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"Compatible Beliefs and Equilibrium"

by

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Compatible Beliefs and Equilibrium*

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Abstract

In this paper I investigate the nature of the beliefs which agents must hold (at least implicitly) in order to justify their considering various alternatives, in two distinct settings: the Walrasian model without production (with competitive equilibrium), and the sell-all version of the Shapley-Shubik market game (with Nash equilibrium). For this purpose I introduce a weak consistency requirement on behavior, one which I refer to as (having) compatible beliefs. My main conclusion is that, in this respect, these two versions of market allocation are essentially identical. For both, contemplating different choices requires varying the associated set of values of the variables defining compatible beliefs. And – though prima facie very different – it turns out that both equilibrium concepts can be recast entirely in terms of having compatible beliefs. My analysis also leads unequivocally to the interesting conclusion that, in the Walrasian model (even elaborated to encompass production, financial markets, and so on), budget constraints must hold, ab initio, with equality. This has one very important consequence: the First Basic Welfare Theorem, as usually stated, is false, as I demonstrate with two distinct counterexamples, the second of which is (in classical terms) unexceptional.

*This paper is dedicated, with affection, to my good friend and fellow equilibrium theorist Roko Aliprantis on the occasion of his 60th birthday celebration. In writing it, conversations with a number of people, in particular, Rich McLean, Anna Pavlova, Karl Shell, Paolo Siconolfi, and Alfredo Di Tillio, as well as reactions from the participants in my advanced theory seminar during the spring of 2004 have been very illuminating and useful. Rich McLean has been especially helpful in forcing me to be explicit about what I now refer to as the overarching model, Anna Pavlova in cajoling me to illuminate cryptic exposition, Karl Shell in guiding me through the intricacies of the Shapley-Shubik market game, and Ben Lester (from my seminar) in assisting me to double-check some details in the analysis. More generally, I am greatly indebted to Catherine Rouzaud for countless fruitful and stimulating (sometimes heated) discussions about the basis of the theory of competitive equilibrium. While all bear some responsibility for encouraging my idiosyncratic (sometimes heretical) views, I alone am accountable for their specific embodiment here.

∂I. Introduction

For a long time I have taught that the concept of competitive equilibrium in the Walrasian model (as well, for example, in its extension to include financial markets) is based on the following, two-part *competitive hypothesis*: all agents believe that (i) they can transact without affecting market prices, and that (ii) within solvency constraints (meaning households' budget constraints and firms' profitability constraints), they can transact any quantities of the objects for which markets exist.[1] On the face of it there seems to be a serious difficulty with the second kind of belief, since it contradicts market clearing. That is, given all other agents' actions, a particular agent's action satisfies market clearing if and only if he takes the opposite side on all markets.

In this paper I will argue that this apparent inconsistency is completely beside the point. In fact, under a weak, but plausible consistency requirement on agents' behavior, what I will refer to as (having) *compatible beliefs*, the competitive hypothesis is in fact tenable. Moreover, imposing the same weak consistency requirement in interpreting Nash equilibrium, it turns out that the two equilibrium concepts share two crucial properties: First, even "at equilibrium,"[2] different feasible choices must be associated with different sets of values for the variables defining compatible beliefs. Second, the essential defining property of both competitive equilibrium and Nash equilibrium can be formulated or, perhaps better, reformulated purely in terms of compatible beliefs.

Roughly speaking, what I mean by compatible beliefs is that agents' (resp. players') beliefs about their feasible choices (resp. strategies) are consistent with what they can actually know about other agents' unobserved choices. The argument is carried out in terms of equilibrium in two comparable models, the Walrasian model without production, and the sell-all version of the Shapley-Shubik market game, an extreme case of what Peck, Shell, and Spear (1992) call the offer-constrained game. For simplicity I will refer to these as the Walrasian model and the market game, respectively.

Since I claim that my reasoning is valid for all applications of the competitive hypothesis, I will also briefly outline the more or less obvious extension of my argument to encompass production as well as distribution (and may eventually, in a subsequent paper, detail its less obvious extension to encompass financial markets as well as commodity markets). I could also easily apply my reason-

[1] The arbitrageurs (and their academic experts) responsible for operating Long-Term Capital Management would have been well-advised to heed the obvious fact that, for their purposes, such beliefs were not justified. For an educated and fascinating account of the LTCM fiasco, see Lowenstein (2000) – recommended to me by Anna Pavlova.

[2] For competitive equilibrium, "at equilibrium" is unambiguous: it means "at equilibrium prices." On the other hand, for Nash equilibrium, "at equilibrium" is more model specific: in the abstract it means something like "(for a particular agent) given the relevant outcome of equilibrium actions (of all other agents)." Note that it is conventional in noncooperative (hereafter I omit this qualifier) game theory to refer to a model as a game (and to the agents as players, their choices or actions as strategies), since attention is restricted to models with what amounts to extraordinarily special structure, as described, for example, in Fudenberg and Tirole (1991, p. 4) – a widely used graduate level textbook. I will have much more to say about this.

ing, for example, to the market game with both bids and offers – or the few other games whose details I more or less master – but have chosen not to. My primary interest is in equilibrium theory, not game theory, so this task I am quite willing to leave to others. And I think there are at least two good reasons for my doing so. As I understand it, by their very nature, the vast majority of games concern isolated, limited interaction, so intrinsically involve very "partial equilibrium," and I find this by itself extremely distasteful. More importantly, (in the way games are conventionally specified) the only interaction is through preferences not opportunities, that is, strategy sets are independent, so that while the concept of compatible beliefs makes sense, it buys nothing interesting. In particular, these criticisms certainly apply to the canonical strategic or normal form game, as well as its many convolutions (for example, in repeated games). I will also have more to say about this whole issue in the concluding section.

With respect to competitive equilibrium (but also Nash equilibrium in the market game), compatible beliefs entail one very significant conclusion: Households' budget constraints must hold with equality. While I have been asserting this for many years (on the principle that, after all, competitive equilibrium is a property of closed economic systems), I must admit that I have done so with some trepidation, and lots of waffling – for two basic reasons. First, it flies in the face of the gospel according to Arrow and Hahn (1971, p. 79), Debreu (1959, p. 62), and McKenzie (2002, p. 3), dutifully recorded without comment in Mas-Colell, Whinston, and Green (1995, p. 50) – also a widely used graduate level textbook. Second, if budget constraints must hold with equality, then, as usually stated, the First Basic Welfare Theorem is false, as I demonstrate with two distinct counterexamples, the second of which is (in classical terms) unexceptional. But, on the basis of my analysis here, I am now of the strong opinion that it is incumbent on equilibrium theorists to fully explore the ramifications of equality rather than inequality in budget constraints. Serious work on this problem has already been undertaken by Polemarchakis and Siconolfi (1993) (with emphasis on questions of existence), and here I follow their lead (with emphasis on questions of optimality).

Finally, I must mention that almost all my arguments are very elementary, requiring no more than rudimentary algebra and calculus, and two-dimensional diagrams. When it comes to conceptual matters, I view such simplicity as a great virtue. For instance, just consider the revolution in understanding which has stemmed from Arrow (1963-4) and Debreu's (1953) neat importation of the Wald-Savage approach to modeling uncertainty into economics proper!

In the following two sections I present the two applications of the concept of compatible beliefs, first, to competitive equilibrium in the Walrasian model, and then second, to Nash equilibrium in the market game. In the concluding section I discuss various odds and ends which seem to me best separated from the body of the paper. Since my mathematical reasoning is so uncomplicated, it is presented in the text – not, as has become all too common, relegated to a typically unread appendix.

II. Competitive Equilibrium

A. The Walrasian Model without Production

There are $2 \leqq C < \infty$ commodities, indexed by $c \in \mathcal{C} = \{1, 2, \ldots, C\}$, and $2 \leqq H < \infty$ households, indexed by $h \in \mathcal{H} = \{1, 2, \ldots, H\}$. A typical household h is described by its consumption vector $x_h = (x_h^1, x_h^2, \ldots, x_h^C) \in \mathbb{R}_+^C$, consumption set $X_h \subset \mathbb{R}_+^C$, utility function $u_h : X_h \to \mathbb{R}$, and commodity endowment $e_h = (e_h^1, e_h^2, \ldots, e_h^C) \in E = \mathbb{R}_+^{\mathcal{C}} \setminus \{0\}$. Economy-wide consumption and endowments are denoted $x = (x_1, x_2, \ldots, x_H)$ and $e = (e_1, e_2, \ldots, e_H)$, respectively, while corresponding aggregates are denoted

$$\boldsymbol{x} = \sum_h x_h \text{ and } \boldsymbol{e} = \sum_h e_h.$$

It will also be very convenient to define excess demand by $z_h = x_h - e_h$, with $z = (z_1, z_2, \ldots, z_H)$ and

$$\boldsymbol{z} = \sum_h z_h.$$

Finally, market prices are denoted $p = (p^1, p^2, \ldots, p^C) \in P = \mathbb{R}_+^C \setminus \{0\}$.

I employ two particular notational conventions: First, p is a row vector (so that I can write, for example,

$$pz_h = \sum_c p^c z_h^c \text{).}$$

Second, partly borrowing from common practice in game theory, I use, for instance, for some $h' \in \mathcal{H}$,

$$z_{\setminus h'} = (z_1, z_2, \ldots, z_{h'-1}, z_{h'+1}, \ldots, z_H)\text{, and}$$

$$\boldsymbol{z}_{\setminus h'} = \sum_{h \neq h'} z_h.$$

Notice, in particular, that consumption is assumed to be nonnegative, and prices nonnegative and nontrivial; these assumptions underline the fact that this model is intended to be a useful abstraction. It also makes sense for my purposes here to assume, at the outset, that u_h exhibits local nonsatiation (i.e., has no local maxima).

Formally, (p^*, x^*) is a *competitive equilibrium* (for this model) – the standard definition – if (i) households optimize, that is, for $h \in \mathcal{H}$, given $p^* \in P$, x_h^* is an optimal solution to the problem

$$\begin{array}{c} \text{maximize } u_h(x_h) \\ \text{subject to } p^* z_h \leqq 0 \\ \text{and } x_h \in X_h, \end{array} \tag{1}$$

while (ii) markets clear, that is,

$$\boldsymbol{z}^* = 0. \tag{2}$$

For future reference, define the budget set corresponding to the constraints in (1) (but for arbitrary $p \in P$) by

$$B_h(p, e_h, X_h) = \left\{ x_h \in \mathbb{R}^C_+ : pz_h \leqq 0 \text{ and } x_h \in X_h \right\}. \tag{3}$$

B. Compatible Beliefs

The basis for compatible beliefs is what a household knows for sure. This consists of information about the economic environment, itself, and others.

- **The economic environment**. Commodities, institutions (private property and commodity markets), and scarcity (materials balance).
- **The household itself**. Its consumption set, utility function, and commodity endowment.
- **All other households**. That they consume commodities – in nonnegative quantities – and possess endowments – in nonnegative, nontrivial quantities.

Obviously, the two economic institutions of private property and commodity markets have many facets, but the particular features which are of importance in the theory of value are (i) the households' exclusive rights to their endowment (initially) and their consumption (finally), (ii) the requirement of financial solvency, and (iii) the possibility of trading all commodities at market prices.

In order to make considered, rational choices, a household also needs operational beliefs about market prices and other households' choices – our concern here. It is worth emphasizing that its beliefs about other households' choices need only be implicit in its behavior.

So consider a particular household $h' \in \mathcal{H}$ who believes that market prices are

$$\hat{p}_{h'} \in P.^3 \tag{4}$$

Then the household has compatible beliefs for a feasible choice

$$x_{h'} \in B_{h'}(\hat{p}_{h'}, e_{h'}, X_{h'})$$

if there are some values for other households' choices $\hat{x}_{h',h}$ and endowments $\hat{e}_{h',h}, h \neq h',$[4] which satisfy the condition that

$$(\hat{z}_{h',h}, h \neq h') \in \hat{\phi}_{h'}\left(\hat{p}_{h'}, z_{h'}\right), \tag{5}$$

[3] The notation "^" refers to beliefs, "h'" to the household who has the beliefs (don't worry, it will be somewhat simplified shortly!). Here I'm adopting the standard approach of first considering household behavior by itself, in isolation from equilibrium (where it would be conventional to write simply p, as in (3) above). Of course, in equilibrium, $\hat{p}_{h'}$ takes the value p^* common to all households because it is observed. It will be obvious in what follows that my analysis could just a well be carried out fixing $\hat{p}_{h'} = p^*$ at the outset.

[4] To be precise, $h \neq h',\ h \in \hat{\mathcal{H}}_{h'} = \{1, 2, \ldots, \hat{H}_{h'}\}$; the household must also have beliefs, however amorphous, about "who" the other households are. These beliefs play a role later on, but only in terms of numbers - which I will acknowledge by writing $\hat{H}_{h'}$, while, for simplicity, also writing just $h \neq h'$.

where the *belief correspondence*

$$\hat{\phi}_{h'} : \left\{(p, z_{h'}) \in P \times \mathbb{R}^C : x_{h'} \in B_{h'}(p, e_{h'}, X_{h'}) \text{ and } z_{h'} = x_{h'} - e_{h'}\right\} \rightrightarrows \mathbb{R}^{(\hat{H}_{h'}-1)C}$$

is such that

$$(p, z_{h'}) \mapsto \{z_{\setminus h'} \in \mathbb{R}^{(\hat{H}_{h'}-1)C} : \boldsymbol{z} = 0 \text{ and } pz_h \leqq 0, h \neq h'\}.$$

(5) means nothing more than that the household believes that other households can make choices for which materials balance, knowing only that these choices must be nonnegative, i.e., that $\hat{x}_{h',h} \in (\hat{X}_{h',h} \subset)\mathbb{R}^C_+, h \neq h'$, and must lie in budget sets consistent with its beliefs about prices and knowledge that endowments must be nonnegative and nontrivial, i.e., that $\hat{e}_{h',h} \in E, h \neq h'$. Using the fact that, for $z_h \in \mathbb{R}^C$, if

$$e_h > (\max\ \{0, -z_h^c\}, c \in C),$$

then there are $x_h = z_h + e_h \geqq 0$ and $e_h > 0$, it follows that the household's compatible beliefs can be expressed entirely in terms of the other households' excess demands, $\hat{z}_{h',h}$. Since I will mostly focus on values of this household's beliefs, in order to avoid unnecessary notational clutter, when there is no ambiguity I will simply write $\hat{p}, \hat{z}_h$ (bearing in mind that $\hat{z}_h = \hat{x}_h - \hat{e}_h$), $\hat{\phi}$, $\hat{H}$, and $\hat{\mathcal{H}}$.

The belief correspondence has three properties of special interest. The first concerns the question of whether or not the concept of compatible beliefs is vacuous, and describes why it isn't in practical terms.

$\hat{\phi}$ **is Nonempty-Valued**. *Suppose that* $\hat{p} \in P$ *and* $x_{h'} \in B_{h'}(\hat{p}, e_{h'}, X_{h'})$. *Then there is* $\hat{z}_{\setminus h'} \in \hat{\phi}(\hat{p}, z_{h'})$ *iff* $\hat{p}z_{h'} = 0$.

In other words, *compatible beliefs are tantamount to assuming that the household must take its budget constraint to hold with equality.* So from here on I will restrict attention to the subset of the households' original budget sets such that, for $h \in \mathcal{H}$,

$$\hat{B}_h(p, e_h, X_h) = \left\{x_h \in \mathbb{R}^C_+ : pz_h = 0 \text{ and } x_h \in X_h\right\}. \tag{6}$$

Proof. Necessity follows from the simple observation that, for $\hat{z}_{\setminus h'} \in \hat{\phi}(\hat{p}, z_{h'})$,

$$z_{h'} + \hat{\mathbf{z}}_{\setminus h'} = 0 \Longrightarrow \hat{p}z_{h'} + \sum_{h \neq h'} \hat{p}\hat{z}_h = 0,$$

and sufficiency from the fact that

$$\hat{p}z_{h'} = 0 \text{ and } \hat{z}_h = -z_{h'}/(\hat{H}-1),\ h \neq h' \Longrightarrow \hat{p}\hat{z}_h = 0,\ h \neq h' \text{ and } z_{h'} + \hat{\boldsymbol{z}}_{\setminus h'} = 0. \blacksquare$$

The other two properties describe the most important aspects of the structure of compatible beliefs. The first concerns how they vary as feasible actions vary (mathematically – but not substantively – a triviality), the second how extensive they are for a given feasible choice.

$\hat{\phi}$ **is Nonintersecting-Valued**. *If* $\hat{p} \in P$ *and* $\tilde{x}_{h'}, \tilde{\tilde{x}}_{h'} \in \hat{B}_{h'}(\hat{p}, e_{h'}, X_{h'})$ *with* $\tilde{\tilde{x}}_{h'} \neq \tilde{x}_{h'}$, *then* $\hat{\phi}(\hat{p}, \tilde{\tilde{z}}_{h'}) \cap \hat{\phi}(\hat{p}, \tilde{z}_{h'}) = \emptyset$.

Proof. This is obvious, since $\hat{\boldsymbol{z}}_{\setminus h'} = -z_{h'}$.■

$\hat{\phi}$ **is Affine-Valued**. *If* $\hat{p} \in P$ *and* $x_{h'} \in \hat{B}_{h'}(\hat{p}, e_{h'}, X_{h'})$, *then* $\hat{\phi}(\hat{p}, z_{h'})$ *is an affine set with dimension* $(\hat{H} - 2)(C - 1) \geqq 0$.

Proof. To begin with, observe that if $\hat{z}_{\setminus h'} \in \hat{\phi}(\hat{p}, z_{h'})$, then, given $\hat{p}$ and $z_{h'}$, $\hat{z}_{\setminus h'}$ must be a solution to the system of $J = C + (\hat{H} - 1)$ linear equations

$$\begin{aligned} \hat{\boldsymbol{z}}_{\setminus h} &= -z_{h'}, \text{ and} \\ \hat{p}\hat{z}_h &= 0, \ h \neq h' \end{aligned} \tag{7}$$

in the $K = (\hat{H} - 1)C$ variables $\hat{z}_{\setminus h'}$. Without loss of generality, take $h' = 1$. Then writing out (7) explicitly in the form $Ax = a$ yields

$$\begin{bmatrix} I & \cdots & I & \cdots & I \\ \hat{p} & & 0 & & 0 \\ & \ddots & & & \\ 0 & & \hat{p} & & 0 \\ & & & \ddots & \\ 0 & & 0 & & \hat{p} \end{bmatrix} \hat{z}_{\setminus 1} = \begin{pmatrix} -z_1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where I is a C^2-dimensional identity matrix. But

$$(\hat{p}, -1, \ldots, -1, \ldots, -1)A = 0 \text{ and } (\hat{p}, -1, \ldots, -1, \ldots, -1)a = 0,$$

while, for $v^T \in \mathbb{R}^J$ with $v_j = 0$ for some $j > C$,

$$vA = 0 \text{ iff } v_{\setminus j} = 0.$$

Hence, for any given subset of $K - (J - 1) = (\hat{H} - 2)(C - 1)$ variables $\hat{z}_{\setminus 1}$, excluding those for some $\tilde{h} > 1$ and for some $\tilde{c}_h \in C$, for $h \neq 1, \tilde{h}$, the equations uniquely determine the remaining variables.■

The dimensionality of $\hat{\phi}(\hat{p}, z_{h'})$ is not a mere trifle. The household may have further, more specific information about some – presumably not all – other households, say $\mathcal{H}' \subset \hat{\mathcal{H}}$ with $\mathcal{H}' \neq \hat{\mathcal{H}}$, which would take the form of additional restrictions on some – again, presumably not all – variables, say, $(\hat{x}_h^c, \hat{e}_h^c), c \in \mathcal{C}_h' \subset \mathcal{C}$, with $\mathcal{C}_h' \neq \mathcal{C}$, $h \neq h', h \in \mathcal{H}'$. (Here, of course, I assume that the household doesn't entertain any concrete views about the "rationality" of the behavior of other households.) In light of the foregoing argument, any such additional hard intelligence can be easily accommodated. Note that nothing at all here depends on the long-standing idea that, for the competitive hypothesis to to be tenable, there must necessarily be "many, many" households – though households having a common belief that there are not insignificant numbers of other households certainly makes the hypothesis much more plausible (in fact, all but incontrovertible).

Compatible beliefs in the Walrasian model formalize the competitive hypothesis – the raison d'etre for the analysis in this paper. It is therefore interesting, and I stress, that the essential defining property of competitive equilibrium itself, market clearing, can also be formulated entirely in terms of this concept: Just substitute for the original definition of market clearing the consistency requirement, say, "(ii) *beliefs mesh*, that is, for some $h' \in \mathcal{H}$,

$$z^*_{\backslash h'} \in \hat{\phi}_{h'}(p^*, z^*_{h'})." \tag{$\hat{2}$}$$

($\hat{2}$) is obviously equivalent to the symmetric condition that, for $h \in \mathcal{H}$,

$$z^*_{\backslash h} \in \hat{\phi}_h(p^*, z^*_h).$$

C. Implications of Equality in the Budget Constraints

In this section I draw some of the major conclusions which follow from the fact that compatible beliefs require that (3) be replaced by (6). In this connection, recall that I've taken as a maintained assumption that all households are locally nonsatiated. Also, since the focus here is on competitive equilibria (p^*, x^*), I will abbreviate $B_h(p^*, e_h, X_h)$ and $\hat{B}_h(p^*, e_h, X_h)$ by writing B^*_h and $\hat{B}^*_{h,}$ respectively.

Classical competitive equilibrium theory applied to the Walrasian model (here, for simplicity, ignoring production) utilizes the assumption that $x_h \in B^*_h$, rather then $x_h \in \hat{B}^*_h$, in two distinct ways:

- **For optimality**. To establish the First Basic Welfare Theorem, by appeal to the fact that, in a competitive equilibrium (p^*, x^*), for $h \in \mathcal{H}$, if $x_h \in X_h$ and $u_h(x_h) \geqq u_h(x^*_h)$, then

$$p^* z_h \left\{ \begin{array}{c} > \\ \geqq \end{array} \right\} 0 \text{ according as } u_h(x_h) \left\{ \begin{array}{c} > \\ = \end{array} \right\} u_h(x^*_h).$$

  But nothing at all can be inferred from $x_h \in X_h$ and $u_h(x_h) \geqq u_h(x^*_h)$ concerning the sign of $p^* z_h$ when $\hat{B}^*_h$ is imposed to begin with.

- **For existence**. To establish existence of a competitive equilibrium (p^*, x^*) (under standard assumptions) for the Walrasian model, by appeal to the fact that

$$x^*_h \in \underset{x_h \in B^*_h}{\arg\max}\ u_h(x_h) \Longrightarrow x^*_h \in \underset{x_h \in \hat{B}^*_h}{\arg\max}\ u_h(x_h).$$

But this result simply does not address the question of whether – under the stronger assumption that, for $h \in \mathcal{H}$, $x_h \in \hat{B}_h^*$ – there may be other competitive equilibria such that, for some $h' \in \mathcal{H}$,

$$x_{h'}^* \in \underset{x_{h'} \in \hat{B}_{h'}^*}{\arg\max}\ u_h(x_h) \text{ but } x_{h'}^* \notin \underset{x_{h'} \in B_{h'}^*}{\arg\max}\ u_h(x_h). \tag{8}$$

Though Polemarchakis and Siconolfi consider a model in which trade only takes place through incomplete asset markets, and so is not directly comparable to the Walrasian model, I adopt their terminology: a competitive equilibrium as originally defined with the budget sets (3) is a *strong* competitive equilbrium.[5] In the balance of the section I provide two counterexamples. The first shows that, without additional structure, the First Basic Welfare Theorem is false: no competitive equilibrium yields a Pareto optimal allocation (and hence no competitive equilibrium is a strong competitive equilibrium). While this counterexample is very simple, it has some other peculiar properties. So I add assumptions which guarantee that there is some strong competitive equilibrium. Nevertheless, the second counterexample shows that, even with such additional structure, there may be other competitive equilibria for which allocation is not Pareto optimal (and hence must be competitive equilibria which are not also strong competitive equilibria).

Example 1: Nonoptimality of competitive equilibrium.

This is a 2×2 economy, $C = H = 2$. The households are identical: For $h = 1, 2, X_h = \mathbb{R}_+^2$,

$$u_h(x_h) = \begin{cases} k + \left(x_h^1 + x_h^2\right), & \text{for } x_h^1 + x_h^2 < 1 \\ x_h^1 + x_h^2, & \text{for } x_h^1 + x_h^2 \geqq 1 \end{cases} \tag{9}$$

with $k > 0$, and $e_h = (1/2,\ 1/2)$. So both households are locally nonsatiated. Fig. 1 depicts the competitive equilibria in an

[insert Fig. 1]

Edgeworth-Bowley box diagram. They correspond to the allocations represented by the heavy line running from $x_1 = (0, 1)$ to $x_1 = (1, 0)$. In other words, (p^*, x^*) is a competitive equilibrium iff $p^* = (1/2,\ 1/2)$ (adopting the usual normalization that prices lie in the unit simplex) and

$$x_1^* + x_2^* = (1, 1) \text{ and } x^* \geqq 0.$$

[5] For my purposes "strong" refers to the relationship between equilibrium correspondences: assuming local nonsatiation, a strong competitive equilibrium is a competitive equilibrium. However, for Polemarchakis and Siconolfi - who allow negative prices and are primarily concerned with existence - "strong" refers to another implication of the relationship between budget sets: without, in particular, assuming local nonsatiation, competitve equilibria exist when requiring that $x_h \in \hat{B}_h$, but may not exist when only requiring that $x_h \in B_h$. Also, because they take an agnostic position, leading them to label competitive equilibria under the former specification as "weak," I take the unambiguous position that the conceptual development in the preceding section renders such a qualification unnecessary.

Obviously, given the utility functions (9), none of these allocations are Pareto optimal, since moving in either the SW or the NE direction makes both households better off.

This example, which is essentially based on discontinuity of the utility functions, is not very convincing by itself, precisely because it does have other peculiar properties. In particular, every allocation except those associated with the competitive equilibria is Pareto optimal, while, even though for every initial endowment there is some competitive equilibrium, there are no strong competitive equilibria (again refer to Fig. 1). So I now turn to a more conventional counterexample.

Example 2: Coexistence of both strong competitive equilibria and competitive equilibria which do not yield Pareto optimal allocations.

This is again a 2×2 economy, $C = H = 2$. Now, however, Ms. 1 and Mr. 2 differ: For $h = 1$,

$$\begin{array}{l} X_1 = \left\{x_1 \in \mathbb{R}^2 : x_1^2 \geqq 1/x_1^1 \text{ and } x_1^1 > 0\right\}, \\ u_1(x_1) = -x_1^1, \text{ and} \\ e_1 >> (1,1), \end{array} \tag{10}$$

while for $h = 2$,

$$\begin{array}{l} X_2 = \mathbb{R}^2_+, \\ u_2(x_2) = x_2^1, \text{ and} \\ e_2 >> 0. \end{array} \tag{11}$$

As before, both households are locally nonsatiated. In addition, for $h = 1, 2,\ u_h$ is C^∞ and quasi-concave. Fig. 2 depicts the competitive

[insert Fig. 2]

equilibria, which correspond to the allocations represented by the heavy line parallel to the axis for commodity 2 and the heavy dot where the boundary of Ms. 1's consumption set intersects Mr. 2's axis for commodity 1. In other words, there are now two sorts of equilibria: a continuum of competitive equilibria in which $p^* = (1, 0)$, and a unique strong competitive equilibrium in which $p^* >> 0$. Obviously, given the two utility functions defined in (10) and (11), none of the former yield Pareto optimal allocations.

This is a nice, clean example, since its primitives satisfy assumptions under which strong competitve equilibria exist. That is, (i) for $h \in \mathcal{H}$, X_h is closed, convex, and bounded below; u_h is C^o, without local maxima, and quasi-concave; and $e_h \in \text{int}\, X_h$, and (ii) for some $h' \in \mathcal{H}$, $X_{h'}$ is unbounded above (i.e., $\widetilde{x}_{h'} \in X_{h'}$ and $\widetilde{\widetilde{x}}_{h'} \geqq \widetilde{x}_{h'} \Longrightarrow \widetilde{\widetilde{x}}_{h'} \in X_h$) and $u_{h'}$ is increasing.[6] So the question naturally arises: what further structure is sufficient to rule out competitive

[6] The proof of existence proceeds along well-known lines (by imposing artificial bounds on households' consumption $\bar{x}_h >> \boldsymbol{e}$, $h \in \mathcal{H}, \ldots$) until the last step, in which the distinguished household h' is used to absorb negative excess demands (i.e., at the fixed point, all $\boldsymbol{z}^c < 0$ with $p^{c*} = 0$), and thus insure market clearing.

equilibria which are not strong competitive equilibria? One such assumption is that, for the same h', $u_{h'}$ is strictly increasing, since then $p^* >> 0$, and (8) is impossible. I have not investigated whether there are alternative, weaker assumptions which also suffice. (To salvage just the First Basic Welfare Theorem, one only needs to add the hypothesis that the competitive equilibrium is also a strong competitive equilibrium – without getting to the root of the matter.)

D. The Walrasian Model with both Production and Distribution

Now, in addition to households, there are $1 \leqq F < \infty$ firms, indexed by $f \in \mathcal{F} = \{1, 2, \ldots, F\}$. Because it stresses the close parallel between market prices and firm profits, I adopt McKenzie's (1959, pp. 66-67) clever view of the firm. Thus, a typical firm f is described by its input-output vector $y_f \in \mathbb{R}^C$, production set $Y_f \subset \mathbb{R}^C$, and entrepreneurial factor $c_f \in \mathcal{C}$. Following McKenzie's line of reasoning further, its production set exhibits constant returns to scale (i.e., is a cone) with two properties related to its entrepreneurial factor: First, c_f is in fact an input, that is, $y_f \in Y_f \implies y_f^{c_f} \leqq 0$, and second, c_f is also indispensable, that is, $y_f \in Y_f$ and $y_f^{c_f} = 0 \implies y_f = 0$.[7] Given this formulation, and again focusing on feasible choices, defining compatible beliefs – for both households and firms – is straightforward.

- **For a particular household** $h' \in \mathcal{H}$. The household's beliefs now include firms' input-output vectors, $\hat{y}_f \in \mathbb{R}^C$, which must satisfy the solvency constraint $\hat{p}\hat{y}_f \geqq 0$, $f \in \mathcal{F}$. Furthermore, from the household's viewpoint, materials balance now takes the form

$$z_{h'} = -\hat{\boldsymbol{z}}_{\setminus h'} + \hat{\boldsymbol{y}}.$$

- **For a particular firm** $f' \in \mathcal{F}$. The firm's beliefs encompass (i) market prices $\hat{p} \in P$, (ii) excess demands $\hat{z}_h \in \mathbb{R}^C$ such that $\hat{x}_h \in B_h(\hat{p},\ \hat{e}_h,\ \mathbb{R}^C_+)$, $h \in \mathcal{H}$, (iii) other firms' input-output vectors $\hat{y}_f \in \mathbb{R}^C$ such that $\hat{p}\hat{y}_f \geqq 0$, $f \neq f'$, and (iv) materials balance

$$y_{f'} = \hat{\boldsymbol{z}} - \hat{\boldsymbol{y}}_{\setminus f'}.$$

It is easily verified that each type of agent's belief correspondence enjoys the properties of being Nonempty-Valued, Nonintersecting-Valued, and Affine-Valued (with dimension $(\hat{H} + \hat{F} - 2)(C - 1)$). Thus, in particular, households' budget constraints must again hold with equality, and firms' solvency constraints must now also hold with equality – which, of course, is dictated by constant returns to scale anyway.

[7] This conception of the firm has yet other possibilities (and consequences) which we needn't be concerned with here. Note that $e_h^{c_f}$ corresponds to what is usually interpreted as household $h's$ share of firm $f's$ profits, so that, by definition,

$$\sum_h e_h^{c_f} = 1,\ f \in \mathcal{F}.$$

A further point is worth making explicitly. In terms of competitive equilibrium, firms optimize, that is, for $f \in \mathcal{F}$, given $p^* \in P$, y_f^* is an optimal solution to the problem

$$\begin{gathered}\text{maximize } p^* y_f \\ \text{subject to } p^* y_f \geqq 0 \\ \text{and } y_f \in Y_f.\end{gathered}$$

Again following McKenzie, profit maximization is justified by the natural assumption that the entrepreneurial factors share profits (which are zero in equilibrium), as well as earn returns. So, the hypothesis is virtually identical to that usually justified by the assumption that households share profits (and therefore unanimously prefer maximum profit). The beauty of McKenzie's approach lies precisely in the fact that, since firm profits are thus interpreted as entrepreneurial returns, they have the exactly the same logical footing as the price of any other commodity.

Finally, it is a straightforward exercise to verify that the analogue of $(\hat{2})$ (for appropriately defined belief correspondences) is equivalent to market clearing.

III. Nash Equilibrium

A. The Sell-All Market Game

The households' primitives are the same as in the Walrasian model. However, the rules of the market and attendant behavior are somewhat different. Households consign their endowments to a *market maker* – a more concrete personage than the shadowy Walrasian auctioneer. They then buy back commodities by making bids, commodity-by-commodity, in units of account. The market maker records the bids and, simultaneously, credits each household the proportion of total bids, again commodity-by-commodity, equal to its endowment relative to total endowments – which determines its income from each market. A household's total bids cannot exceed its total income, the solvency constraint. Finally, each household receives back the proportion of total endowments, commodity-by-commodity, equal to its bid relative to total bids.

What makes this a game[8] is that, given its perception of the bids by other households, a household chooses its own bids (and hence its consumption vector) recognizing its direct effect on total bids. Let $b_h = \left(b_h^1, b_h^2, \ldots, b_h^C\right) \in \mathbb{R}_+^C$ denote a typical household's bids, and $\beta_h = (\beta_h^1, \beta_h^2, \ldots, \beta_h^C) \in \mathbb{R}_+^C$ the exogenous variables reflecting its perception of the other households' total bids, with $b = (b_1, b_2, \ldots, b_H)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_H)$, and

$$\boldsymbol{b} = \sum_h b_h.$$

So β plays the same role here as p does in the Walrasian model. Thus, formally, (β^*, b^*, x^*) is a Nash equilibrium (for this game) if (i) households optimize, that

[8] More precisely, using Debreu's (1952) term, this is a *generalized game,* since strategy sets are interdependent. I will elaborate on this distinction in the concluding section.

is, for $h \in \mathcal{H}$, given $\beta_h^* \in \mathbb{R}_+^C$, (b_h^*, x_h^*) is an optimal solution to the problem

$$\begin{aligned} &\text{maximize } u_h(x_h) \\ &\text{subject to } x_h^c = [b_h^c/(b_h^c + \beta_h^{c*})]\boldsymbol{e}^c, c \in \mathcal{C}, \\ &\sum_c b_h^c = \sum_c (e_h^c/\boldsymbol{e}^c)(b_h^c + \beta_h^{c*}), \\ &\text{and } (b_h, x_h) \in \mathbb{R}_+^C \times X_h, \end{aligned} \tag{12}$$

and, say, (ii) *perceptions consist*, that is, for $h \in \mathcal{H}$,

$$\boldsymbol{b}_{\setminus h}^* = \beta_h^*. \tag{13}$$

For future reference, define the set corresponding to the restrictions on bids in (12) (but for arbitrary $\beta_h \in \mathbb{R}_+^C$), say, the *solvency set*, by

$$S_h(\beta_h) = \{b_h \in \mathbb{R}^C : \sum_c b_h^c = \sum_c (e_h^c/\boldsymbol{e}^c)(b_h^c + \beta_h^c) \text{ and } b_h \in \mathbb{R}_+^C)\}. \tag{14}$$

Also, note that in game theory it is conventional to formulate the definition of Nash equilibrium more directly, by having already substituted from (13) into (12) – a maneuver which completely obscures the essential structure of Nash equilibrium, and therefore completely defeats the whole purpose of this analysis.

For simplicity I have assumed at the outset that (i) households know total endowments $\boldsymbol{e}$ – otherwise they would also have to have beliefs $\hat{\boldsymbol{e}}_{\setminus h}$ which, in equilibrium, would have to mesh, so that, for $h \in \mathcal{H}$, $\hat{\boldsymbol{e}}_{\setminus h} = \boldsymbol{e}_{\setminus h}$; and that (ii) the solvency constraint in (12) holds with equality – otherwise, as for the Walrasian model, this would be a consequence of compatible beliefs.

One further issue must be resolved in order that this model be well-defined, the appropriate convention regarding the ratio $b_h^c/(b_h^c + \beta_h^c)$ when $b_h^c + \beta_h^c = 0$. For participation in the market to be individually rational, the natural choice is to replace the first constraint in (12) (again for arbitrary $\beta_h \in \mathbb{R}_+^C$) by its refined counterpart

$$x_h^c = \begin{cases} e_h^c & \text{, for } b_h^c + \beta_h^c = 0 \\ [b_h^c/(b_h^c + \beta_h^c)]\boldsymbol{e}^c & \text{, for } b_h^c + \beta_h^c > 0,\ c \in \mathcal{C}; \end{cases}$$

if there are no bids on a market, then consignments are returned to their owners.[9] Hereafter I adopt this convention, and assume that (12) has been modified accordingly.

B. Compatible Beliefs

[9]It is easily verified that, under this convention, the bids

$$b_h^c = \begin{cases} 0 & \text{, for } \beta_h^c = 0 \\ \beta_h^c e_h^c/\boldsymbol{e}_{\setminus h}^c & \text{, for } \beta_h^c > 0,\ c \in \mathcal{C} \end{cases}$$

satisfy the typical household's solvency constraint and yield the consumption vector $x_h = e_h$, provided that that $\boldsymbol{e}_{\setminus h} \gg 0$, or, equivalently, $e_h \ll e, h \in \mathcal{H}$, a pretty innocuous assumption. (The assumption merely permits the typical household to repurchase its own consignments with finite bids.)

The development here follows the same pattern as for the Walrasian model, so I will be fairly terse.

Again consider a particular household h' and, as before, denote its beliefs by "^" (without identifying the household explicitly).[10] For the market game, compatible beliefs involve only bids, not consumption. This follows from three observations: it must be the case, first, that, as in the Walrasian model, $\hat{X}_h = \mathbb{R}^C_+, h \neq h'$, second, that the household's beliefs and perceptions are consistent $\hat{\boldsymbol{b}}_{\setminus h'} = \hat{\beta}_{h'}$, and, third, that the household's effective market prices $(b^c_{h'} + \hat{\beta}^c_{h'})/\boldsymbol{e}^c, c \in \mathcal{C}$ are presumed taken as given by all the other households. (I will also elaborate on this last interpretation in the concluding section.) Thus, the first constraint in (12), rewritten in the latter terms,

$$\hat{x}^c_h = \begin{cases} e^c_h & , \text{ for } b^c_{h'} + \hat{\beta}^c_{h'} = 0 \\ [\hat{b}^c_h/(b^c_{h'} + \hat{\beta}^c_{h'})]\boldsymbol{e}^c & , \text{ for } b^c_{h'} + \hat{\beta}^c_{h'} > 0, c \in \mathcal{C}, h \neq h', \end{cases}$$

entails that, for $b^c_{h'} + \hat{\beta}^c_{h'} = 0$,

$$x^c_{h'} + \hat{\boldsymbol{x}}^c_{\setminus h'} = e^c_{h'} + \boldsymbol{e}^c_{\setminus h'} = \boldsymbol{e}^c,$$

and that, for $b^c_{h'} + \hat{\beta}^c_{h'} > 0$,

$$\begin{aligned} x^c_{h'} + \hat{\boldsymbol{x}}^c_{\setminus h'} &= [b^c_{h'}/(b^c_{h'} + \hat{\beta}^c_{h'})]\boldsymbol{e}^c + \sum_{h \neq h'} [\hat{b}^c_h/(b^c_{h'} + \hat{\beta}^c_{h'})]\boldsymbol{e}^c \\ &= [(b^c_{h'} + \hat{b}^c_{\setminus h'})/(b^c_{h'} + \hat{\beta}^c_{h'})]\boldsymbol{e}^c \\ &= [(b^c_{h'} + \hat{\beta}^c_{h'})/(b^c_{h'} + \hat{\beta}^c_{h'})]\boldsymbol{e}^c = \boldsymbol{e}^c, \end{aligned}$$

so that

$$z^c_{h'} + \hat{\boldsymbol{z}}^c_{\setminus h'} = 0, c \in \mathcal{C}.$$

In other words, it it is unnecessary to account explicitly for materials balance. This means that there are just two conditions, both relating only to other households' bids, which define compatible beliefs: given $\hat{\beta}_{h'} \geqq 0$ and $b_{h'} \in S_{h'}(\hat{\beta}_{h'})$, (i) (as already mentioned) the household believes that the other households' total bids are consistent (for short, consist) with its perception of their total bids, i.e.,

$$\hat{\boldsymbol{b}}_{\setminus h'} = \hat{\beta}_{h'}, \tag{15}$$

and (ii) the household believes that the other households are solvent (at the household's effective market prices), i.e.,

$$\hat{b}_h \in S_h(b_{h'} + \hat{\beta}_{h'} - \hat{b}_h)$$

[10] Again for simplicity, from here on I assume that the household knows the distribution of other households' endowments $e_{\setminus h'}$ (and, a fortiori, H), as well as their total endowment $\boldsymbol{e}_{\setminus h'}$. The calculations based on beliefs $\hat{e}_{\setminus h'}$ (and $\hat{H}$) are similar to those I will detail, only involving more variables (while maintaining the linear structure of compatible beliefs). In any case, some beliefs about $\boldsymbol{e}_{\setminus h}$ are necessary for (12) to make any sense (which is why I'd already assumed that $\hat{\boldsymbol{e}}_{\setminus h} = \boldsymbol{e}_{\setminus h}, h \in \mathcal{H}$).

or

$$\sum_c \hat{b}_h^c = \sum_c (e_h^c/\boldsymbol{e}^c)\,(b_{h'}^c + \hat{\beta}_{h'}^c), h \neq h'. \tag{16}$$

As before, these two conditions constitute a system of $J = C + (H-1)$ linear equations in $K = C(H-1)$ variables, here $\hat{b}_{\setminus h}$. (15)-(16) clearly motivate the appropriate definition of the household's belief correspondence, here $\hat{\psi}_{h'}$:

$$\hat{\psi}_{h'} : \mathbb{R}_+^C \times \mathbb{R}_+^C \rightrightarrows \mathbb{R}_+^{(H-1)C}$$

is such that

$$(\beta_{h'}, b_{h'}) \mapsto \{b_{\setminus h} \in \mathbb{R}^{(H-1)C} : \boldsymbol{b}_{\setminus h} = \beta_{h'} \text{ and } b_h \in S_h(b_{h'} + \beta_{h'} - b_h), h \neq h'\}.$$

The single most important attribute of compatible beliefs for Nash equilibrium in the market game is also the most obvious. This is the property that the belief correspondence is nonintersecting-valued: if the right-hand side of the system (15)-(16) changes, then so must its left-hand side. Why is this of such importance? Because it gives lie to a possible objection that it is only competitive equilibrium in which compatible beliefs are not, in the strongest sense, consistent beliefs. Even absent this argument, however, such objection has little merit. Without further concrete information, from the household's viewpoint the different values of its beliefs corresponding to different values of its choices are equally "likely." The argument here only justifies the additional, stronger claim that this property transcends equilibrium concepts.

But what about other properties of the belief correspondence in the market game? The requirement that bids be nonnegative makes the analysis of the existence and other structure of compatible beliefs a bit more subtle than it was for the Walrasian model. I present two results concerning such issues. The first establishes the maximum dimension of the convex set of solutions to (15)-(16), the second that this system of linear equations must, in fact, have a nonnegative solution (one which will also be nontrivial when $(\hat{\beta}_{h'}, b_{h'}) > 0$).

So, to derive the first property, suppose that there is a strictly positive solution, say, for $(\hat{\beta}_{h'}, b_{h'}) = (\beta_{h'}^*, b_{h'}^*) \gg 0$ (so that $b_{h'} + \hat{\beta}_{h'} = b_{h'}^* + \beta_{h'}^* \gg 0$), $\hat{b}_{\setminus h'} = b_{\setminus h'}^* \gg 0$ (for one justification of this supposition, again see Peck, Shell, and Spear (1992)). As before, taking $h' = 1$, and then rewriting (15)-(16) in the general form $Ay = a$ now yields

$$\begin{bmatrix} I & \cdots & I & \cdots & I \\ \mathbf{1} & & 0 & & 0 \\ \vdots & \ddots & & & \\ 0 & & \mathbf{1} & & 0 \\ \vdots & & & \ddots & \\ 0 & & 0 & & \mathbf{1} \end{bmatrix} \hat{b}_{\setminus 1} = \begin{pmatrix} \hat{\beta}_1 \\ \sum_c (e_2^c/\boldsymbol{e}^c)(b_1^c + \hat{\beta}_1^c) \\ \vdots \\ \sum_c (e_h^c/\boldsymbol{e}^c)(b_1^c + \hat{\beta}_1^c) \\ \vdots \\ \sum_c (e_H^c/\boldsymbol{e}^c)(b_1^c + \hat{\beta}_1^c) \end{pmatrix}, \tag{17}$$

where I is a C^2-dimensional identity matrix and $\mathbf{1}^T$ is a C-dimensional vector of ones. And now

$$(\mathbf{1}, -1, \ldots, -1, \ldots, -1)A = 0 \text{ and } (\mathbf{1}, -1, \ldots, -1, \ldots, -1)a = 0^{11} \tag{18}$$

while again, for $v^T \in \mathbb{R}^J$ with $v_j = 0$ for some $j > C$,

$$vA = 0 \text{ iff } v_{\setminus j} = 0.$$

We already know two basic facts about the set of nonnegative solutions to (17), first, by linearity of the left-hand side of the system of equations, that it is convex, and second, by hypothesis, that, for $(\hat{\beta}_1, b_1) = (\beta_1^*, b_1^*) \gg 0$, it contains $\hat{b}_{\setminus 1} = b_{\setminus 1}^* \gg 0$. So (as in establishing affine-valuedness for $\hat{\phi}_{h'}$ earlier) we can conclude from the rank properties of A that it contains a $(C-1)(H-2)$-dimensional set.[12] What this means, basically, is that the set of solutions has at most this dimension.

Turning now to the question of whether (17) even has a nonnegative solution, recall that if it doesn't, then by Farkas lemma, the inequalities

$$\begin{array}{c} vA \geqq 0 \text{ and} \\ va < 0, \end{array} \tag{19}$$

where $v^T \in \mathbb{R}^J$, must have a solution. I show that this is not possible.

It follows from the first $(H-1)C$ inequalities in (19) that

$$\begin{array}{c} v^j + v^{C+i} \geqq 0 \text{ or} \\ v^j \geqq -v^{C+i}, i = 1, 2, \ldots .H-1, j = 1, 2, \ldots, C. \end{array}$$

So, let

$$\alpha = \min_i \{v^{C+i}, i = 1, 2, \ldots .H-1\},$$

so that

$$v^j \geqq -\alpha \geqq -v^{C+i}, i = 1, 2, \ldots .H-1, j = 1, 2, \ldots, C. \tag{20}$$

[11] The second claim follows from the fact that, for

$$b_1 \in S_1(\hat{\boldsymbol{\beta}}_1),$$

$$\begin{array}{c} (\mathbf{1}, -1, \ldots, -1, \ldots, -1)a = \\ \sum_c \hat{\beta}_1^c - \sum_{h \geqq 2} \sum_c (e_h^c / \boldsymbol{e}^c)(b_1^c + \hat{\beta}_1^c) = \\ \sum_c \hat{\beta}_1^c - \sum_c (\sum_{h \geqq 2} e_h^c / \boldsymbol{e}^c)(b_1^c + \hat{\beta}_1^c) = \\ \sum_c \hat{\beta}_1^c - \sum_c (1 - e_1^c / \boldsymbol{e}^c)(b_1^c + \hat{\beta}_1^c) = \\ \sum_c [(e_1^c / \boldsymbol{e}^c)(b_1^c + \hat{\beta}_1^c) - b_1^c] = 0. \end{array}$$

[12] To see this, notice that, after fixing $(\hat{\beta}_1, b_1) = (\beta_1^*, b_1^*)$, we can generate solutions to (17) by perturbing just $(H-2)(C-1)$ of the $(H-1)C$ variables $\hat{b}_{\setminus 1}$ (chosen appropriately).

Since $a \geqq 0$, using (20) it is easily verified that $v^{\alpha} = -\alpha(\mathbf{1}, -1, \ldots, -1, \ldots, -1)$ must also be a solution to (19) (with $v^{a}A = 0$). But from the last inequality in (19)

$$v^{\alpha}a = -\alpha(\mathbf{1}, -1, \ldots, -1, \ldots, -1)a < 0,$$

contradicting (18). Thus, finally, I can conclude that it must be the case that (17) has a nonnegative solution.

In terms of reformulating the concept of Nash equilibrium in terms of $\hat{\psi}_h$, simply substitute for the original definition of perceptions consisting, say, again, "(ii) beliefs mesh, that is, for some $h' \in \mathcal{H}$,

$$(\beta^{*}_{\setminus h'}, b^{*}_{\setminus h'}) \in \hat{\psi}_{h'}(\beta^{*}_{h'}, b^{*}_{h'}).\text{"} \qquad (\widehat{13})$$

As with the reformulation of competitive equilibrium for the Walrasian model, $(\widehat{13})$ is equivalent to the symmetric condition that, for $h \in \mathcal{H}$,

$$(\beta^{*}_{\setminus h}, b^{*}_{\setminus h}) \in \hat{\psi}_{h}(\beta^{*}_{h}, b^{*}_{h}).$$

Finally, I want to reemphasize the most crucial parallel between compatible beliefs for the Walrasian model and compatible beliefs for the market game: both require that different choices in the household's constraint set entail different values of supporting beliefs. This property appears to be a basic feature of "general" equilibrium – in both its competitive and Nash persuasions.

IV. Concluding Comments

The Walrasian model and the market game themselves have much more in common than appears from a superficial comparison. Ignoring for a moment the "zero-in-the-denominator problem" where, for some $c \in \mathcal{C}$, $b^{c}_{h} + \hat{\beta}^{c}_{h} = 0$, the former has essentially the same basic structure as the latter – except that households do not take account of their actions' effect (direct or otherwise) on the effective market price. To see this, (i) replace $b^{c}_{h} + \boldsymbol{\beta}^{c*}_{h}$ by $\boldsymbol{b}^{c*}$ in the constraints in (12), (ii) substitute for b^{c}_{h} in the second constraint from the first, and then (iii) simply define $p^{c*} = \boldsymbol{b}^{c*}/\boldsymbol{e}^{c}$. By this maneuver, voila! (12) is transformed into (1) with B^{*}_{h} replaced by $\hat{B}^{*}_{h}$. For the Walrasian model, the "zero-in-the-denominator problem" is solved just by avoiding it completely. But notice that the market maker is still around – at least if one insists that equilibrium market prices, as must aggregated bids, "come from somewhere." Personally, I don't. For this reason I find the Walrasian model, based on the competitive hypothesis, by far the most appealing and compelling artifact for helping me conceptualize the constraints on my choices between Red Bull and True cigarettes at the local convenience store.

This transformation is, of course, ultimately based on the two different, in fact polar views of equilibrium associated with the two formulations. However, understanding many interesting economic phenomena often involves a mongrel, not the purebreds. Two well-known examples come immediately to mind, the Walrasian model extended to include consumption externalities, and the Cournot model (in the form taught in undergraduate microeconomic theory courses) when cost functions are interpreted (as they almost must be) as an implication of neoclassical firms dealing on "competitive" factor markets. And for such hybrids one still needs a convincing rationale justifying the specification of agents' feasible choices. The Cournot model also makes very concrete one major difficulty I have with the "partial equilibrium" nature of most games. What in the world is the demand curve meant to represent? More generally, how can such (usually implicit) assumptions be convincingly justified, and then explicitly formalized? Fortunately, this is not my problem. From the perspective of the analysis in this paper, the Cournot model also raises a second major difficulty I have with many other games (as delimited by most game theorists themselves). Put aside the issue of partial vs. general equilibrium, that is, the issue of where the demand curve – or, for that matter, where the firm's cost function comes from. Since a typical firm's output can be chosen as any nonnegative quantity (perhaps with a fixed upper bound), a particular firm's beliefs about other firms' total output can also be any nonnegative quantity. Admittedly, for the Cournot model itself this makes some sense. But, in particular, in order to cast the generalized market game in the same mold requires a very unnatural maneuver: the household can only, presumptively, choose any bid $b_h \in R^c_+$ provided that its payoff function be specified, say (there are other, equally objectionable, specifications, but this one preserves convexity), by the formula

$$v_h(b) = \begin{cases} u_h(x_h) & \text{, for } x_h \in X_h(b) \\ -\infty & \text{, otherwise,} \end{cases}$$

where

$$X_h(b) = \{x_h \in \mathbb{R}^C_+ : x^c_h = \begin{cases} e^c_h & \text{, for } \boldsymbol{b} = 0 \\ (b^c_h/\boldsymbol{b})\boldsymbol{e}^c & \text{, for } \boldsymbol{b} > 0, c \in \mathcal{C}, \end{cases}$$
$$\text{and } \sum_c b^c_h = \sum_c (e^c_h/\boldsymbol{e}^c)\boldsymbol{b}\}, \text{ for } b \in \mathbb{R}^{HC}_+.$$

(Note that for this purpose I have greatly simplified by completely ignoring all the subtleties associated with the concept of compatible beliefs, instead following standard practice in game theory.) Moreover, for the purposes of actually analyzing the structure of Nash equilibrium in the market game, this maneuver simply has to be undone, there are no two ways about it!

Turning now to the concept of compatible beliefs itself, a fundamental aspect – and one shared by both the Walrasian model and the market game – is that it only concerns the households' opportunities, not their preferences, or more broadly, their actual behavior. So, in both models, borrowing, for emphasis, from Postlewaite and Schmeidler (1978), the concept is valid "... regardless of whether or not the traders behave intelligently" when making a concrete choice.[13] This also means that, for the purposes of analyzing the structure of

[13] The complete quote is an assertion of the purported superiority of the market game over the Walrasian model: "This model has several interesting features when compared to the Walrasian model. First, the rules of the market yield a well-defined outcome regardless of whether or not traders behave intelligently." This seems to me to reflect profound misunderstanding.

equilibrium (however defined), individual choices can be isolated from considerations of aggregate consistency.

In this connection, a perceptive reader may have wondered why I restricted the other households $h \neq h'$ to have the same beliefs about prices as the particular household h' in defining the belief correspondence for the Walrasian model. The answer is straightforward. In deriving the principal properties of $\hat{\phi}_{h'}$, I also derive properties of the expanded belief correspondence (defined in such a way that the other households' idiosyncratic beliefs about prices are accounted for, say, $\hat{\Phi}_{h'}$) along its "diagonal" (where $\hat{p}_h = \hat{p}_{h'}, h \neq h'$). And this is the only relevant consideration provided that households know that they all participate in the same market, and therefore face the same prices. A similar observation applies for the market game.

It is also worth mentioning that the logic behind my analysis seems relevant for understanding even the most abstract of games – covering both the strategic form and extensive form (including refinements and all). Such a formulation is, in fact, completely missing from Fudenberg and Tirole (1991), though I'm pretty confident that it must appear elsewhere in the (truly theoretical) game theory literature. What I have in mind here is the overarching game (better, model) in which players (agents) are identified by $i \in \mathcal{I}$, some abstract index set, strategies (choices) are described by $s_i \in S_i, i \in \mathcal{I}$, some abstract ambient spaces, and restrictions on individual choices and aggregate outcomes are represented by $s = (s_i, i \in \mathcal{I}) \in S \subset \underset{i}{\times} S_i$, a carefully specified subset. To achieve full generality – in particular, to deal adequately with the "partial equilibrium problem" – this last will necessarily involve introducing auxiliary variables analogous to market prices, as well as underlying parameters analogous to consumption sets and commodity endowments, explicitly. The formalization of compatible beliefs in such a general setting seems to me well worth undertaking.

Finally, there are a number of other avenues for research suggested by my initial foray. Among these are

- investigating, in greater depth, the properties of competitive equilibrium with equalities replacing inequalities in the households' budget constraints;
- extending the analysis of the compatible beliefs underlying competitive equilibrium to cover financial markets, and firm and household behavior under uncertainty; and
- completing the analysis of the market game by encompassing the situation where the household has beliefs rather than knowledge about other house-

In the first place, by assuming that the Walrasian auctioneer is just as active as the market maker, so that, for example (there are many other possibilities here), given z, he reallocates endowments according to the rule

$$x_h = \begin{cases} z_h + e_h & \text{, for } \boldsymbol{z} = 0 \\ e_h & \text{, otherwise, } h \in \mathcal{H}, \end{cases}$$

the Walrasian model shares precisely the same property. But, in the second place, and much more important, having such tidy rules is completely misleading: in both cases it is only in equilibrium that the households would not be very, very surprised by their application!

holds' endowments – and then going beyond, by relaxing the simplifying assumption that all endowments must be consigned.

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