Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://economics.sas.upenn.edu/pier

PIER Working Paper 11-016

"Achievable Outcomes of Dynamic Contribution Games"
Second Version

by

Steven A. Matthews

http://ssrn.com/abstract=1871961

Achievable Outcomes of Dynamic Contribution Games

Steven A. Matthews*

June 20, 2011

Abstract

This paper concerns multistage games, with and without discounting, in which each player can increase the level of an action over time so as to increase the other players' future payoffs. An action profile is *achievable* if it is the limit point of a subgame perfect equilibrium path. Necessary conditions are derived for achievability under relatively general conditions. They imply that any efficient profile that is approximately achievable must be in the core of the underlying coalitional game. In some but not all games with discounting, the necessary conditions for achievability are also sufficient for a profile to be the limit of achievable profiles as the period length shrinks to zero. Consequently, in these games when the period length is very short, (i) the set of achievable profiles does not depend on the move structure; (ii) an efficient profile can be approximately achieved if and only if it is in the core; and (iii) any achievable profile can be achieved almost instantly.

KEYWORDS: *dynamic games, monotone games, core, public goods, voluntary contribution, gradualism*

JEL NUMBERS: C7

$\mathbb{R}$

*Department of Economics, University of Pennsylvania, Philadelphia, PA 19104. stevenma@econ.upenn.edu.
This paper extends and replaces "Achievable Outcomes in Smooth Dynamic Contribution Games," PIER Working Paper No. 08-028. I thank Leslie Marx for conversations and joint work on which this paper builds. I thank for their comments Parkash Chander, Prajit Dutta, Douglas Gale, George Mailath, Andrew Postlewaite, Larry Samuelson, Peyton Young, and several seminar audiences. I am grateful for the hospitality of the Eitan Berglas School of Economics, where some of this work was done. Partial support was received from NSF Grant No. 0079352.

1. Introduction

A dynamic contribution game is defined broadly here to be a multiperiod game in which each player can increase the level of an action incrementally, thereby increasing the other players' future payoffs. Such games exhibit positive spillovers that do not diminish with time. A leading application is one in which the actions are cumulative contributions of a private good to the production of a durable public good. The game is then a model of a fund drive, or sequence of fund drives, such as those held to finance church or university building projects or public radio programs.[1] Another application is to adoption and entry: agents decide when to invest in a new technology, and the future returns from adoption increase as the number of adopters grows.[2] Another application is to holdup: a seller and buyer make pre-trade investments over time in an asset's quality, and perhaps periodic payments to each other.[3] Yet another is to partnership: partners contribute effort over time in order to increase a common capital stock.

Some dynamic contribution games have unique equilibria that can be characterized by backwards induction. For example, this is true in Admati and Perry (1991) and Compte and Jehiel (2003) because of their restriction to a binary public good – backward induction starts in the period in which the threshold provision point is reached. If instead the payoff functions are smooth, backwards induction generally cannot be used and multiple equilibria exist. Results in this case have been fragmentary. Typically, for example, the existence of an equilibrium that achieves an (approximately) efficient outcome is established by construction, as in Marx and Matthews (2000), Lockwood and Thomas (2002), and Pitchford and Snyder (2004), without a systematic exploration of other equilibria.

The goal of this paper is to characterize as fully as possible the set of equilibrium outcomes of a range of dynamic contribution games. Even the size of this set of outcomes is an issue. As in a repeated game, it might be large because current deviations may by severely punished by triggering a decrease in the future contributions of the other players.[4] On the other hand, it might be small because the ability to punish deviations is diminished once sunk contributions

[1] Bagnoli and Lipman (1989), Fershtman and Nitzan (1991), Admati and Perry (1991), Marx and Matthews (2000), Compte and Jehiel (2003), Yildirim (2006), Duffy, Ochs, and Vesterlund (2007), and Battaglini, Nunnari, and Palfrey (2010) study dynamic contribution games to fund a public project.

[2] Gale (1995), Choi, Gale, and Kariv (2008), and Ochs and Park (2010) study dynamic adoption games.

[3] Pitchford and Snyder (2004) and Che and Sakovics (2004) study dynamic holdup games.

[4] The folk theorem of Dutta (1995) for stochastic games does not apply to the games of this paper because they do not satisfy its "asymptotic state independence" assumptions, (A1) and (A2). Indeed, we shall see that a folk theorem does not hold for them.

have become large. This effect of prior actions on security payoffs is not present in a repeated game, and it can result in "strategic gradualism," the property that contributions must be raised slowly over time in equilibrium.[5]

Overview of the Model and Results

A player's action/contribution in the games to be studied is a nonnegative real number that can be raised in any period in which the player can move. The only maintained assumption on the move structure is that each player can move in an infinite number of periods. Payoffs are given by either a discounted sum of stage game payoffs or, in the no-discounting case, by the lower limit of the sequence of stage game payoffs. Payoffs exhibit a weak positive spillovers property, and may have discontinuities due to the presence of thresholds in the provision of discrete public goods. All past actions are observable.

Every pure strategy subgame perfect equilibrium generates a convergent path of contribution profiles. The profile to which the path converges is said to be *achieved* by the equilibrium. The first set of results consists of necessary conditions that equilibrium paths and achievable profiles must satisfy.

The most novel necessary condition is that any achievable profile must be in a particular set, the *undercore*. Its definition is similar to that of the core, and does not depend on the dynamic structure of the game. A profile x is said to be *underblocked* by a coalition of players if there exists a smaller profile $z \leq x$ that prescribes zero contributions for the non-coalition players, and that each coalition member i prefers to x once she raises z_i to the level she most prefers, holding z_{-i} fixed. A *satiation profile* is preferred by each player to any other profile obtained by raising just her contribution. The undercore is then the set of satiation profiles that are not underblocked. Theorem 1 establishes that all achievable profiles are in the undercore.

An interpretation of Theorem 1 is that an achievable profile must satisfy a certain fairness property: it must not require any coalition's contribution to be too large. Proposition 1 establishes that the core, as typically defined in similar settings (e.g., Foley (1970)), is precisely the set of efficient profiles in the undercore. Theorem 1 thus tells us that any efficient profile that is not in the core is unachievable.

Theorem 2 establishes another necessary condition for achievability. In the discounting case, if the stage game payoff functions are differentiable and satisfy a strict positive spillovers property, then all achievable profiles are inefficient. In these games a core profile can at best be

[5] Strategic gradualism has been explored, e.g., in Marx and Matthews (2000), Lockwood and Thomas (2002), and most generally in Compte and Jehiel (2004).

the limit of achievable profiles as the discount factor converges to one.

In some games with discounting, many undercore and core profiles are neither achievable nor limits of achievable profiles as the discount factor converges to one. This is dramatically illustrated by the binary public good game of Compte and Jehiel (2003). This game has a unique achievable profile, given a sufficiently large discount factor, even though the undercore and core are continua of profiles. However, all undercore profiles are achievable in the no-discounting version of the game. Thus, in some games the set of achievable profiles expands discontinuously at $\delta = 1$.

Theorem 3 identifies a familiar class of games in which this discontinuity is absent. In these games the aggregate contribution determines a public good quantity, and each player's payoff is quasilinear in her own contribution and smooth and strictly concave in the public good. Furthermore, a *prisoners' dilemma* (PD) property holds: starting from any profile, not raising her contribution further is each player's dominant strategy in the stage game. Lastly, the move structure is assumed to satisfy a weak cyclicity property which is satisfied by all commonly assumed move structures, such as the simultaneous and round robin ones. Under these assumptions, Theorem 3 shows that any neighborhood of any undercore profile contains a profile that is achievable if the discount factor is sufficiently large. The undercore is thus equal to both the closure of the set of profiles that are achievable for some $\delta < 1$, and to the closure of the set of profiles that are achievable for $\delta = 1$. Since the definition of the undercore is independent of the move structure, this result implies that in this class of games, the limiting set of achievable profiles is independent of the move structure.

The proof of Theorem 3 shows that any neighborhood of almost any undercore profile contains the limit of a sequence of profiles that is an equilibrium path for all large discount factors. Thus, if the period length is small, it takes very little actual time for the path of contributions to reach any neighborhood of the profile being achieved. Strategic gradualism may be necessary in the sense that contributions cannot be raised to the ultimate goal in a finite number of periods, but it is not necessary in the sense that it must take a long time to approximately reach the goal.

The final result is Corollary 2, which derives three implications of the previous results for equilibrium payoffs. The first one is that any equilibrium payoff is weakly Pareto dominated by an undercore payoff. The second is that any equilibrium payoff that is efficient must be the payoff generated by a core profile. The third implication is that under the conditions of Theorem 3, any neighborhood of an undercore (and hence core) payoff contains an equilibrium payoff.

Related Literature

Gale (2001) studies dynamic contribution games in which the players do not discount. These games differ from those of this paper in that the stage-game payoff functions are assumed to be continuous and the actions multidimensional. The main result, Theorem 1, is that a profile is achievable if and only if it is "approachable", i.e., it is the limit of a feasible path of profiles and gives each player at least as large a payoff as she can obtain on her own starting from any point on the path. Two lemmas in the present paper extend Gale's result to cases with discounting and discontinuous payoffs. Lemma 2 shows that approachability is necessary for achievability, and Lemma 5 shows that a generalization of approachability is sufficient for achievability if the prisoners' dilemma property holds.

Gale (2001) also has a sufficient condition for a profile to be achievable that does not refer to a path: any "strongly minimal positive satiation point" is achievable. Proposition 4 of this paper establishes conditions under which the same is true in the discounting case as the discount factor goes to one.

Also related is Lockwood and Thomas (2002), which considers two-player games with discounting and continuous symmetric payoff functions satisfying the prisoners' dilemma property. When payoffs are differentiable, the profile achieved by the most efficient symmetric equilibrium is shown to be inefficiently small. Our Theorem 2 generalizes this result to any equilibrium, multiple players, and non-symmetric payoff functions. Lockwood and Thomas (2002) also show, in the differentiable case, that the most efficient symmetric equilibrium outcome converges to the symmetric efficient outcome as the discount factor goes to one, whether the players move simultaneously or alternately. (Pitchford and Snyder (2004) obtain a similar result.) This is a small hint of Corollary 2 (iii), that under the conditions of Theorem 3, any core payoff is the limit of equilibrium payoffs as the discount factor goes to one, regardless (almost) of the move structure.

Lastly, Bagnoli and Lipman (1989) is somewhat related. It describes a mechanism that fully implements the core in a discrete public good setting, via a refinement of subgame perfect equilibrium. The mechanism is similar to the dynamic contribution games studied here, except that it refunds the contributions each period that exceed the largest threshold point reached so far, and it stops the game in the first period in which the next threshold is not reached.

Organization

The model is set out in Section 2. Examples that motivate the questions and results are collected in Section 3. Necessary conditions for a path to be an equilibrium path and for a profile to be achievable are derived in Section 4. The structure of the undercore and core are delineated in Section 5. Sufficient conditions for a profile to be achievable are derived in Section 6. Implications for equilibrium payoffs are in Section 7, and concluding comments in Section 8. Appendices A−D contains proofs missing from Sections 4 − 7, respectively.

2. Model

The set of players is $N = \{1, \ldots, n\}$, with $n \geq 2$. At each date $t = 1, 2, \ldots,$ player i chooses a number, $x_i^t \in \mathbb{R}_+$. For concreteness, we refer to x_i^t as the player's (*cumulative*) *contribution*. The contribution profile chosen in period t is denoted x^t. A *path,* $\vec{x} = \{x^t\}_{t=0}^{\infty}$, is a sequence starting with $x^0 = (0, \ldots, 0)$. Past actions are publicly observed.

The game satisfies a monotonicity property: for $t \geq 1$ and any previously chosen x^{t-1}, the players in period t can only choose a profile x^t for which $x^t \geq x^{t-1}$ holds.[6]

The *move structure* is a sequence of subsets of players, $\vec{N} = \{N_t\}_{t=1}^{\infty}$. Only players in N_t can raise their contributions in period t. The move structure is assumed to satisfy $\cup_{\tau \geq t} N_\tau = N$ for all $t \geq 1$, so that each player is able to move infinitely often. A path is *feasible* if it is nondecreasing and satisfies $x_i^t = x_i^{t-1}$ for all $t \geq 1$ and $i \notin N_t$.

The stage-game payoff function is $u : \mathbb{R}_+^n \to \mathbb{R}^n$. Both discounting and no-discounting cases are considered. In the discounting case, a path $\vec{x}$ generates a continuation payoff in period t that is the usual weighted average of present and future stage-game payoffs:

$$U^t(\vec{x}, \delta) := (1 - \delta) \sum_{s \geq t} \delta^{s-t} u(x^s), \tag{1}$$

where $\delta \in (0, 1)$ is the common discount factor. In the no-discounting ($\delta = 1$) case, payoffs are given by

$$U^t(\vec{x}, 1) := \lim \inf_{s \to \infty} u(x^s). \tag{2}$$

Payoffs for the game as a whole are denoted without a superscript: $U(\vec{x}, \delta) := U^1(\vec{x}, \delta)$. If the discount factor is not explicitly mentioned in a result, the result holds for all $\delta \in (0, 1]$.

The maintained assumptions about u begin with it taking the form

$$u(x) = \hat{u}(f(X), x),$$

where $X = \sum_{i \in N} x_i$ is the *aggregate* contribution, $\hat{u} : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$, and $f : \mathbb{R}_+ \to \mathbb{R}_+$. An interpretation is that f is a production function that uses the aggregate X to produce an amount $y = f(X)$ of a public good that may have threshold provision points. Accordingly, f

[6] Here, $x \geq x'$ means $x_i \geq x_i'$ for all i; $x > x'$ means $x \neq x'$ and $x \geq x'$; and $x \gg x'$ means $x_i > x_i'$ for all i.

is assumed to be nondecreasing and right continuous. Refer to a profile that has an aggregate at which f is discontinuous as a *threshold profile*.

The function $\hat{u}$ is assumed to be continuous, with each $\hat{u}_i(y, x_i, x_{-i})$ strictly increasing in y and strictly decreasing in x_i. (These assumptions, together with f nondecreasing and right continuous, imply that u_i is upper semicontinuous.) In general $\hat{u}_i$ may increase or decrease in x_{-i}, representing positive or negative direct externalities. However, the sum of the direct and indirect (via $y = f(X)$) effects is assumed to be nonnegative, and hence u satisfies a weak *positive spillovers* property:

(PS) $u_i(\cdot)$ is nondecreasing in x_j, for all $i \neq j \in N$.

A profile $x \in \mathbb{R}^n_+$ is *efficient* if no $z \in \mathbb{R}^n_+$ satisfying $u(z) > u(x)$ exists. The origin, $x = 0$, is assumed to be inefficient. For convenience we make the mild assumption that $u(x) \neq u(\hat{x})$ for any two efficient profiles x and $\hat{x}$.

Lastly, in order to insure that best replies exist and equilibrium paths converge, u is taken to satisfy a mild *boundedness assumption:*

(BA) for any unbounded $\{x^k\}_{k=1}^{\infty} \subset \mathbb{R}^n_+$, $u_i(x^1) > \lim_{k \to \infty} \sup u_i(x^k)$ for some $i \in N$.

The assumptions made so far are maintained throughout the paper. The resulting extensive form game is denoted as $\Gamma(\delta, \vec{N})$.

At times attention shall be restricted to payoffs that arise in a public good setting in which direct externalities are absent, i.e., $\hat{u}_i(y, x_i, x_{-i})$ does not actually depend on x_{-i}. Two such settings that are of particular interest are the following:

- *Binary setting.* For all $i \in N$, $u_i(x) = 0$ if $X < X^*$, and $u_i(x) = V_i$ if $X \geq X^*$, where X^* is a threshold provision point. When referring to this setting, $0 < V_i < X^*$ for each i, and $0 < X^* < \sum_i V_i$, shall always be assumed.

- *Neoclassical setting.* For all $i \in N$, $u_i(x) = v_i(X) - x_i$, where the valuation function v_i satisfies $v_i(0) = 0$ and is continuously differentiable, strictly increasing, and strictly concave. When referring to this setting,

$$\lim_{X \to \infty} \sum_{i \in N} v_i'(X) < 1 < \sum_{i \in N} v_i'(0). \tag{3}$$

shall always be assumed. Both (BA) and (PS) hold in this setting, the latter strictly.[7]

[7] To prove (BA) holds, note that concavity and (3) imply that for any unbounded $\{x^k\}$, $\sum_i u_i(x^k) = \sum_i v_i(X^k) - X^k \to -\infty$ as $k \to \infty$. Hence, i exists such that $u_i(x^k) \to -\infty$, and so $u_i(x^1) > \lim_{s \to \infty} \sup u_i(x^s)$.

A range of timing and economic scenarios give rise to games with the formal structure of $\Gamma(\delta, \vec{N})$. The following three are illustrative.

Scenario 1: Random Terminal Date

In this scenario the game ends at a random date $\tilde{T}$, where $\Pr(\tilde{T} = T) = (1-\delta)\delta^{T-1}$. Consumption occurs only at the terminal date. At date t a player's expected continuation payoff is $\sum_{s \geq t} \Pr(\tilde{T} = s | \tilde{T} \geq t) u(x^s)$, which is precisely as shown in (1). This scenario arises by allowing the players of the static normal form game defined by u to raise their actions incrementally period by period, subject to the specified random stopping rule that determines when the payoffs will be realized.

Scenario 2: Endogenous Terminal Date

In this scenario the terminal date is determined by the history of play. A preeminent example is contribution to a binary public project by impatient players, studied by, e.g., Admati and Perry (1991) and Compte and Jehiel (2003). The project is completed once the aggregate reaches a threshold X^*, at which date player i receives a value V_i. Players bear the cost of each incremental contribution, $x_i^t - x_i^{t-1}$, when it is made in period t. A path $\vec{x}$ that completes the project at date T gives player i the payoff

$$\delta^{T-1} V_i - \sum_{s=1}^{\infty} \delta^{s-1} \left(x_i^s - x_i^{s-1} \right) = (1-\delta) \sum_{s=1}^{\infty} \delta^{s-1} \left[v_i(X^s) - x_i^s \right],$$

where $v_i(X) = V_i 1_{\{X \geq X^*\}}$. This yields our binary setting.[8]

An equivalent formulation is for the project to generate a flow of benefits, $(1-\delta)V_i$ per period, subsequent to completion, rather than the one-period benefit V_i upon completion. This brings us to the next scenario.

Scenario 3: Public Capital

Contributions in this scenario become the non-depreciating capital of one or more projects that produce a flow of future benefits over the infinite future. For example, suppose contributions can be made at dates $\Delta, 2\Delta, \ldots$, where $\Delta > 0$. At date $t\Delta$ player i contributes $x_i^t - x_i^{t-1} \geq 0$, which is instantly converted into capital on a one-to-one basis. So x^t is the vector of capital available to produce benefits in the time interval $[t\Delta, (t+1)\Delta)$. Player i values these benefits at rate $\hat{v}_i(x^t)$. The players discount payoffs at rate $r > 0$, and their discount factor is $\delta = e^{-r\Delta}$.

[8] In Admati and Perry (1991) the cost of contributing $x_i^s - x_i^{s-1}$ in period s is $w_i\left(x_i^s - x_i^{s-1}\right)$, where w_i is strictly convex. This convexity generates a non-incentive reason for contributions to be made incrementally. Only if w_i is linear is the Admati-Perry game of the form $\Gamma(\delta, \vec{N})$.

Letting $v_i(x) = r^{-1}\hat{v}_i(x)$, the continuation payoff of player i at date $t\Delta$ is then

$$\sum_{s \geq t} \delta^{s-t} \left[\int_0^{\Delta} \hat{v}_i(x^s) e^{-r\tau} d\tau - (x_i^s - x_i^{s-1}) \right] = (1-\delta) \sum_{s \geq t} \delta^{s-t} \left[v_i(x^s) - x_i^s \right] - x_i^{t-1}. \tag{4}$$

This payoff is as in (1), with $u_i(x) = v_i(x^s) - x_i^s$, less the constant (at time t) x_i^{t-1}.

Another application within this scenario is to relational contracting in a firm.[9] Suppose player 1 owns a firm and players $i > 1$ are the workers. Each worker chooses a non-contractible effort level each period. The quality of the firm's productive assets in a period increases in the cumulation of the workers' prior efforts. The rate of flow of revenue in period t to the owner is thus an increasing function of the workers' cumulative efforts, $\hat{v}_1(x_{-1}^t)$. The owner pays the workers $x_1^t - x_1^{t-1}$ in period t. The owner's continuation payoff is then as shown in (4), with $v_i(x^s) = r^{-1}\hat{v}_1(x_{-1}^s)$. A worker's stage-game payoff in a period is a share $\alpha_i \in [0,1]$ of the wages paid in that period, less the effort she takes. (The shares α_i sum to one, and are determined ex ante.) A worker's continuation payoff is then

$$\sum_{s \geq t} \delta^{s-t} \left[\alpha_i (x_1^s - x_1^{s-1}) - (x_i^s - x_i^{s-1}) \right] = (1-\delta) \sum_{s \geq t} \delta^{s-t} u_i(x_1^s, x_i^s) - u_i(x_1^{t-1}, x_i^{t-1}),$$

where $u_i(x_1, x_i) = \alpha_i x_1 - x_i$. This payoff is as in (1), modulo the constant (at time t) $u_i(x_1^{t-1}, x_i^{t-1})$.

If a scenario like this is the one of interest, it is important to interpret $\delta \to 1$ as taking the period length rather than the discount rate to zero, since $v_i = r^{-1}\hat{v}_i$. If r were taken to zero, the present value of future benefits would go to infinity and the free rider problem would vanish.

3. Equilibrium Examples

In this paper, an unmodified "equilibrium" always denotes a pure strategy subgame perfect equilibrium. Refer to the outcome of an equilibrium is an *equilibrium path*. A profile is *achievable* if it is the limit of an equilibrium path. The examples of this section are intended to motivate and illustrate upcoming results and arguments.

Example 1. Binary Threshold

Consider the binary setting with two players and $V_1 < V_2$. The efficient individually rational profiles satisfy $X = X^*$ and $x_i \leq V_i$. Let the move structure be the alternating one in which only player 1 (2) is able to move in odd (even) numbered periods.

[9] This is somewhat similar to the hold-up model of Pitchford and Snyder (2004), although the discounting there is the result of a random terminal date as in the first scenario described above.

Whether there is discounting makes a radical difference in this example. In the no-discounting case, any efficient individually rational profile is achievable. For instance, let x be such a profile, and define a Markovian strategy profile as follows: if player i can move in period t, she plays $x_i^t = \sigma_i(x^{t-1})$, where

$$\sigma_1(x^{t-1}) := \begin{cases} x_1 - x_2^{t-1} & \text{if } X^{t-1} \in [0, x_1] \\ 0 & \text{if } X^{t-1} \notin [0, x_1] \end{cases}, \qquad \sigma_2(x^{t-1}) := \begin{cases} 0 & \text{if } X^{t-1} \notin [x_1, X^*] \\ X^* - x_1^{t-1} & \text{if } X^{t-1} \in [x_1, X^*] \end{cases}. \tag{5}$$

These strategies are characterized by two contribution goals. Player 1 is responsible for bringing the aggregate from 0 up to the first goal, x_1, and until she does so player 2 does nothing. Player 2 is then responsible for bringing the aggregate up to the second and final goal, X^*. The equilibrium path is $x^1 = (x_1, 0)$ and $x^t = x$ for $t > 1$.[10]

In stark contrast, the discounting game with a sufficiently large δ has a unique equilibrium, the one just described with $x = (X^* - V_2, V_2)$. This is the result of Compte and Jehiel (2003).[11] The set of achievable profiles in this binary setting thus expands discontinuously at $\delta = 1$.

Remark 1. *This discontinuity can even occur if payoffs are continuous. Suppose* $u_i(x) = v_i(X) - x_i$, *where* $v_i(X) = V_i - V_i\sqrt{1-(X/X^*)}$ *for* $X \le X^*$, *and* $v_i(X) = V_i$ *for* $X > X^*$. *The extreme nonconcavity at* X^* *acts like a threshold. An argument like that of Compte and Jehiel (2003) shows that the game has a unique equilibrium if* δ *is large enough,* $V_1 < V_2 < X^*$, *and* $(V_1)^2/X^* + V_2 > X^*$.

Example 2. Gradualism

Consider a two-player game in a neoclassical setting, with each $v_i' < 1$ and the alternating move structure. Under these assumptions, in the no-discounting case, any x satisfying individual rationality, $x_i \le v_i(X)$, and no over-production, $v_1'(X) + v_2'(X) \le 1$, is achievable. This closed

[10] Note that σ gives a payoff of 0 to player 1 if $x = (V_1, X^* - V_1)$, and a payoff of 0 to player 2 if $x = (X^* - V_2, V_2)$. A strategy profile that requires both players to contribute zero and punishes any unilateral deviation by the play of the appropriate one of these punishing equilibria is thus an equilibrium that achieves $x = 0$.

[11] If $\delta < 1$, the strategy σ defined by (5) for any efficient individually rational x with $x_2 < V_2$ is not subgame perfect. For, in a subgame starting in an even period t and $x^{t-1} = (x_1 - \varepsilon, 0)$, player 2 would deviate by raising X^{t-1} to X^* immediately instead of waiting two periods to do so, provided $\varepsilon < (1-\delta^2)(V_2 - x_2)$.

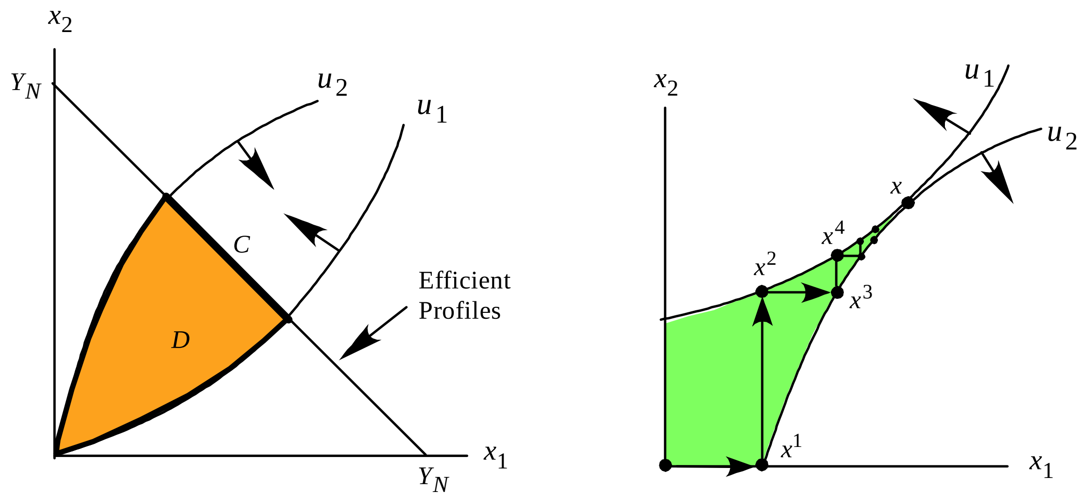

Figure 1 (a). The closed shaded region is the set of achievable profiles if $\delta = 1$.

Figure 1 (b). An equilibrium path that achieves an efficient profile if $\delta = 1$.

set of profiles is the shaded region in Figure 1 (a). (The labelings will be explained later.) Any feasible individually rational payoff is thus an equilibrium payoff of this game.

For instance, consider the efficient individually rational profile shown in Figure 1 (b). The shaded region is the set of all profiles below x that are worse for both players than x. It contains the origin because x is individually rational. The indicated path $\vec{x}$ converges to x. It is constructed by first having player 1 raise her contribution enough so that the resulting profile, $x^1 = (x_1^1, 0)$, gives player 2 the payoff $u_2(x)$. Then player 2 raises hers enough to give player 1 the payoff $u_1(x)$, and so on. The trigger strategy profile in which any deviation from this path triggers the play of the passive strategies, which are those that call for each player to never raise her contribution after any history, is an equilibrium when $\delta = 1$. Obviously, no player can gain by deviating from the path. Off the path, the passive strategy profile is an equilibrium because neither player can gain by unilaterally raising her contribution.[12]

Any equilibrium path that achieves the x of Figure 1 (b) must exhibit gradualism in the sense that it achieves x only asymptotically. Neither player can increase her contribution too much in any period because doing so would result in a profile that the other player prefers to x, and thus that player could profitably deviate by never raising her contribution again. Any equilibrium path achieving x must therefore stay in the shaded region, ensuring gradualism. Note, however, that this gradualism has no welfare cost because of the lack of discounting.

In contrast to the binary Example 1, discounting creates no discontinuity in this neoclassical

[12]The equilibrum path in Figure 1 (b) is also generated by a Markov perfect equilibrium, a contribution goal equilibrium defined as in (5) but with an infinite sequence of goals.

example. As shall be shown, discounting shrinks the set of achievable profiles in a continuous way – there is no discontinuity at $\delta = 1$. While no efficient profile is achievable if $\delta < 1$, every neighborhood of an individually rational efficient profile contains an achievable profile if δ is sufficiently large.

Example 3. No Folk Theorem

The previous example suggests that under its payoff assumptions, any individually rational payoff vector should be achievable if $\delta = 1$, and be the limit of achievable payoff vectors as $\delta \to 1$. This conjecture is false, however, if the number of players is larger than two.

To construct a counterexample, let $n = 3$ and $v_i(X) = 1 - (X+1)^{-1}$. Note that the maximizer of $v_1(Y) + v_2(Y) - Y$ is $Y_{\{1,2\}} := \sqrt{2} - 1$. Let x be any efficient and strictly individually rational profile satisfying $x_3 = 0$ and $X > Y_{\{1,2\}}$.[13] We claim x is unachievable. To prove this, suppose $\vec{x}$ is an equilibrium path achieving x. If $X^{t-1} < X^t = X$ for some $t < \infty$, then a player i for whom $x_i^{t-1} < x_i^t$ would gain by not raising her contribution in period t or thereafter: she would then obtain a continuation payoff of at least $u_i(x_i^{t-1}, x_{-i}^t) = u_i(x_i^{t-1}, x_{-i})$, and this exceeds her continuation payoff of $u_i(x)$ from not deviating because $v_i' < 1$. The convergence is therefore asymptotic. This implies that for some t, $Y_{\{1,2\}} < X^t < X$. The definition of $Y_{\{1,2\}}$ and strict concavity imply that

$$\begin{aligned} u_1(x^t) + u_2(x^t) &= v_1(X^t) + v_2(X^t) - X^t \\ &> v_1(X) + v_2(X) - X = u_1(x) + u_2(x). \end{aligned}$$

We conclude that $u_i(x^t) > u_i(x)$ for some $i \in \{1, 2\}$. Fixing this i and letting x^τ be the largest maximizer of u_i in the set $\{x^s\}$, we have $u_i(x^\tau) > u_i(x^s)$ for all $s > \tau$. Therefore, the continuation payoff of player i in period $\tau + 1$ if she deviates by never raising her contribution again, which is at least $u_i(x^\tau)$, exceeds her continuation payoff from not deviating. So $\vec{x}$ cannot be an equilibrium path.

The reason why x cannot be achieved in this example is that it specifies an overly large contribution from players 1 and 2. Because their joint contribution exceeds $Y_{\{1,2\}}$ and they are the only ones contributing, they both could be made better off in an incentive-free world by reducing their contributions. As we shall see, this is the condition which implies that given any feasible path converging to x, at least one of these players can profitably deviate. Which of them it is depends upon the path, and so it is necessary to consider the coalition $\{1, 2\}$ of players as the entity able to "block" x from being achieved.

[13]For example, $x = (.5Y_N, .5Y_N, 0)$, for $Y_N = \sqrt{3} - 1$, is efficient and strictly individually rational.

This example has features that simplify the argument but are not required. Its generalization in Theorem 1 below does not need x to be on the boundary, nor any u_i to be concave or continuous.

4. Necessary Conditions

General necessary conditions are derived in this section for a path to be an equilibrium path, and for a profile to be achievable. Their derivations require two definitions.

First, a player's *passive strategy* is the strategy specifying that she not raise her contribution after any history. It is obviously a feasible strategy in any subgame, regardless of the move structure. Because of (PS), a player imposes the most severe punishment possible upon the other players by playing her passive strategy.

Second, the *security payoff function* u^* is defined as

$$u_i^*(x) := \max_{x_i' \geq x_i} u_i(x_i', x_{-i}). \tag{6}$$

(Lemma A1 in Appendix A establishes that this program has a solution.) In a subgame that starts from a profile x, player i can obtain a continuation payoff of at least $u_i^*(x)$ by playing her myopic best reply to x and passively thereafter. Note that u_i^* is nonincreasing in x_i and, since u_i satisfies (PS), nondecreasing in x_{-i}.

Necessary Conditions for Equilibrium Paths

Consider an equilibrium that generates a path $\vec{x}$. Suppose player i deviates by playing $\hat{x}_i$ in period t, and her passive strategy thereafter. Since the contributions of the other players in each period $s \geq t$ can be no lower than x_{-i}^t, (PS) implies that following this deviation, player i's stage-game payoffs, and hence her continuation payoff, are no less than $u_i(\hat{x}_i, x_{-i}^t)$. This payoff therefore must not exceed her equilibrium continuation payoff, $U_i^t(\vec{x}, \delta)$. As this is true for any $\hat{x}_i \geq x_i^{t-1}$ if the player is able to move in period t, $\vec{x}$ must satisfy the following condition:

$$u_i^*(x_i^{t-1}, x_{-i}^t) \leq U_i^t(\vec{x}, \delta) \text{ for all } t \geq 1,\ i \in N_t. \tag{7}$$

Another useful condition is obtained by considering an immediate deviation by player i to her passive strategy in period t. This deviation is feasible even if $i \notin N_t$, and it yields a continuation payoff no less than $u_i(x_i^{t-1}, x_{-i}^t)$. Thus, $\vec{x}$ is an equilibrium path only if

$$u_i(x_i^{t-1}, x_{-i}^t) \leq U_i^t(\vec{x}, \delta) \text{ for all } t \geq 1,\ i \in N. \tag{8}$$

The following lemma uses (8) and (BA) to show that equilibrium paths converge. It also establishes that if an equilibrium path that does not converge in a finite number of periods, the profile it achieves is not a threshold. This is because once a path comes close to a threshold, some player would want to deviate by raising her contribution enough to reach the threshold.

Lemma 1. *Equilibrium paths converge. An equilibrium path that converges to a threshold profile does so in a finite number of periods.*

We can now observe that in the no-discounting case, any continuation equilibrium payoff is equal to the payoff generated by the profile being achieved:

$$U^t(\vec{x}, 1) = u(x) \text{ for all } t \geq 0. \tag{9}$$

For, by Lemma 1, x is either a profile at which u is continuous, or it is achieved in a finite number of periods. In either case (9) follows from (2).

The path necessary conditions, (7) and (8), are used to prove the following lemma, which establishes that conditions like (9) hold regardless of the discount factor.[14]

Lemma 2. *If x is achieved by an equilibrium path $\vec{x}$, then*

$$\lim_{t\to\infty} U^t(\vec{x}, \delta) = \lim_{t\to\infty} u(x^t) = u(x). \tag{10}$$

Furthermore, for all $t > 0$ and $i \in N$,

$$\max\{U_i^t(\vec{x}, \delta),\ u_i(x^{t-1}),\ u_i^*(x_i^{t-1}, x_{-i}^t)\} \leq u_i(x). \tag{11}$$

The next lemma is a simple consequence of (11). Suppose x is achieved by an equilibrium path $\vec{x}$. Suppose also that a profile z and a player i exist such that by some date all the other players have raised their contributions above what z specifies, but that at date $\tau - 1$ player i has not. Then $u_i^*(z) \leq u_i(x)$. If the opposite held, player i would want to deviate from the path at date τ.

Lemma 3. *If x is achieved by an equilibrium path $\vec{x}$, then there does not exist a triple (z, i, τ) that satisfies (a) $u_i^*(z) > u_i(x)$, (b) $z_i \geq x_i^{\tau-1}$, and (c) $z_{-i} \leq x_{-i}^{\tau}$.*

Proof. Since $u_i^*(\hat{x}_i, \hat{x}_{-i})$ is nonincreasing in $\hat{x}_i$ and nondecreasing in $\hat{x}_{-i}$, (b) and (c) imply $u_i^*(x_i^{\tau-1}, x_{-i}^{\tau}) \geq u_i^*(z)$. Thus, (a) implies $u_i^*(x_i^{\tau-1}, x_{-i}^{\tau}) > u_i(x)$, violating the necessary condition (11). ∎

[14] Gale (2001) defines a profile x to be *approachable* if it is the limit of a feasible path $\vec{x}$ such that $u_i^*(x_i^{t-1}, x_{-i}^t) \leq u_i(x)$ for every t and $i \in N$. He shows, in the no-discounting case with a continuous u, that any achievable profile is approachable. The second statement in Lemma 2 generalizes this to the discounting case and to payoffs with some discontinuities.

Necessary Conditions for Achievable Profiles

We now seek necessary conditions for achievability that do not refer to a feasible path. Path-free conditions are useful because they require less data to check. Furthermore, they do not depend on the nature of the game's move structure (except for its property that each player can move infinitely often).

Two necessary conditions are fairly obvious. Say that a profile x is a *satiation profile* if $u^*(x) = u(x)$, and that it is *individually rational* if $u^*(0) \leq u(x)$.

Lemma 4. *Any achievable profile is an individually rational satiation profile.*

We prove here in the text the necessity of individual rationality, as the proof is both simple and an introduction to the more general argument used below. So, suppose x is a profile for which $u_i^*(0) > u_i(x)$ for some player i. Let τ be the first period in which player i can move. Then, with respect to any feasible path that converges to x, the triple $(0, i, \tau)$ satisfies (a)-(c) of Lemma 3. This proves x is unachievable. Essentially, behind the formality, player i can gain be deviating as soon as possible from any path that converges to x, raising her contribution to whatever maximizes $u_i(\cdot, x_{-i}^{\tau})$ and then never raising it again.

We now formulate a condition more general than individual rationality that any achievable profile must satisfy. As the condition utilizes a concept related to that of "blocking" in cooperative game theory, it is natural to adopt a similar terminology. Refer to a nonempty subset of players as a *coalition.* Then, a profile x is *underblocked* by a coalition S if $z \leq x$ exists such that $z_{-S} = 0$ and $u_S^*(z) \gg u_S(x)$. This definition generalizes that of individual rationality, since a profile is individually rational if and only if it is not underblocked by a singleton coalition.[15]

Underblocked profiles are unachievable. The precise argument is given below in the proof of Theorem 1, but here is the gist of it. Suppose x is underblocked, say by coalition S using profile z. Let $\vec{x}$ be any feasible path converging to x. Let τ be the first date at which x^t exceeds z. The definition of τ insures that $z_i \geq x_i^{\tau-1}$ for some coalition member $i \in S$. (Which coalition member this is may depend on the path, unless S is a singleton). This construction yields a triple, (z, i, τ), satisfying (a)-(c) of Lemma 3. Thus, x is not achievable.

We have now two necessary conditions for achievability, being a satiation profile and not being underblocked. Define the *undercore* to be the set of satiation profiles that are not underblocked, and denote it as D. The following is our first main result.

---

[15]If $u_i^*(0) > u_i(x)$, then $\{i\}$ underblocks x using $z = 0$. Conversely, if $\{i\}$ underblocks x using z, then $z = (z_i, 0_{-i})$, and so $u_i^*(0) \geq u_i^*(z) > u_i(x)$.

Theorem 1. *All achievable profiles are in the undercore.*

Proof of Theorem 1. Let x be achievable. Then it is a satiation profile. Assume x is underblocked. Hence, a coalition S and profile $z \le x$ exist such that $z_{-S} = 0$ and $u_S^*(z) \gg u_S(x)$. For $i \in S$ we have $u_i^*(x_i, z_{-i}) \le u_i^*(x) = u_i(x)$, since $z_{-i} \le x_{-i}$ and x is a satiation profile. This proves $z_i \ne x_i$, and so $z_S \ll x_S$.

Let $\vec{x}$ be an equilibrium path that achieves x. Since $z_S \ll x_S$, a smallest date exists at which x_S^t is strictly larger than z_S : there exist $\tau \ge 1$ and $i \in S$ such that

$$z_i \ge x_i^{\tau-1} \text{ and } z_S \ll x_S^{\tau}. \tag{12}$$

Observe that (z, i, τ) satisfies the conditions of Lemma 3 with respect to $\vec{x}$. It satisfies $u_i^*(z) > u_i(x)$ because $i \in S$. It satisfies $z_i \ge x_i^{\tau-1}$ by the first part of (12). It satisfies $z_{-i} \le x_{-i}^{\tau}$ by the second part of (12) and the fact that $z_{-S} = 0 \le x_{-S}^{\tau}$. Lemma 3 thus implies that $\vec{x}$ does not achieve x, a contradiction. This proves x is not underblocked, and so $x \in D$. ■

The consequences of Theorem 1 are explored in the next section by examining the structure of the undercore. We end this section with a final necessary condition: in the discounting case, every achievable profile is inefficient if the payoffs are continuously differentiable and satisfy a strict version of (PS).[16,17] Essentially, the sum of the player's gains from deviating are first order in the remaining amount to contribute, but the sum of their time-average future benefits from not deviating is second order in this amount.

Theorem 2. *Suppose $\delta < 1$, and u is continuously differentiable and satisfies $\partial u_i(x)/\partial x_j > 0$ for all $i \ne j$. Then any achievable profile is inefficient.*

5. The Undercore

The undercore contains all achievable profiles by Theorem 1 and, as is shown in the next section, the reverse inclusion holds in a limiting sense in some settings. Uncovering the structure of the undercore will thus be useful for understanding the nature of achievable profiles. The first step is Lemma B1 in Appendix B, which shows that D is a compact set under the maintained assumptions.

---

[16] Related results are obtained for special cases by Marx and Matthews (2000), Lockwood and Thomas (2002), and Pitchford and Snyder (2004).

[17] Achievable profiles may be efficient if payoffs are not differentiable, even if $\delta < 1$. This is the case in Example 1 of Section 3, and in other exampes in Marx and Matthews (2000) and Lockwood and Thomas (2002).

The undercore generally contains some but not all efficient profiles. Define the *core*, C, to be the set of profiles that are not blocked, where a profile x is *blocked* by a coalition S if and only if a profile z exists such that $z_{-S} = 0$ and $u_S(z) > u_S(x)$.[18] As N blocks inefficient profiles, core profiles are efficient. The following proposition shows that the core consists precisely of the efficient profiles in the undercore.

Proposition 1. *The core is the subset of profiles in the undercore that are efficient:*

$$C = \{x \in D : x \text{ is efficient}\}.$$

In a binary setting, straightforward arguments show that the core is the entire set of efficient individually rational profiles:

$$C = \{x \in \mathbb{R}^n_+ : X = X^*,\ x_i \le V_i \text{ for all } i \in N\}.$$

The undercore differs only by containing the origin: $D = C \cup \{0\}$. Thus, in a binary setting with discounting, much of the undercore may be unachievable. Recall that in Example 1, just one profile is achievable when $\delta < 1$, but the entire undercore is achievable when $\delta = 1$.

We end this section with a characterization of the core and undercore in neoclassical settings. In these settings the *surplus function* for a coalition S,

$$f_S(X) := \sum_{i \in S} v_i(X) - X,$$

plays a central role. Since each v_i is strictly concave increasing, (3) implies that f_S has a unique maximizer, which we denote as Y_S. Note that for any coalitions S and $T \subset S$, $Y_T \le Y_S$, and $Y_T < Y_S$ if $Y_S > 0$. (But for convenience, set $Y_\varnothing := \infty$.) Letting $\bar{Y} := \max_i Y_{\{i\}}$, the concavity of each v_i implies that x is a satiation profile if and only if $X \ge \bar{Y}$.

The *value* of a coalition S is $V(S) := f_S(Y_S)$. For any profile x, let $X_S := \sum_{i \in S} x_i$. The following familiar proposition states that a profile is in the core if and only if the sum of payoffs it gives any coalition is no less than its value – what it could obtain “on its own”. (Its proof may be less familiar because of the $x \ge 0$ constraint.)

Proposition 2. *In a neoclassical setting, the core is the set of satiation profiles satisfying, for all coalitions S,*

$$\sum_{i \in S} v_i(X) - X_S \ge V(S). \tag{13}$$

[18] This is a typical definintion of the core in public good settings, e.g., Foley (1970).

Roughly speaking, a coalition S cannot underblock a satiation profile x if either it cannot block it, or if X is sufficiently small that S can not block it using any $z \leq x$. This intuition is formalized in the first part of the following proposition.

Proposition 3. *In a neoclassical setting, the undercore is the set of satiation profiles satisfying, for all coalitions S,*

$$X < Y_S \text{ or } \sum_{i \in S} v_i(X) - X_S \geq V(S). \tag{14}$$

Equivalently, the undercore is the set of satiation profiles satisfying, for all coalitions S,

$$X_S \leq \max\left(Y_S,\ \sum_{i \in S} v_i(X) - V(S)\right). \tag{15}$$

For a given aggregate X, the inequalities in (15) impose upper bounds on each coalition's contribution. That is, an undercore profile must not require any coalition to contribute too much.

The inequalities determining the undercore are less restrictive for profiles with smaller aggregates. For example, (14) implies that if x is a satiation profile satisfying $X < Y_S$ for every non-singleton coalition, then $x \in D$ if and only if it is individually rational. However, if $X = Y_N$, and so $X \geq Y_S$ for all coalitions, then (14) implies that $x \in D$ if and only if it satisfies (13) for all coalitions. The core is therefore $\{x \in D : X = Y_N\}$.

The following corollary implies that in a neoclassical setting, the aggregate of any undercore profile x is no greater than the amount that maximizes the surplus of the contributing coalition, $N(x) := \{i \in N : x_i > 0\}$. As $Y_{N(x)} \leq Y_N$, this implies that the core is the northeast surface of the undercore. Part (ii) of the corollary, which will be used in the next section, establishes that the undercore contains a particular line segment of strictly positive profiles.

Corollary 1. *In a neoclassical setting,*

(i) *any $x \in D$ satisfies $X \leq Y_{N(x)}$, and*

(ii) $\left(v_1'(Y_N)Y, \ldots, v_n'(Y_N)Y\right) \in D$ *for any $Y \in [\bar{Y}, Y_N]$.*

Figure 1 (a) illustrates a two-person example of the core and undercore in a neoclassical setting. In this example $\bar{Y} = 0$, so that all profiles are satiation profiles. Note that the set $u(D)$ of undercore payoffs is the entire set of feasible individually rational payoffs.

This is not generally true. Figure 2 depicts a two-person neoclassical setting in which $Y_{\{2\}} > Y_{\{1\}} = 0$. The undercore does not include individually rational profiles like the indicated x because they are not satiation profiles. Such a profile cannot be achieved for any $\delta \leq 1$ because, once a profile sufficiently close to x is reached, player 2 would deviate by raising the aggregate up to $Y_{\{2\}} > X$. The payoff $u(x)$ is not also generated by any undercore profile.

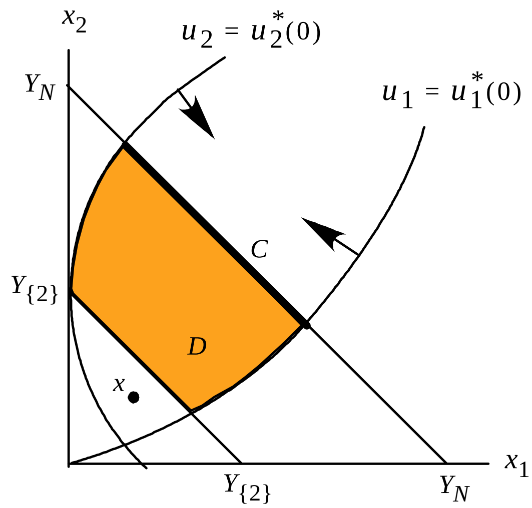

Figure 2.

In the following numerical example, the undercore payoffs are again a strict subset of the feasible individually rational payoffs. The core payoffs are a strict subset of the individually rational efficient payoffs. This is generally the case when there are more than two players.

Example 4. *Let* $n = 3$ *and* $v_i(X) = 2\sqrt{X}$. *The optimal coalitional contributions are then* $Y_{\{i\}} = 1$, $Y_{\{i,j\}} = 4$, *and* $Y_N = 9$. *The satiation profiles are those with* $X \geq 1$, *and the set of individually rational profiles is* $R = \{x \in \mathbb{R}^3_+ : x_i \leq 2\sqrt{X} - 1\}$. *The undercore is the union of two sets,* $D = D_1 \cup D_2$, *where*

$$D_1 = \{x \in R : 1 \leq X \leq 4\},$$
$$D_2 = \{x \in R : 4 < X \leq 9,\ x_i + x_j \leq 4\sqrt{X} - 4\}.$$

The set of undercore payoffs is $u(D) = u(D_1) \cup u(D_2)$, *where*

$$u(D_1) = \left\{\tilde{u} \in \mathbb{R}^3 : 5 \leq \sum \tilde{u}_i \leq 8,\ 1 \leq \tilde{u}_i \leq 4\right\},$$
$$u(D_2) = \left\{\tilde{u} \in \mathbb{R}^3 : 8 < \sum \tilde{u}_i \leq 9,\ 4 \leq \tilde{u}_i + \tilde{u}_j,\ 1 \leq \tilde{u}_i\right\}.$$

Observe that $u(D)$ *is a strict subset of the set of individually rational payoffs that arise from satiation profiles,* $\left\{\tilde{u} \in \mathbb{R}^3 : 5 \leq \sum \tilde{u}_i \leq 9,\ 1 \leq \tilde{u}_i\right\}$.

The core consists of the undercore profiles for which $X = 9$, *which can be written as* $C = \left\{x \in \mathbb{R}^3 : X = 9,\ 1 \leq x_i \leq 5\right\}$. *Note that the set of core payoffs,*

$$u(C) = \left\{\tilde{u} \in \mathbb{R}^3 : \sum \tilde{u}_i = 9,\ 1 \leq \tilde{u}_i \leq 5\right\},$$

is a strict subset of the individually rational efficient payoffs, $\left\{\tilde{u} \in \mathbb{R}^3 : \sum \tilde{u}_i = 9,\ 1 \leq \tilde{u}_i\right\}$.

6. Sufficient Conditions

The main result of this section is that under certain conditions in a neoclassical setting, almost any undercore profile is achievable if the discount factor is close enough to one. Note that some restriction of the setting is required, as the result is untrue in general. Recall that only one of the continuum of undercore profiles in Example 1 is achievable if $\delta < 1$, but they are all achievable if $\delta = 1$.

Sufficient Conditions for Equilibrium Paths

The first step is to find a condition under which (7) is sufficient as well as necessary for a path to be an equilibrium path. This will be useful because it allows the analysis to focus on paths, which are much simpler than strategies.

Recall that (7) requires, for $i \in N_t$, that the continuation payoff $U_i^t(\vec{x}, \delta)$ from not deviating be no less than the security payoff $u_i^*(x_i^{t-1}, x_{-i}^t)$. Thus, if $\vec{x}$ satisfies (7), player i will not want to deviate from the path at date t if the strategies that will then be played give her a continuation payoff no greater than $u_i^*(x_i^{t-1}, x_{-i}^t)$. This is the case if deviations trigger the passive strategies.[19] The strategy profile in which $\vec{x}$ is played and any deviation triggers the passive strategies is therefore an equilibrium that generates $\vec{x}$, provided that the passive strategy profile is itself an equilibrium of any subgame. This is true if (and only if) every profile is a satiation profile. Accordingly, (7) is a sufficient condition for $\vec{x}$ to be an equilibrium path if the following "Prisoners' Dilemma" property holds:[20]

(PD) $\quad u^* = u.$

Commonly assumed, (PD) implies that each player's dominant strategy in any stage game is to not raise her contribution. The following lemma records the result just proved.

Lemma 5. *If u satisfies* (PD), *then for any $\delta \in (0, 1]$, a feasible $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N})$ if*

$$u_i(x_i^{t-1}, x_{-i}^t) \leq U_i^t(\vec{x}, \delta) \text{ for all } t \geq 1,\ i \in N_t. \tag{16}$$

[19] If player i unilaterally deviates from $\vec{x}$ at date t to some $z_i \geq x_i^{t-1}$, and this triggers the passive strategy profile, her contiuation payoff will be $u_i(z_i, x_{-i}^t)$. This payoff, by definition, is not more than $u_i^*(x_i^{t-1}, x_{-i}^t)$.

[20] Equivalent to (PD) is the property that each u_i is nonincreasing in x_i. Note too that (PD) implies the absence of thresholds, i.e., f and hence u are continuous. In a neoclassical setting, (PD) is equivalent to $v_i'(0) \leq 1$ for all $i \in N$, since each v_i is concave.

Remark 2. *In the no-discounting case,* (PD) *in Lemma 5 can be weakened to the assumption that u is continuous. This is established by Theorem 1 of Gale (2001). The key step in proving this is to show that when $\delta = 1$, any subgame starting from any profile x has, for any player i, an equilibrium giving player i her security payoff $u_i^*(x)$. These maximally punishing equilibria can then be used instead of the passive strategies to prove the sufficiency of (7). It is an open question how much* (PD) *can be weakened in Lemma 5 when $\delta < 1$.*

Due to discounting, one more mild assumption shall be made. Discounting implies that rewards and punishments can influence current behavior only if they are not delayed too long. Hence, the interval between the times at which a player can move should not grow too quickly as the game progresses. This is ensured if the move structure $\vec{N}$ satisfies a *cyclicity* property:

(CY) integer $m > 0$ exists such that $i \in N_{(nk+i)m}$ for all $i \in N$ and $k \geq 0$.

This property specifies that player 1 is able to move at date m, player 2 at date $2m$, and so on until the pattern repeats with player 1 able to move at date $(n+1)m$. There are no restrictions on who else can move at dates that are multiples of m, nor on who can move at any other date. Familiar move structures satisfy (CY). With $m = 1$, it is satisfied by both the simultaneous move structure and the round-robin structure defined by

$$N_t^R := \{t \bmod n + 1\} \text{ for all } t \geq 1.$$

The following lemma establishes that if $\vec{N}$ satisfies (CY), then any equilibrium path of the round-robin game passes through the same profiles as does an equilibrium path of a game that has the move structure $\vec{N}$ and a certain larger discount factor. This result will allow attention to be restricted to the round-robin structure.

Lemma 6. *Suppose u satisfies* (PD), *$\vec{N}$ satisfies* (CY), *and $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N}^R)$ for some $\delta \in (0, 1]$. Then $\Gamma(\delta^{1/m}, \vec{N})$ has an equilibrium path $\vec{z}$ that passes through the same profiles as does $\vec{x}$.*

The path $\vec{z}$ in Lemma 6 is obtained by slowing down the round-robin path $\vec{x}$: player 1 moves in period m instead of period 1, player 2 moves in period $2m$ instead of period 2, and so on. Property (CY) insures that this new path is feasible for $\vec{N}$. Along this new path the future reward a player receives for raising her contribution in the current period is postponed, but raising the discount factor to $\delta^{1/m}$ increases its present value enough to restore incentives.

Sufficient Conditions for Achievability

Before presenting the sufficiency result for the discounting case, it is useful to consider the analogous result obtained by Gale (2001) for the no-discounting case. Define a profile x to be *strongly minimal* if there does not exist a coalition S and a profile $z < x$ such that $z_{-S} = 0$ and $u^*_S(z) \geq u_S(x)$.[21] Then, let D_0 be the set of satiation profiles that are strongly minimal. For a continuous u, Gale's result (Lemma 5 and Theorem 1) is that any strictly positive profile in D_0 is achievable in the no-discounting case.

The set D_0 of strongly minimal satiation profiles plays a role in the discounting case as well. Observe that its definition is nearly the same as that of the undercore. Since any underblocked satiation profile is not strongly minimal, $D_0 \subseteq D$ is always true. In most settings of interest, D is in fact the closure of D_0, so that the two sets are essentially the same. This is true in both the binary and, as the next lemma implies, neoclassical settings.[22]

Lemma 7. *In a neoclassical setting,* $c\ell \left\{x \in D_0 : X < Y_{N(x)}\right\} = D$.

The following proposition is the central result of this section. It establishes that if (PD) and (CY) hold in a neoclassical setting, then essentially any profile in D_0 is achievable for all $\delta < 1$ sufficiently large. Furthermore, the same equilibrium path achieves the profile for all large discount factors.

Proposition 4. *For any neoclassical setting satisfying* (PD) *and* $\vec{N}$ *satisfying* (CY), *suppose* $x \in D_0$ *satisfies* $X < Y_{N(x)}$. *Then there exists a path* $\vec{x}$ *converging to* x, *and a discount factor* $\underline{\delta} < 1$, *such that* $\vec{x}$ *is an equilibrium path of* $\Gamma(\delta, \vec{N})$ *for all* $\delta \in [\underline{\delta}, 1]$.

Before discussing the structure of the proof of this proposition, we first consider some its implications for the set of achievable allocations. Denote the set of achievable profiles in $\Gamma(\delta, \vec{N})$ as $A(\delta, \vec{N})$, and let

$$A(\vec{N}) := c\ell \left\{x \in \mathbb{R}^n_+ : \underline{\delta} < 1 \text{ exists such that } x \in A(\delta, \vec{N}) \text{ for all } \delta \in \left(\underline{\delta}, 1\right)\right\}.$$

That is, $A(\vec{N})$ is the closure of the set of profiles that can be achieved for all large discount factors less than one. The analogous set in the no-discounting case is $A_1(\vec{N}) := c\ell A(1, \vec{N})$. Both sets are always in the undercore, by Theorem 1. (The result of Gale (2001) also implies

[21] The definition in Gale (2001) of strong minimality differs slightly by requiring $z \neq 0$. It is useful here to allow the possibility that $z = 0$, so that any strongly minimal $x > 0$ is strictly individually rational: $u^*(0) \ll u(x)$.

[22] It is easy to show that in a binary setting, $D_0 = (x \in D : x_i < V_i \text{ for all } i \in N\}$, which implies $c\ell D_0 = D$.

$A_1(\vec{N}) \subseteq D$, by Lemma 7.) The reverse inclusions are true under the conditions of Proposition 4, so that the undercore is essentially the set of profiles that can be achieved for both $\delta = 1$ and for sufficiently large $\delta < 1$. This is established by the following theorem, which follows immediately from Theorem 1, Lemma 7, and Proposition 4.

Theorem 3. *In a neoclassical setting in which* (PD) *holds,* $A(\vec{N}) = A_1(\vec{N}) = D$ *for all move structures* $\vec{N}$ *that satisfy* (CY).

Theorem 3 and Proposition 4 have, under their assumptions, three notable economic consequences. The first bears on the nature of the efficient profiles that can be approximately achieved if $\delta = 1$ or as $\delta \to 1$. Since the core is the subset of profiles in D that are efficient, an efficient profile can be approximately achieved if and only if it is in the core.

The second consequence bears on the issue of gradualism. Under the assumptions of these results, almost any achievable profile is achieved by the same equilibrium path for all large discount factors. Now, the time it takes a fixed path to reach any given neighborhood of its limiting profile becomes negligible as the period length becomes small. Thus, essentially all achievable profiles can be achieved instantaneously in the limit as the period length goes to zero. Even though (PD) implies that strategic gradualism is necessary in the sense that no equilibrium path achieves a non-zero profile in a finite number of periods, there is no real-time gradualism if the period length is arbitrarily short.

The third consequence bears on the relevance of the move structure. The result that both $A_1(\vec{N})$ and $A(\vec{N})$ are equal to the set D, which does not depend on $\vec{N}$, tells us that the set of profiles that can be achieved for either $\delta = 1$ or as $\delta \to 1$ is independent of the move structure. Both the simultaneous and round-robin structures, for example, give rise to the same set of limiting achievable profiles. Of course, for a fixed $\delta < 1$ the set of achievable profiles does generally depend on the move structure.

We end this section with an overview of the proof of Proposition 4. In light of Lemma 6, it only needs to be proved for the round-robin structure. Consider a nonzero $x \in D_0$ satisfying $X < Y_{N(x)}$. The proof begins by finding two profiles, $\bar{x}$ and $\hat{x}$, that satisfy $\bar{x} < \hat{x} < x$ and $u(\bar{x}) \ll u(\hat{x}) \ll u(x)$. The proof that these profiles exist depends on the assumption $X < Y_{N(x)}$. Because v is strictly concave, $\hat{x}$ can be chosen so that it too is in D_0. The proof then has three steps.

Step 1 consists of the construction of a round-robin path starting at $\bar{x}$ and converging to x. Each player raises her contribution the same proportional amount towards x when it is her turn to move. The increases are made small enough that $u(x) - u(x^t)$ is always positive. But this difference shrinks to zero so quickly that for all large δ, player i's continuation payoff is close enough to $u_i(x)$ that she is willing to raise her contribution in the current period. This step uses $X < Y_{N(x)}$ and the concavity of υ.

Step 2 uses the fact that $\hat{x}$ is strongly minimal. Adapting an argument in Gale (2001), a finite, decreasing sequence from $\bar{x}$ to the origin is constructed, along which the players' payoffs never exceed $u(\hat{x})$. The first profile of the sequence is obtained by lowering the contribution of player 1 from $\bar{x}_1$ as much as possible without allowing her payoff to exceed $u_1(\hat{x})$. The second profile is then obtained by lowering the contribution of player 2 in the same manner. Continuing in round-robin fashion yields a decreasing sequence of profiles that each generate a payoff no greater than $u(\hat{x})$. The sequence converges to some $z < x$ for which $u_i(z) = u_i(\hat{x})$ for any i such that $z_i > 0$. This implies, since $\hat{x}$ is strongly minimal, that $z = 0$. The convergence occurs in a finite number of steps because, once the sequence is close enough to the origin, a player's contribution can be lowered all the way to zero without raising her payoff above $u_i(\hat{x})$.

Step 3 puts together the sequences obtained in the previous steps to yield a path $\vec{x}$ that converges to x and is feasible for $\vec{N}^R$. For $x^t \geq \bar{x}$, the construction of Step 1 insures that the remainder of the path is a continuation equilibrium path if δ is large. For $x^t < \bar{x}$, $u(x^t)$ is bounded strictly below $u(x)$, and so again the continuation payoffs from $\vec{x}$ exceed any deviation payoff if δ is large. The path $\vec{x}$ is thus an equilibrium path for large δ.

7. Equilibrium Payoffs

The results obtained so far about achievable profiles have implications for equilibrium payoffs. The following corollary is about the limits of sequences of equilibrium payoffs for discount factors less than one. For a move structure $\vec{N}$, this set of payoffs is

$$P(\vec{N}) := c\ell \left\{ \tilde{u} \in \mathbb{R}^n : \tilde{u} = U(\vec{x}, \delta) \text{ for some equilibrium path } \vec{x} \text{ and } \delta \in (0,1) \right\}.$$

Of natural interest are the *efficient payoffs* in this set, those for which $\tilde{u} = u(x)$ for some efficient profile x.

Corollary 2.

(i) *For any $\tilde{u} \in P(\vec{N})$, an undercore payoff $u' \in u(D)$ exists such that $\tilde{u} \leq u'$.*

(ii) *Any efficient $\tilde{u} \in P(\vec{N})$ is a core payoff: $\tilde{u} \in u(C)$.*

(iii) *In a neoclassical setting satisfying* (PD), *$P(\vec{N})$ contains all undercore (and hence core) payoffs for all $\vec{N}$ satisfying* (CY).

Part (i) shows that equilibrium payoffs are bounded above by undercore payoffs. This and the fact that the only efficient undercore profiles are core profiles implies (ii), that any efficient payoff that approximates an equilibrium payoff is a core payoff. Part (iii) establishes that all core payoffs are approximate equilibrium payoffs in a neoclassical setting satisfying (PD), if the move structure is cyclical.

Remark 3. *Similar results hold in the no-discounting case. By Theorem 1, any equilibrium payoff of* $\Gamma(1, \vec{N})$ *is in* $u(D)$. *Proposition 1 thus implies that any equilibrium payoff that is efficient is a core payoff. By Theorem 3,* $u(D)$ *is equal to the closure of the set of equilibrium payoffs in a neoclassical setting given* (PD) *and* (CY).

8. Conclusion

The goal of this paper has been to describe the achievable profiles and equilibrium payoffs of a range of dynamic contribution games. The central construct was the undercore, a set of profiles determined by the payoff functions independently of the dynamic structure of the game. The most general result obtained was that for any dynamic contribution game, only profiles in this set are achievable. This theorem has welfare implications: the only efficient payoffs that are even approximately achievable are the core payoffs. It also has theoretical implications: there is no folk theorem for this class of games. Lastly, it may have empirical implications: since the undercore is often readily characterized, whether only undercore profiles are achieved should be testable in the field or laboratory.

In the discounting case, generally not all undercore profiles are achievable. But in some settings they are, such as the neoclassical public good settings satisfying the prisoners' dilemma property. In these settings, if the move structure is cyclical, the entire undercore is the limit of the set of achievable profiles as the discount factor increases to one. One implication is that all commonly assumed move structures yield this same set of achievable profiles. Another implication is a lack of gradualism: almost any achievable profile can be approximately reached arbitrarily quickly as the period length shrinks to zero. One task for the future is to determine the extent to which these results hold for other payoff functions.

Appendices

A. Proofs Missing from Section 4

Lemma A1. *For each $i \in N$ and $x_{-i} \in \mathbb{R}_+^{n-1}$, program* (6)*, which is*

$$u_i^*(x) := \max_{x_i' \geq x_i} u_i(x_i', x_{-i}),$$

has a solution. Furthermore, $u_i^*(\cdot, x_{-i})$ is right continuous.[23]

Proof. Since $\hat{u}_i$ is continuous and f is nondecreasing right continuous, u_i is upper semicontinuous. Hence, for any positive integer s, $u_i(\cdot, x_{-i})$ has a maximizer on $[x_i, x_i + s]$. Thus, assuming (6) has no solution, an unbounded sequence $\{x^k = (x_i^k, x_{-i})\}$ exists such that $x_i^k \leq x_i^{k+1}$ and $u_i(x^k) \leq u_i(x^{k+1})$. By (PS), $u_j(x^k) \leq u_j(x^{k+1})$ for all $j \neq i$. This contradicts (BA). Hence, (6) has a solution.

Now let $\{x_i^k\}$ be a decreasing sequence such that $x_i^k \to x_i$. Since $u_j^*(\cdot, x_{-i})$ is nonincreasing, $u_i^*(x_i^k, x_{-i}) \leq u_i^*(x)$ for each k, and so

$$\limsup u_i^*(x_i^k, x_{-i}) \leq u_i^*(x).$$

Let $b_i(x)$ solve (6). The right continuity of f and the continuity of $\hat{u}_i$ imply that $u_i(\cdot, x_{-i})$ is right continuous. Hence, if $b_i(x) = x_i$ (and so $u_i^*(x) = u_i(x)$), we have

$$\begin{aligned} \liminf u_i^*(x_i^k, x_{-i}) &\geq \liminf u_i(x_i^k, x_{-i}) \\ &= \lim u_i(x_i^k, x_{-i}) = u_i(x) = u_i^*(x). \end{aligned}$$

The other case to consider is $b_i(x) > x_i$. In this case, for large k, we have $z_i^k < b_i(x)$, and so $u_i^*(x_i^k, x_{-i}) \geq u_i(b_i(x), x_{-i})$. Thus,

$$\liminf u_i^*(x_i^k, x_{-i}) \geq u_i(b_i(x), x_{-i}) = u_i^*(x).$$

We conclude that in either case, $\liminf u_i^*(x_i^k, x_{-i}) \geq u_i^*(x) \geq \limsup u_i^*(x_i^k, x_{-i})$, and so $u_i^*(x_i^k, x_{-i}) \to u_i^*(x)$. ■

Proof of Lemma 1. Let $\vec{x}$ be an equilibrium path, and assume it does not converge. Then, since it is nondecreasing, it is unbounded. By (BA), i exists such that $u_i(x^1) > \limsup_{s\to\infty} u_i(x^s)$. Thus, u_i has a positive, finite number of maximizers on the set $\{x^s\}_{s\geq 0}$. Let $x^{\tau-1}$ be the maximizer with the largest superscript. Then for $\delta < 1$ we have

$$u_i(x^{\tau-1}) - U_i^\tau(\vec{x}, \delta) = (1-\delta) \sum_{s\geq\tau} \delta^{s-\tau} \left[u_i(x^{\tau-1}) - u_i(x^s)\right] > 0,$$

[23]In fact u_i^* is continuous, but we only need its right continuity in x_i.

and for $\delta = 1$ we have

$$u_i(x^{\tau-1}) - U_i^\tau(\vec{x}, 1) \geq u_i(x^t) - \limsup_{s\to\infty} u_i(x^s) > 0.$$

Hence, since $x_{-i}^\tau \geq x_{-i}^{\tau-1}$, in either case (PS) implies $u_i(x_i^{\tau-1}, x_{-i}^\tau) \geq u_i(x^{\tau-1}) > U_i^\tau(\vec{x}, \delta)$. This contradicts the necessary condition (8). Therefore $\vec{x}$ must converge.

Now let x be a profile achieved asymptotically by an equilibrium path $\vec{x}$, so that $X^s < X$ for all s. Then, $f(X^s)$ converges to the left-hand limit $f(X-)$, and $u(x^s)$ converges to $\hat{u}(f(X-), x)$. For $\delta \in (0, 1]$ we have

$$\lim_{t\to\infty} U^t(\vec{x}, \delta) = \lim_{t\to\infty} u(x^t) = \hat{u}(f(X-), x).$$

Fix $i \in N$ and let $\hat{x}_i^t = X - X_{-i}^t$. If $i \in N_t$, then

$$\hat{u}_i(f(X), \hat{x}_i^t, x_{-i}^t) = u_i(\hat{x}_i^t, x_{-i}^t) \leq u_i^*(x_i^{t-1}, x_{-i}^t) \leq U_i^t(\vec{x}, \delta),$$

where the first inequality follows from $\hat{x}_i^t > x_i^{t-1}$, and the second from (7). Since $(\hat{x}_i^t, x_{-i}^t) \to x$, taking the limit along the infinite subsequence of dates t that satisfy $i \in N_t$ yields $\hat{u}_i(f(X), x) \leq \hat{u}_i(f(X-), x)$. This implies $f(X) \leq f(X-)$, as $\hat{u}_i(\cdot, x)$ is a strictly increasing function. Hence, since f is nondecreasing, $f(X-) = f(X)$. This proves f is continuous at X, and so x is not a threshold profile. ■

Proof of Lemma 2. If $\vec{x}$ converges to x in a finite number of periods, (10) holds trivially. If $\vec{x}$ instead converges asymptotically, then by Lemma 1 we know x is not a threshold profile. Since f is thus continuous at X, and $\hat{u}$ is continuous, u is continuous at x. This and the convergence of $\vec{x}$ to x imply (10).

Since $\vec{x}$ converges, $\{x^s\}_{s\geq 0} \cup \{x\}$ is compact. Because u_i is upper semicontinuous, it has a maximizer in this set. Assume x is not a maximizer. Then, since $u_i(x^s) \to u_i(x)$, u_i has a positive, finite number of maximizers in $\{x^s\}_{s\geq 0}$. This implies, by the argument given to prove Lemma 1, that τ exists such that the necessary condition (8) is violated at (i, τ). This contradiction proves that x is in fact a maximizer, i.e., $u_i(x^{t-1}) \leq u_i(x)$ for all $t > 0$. This and (1) or (2) now imply $U^t(\vec{x}, \delta) \leq u(x)$ for all $t > 0$.

It remains to prove that for any $t > 0$, $u_i^*(x_i^{t-1}, x_{-i}^t) \leq u_i(x)$. Fix $i \in N$. Let $\tau \geq t$ be the smallest date no less than t for which $i \in N_\tau$. From (7) and $U^t(\vec{x}, \delta) \leq u(x)$, we obtain $u_i^*(x_i^{\tau-1}, x_{-i}^\tau) \leq u_i(x)$. We have $u_i^*(x_i^{t-1}, x_{-i}^t) \leq u_i^*(x_i^{\tau-1}, x_{-i}^\tau)$ because $x_i^{t-1} = x_i^{\tau-1}$, $x_{-i}^t \leq x_{-i}^\tau$, and u^* satisfies (PS). Hence, $u_i^*(x_i^{t-1}, x_{-i}^t) \leq u_i(x)$. ■

Proof of Lemma 4. Suppose x is achievable. The argument in the text proves it is individually rational. To show that it is a satiation profile, let $\vec{x}$ be an equilibrium path that achieves x, and fix $i \in N$. Let b_i maximize $u_i(\cdot, x_{-i})$ on $[x_i, \infty)$, and choose $\varepsilon > 0$. Because u_i is right continuous and has at most a countable number of discontinuities, $b_i' \geq b_i$ exists such that u_i is continuous at (b_i', x_{-i}), and

$$u_i^*(x) = u_i(b_i, x_{-i}) \leq u_i(b_i', x_{-i}) + \tfrac{1}{2}\varepsilon.$$

As u_i is continuous at (b_i', x_{-i}), there exists T such that for all $t \geq T$,

$$u_i(b_i', x_{-i}) \leq u_i(b_i', x_{-i}^t) + \tfrac{1}{2}\varepsilon.$$

Since $b_i' \geq b_i \geq x_i \geq x_i^{t-1}$, the definition of u_i^* implies

$$u_i(b_i', x_{-i}^t) \leq u_i^*(x_i^{t-1}, x_{-i}^t).$$

Putting the displayed inequalities together yields $u_i^*(x) \leq u_i^*(x_i^{t-1}, x_{-i}^t) + \varepsilon$ for $t \geq T$. Hence, by Lemma 2, $u_i^*(x) \leq u_i(x) + \varepsilon$. As this is true for all $\varepsilon > 0$, and $u_i^*(x) \geq u_i(x)$, we conclude that $u_i^*(x) = u_i(x)$. ■

Proof of Theorem 2. To simplify notation, let $u_{ij}(x) := \partial u_i(x)/\partial x_j$. Let x^* be an efficient profile. Then it maximizes $u_1(x)$ subject to $x \in \mathbb{R}_+^n$ and $u_j(x) \geq u_j(x^*)$ for $j > 1$. By the Fritz John Theorem, multipliers $(\lambda, \alpha) \in \mathbb{R}_+^{2n} \backslash \{0\}$ exist such that $\sum_i \lambda_i u_{ij}(x^*) + \alpha_j = 0$ for all $j \in N$. If some $\lambda_j = 0$, then $\sum_{i \neq j} \lambda_i u_{ij}(x^*) + \alpha_j = 0$, which implies $(\lambda, \alpha) = 0$ since $u_{ij} > 0$ for all $i \neq j$. This contradiction proves $\lambda \gg 0$. Without loss of generality we can assume each $\lambda_i = 1$ (normalize by multiplying each u_i by λ_i). Hence, defining $W(x) := \sum_i u_i(x)$, we have $W_j(x^*) = -\alpha_j \leq 0$ for all $j \in N$ (where $W_j = \partial W/\partial x_j$). We also have $u_{jj}(x^*) = -\sum_{i \neq j} u_{ij}(x^*) - \alpha_j < 0$.

Since it is efficient, $x^* \neq 0$. Assume it is achieved by an equilibrium path $\vec{x}$. Suppose $t > 1$ exists such that $x^{t-1} < x^t = x^*$. Then, in period t some player i is raising her contribution from x_i^{t-1} to $x_i^* > x_i^{t-1}$ to obtain a continuation payoff of $U_i^t(\vec{x}, \delta) = u_i(x^*)$. But since $u_{ii}(x^*) < 0$, there exists $x_i \in \left(x_i^{t-1}, x_i^*\right)$ such that she could obtain a continuation payoff of at least $u_i(x_i, x_{-i}^*) > u_i(x^*)$ by raising her contribution only to x_i in period t and subsequently playing passively. This is contrary to $\vec{x}$ being an equilibrium path. We conclude that $\vec{x}$ converges asymptotically to x^*.

Now, for any $i \in N$ and $t \geq 1$, from (8) we obtain the inequality

$$\sum_{s \geq t} \delta^{s-t} \left[u_i(x^s) - u_i(x_i^{t-1}, x_{-i}^t)\right] \geq 0.$$

Thus, by the Mean Value Theorem, for each $s \geq t$ the line segment between (x_i^{t-1}, x_{-i}^t) and x^s contains a point y^{is} such that

$$\sum_{s \geq t} \delta^{s-t} \left[u_{ii}(y^{is})(x_i^s - x_i^{t-1}) + \sum_{j \neq i} u_{ij}(y^{is})(x_j^s - x_j^t) \right] \geq 0.$$

Sum these inequalities over $i \in N$ to obtain

$$\sum_{s \geq t} \delta^{s-t} \left[\sum_i u_{ii}(y^{is})(x_i^s - x_i^{t-1}) + \sum_i \sum_{j \neq i} u_{ij}(y^{is})(x_j^s - x_j^t) \right] \geq 0.$$

Add and subtract $\sum_i u_{ii}(y^{is})(x_i^s - x_i^t)$ to the square-bracketed term to obtain

$$\sum_{s \geq t} \delta^{s-t} \left[\sum_i u_{ii}(y^{is})(x_i^t - x_i^{t-1}) + \sum_i \sum_j u_{ij}(y^{is})(x_j^s - x_j^t) \right] \geq 0.$$

Reverse the summation order in the double sum to obtain

$$\sum_{s \geq t} \delta^{s-t} \left[\sum_i u_{ii}(y^{is})(x_i^t - x_i^{t-1}) + \sum_j \left\{ (x_j^s - x_j^t) \sum_i u_{ij}(y^{is}) \right\} \right] \geq 0. \tag{17}$$

Now, let $\varepsilon > 0$, and let B be an open ball centered at x^* such that for any $x \in B$ and $i, j \in N$,

$$-\varepsilon > u_{ii}(x) \text{ and } u_{ij}(x^*) + \tfrac{1}{n}(1-\delta)\delta^{-1}\varepsilon > u_{ij}(x).$$

(Such a ball exists because each $u_{ii}(x^*) < 0$ and each u_{ij} is continuous.) Since $x^t \to x^*$, date $T \geq 1$ exists such that $x^s \in B$ for all $s \geq T$. Choose $t > T$ such that $x^{t-1} < x^t$ (which can be done because the convergence is asymptotic). Since each y^{is} is between (x_i^{t-1}, x_{-i}^t) and x^s, and both these points are in B, $y^{is} \in B$. Accordingly, for all $i, j \in N$ and $s \geq t$, we have

$$-\varepsilon > u_{ii}(y^{is}) \tag{18}$$

and, since $W_j(x^*) \leq 0$,

$$\begin{aligned} (1-\delta)\,\delta^{-1}\varepsilon &\geq W_j(x^*) + (1-\delta)\,\delta^{-1}\varepsilon \\ &= \sum_i \left[u_{ij}(x^*) + \tfrac{1}{n}(1-\delta)\delta^{-1}\varepsilon \right] \\ &> \sum_i u_{ij}(y^{is}). \end{aligned} \tag{19}$$

Now, because $x_i^t - x_i^{t-1} \geq 0$ and $x_j^s - x_j^t \geq 0$ for $s \geq t$, (17)–(19) imply

$$\sum_{s \geq t} \delta^{s-t} \left[\sum_i (-\varepsilon)\,(x_i^t - x_i^{t-1}) + \sum_j \left\{ (x_j^s - x_j^t)\,(1-\delta)\,\delta^{-1}\varepsilon \right\} \right] \geq 0.$$

Simplify this, using $\sum_i x_i^s = X^s$, to obtain

$$(1-\delta)^2\,\delta^{-1}\sum_{s\geq t}\delta^{s-t}(X^s - X^t) \geq X^t - X^{t-1}.$$

Use the identity $(1-\delta)\sum_{s\geq t}\delta^{s-t}(X^s - X^t) = \delta\sum_{s\geq t}\delta^{s-t}(X^{s+1} - X^s)$ to obtain

$$(1-\delta)\sum_{s\geq t}\delta^{s-t}(X^{s+1} - X^s) \geq X^t - X^{t-1}.$$

Since the left side of this inequality is a convex combination of terms, one of them must weakly exceed the right side. That is, $t_1 > t$ exists such that $X^{t_1} - X^{t_1-1} \geq X^t - X^{t-1}$. Since $t_1 > T$, we can repeat the argument to find $t_2 > t_1$ such that $X^{t_2} - X^{t_2-1} \geq X^{t_1} - X^{t_1-1}$. Proceeding recursively yields a subsequence $\{t_k\}$ such that the differences $X^{t_k} - X^{t_k-1}$ are positive (since $X^t - X^{t-1} > 0$) and nondecreasing. This is impossible, since $x^s \to x^*$. The profile x^* is therefore not achievable. ■

B. Proofs Missing from Section 5

Lemma B1. *D is closed and bounded.*

Proof. Assume D contains an unbounded sequence $\{x^k\}_{k=2}^{\infty}$. Let $x^1 = 0$. Then (BA) implies that $i \in N$ exists such that $u_i(0) > \lim_{k\to\infty}\sup u_i(x^k)$. Thus, $u_i(0) > u_i(x^k)$ for large k. This implies x^k is not individually rational, i.e., it is underblocked by a singleton coalition. This contradicts $x^k \in D$. Hence, D is bounded.

To show that D is closed, let $\{x^k\}$ be a convergent sequence in D, with limit x. We first show that x is a satiation profile. Assume not. Then $i \in N$ and $b_i > x_i$ exist such that $u_i(b_i, x_{-i}) > u_i(x)$. Let $\hat{X} = b_i + X_{-i}$ and $b_i^k = \hat{X} - X_{-i}^k$. Then $b_i^k \to b_i > x_i$. Thus, for large k we have $b_i^k > x_i^k$. It follows that for large k,

$$u_i^*(x^k) \geq u_i(b_i^k, x_{-i}^k) = \hat{u}_i(f(\hat{X}), b_i^k, x_{-i}^k).$$

Hence, as $\hat{u}_i$ is continuous, we have

$$\begin{aligned}
\limsup u_i^*(x^k) &\geq \lim \hat{u}_i(f(\hat{X}), b_i^k, x_{-i}^k) \\
&= \hat{u}_i(f(\hat{X}), b_i, x_{-i}) = u_i(b_i, x_{-i}) \\
&> u_i(x) \geq \limsup u_i(x^k),
\end{aligned}$$

where the last inequality holds because u_i is upper semicontinuous. This implies that $u_i^*(x^k) > u_i(x^k)$ for large k. This is impossible, since $x^k \in D$ implies that x is a satiation profile. Thus, x is a satiation profile.

It remains to show that x is not underblocked, as this will now imply $x \in D$. Assume x is underblocked. Then a coalition S and a profile $z \leq x$ exist such that $z_{-i} = 0$ and $u^*_S(z) \gg u_S(x)$. Suppose $z_i = x_i$ for some $i \in S$. Then, since $z_{-i} \leq x_{-i}$ and x is a satiation profile,

$$u^*_i(z) = u^*_i(x_i, z_{-i}) \leq u^*_i(x_i, x_{-i}) = u_i(x).$$

This contradiction implies $z_S \ll x_S$. For any $i \in N$ we have, since u_i is upper semicontinuous,

$$u_i(x) \geq \limsup_{k\to\infty} u_i(x^k).$$

Hence, for large k we have $z_S \ll x^k_S$, $z_{-S} = 0$, and $u^*_S(z) \gg u_S(x^k)$. This implies $x^k \notin D$ for large k, contrary to the assumption $\{x^k\} \subset D$. So x is not underblocked. ■

Proof of Proposition 1. Let $x \notin D$. Then a coalition S and profile z exist such that $u^*_S(z) > u_S(x)$ and $z_{-S} = 0$. For $i \in S$, let $\tilde{z}_i \geq z_i$ solve (6), and set $\tilde{z} = (\tilde{z}_S, 0_{-S})$. Then $u_i(\tilde{z}) \geq u_i(\tilde{z}_i, z_{-i}) = u^*_i(z)$ for $i \in S$, where the inequality is implied by (PS). Hence, $u_S(\tilde{z}) > u_S(x)$. This proves $x \notin C$. We conclude that $C \subset D$. Since core profiles are efficient, we have $C \subseteq \{x \in D : x \text{ is efficient}\}$.

To prove the reverse, let $x \in D$ be efficient, and assume $x \notin C$. Then (S, z) exists such that $z_{-S} = 0$ and $u_S(z) > u_S(x)$. Since $x \in D$, S does not underblock x using z, and hence

$$T := \{i \in S : z_i > x_i\} \neq \varnothing.$$

Let $\hat{x} := (z_T, x_{-T})$. By (PS), $u_{-T}(\hat{x}) \geq u_{-T}(x)$. Hence, since x is efficient, $u_T(\hat{x}) \not> u_T(x)$. If $u_T(\hat{x}) = u_T(x)$, then $\hat{x}$ would also be efficient and $u(\hat{x}) = u(x)$. This is not possible because of the assumption that distinct efficient profiles generate distinct payoffs. We conclude that for some $j \in T$, $u_j(\hat{x}) < u_j(x)$. Since $z_{-T} \leq x_{-T}$, (PS) implies $u_j(z) \leq u_j(\hat{x})$. Thus, $u_j(z) < u_j(x)$. This contradicts $u_S(z) > u_S(x)$. ■

Proof of Proposition 2. If S blocks x using z, then summing $u_i(z)$ and $u_i(x)$ over $i \in S$ yields

$$\sum_{i\in S} v_i(Z) - Z > \sum_{i\in S} v_i(X) - X_S,$$

since $Z = Z_S$. As the left side of this inequality is no greater than $V(S)$, this proves that if (13) holds for all coalitions S, then $x \in C$.

To prove the converse, suppose $x \in C$, but (13) does not hold for some coalition S. Then,

$$\Delta := \frac{V(S) - \left[\sum_{i\in S} v_i(X) - X_S\right]}{|S|} > 0.$$

Define $z \in \mathbb{R}^n$ by $z_{-S} = 0$, and $z_i := x_i - \Delta - v_i(X) + v_i(Y_S)$ for $i \in S$. Then, summing z_i over S yields $Z = Y_S$. This implies that $\hat{S} := \{i \in S : z_i \geq 0\}$ is nonempty. Define $\hat{z} \in \mathbb{R}^n_+$ by $\hat{z}_i := \max(0, z_i)$. Then $\hat{z} \in \mathbb{R}^n_+$ and $\hat{z}_{-\hat{S}} = 0$. Because $\hat{Z} \geq Z = Y_S$, and $\hat{z}_i = z_i$ for $i \in \hat{S}$, we have

$$v_i(\hat{Z}) - \hat{z}_i \geq v_i(Y_S) - z_i = v_i(X) - x_i + \Delta > v_i(X) - x_i$$

for all $i \in \hat{S}$. Hence, $\hat{S}$ blocks x using $\hat{z}$. This contradiction of $x \in C$ shows that if $x \in C$, then (13) holds for all coalitions S. ■

Lemma B2. *In a neoclassical setting, for any satiation profile x and coalition S that underblocks x, a profile $z < x$ exists such that $u^*_S(z) = u_S(z)$, and S underblocks x using z.*

Proof. As S underblocks x, there exists $\hat{z} \leq x$ such that $\hat{z}_{-S} = 0$ and $u^*_S(\hat{z}) \gg u_S(x)$. Let $i \in S$ be a player such that $Y_{\{i\}} \geq Y_{\{j\}}$ for all $j \in S$. We can assume $\hat{Z} < Y_{\{i\}}$, as otherwise $u^*_S(\hat{z}) = u_S(\hat{z})$, and the result holds with $z = \hat{z}$. Define z by $z_{-i} = \hat{z}_{-i}$ and

$$z_i = \arg\max_{z'_i \geq \hat{z}_i} v_i(z'_i + Z_{-i}) - z'_i = Y_{\{i\}} - Z_{-i},$$

where the second equality holds because v_i is strictly concave, $v'_i(Y_{\{i\}}) = 1$, and $\hat{Z} < Y_{\{i\}}$. Hence, $Z = Y_{\{i\}} \geq Y_{\{j\}}$ for all $j \in S$. This implies $u^*_S(z) = u_S(z)$.

Now, note that

$$u^*_i(z) = u_i(z_i, \hat{z}_{-i}) = u^*_i(\hat{z}) > u_i(x), \text{ and}$$
$$u^*_j(z) \geq u^*_j(\hat{z}) > u_j(x) \text{ for all } j \in S \setminus \{i\},$$

where the weak inequality holds because u^* satisfies (PS). Hence, $u^*_S(z) \gg u_S(x)$. It follows, once we show that $z < x$, that S underblocks x using z. To prove $z < x$, observe first that $z_{-i} = \hat{z}_{-i} \leq x_{-i}$. For i we have $u_i(z) > u_i(x)$; hence, since $Z = Y_{\{i\}}$,

$$z_i - x_i < v_i(Y_{\{i\}}) - v_i(X) \leq 0,$$

since $Y_{\{i\}} \leq X$ (as x is a satiation profile). Thus, $z < x$. ■

Lemma B3. *In a neoclassical setting, a satiation profile x is underblocked if and only if for some coalition S,*

$$X_S > \max\left(Y_S, \sum_{i \in S} v_i(X) - V(S)\right). \tag{20}$$

Proof. Suppose x is underblocked by S. By Lemma B2, $z < x$ exists such that $z_{-S} = 0$ and $u_S(x) \ll u_S(z)$. Summing these inequalities over S and using $Z_S = Z$ yields

$$\sum_{i \in S} v_i(X) - X_S < f_S(Z). \tag{21}$$

This and $Z \leq X$ imply $Z < X_S$. As $f_S(Z) \leq V(S)$, (21) also implies $X_S > \sum_{i \in S} v_i(X) - V(S)$, which is half of (20). To show the other half, $X_S > Y_S$, assume the opposite. Hence, $Z < X_S \leq Y_S$. Since f_S is strictly increasing on $[0, Y_S]$, this implies $f_S(Z) < f_S(X_S)$.But from $X_S \leq X$ and (21) we obtain $f_S(X_S) < f_S(Z)$. This contradiction proves $X_S > Y_S$.

To prove the converse, suppose (20) holds for coalition S. From this we obtain $v_S(X) \geq v_S(X_S) \gg v_S(Y_S)$ and

$$\Delta := \frac{V(S) - \left[\sum_{i \in S} v_i(X) - X_S\right]}{|S|} > 0.$$

Define $z \in \mathbb{R}^n$ by $z_{-S} = 0$, and $z_i := x_i - \Delta - v_i(X) + v_i(Y_S)$ for $i \in S$. Then $z_S \ll x_S$. Summing z_i over S yields $Z = Y_S$. As $Y_S \geq 0$, this implies that $\hat{S} := \{i \in S : z_i \geq 0\}$ is nonempty. Define $\hat{z} \in \mathbb{R}^n_+$ by $\hat{z}_i := \max(0, z_i)$. Then $\hat{z} \in \mathbb{R}^n_+$, $\hat{z}_{-\hat{S}} = 0$, and $\hat{z} < x$. For $i \in \hat{S}$ we have

$$v_i(\hat{Z}) - \hat{z}_i \geq v_i(Y_S) - z_i = v_i(X) - x_i + \Delta > v_i(X) - x_i$$

where the first inequality follows from $\hat{Z} \geq Z = Y_S$ and $\hat{z}_i = z_i$. This proves that $\hat{S}$ underblocks x using $\hat{z}$. ■

Proof of Proposition 3. If $x \in D$, then x is a satiation profile. As x is not underblocked, Lemma B3 implies that (20) is not satisfied for any S, i.e., (15) is satisfied for every S. Conversely, if x is a satiation profile satisfying (15) for all S, then (20) is not satisfied for any S. Hence, Lemma B3 implies x is not underblocked, and so $x \in D$.

If (x, S) satisfies (14), it obviously satisfies (15), since $X_S \leq X$. Suppose then that (x, S) satisfies (15). If $Y_S > \sum_{i \in S} v_i(X) - V(S)$, then $\sum_{i \in S} v_i(Y_S) > \sum_{i \in S} v_i(X)$, which implies $Y_S > X$ and so (14). If instead $Y_S \leq \sum_{i \in S} v_i(X) - V(S)$, then (15) implies $X_S \leq \sum_{i \in S} v_i(X) - V(S)$, and so (14). This proves that (14) and (15) are equivalent. ■

Proof of Corollary 1. (i) Let $x \in D$. The convention $Y_\varnothing = \infty$ implies the result trivially if $x = 0$. So suppose $x \neq 0$, and let $S = N(x)$. Since $x \in D$, (14) holds for (x, S). Hence, since $X = X_S$,

$$X < Y_S \text{ or } \sum_{i \in S} v_i(X) - X \geq V(S).$$

This implies, if $X > Y_S$, that $f_S(X) \geq V(S) = f_S(Y_S)$, contrary to Y_S being the unique maximizer of f_S. We thus have $X \leq Y_S$.

(ii) Let $x = \left(v_1'(Y_N)Y, \ldots, v_n'(Y_N)Y\right)$ for some $Y \in [\bar{Y}, Y_N]$. Since $\sum_{i \in N} v_i'(Y_N) = 1$, we have $X = Y$. Since $Y \geq \bar{Y}$, x is a satiation profile. To show that $x \in D$, we let S be a coalition and verify that (15) holds. To do this, we can assume $X_S > Y_S$, and show from this that $X_S \leq \sum_{i \in S} v_i(X) - V(S)$. As each v_i is concave, we have

$$\sum_{i \in S} [v_i(X) - v_i(Y_S)] \geq (X - Y_S) \sum_{i \in S} v_i'(X).$$

Since each v_i is concave and $Y_S < X \leq Y_N$, we have

$$(X - Y_S) \sum_{i \in S} v_i'(X) \geq (X - Y_S) \sum_{i \in S} v_i'(Y_N).$$

From $X_S = X \sum_{i \in S} v_i'(Y_N)$ and $\sum_{i \in S} v_i'(Y_N) \leq 1$, we obtain

$$(X - Y_S) \sum_{i \in S} v_i'(Y_N) \geq X_S - Y_S.$$

These three displayed inequalities together yield

$$\sum_{i \in S} [v_i(X) - v_i(Y_S)] \geq X_S - Y_S,$$

which rearranges to the desired $X_S \leq \sum_{i \in S} v_i(X) - V(S)$. ■

C. Proofs Missing from Section 6

Proof of Lemma 6. Define $\vec{z}$ by letting the players move as in $\vec{x}$, but only at dates that are multiples of m. That is, let $z^t = 0$ for $t = 0, \ldots, m-1$, and for $t \geq m$ let $z^t = x^{nk+i}$, where k and i are the unique integers satisfying $k \geq 0$, $i \in N$, and

$$(nk + i)m \leq t < (nk + i + 1)m.$$

In $\vec{z}$ player i moves only at dates $(nk + i)m$, since in $\vec{x}$ she moves only at dates $nk + i$. The path $\vec{z}$ is feasible for $\vec{N}$ by (CY), since $i \in N_{(nk+i)m}$. Since $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N}^R)$, it achieves some profile x by Lemma 1). Thus, $\vec{z}$ also converges to x. We use Lemma 5 to prove that $\vec{z}$ is an equilibrium path of $\Gamma(\delta^{1/m}, \vec{N})$.

Consider first the case $\delta = 1$. Fix $i \in N$ and $t \geq 1$. Since $\vec{x}$ is an equilibrium path of $\Gamma(1, \vec{N}^R)$, from (9) we have $U_i^t(\vec{x}, 1) = u_i(x)$. The derivation of $\vec{z}$ therefore implies $U_i^t(\vec{z}, 1) = u_i(x)$. The construction of $\vec{z}$ implies that τ exists such that $(z_i^{t-1}, z_{-i}^t) = x^\tau$ or $(z_i^{t-1}, z_{-i}^t) = (x_i^{\tau-1}, x_{-i}^\tau)$. Since $\vec{x}$ is an equilibrium path, Lemma 2 implies that $u_i(x^\tau) \leq u_i(x)$ and $u_i(x_i^{\tau-1}, x_{-i}^\tau) \leq u_i(x)$. We thus have $u_i(z_i^{t-1}, z_{-i}^t) \leq u_i(x) = U_i^t(\vec{z}, 1)$. Now Lemma 5 implies $\vec{z}$ is an equilibrium path of $\Gamma(1, \vec{N})$.

We now turn to the case $\delta < 1$, and let $\hat{\delta} = \delta^{1/m}$. Fix $t \geq 1$ and $i \in N_t$. By Lemma 5, $\vec{z}$ is an equilibrium path of $\Gamma(\hat{\delta}, \vec{N})$ if

$$u_i(z_i^{t-1}, z_{-i}^t) \leq (1 - \hat{\delta}) \sum_{s \geq t} \hat{\delta}^{s-t} u_i(z^s), \tag{22}$$

which we now show. If $z_i^s = z_i^{t-1}$ for all $s \geq t$, then (PS) implies (22). So suppose a date $\tau \geq t$ exists such that $z_i^{t-1} = z_i^{\tau-1} < z_i^{\tau}$. This date is a multiple of m, say $\tau = pm$. Furthermore, $z^{\tau} = x^p$ and $z^{\tau-1} = z^{t-1} = x^{p-1}$. Observe that

$$\begin{aligned}(1-\hat{\delta})\sum_{s\geq t}\hat{\delta}^{s-t}u_i(z^s) &= (1-\hat{\delta})\sum_{s=t}^{\tau-1}\hat{\delta}^{s-t}u_i(z_i^{t-1}, z_{-i}^s) + \hat{\delta}^{\tau-t}(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s) \\ &\geq (1-\hat{\delta}^{\tau-t})u_i(z_i^{t-1}, z_{-i}^t) + \hat{\delta}^{\tau-t}(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s),\end{aligned}$$

since $u_i(z_i^{t-1}, z_{-i}^s) \geq u_i(z_i^{t-1}, z_{-i}^t)$ for each $s \geq t$ by (PS). Hence, (22) holds if

$$(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s) \geq u_i(z_i^{t-1}, z_{-i}^t), \tag{23}$$

which we now show. The definitions of $\vec{z}$ and $\hat{\delta}$ imply

$$\begin{aligned}(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s) &= (1-\hat{\delta})\sum_{k=0}^{\infty}\sum_{s=\tau+km}^{\tau+(k+1)m-1}\hat{\delta}^{s-\tau}u_i(z^s) \\ &= (1-\hat{\delta})\sum_{k=0}^{\infty}\hat{\delta}^{km}u_i(x^{p+k})\sum_{s=\tau+km}^{\tau+(k+1)m-1}\hat{\delta}^{s-\tau-km} \\ &= (1-\hat{\delta}^m)\sum_{k=0}^{\infty}\hat{\delta}^{km}u_i(x^{p+k}) \\ &= (1-\delta)\sum_{k=0}^{\infty}\delta^k u_i(x^{p+k}).\end{aligned}$$

Because $(\vec{x}, \delta)$ satisfies (8) at date p, we have

$$\begin{aligned}(1-\delta)\sum_{k=0}^{\infty}\delta^k u_i(x^{p+k}) &= (1-\delta)\sum_{s\geq p}\delta^{s-p}u_i(x^s) \\ &\geq u_i(x_i^{p-1}, x_{-i}^p) \\ &= u_i(z_i^{t-1}, z_{-i}^{\tau}).\end{aligned}$$

Putting the two previous displays together yields

$$(1-\hat{\delta})\sum_{s\geq\tau}\hat{\delta}^{s-\tau}u_i(z^s) \geq u_i(z_i^{t-1}, z_{-i}^{\tau}) \geq u_i(z_i^{t-1}, z_{-i}^t),$$

where the second inequality follows from (PS) and $z_{-i}^{\tau} \geq z_{-i}^t$. This proves (23). ■

Proof of Lemma 7. Let $x^* \in D$. Then $X^* \in [\bar{Y}, Y_N]$, by Corollary 1 (i) and the fact that x^* is a satiation profile. Our definition of a neoclassical setting implies $\bar{Y} < Y_N$. Choose a number $\hat{X}$ as follows:

(a) If $X^* = Y_N$, choose $\hat{X} \in \left(\bar{Y}, Y_N\right)$ so that for all coalitions $S \neq N$, $\hat{X} > Y_S$.

(b) If $X^* < Y_N$, choose $\hat{X} \in (X^*, Y_N)$ so that for all coalitions S, if $\hat{X} > Y_S$ then $X^* \geq Y_S$.

Define $\hat{x}$ by $\hat{x}_i := v_i'(Y_N)\hat{X}$. (Since $\sum v_i(Y_N) = 1$, the aggregate of $\hat{x}$ is indeed $\hat{X}$) Because $\hat{X} \in \left(\bar{Y}, Y_N\right)$, Corollary 1 (ii) implies $\hat{x} \in D$. Let $\lambda \in (0,1)$ and $x = \lambda x^* + (1-\lambda)\hat{x}$. Then $X = \lambda X^* + (1-\lambda)\hat{X} \in (\bar{Y}, Y_N)$. Thus, x is a satiation profile, and its aggregate satisfies $\hat{X} < Y_N = Y_{N(x)}$. We now show x is strongly minimal. Since $x \to x^*$ as $\lambda \to 1$, this will prove $x^* \in c\ell \left\{x \in D_0 : X < Y_{N(x)}\right\}$.

Assume x is not strongly minimal. Then a coalition S and profile $z < x$ exist such that $z_{-S} = 0$ and $u_S(x) \leq u_S^*(z)$. Since x is a satiation profile, the argument used to prove Lemma B2 shows that we can find such a z such that $u_S^*(z) = u_S(z)$. Using this z, we have $u_S(x) \leq u_S(z)$. Summing these inequalities over S yields

$$\sum_{i \in S} v_i(X) - X_S \leq f_S(Z), \tag{24}$$

and this implies

$$\sum_{i \in S} v_i(X) - X_S \leq V(S). \tag{25}$$

Because $X_S \leq X$, (24) also implies $f_S(X) \leq f_S(Z)$. This, since $Z < X$ and f_S is strictly increasing on $[0, Y_S]$, implies

$$X > Y_S. \tag{26}$$

This proves $S \neq N$, since $X < Y_N$. The remainder of the proof depends on the case.

Case (a). In this case $X^* = Y_N > Y_S$. Furthermore, since $S \neq N$, the way $\hat{X}$ was chosen implies $\hat{X} > Y_S$. Hence, because $x^* \in D$ and $\hat{x} \in D$, the first part of Proposition 3 implies

$$\sum_{i \in S} v_i(X^*) - X_S^* \geq V(S) \quad \text{and} \quad \sum_{i \in S} v_i(\hat{X}) - \hat{X}_S \geq V(S). \tag{27}$$

Now, since each v_i is strictly concave, $\lambda \in (0,1)$, and $\hat{X} \neq X^*$, we have

$$\begin{aligned} \sum_{i \in S} v_i\,(X) - X_S &= \sum_{i \in S} v_i \left[(1-\lambda)\hat{X} + \lambda X^*\right] - \left[(1-\lambda)\hat{X}_S + \lambda X_S^*\right] \\ &> (1-\lambda)\left[\sum_{i \in S} v_i(\hat{X}) - \hat{X}_S\right] + \lambda \left[\sum_{i \in S} v_i(X^*) - X_S^*\right]. \end{aligned}$$

This and (27) imply $\sum_{i \in S} v_i\,(X) - X_S > V(S)$, contrary to (25). So x is strongly minimal.

Case (b). In this case $\hat{X} > X$, and so (26) implies $\hat{X} > Y_S$. This and the way $\hat{X}$ was chosen imply $X^* \geq Y_S$. The fact that $\hat{X} > Y_S$ and $\hat{x} \in D$ again imply the second inequality in (27). The first inequality in (27) also holds, for the same reason if $X^* > Y_S$, and if $X^* = Y_S$ then because

$$\begin{aligned} \sum_{i \in S} v_i(X^*) - X_S^* &= \sum_{i \in S} v_i(Y_S) - Y_S + X^* - X_S^* \\ &= V(S) + X^* - X_S^* \geq V(S). \end{aligned}$$

So (27) again holds, and the remaining proof is the same as in case (a). ■

The following lemma will be used to prove Proposition 4.

Lemma C1. *In a neoclassical setting satisfying* (PD), *for any $x \in D_0$, a neighborhood of x exists such that every $\hat{x}$ in it that satisfies $\hat{x} < x$ is also in D_0.*

Proof. Assume the lemma is false. Then an infinite sequence $\{x^k\}$ exists such that $x^k \to x$, $x^k < x$, and $x^k \notin D_0$. Since (PD) implies each x^k is a satiation profile, each x^k must not be strongly minimal. Thus, for each k a coalition S^k and a profile $z^k < x^k$ exist such that $z^k_{-S^k} = 0$ and $u^*_{S^k}(z^k) \geq u_S(x^k)$. By taking a subsequence we may assume $N(x^k) = N(x)$ and $S^k = S$ for all k, and that $\{z^k\}$ converges to a profile z (as each z^k is in the compact set $[0, x]^n$). Taking $k \to \infty$ in the inequalities $z^k < x$ and $u^*_S(z^k) \geq u_S(x^k)$ yields $z \leq x$ and $u^*_S(z) \geq u_S(x)$. Since $z^k_{-S} = 0$ for all k, $z_{-S} = 0$. Therefore, since x is strongly minimal, it must not be true that $z < x$. Hence, $z = x$. This implies $N(x) \subseteq S$. Since $N(x^k) = N(x)$, we have $X^k_S = X^k$. Because $u^*_{S^k}(z^k) \geq u_S(x^k)$, (PD) implies $u_S(z^k) \geq u_S(x^k)$. Summing these inequalities over S yields $f_S(Z^k) \geq f_S(X^k)$. Thus, since f_S is strictly increasing on $[0, Y_S]$ and $X^k < X \leq Y_{N(x)} \leq Y_S$, we conclude that $Z^k \geq X^k$. This contradicts $z^k < x^k$. ■

Proof of Proposition 4. Observe first that once we find $\delta \in (0, 1)$ and an equilibrium path of $\Gamma(\delta, \vec{N})$ that converges to x, then $\vec{x}$ is also an equilibrium path of $\Gamma(1, \vec{N})$. For, by Lemma 2, the pair $(\vec{x}, \delta)$ must satisfy (11). Hence, using (PD) and (9), we see that for all $t \geq 1$ and $i \in N$,

$$u_i(x_i^{t-1}, x_{-i}^t) = u_i^*(x_i^{t-1}, x_{-i}^t) \leq u_i(x) = U^t(\vec{x}, 1).$$

Thus, $\vec{x}$ is an equilibrium path of $\Gamma(1, \vec{N})$ by Lemma 5.

Accordingly, we only need to find a path $\vec{x}$ that converges to x and a number $\underline{\delta} < 1$ such that $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N})$ for all $\delta \in [\underline{\delta}, 1)$. By Lemma 6, it suffices to prove this for $\vec{N} = \vec{N}^R$. If $x = 0$ we are done, since (PD) implies that the passive strategy profile is an equilibrium that achieves the origin. So we can assume $x > 0$. Define $d \in \mathbb{R}^n_+$ by $d_i := 0$ if $i \notin N(x)$, and

$$d_i := \frac{v_i'(X)}{\sum_{j \in N(x)} v_j'(X)} \text{ for } i \in N(x).$$

Since $X < Y_{N(x)}$, we have $\sum_{j \in N(x)} v_j'(X) > 1$. Hence, $0 < d_i < v_i'(X)$ for $i \in N(x)$. Choose $\bar{\theta} > 0$ small enough that $\bar{x} := x - \bar{\theta} d \geq 0$. Since x is strongly minimal, Lemma C1 implies the existence of $\hat{\theta} \in (0, \bar{\theta})$ such that $\hat{x} := x - \hat{\theta} d$ is strongly minimal. We have $0 \leq \bar{x} < \hat{x} < x$. We also have $u(\bar{x}) \ll u(\hat{x}) \ll u(x)$, since the concavity of each v_i implies that for any $\theta \geq 0$, $\partial u_i(x - \theta d)/\partial \theta = d_i - v_i'(X - \theta) \leq d_i - v_i'(X) < 0$.

Define $\{x^t\}_{k=0}^{\infty}$ to be a *round-robin sequence* if for each $t > 0$ and $i = t \pmod n$, $x_{-i}^t = x_{-i}^{t-1}$. The rest of the proof consists of three steps.

Step 1. *A discount factor $\delta' < 1$ and a nondecreasing round-robin sequence $\{x^t\}_{t=0}^{\infty}$ exist such that $x^0 = \bar{x}$, $x^t \to x$, and*

$$u_i(x_i^{t-1}, x_{-i}^t) \le (1-\delta) \sum_{s \ge t} \delta^{s-t} u_i(x^s) \tag{28}$$

for all $\delta \in (\delta', 1)$, $t > 0$, and $i = t \pmod n$.

Proof of Step 1. Since $d_i < v_i'(X)$ for all $i \in N(x)$, and $d_i = 0$ for $i \notin N(x)$, we can find positive numbers a and ε such that

$$\frac{(1+\varepsilon)d_i}{v_i'(X)} < a < 1 \tag{29}$$

for all $i \in N$. Define $\{x^t\}_{t=0}^{\infty}$ by $x^0 := \bar{x}$ and, for $t > 0$,

$$x_i^t := \begin{cases} a x_i^{t-1} + (1-a)x_i & \text{if } i = t \pmod n \\ x_i^{t-1} & \text{otherwise.} \end{cases} \tag{30}$$

This $\{x^t\}_{t=0}^{\infty}$ is a round-robin sequence that starts at $\bar{x}$ and converges to x. Fix $t > 0$, and let $i = t \pmod n$. Let $q \ge 0$ be the integer for which $t = i + qn$. At the end of period $t-1$, players $j = 1, \ldots, i-1$ have raised their actions $q+1$ times, and players $j = i, \ldots, n$ have raised theirs just q times. Hence, since $x - \bar{x} = \bar{\theta} d$,

$$x_j^{t-1} = \begin{cases} x_j - \bar{\theta} a^{q+1} d_j & \text{for } 1 \le j < i \\ x_j - \bar{\theta} a^q d_j & \text{for } i \le j \le n. \end{cases} \tag{31}$$

This implies

$$X^{t-1} = X - \bar{\theta} a^q \left[a \sum_{j=1}^{i-1} d_j + \sum_{j=i}^{n} d_j \right]. \tag{32}$$

Similarly, for any $k \ge 1$,

$$x_j^{t+(k-1)n} = \begin{cases} x_j - \bar{\theta} a^{q+k} d_j & \text{for } 1 \le j \le i \\ x_j - \bar{\theta} a^{q+k-1} d_j & \text{for } i < j \le n, \end{cases} \tag{33}$$

and

$$X^{t+(k-1)n} = X - \bar{\theta} a^{q+k-1} \left[a \sum_{j=1}^{i} d_j + \sum_{j=i+1}^{n} d_j \right]. \tag{34}$$

Turning to the desired inequality (28), note that it is equivalent to

$$A := \sum_{s \ge t} \delta^{s-t} \left[u_i(x^s) - u_i(x_i^{t-1}, x_{-i}^t) \right] \ge 0.$$

Observe that $A = \sum_{k=1}^{\infty} \delta^{(k-1)n} A_k$, where

$$A_k := \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[u_i(x^s) - u_i(x_i^{t-1}, x_{-i}^t) \right].$$

Each A_k is a sum over n consecutive dates, and player i moves only at the first one, $t+(k-1)n$. Hence, for each of these dates s, $x_i^s = x_i^{t+(k-1)n}$. This implies that

$$\begin{aligned} A_k &= \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[v_i(X^s) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right] \\ &\geq \sum_{s=t+(k-1)n}^{t+kn-1} \delta^{s-t-(k-1)n} \left[v_i(X^{t+(k-1)n}) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right] \\ &= \left(\frac{1-\delta^n}{1-\delta} \right) \left[v_i(X^{t+(k-1)n}) - v_i(X^{t-1}) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right], \end{aligned}$$

where the inequality follows from $X^s \geq X^{t+(k-1)n}$ for $s \geq t+(k-1)n$. Using now the concavity of v_i and $X^{t-1} \leq X^{t+(k-1)n} \leq X$, we obtain

$$A_k \geq \left(\frac{1-\delta^n}{1-\delta} \right) \left[v_i'(X) \left(X^{t+(k-1)n} - X^{t-1} \right) - \left(x_i^{t+(k-1)n} - x_i^{t-1} \right) \right].$$

This expression can be bounded from below. From (32) and (34) we have

$$\begin{aligned} X^{t+(k-1)n} - X^{t-1} &= \bar{\theta} a^q \left[a \sum_{j=1}^{i-1} d_j + \sum_{j=i}^{n} d_j \right] - \bar{\theta} a^{q+k-1} \left[a \sum_{j=1}^{i} d_j + \sum_{j=i+1}^{n} d_j \right] \\ &= \bar{\theta} a^q \left[a(1-a^{k-1}) \sum_{j=1}^{i-1} d_j + (1-a^k) d_i + (1-a^{k-1}) \sum_{j=i+1}^{n} d_j \right]. \end{aligned}$$

From this, $1-a^k > a(1-a^{k-1})$, and $1-a^{k-1} > a(1-a^{k-1})$, we obtain

$$\begin{aligned} X^{t+(k-1)n} - X^{t-1} &\geq \bar{\theta} a^q \left[a(1-a^{k-1}) \sum_{j=1}^{i-1} d_j + a(1-a^{k-1}) d_i + a(1-a^{k-1}) \sum_{j=i+1}^{n} d_j \right] \\ &= \bar{\theta} a^{q+1} (1-a^{k-1}) \sum_{j=1}^{n} d_j \\ &= \bar{\theta} a^{q+1} (1-a^{k-1}). \end{aligned}$$

From (31) and (33), $x_i^{t+(k-1)n} - x_i^{t-1} = \bar{\theta} a^q \left(1-a^k\right) d_i$. Consequently,

$$A_k \geq \bar{\theta} a^q \left(\frac{1-\delta^n}{1-\delta} \right) \left[v_i'(X) a(1-a^{k-1}) - \left(1-a^k\right) d_i \right].$$

This and (29) imply

$$A_k \geq \bar{\theta} a^q d_i \left(\frac{1-\delta^n}{1-\delta} \right) \left[\varepsilon - a^{k-1}(1+\varepsilon-a) \right].$$

Therefore,

$$\begin{aligned} A &\geq \bar{\theta} a^q d_i \left(\frac{1-\delta^n}{1-\delta}\right) \sum_{k=1}^{\infty} \delta^{(k-1)n} \left[\varepsilon - a^{k-1}(1+\varepsilon-a)\right] \\ &= \bar{\theta} a^q d_i \left(\frac{1-\delta^n}{1-\delta}\right) \left\{\varepsilon \sum_{k=1}^{\infty} (\delta^n)^{k-1} - (1+\varepsilon-a) \sum_{k=1}^{\infty} (a\delta^n)^{k-1}\right\} \\ &= \left(\frac{\bar{\theta} a^q d_i}{1-\delta}\right) \left\{\varepsilon - \left(\frac{1-\delta^n}{1-a\delta^n}\right)(1+\varepsilon-a)\right\}. \end{aligned}$$

Thus, $A \geq 0$ for $\delta \geq \delta' := (1+\varepsilon)^{-1/n}$. As δ' does not depend on t, Step 1 is proved. ■

Step 2. *A finite, nonincreasing round-robin sequence $\{x^k\}_{k=0}^{K}$ exists such that $x^0 = \bar{x}$, $x^K = 0$, and $u(x^k) \leq u(\hat{x})$ for each $k = 0, \ldots, K$.*

Proof of Step 2. Let $x^0 := \bar{x}$. To define x^1, let $x^1_{-1} = x^0_{-1}$. Let $x^1_1 = 0$ if $u_1(0, x^0_{-1}) \leq u_1(\hat{x})$. Otherwise, let x^1_1 be the $\tilde{x}_1$ for which $u_1(\tilde{x}_1, x^0_{-1}) = u_1(\hat{x})$; this equation has a unique solution, and it is in the interval $(0, x^0_1)$, since $u_1(\cdot, x^0_{-1})$ is monotonic and $u_1(x^0) < u_1(\hat{x}) < u_1(0, x^0_{-1})$. Note that $0 \leq x^1 \leq x^0$, $u_1(x^1) \leq u_1(\hat{x})$, and $u_j(x^1) < u_j(\hat{x})$ for $j \neq 1$.

Now suppose that for some $k \geq 1$, profiles $x^0, \ldots, x^k$ have been defined, and they satisfy $0 \leq x^k \leq x^{k-1}$ and $u(x^k) \leq u(\hat{x})$. Let $i = k+1 \pmod n$. Define $x^{k+1}_{-i} := x^k_{-i}$. Let $x^{k+1}_i = 0$ if $u_i(0, x^k_{-1}) \leq u_i(\hat{x})$. Otherwise, let x^{k+1}_i be the unique $\tilde{x}_i \in \left(0, x^k_i\right]$ for which $u_i(\tilde{x}_i, x^k_{-i}) = u_i(\hat{x})$. We now have $u(x^{k+1}) \leq u(\hat{x})$.

This defines a nonincreasing and bounded round-robin sequence $\{x^k\}_{k=0}^{\infty}$. Let z be its limit. We have $z \leq x^k$ for all $k > 0$, and $u(z) \leq u(\hat{x})$.

Assume $z > 0$. In addition, assume $u_i(z) < u_i(\hat{x})$ for some $i \in N(z)$. By continuity, $\tilde{x}_i \in (0, z_i)$ exists such that $u_i(\tilde{x}_i, z_{-i}) < u_i(\hat{x})$. Since $x^k \to z$, there exists k' such that $u_i(\tilde{x}_i, x^k_{-i}) < u_i(\hat{x})$ for all $k > k'$. But then, the construction of the sequence implies that for any $k > k'$ such that $i = k+1 \pmod n$, $x^{k+1}_i < \tilde{x}_i < z_i$. This contradicts $z_i \leq x^{k+1}_i$. Thus, $u_i(z) = u_i(\hat{x})$ for all $i \in N(z)$. Since $z < \hat{x}$ and (PD) holds, we conclude that $\hat{x}$ is not strongly minimal. This contradiction proves that in fact, $z = 0$.

If $u_i(0) \geq u_i(\hat{x})$, then $\hat{x}$ would not be strongly minimal (let $S = \{i\}$ and $z = 0$ in the definition). Hence, $u(0) \ll u(\hat{x})$. Since $x^k \to 0$, this implies that K' exists such that $u_i(0, x^k_{-i}) < u(\hat{x})$ for all $k \geq K'$ and $i \in N$. The construction of the sequence thus implies the existence of $K \leq K' + n$ such that $x^K = 0$. ■

Step 3. *A discount factor $\underline{\delta} < 1$ and a path $\vec{x} \to x$ exist such that $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N}^R)$ for $\delta \in [\underline{\delta}, 1)$.*

Proof of Step 3. Reverse the round-robin sequence obtained in Step 2, and add enough copies of 0 to its beginning and $\bar{x}$ to its end to obtain a finite, nondecreasing round-robin path. This yields a path, $\{z^t\}_{t=0}^T$, from $z^0 = 0$ to $z^T = \bar{x}$, that has player 1 moving first and player n moving last ($z_{-n}^{T-1} = \bar{x}_{-n}$). To the end of of this path add the round-robin sequence obtained in Step 1: $z^{T+s} = x^s$ for all integers $s \geq 0$. This yields a path $\vec{z} = \{z^t\}_{t=0}^{\infty}$ that is feasible for $\vec{N}^R$ and converges to x. To be notationally consistent, relabel it as $\vec{x} := \vec{z}$.

Let $t \geq 1$ and $i \in N_t^R$, so that $i = t \pmod n$. If $t > T$ and $\delta > \delta'$ Step 1 implies

$$u_i(x_i^{t-1}, x_{-i}^t) \leq U_i^t(\vec{x}, \delta). \tag{35}$$

If $t \leq T$, then since $x_{-i}^t = x_{-i}^{t-1}$, Step 2 implies

$$u_i(x_i^{t-1}, x_{-i}^t) = u_i(x^{t-1}) \leq u_i(\hat{x}) < u_i(x).$$

Therefore, since $U_i^t(\vec{x}, \delta) \to u_i(x)$ as $\delta \to 1$, $\delta_t < 1$ exists such that (35) holds for $\delta > \delta_t$. We conclude that (35) holds for all $t \geq 1$, $i \in N_t^R$, and $\delta > \underline{\delta} := \max(\delta', \delta_1, \ldots, \delta_T)$. Lemma 5 now implies that $\vec{x}$ is an equilibrium path of $\Gamma(\delta, \vec{N}^R)$ for all $\delta \in (\underline{\delta}, 1)$. ■

D. Proofs Missing from Section 7

Proof of Corollary 2.

(i) Let $\tilde{u} \in P(\vec{N})$. Then sequences $\{\vec{x}^k\}$ and $\{\delta_k\} \subset (0, 1)$ exist such that $\vec{x}^k$ is an equilibrium path of $\Gamma(\delta_k, \vec{N})$, and $U(\vec{x}^k, \delta_k) \to \tilde{u}$. For each k, let x^k be the profile achieved by path $\vec{x}^k$. Fix $i \in N$. By Lemma 2, $U_i(\vec{x}^k, \delta_k) \leq u_i(x^k)$. By Theorem 1, $x^k \in D$. By Lemma B1, D is compact. Hence, taking a subsequence if necessary, we can assume $\{x^k\}$ converges to some $x \in D$. We now have

$$\tilde{u} = \lim_{k \to \infty} U_i(\vec{x}^k, \delta_k) \leq \limsup_{k \to \infty} u_i(x^k) \leq u_i(x),$$

where the last inequality holds because u_i is upper semicontinuous at x. Thus, letting $u' = u(x)$, we have $\tilde{u} \leq u' \in u(D)$.

(ii) Suppose $\tilde{u} \in P(\vec{N})$ is efficient. Then an efficient $\tilde{x}$ exists such that $\tilde{u} = u(\tilde{x})$. By (i), $x \in D$ exists such that $u(\tilde{x}) \leq u(x)$. Now the efficiency of $\tilde{x}$ implies x is also efficient, and so $\tilde{u} = u(\tilde{x}) = u(x)$. As $x \in D$, Proposition 1 implies $x \in C$. This proves that $\tilde{u} \in u(C)$.

(iii) Let $x \in D_0$. By Proposition 4, $\underline{\delta} < 1$ and $\vec{x}$ exist such that $\vec{x}$ is an equilibrium path converging to x for all $\delta > \underline{\delta}$. Hence, $U(\vec{x}, \delta) \in P(\vec{N})$ for all $\delta > \underline{\delta}$. This implies, since $P(\vec{N})$ is a closed set and $\lim_{\delta \to 1} U(\vec{x}, \delta) = u(x)$, that $u(x) \in P(\vec{N})$. Thus,

$$u(D_0) \subseteq P(\vec{N}). \tag{36}$$

Since u is continuous in a neoclassical setting, and D_0 is dense in D (Lemma 7), and D is compact (Lemma B1), we have $c\ell u(D_0) = u(c\ell D_0) = u(D)$. Taking closures of both sides of (36) now yields $u(D) \subseteq P(\vec{N})$, since $P(\vec{N})$ is closed. This and $C \subset D$ imply $u(C) \subseteq P(\vec{N})$. ■

References

ADMATI, A., AND M. PERRY (1991): "Joint Projects without Commitment," *Review of Economic Studies*, 58, 259–276.

BAGNOLI, M., AND B. L. LIPMAN (1989): "Provision of Public Goods: Fully Implementing the Core through Private Contributions," *Review of Economic Studies*, 56, 583–601.

BATTAGLINI, M., S. NUNNARI, AND T. R. PALFREY (2010): "Political Institutions and the Dynamics of Public Investment," Discussion paper, Princeton University.

CHE, Y.-K., AND J. SAKOVICS (2004): "A Dynamic Theory of Holdup," *Econometrica*, 72, 1063–1103.

CHOI, S., D. GALE, AND S. KARIV (2008): "Sequential Equilibrium in Monotone Games: A Theory-Based JOURNAL = Journal of Economic Theory, YEAR = 2008, VOLUME = 143, PAGES = 302-330," .

COMPTE, O., AND P. JEHIEL (2003): "Voluntary Contributions to a Joint Project with Asymmetric Agents," *Journal of Economic Theory*, 112, 334–342.

——— (2004): "Gradualism in Bargaining and Contribution Games," *Review of Economic Studies*, 71, 975–1000.

DUFFY, J., J. OCHS, AND L. VESTERLUND (2007): "Giving Little by Little: Dynamic Voluntary Contribution Games," *Journal of Public Economics*, 91.

DUTTA, P. K. (1995): "A Folk Theorem for Stochastic Games," *Journal of Economic Theory*, 66, 1–32.

FERSHTMAN, C., AND S. NITZAN (1991): "Dynamic Voluntary Provision of Public Goods," *European Economic Review*, 35, 1057–1067.

FOLEY, D. (1970): "Lindahl Solution and the Core of an Econmy with Public Goods," *Econometrica*, 38, 66–72.

GALE, D. (1995): "Dynamic Coordination Games," *Economic Theory*, 5, 1–18.

——— (2001): "Monotone Games with Positive Spillovers," *Games and Economic Behavior*, 37, 295–320.

LOCKWOOD, B., AND J. P. THOMAS (2002): "Gradualism and Irreversibility," *Review of Economic Studies*, 69, 339–356.

MARX, L. M., AND S. A. MATTHEWS (2000): "Dynamic Voluntary Contribution to a Public Project," *Review of Economic Studies*, 67, 327–358.

OCHS, J., AND I.-U. PARK (2010): "Overcoming the Coordination Problem: Dynamic Formation of Networks," *Journal of Economic Theory*, 145, 689–720.

PITCHFORD, R., AND C. M. SNYDER (2004): "A Solution to the Hold-up Problem Involving Gradual Investment," *Journal of Economic Theory*, 114, 88–103.

YILDIRIM, H. (2006): "Getting the Ball Rolling: Voluntary Contributions to a Large-Scale Public Project," *Journal of Public Economic Theory*, 8, 503–528.