\phi'}[\xi_j | a'_j, a^*_{-j}, \rho_j] \\ \text{for all } \left(a'_j, \rho_j\right) \text{ with } a'_j & \neq & a^*_j \text{ and for all } j \in N. \end{array} \tag{15}$$

Let $u(\delta) \in \mathbb{R}^n$ be the payoff vector to satisfy $v = (1-\delta) g(a^*) + \delta u(\delta)$ for each $\delta \in (0,1)$. Define $w'_\delta : Y \to \mathbb{R}^n$ as follows.

$$w'_\delta(y) := u(\delta) + \frac{1-\delta}{\delta} c\xi(y).$$

Then $\left(a^*, \phi', w'_\delta\right)$ clearly $(1-\delta)\eta'$-enforces the payoff profile $v$ for every $\delta \in (0,1)$ (by (15) and (iii)).

Since $W$ is in the interior of the feasible set,

$$q^* \cdot g(a^*) > q^* \cdot v = q^* \cdot [(1-\delta) g(a^*) + \delta u(\delta)].$$

Hence $q^* \cdot g(a^*) > q^* \cdot u(\delta)$ for any $\delta \in (0,1)$. Since $q^* \cdot \xi(y) = 0$ by construction, this implies the desired inequality:

$$q^* \cdot v > q^* \cdot u(\delta) = q^* \cdot w'_\delta(y) \text{ for all } y \in Y.$$

**Step 2:** We show that $w'_\delta : Y \to intW$ for large enough $\delta$. Fix any $\delta$. Since $v = (1-\delta) g(a^*) + \delta u(\delta)$, $w'_\delta(y)$ can be represented by

$$w'_\delta(y) = \frac{v - (1-\delta) g(a^*)}{\delta} + \frac{1-\delta}{\delta} c\xi(y).$$

Then for any $\delta' \in (\delta, 1)$, we can represent $w'_{\delta'}(y)$ as a positive convex combination of $v$ and $w'_\delta(y)$ as follows.

$$w'_{\delta'}(y) = \frac{\left(1-\delta'\right)\delta}{\delta'\left(1-\delta\right)} w'_\delta(y) + \frac{\delta'-\delta}{\delta'\left(1-\delta\right)} v \text{ for any } y \in Y.$$

Since $Y$ is a finite set and $q^* \cdot v > q^* \cdot w'_\delta(y)$ by step 1, it directly follows from Lemma 9 that $w'_\delta$ takes a value in the interior of $W$ for large enough discount factor.

**Step 3**: Next suppose that $v$ is an interior point of $W$. In this case, it is clear that $v$ is strictly enforceable with some $w'_\delta : Y \to \mathbb{R}^n$ and furthermore $w'_\delta(y)$ is in the interior of $W$ for any $y \in Y$ if $\delta$ is close enough to 1.

**Step 4:** We have shown that, for any $v \in M$, there exists $\delta' \in (0,1)$, $a^* \in A$, $\eta' > 0$, $\phi'$and $w'_{\delta'} : Y \to intW$ such that $\left(a^*, \phi', w'_{\delta'}\right)$ $\left(1-\delta'\right)\eta'$-enforces $v$. We can now choose $\varepsilon > 0$ so that $v' \in U = \{z \in \mathbb{R}^N | ||z - v|| < \varepsilon\}$ implies that $w'_{\delta'}(y) + \frac{v'-v}{\delta'} \in intW$ for each $y$. Then it follows that each $v' \in W \cap U$ is $(1-\delta')\eta'$-enforced by $\left(a^*, \phi', w'_{\delta'} + \frac{v'-v}{\delta'}\right)$ with respect to $W$ and $\delta'$. Hence each $v' \in W \cap U$ is $(1-\delta')\eta'$-decomposable with respect to $W$ and $\delta'$. Define $\eta$ by $\eta := (1-\delta')\eta'$. Then we have $W \cap U \subset B(\delta', W, \eta)$, i.e. the local strict self decomposability is established.
■

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