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by

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Incentive Compatibility and Differentiability: New Results and Classic Applications

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Abstract

This note provides several generalizations of Mailath's (1987) result that incentive compatibility plus separation implies differentiability. The new results extend the theory to classic models in finance such as Leland and Pyle (1977), Glosten (1989), and DeMarzo and Duffie (1999), that were not previously covered.

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1 Introduction

In many problems of asymmetric information, one agent has information upon which she bases her actions, and uninformed agents act based on inferences from these actions. Because the informed agent can reveal information through her actions, she chooses her actions strategically. If the informed agent's action as a function of her private information is one-to-one, then her strategy is said to be *separating* and her actions completely reveal her private information.

If the agent's private information (her "type") is given by a continuously distributed real-valued random variable, incentive-compatible separating strategies in such interactions can easily be characterized by a differential equation, *if* the strategy is known to be differentiable. But exactly because the strategy is not known, differentiability cannot be taken for granted. This obviously poses a serious problem for the determination and uniqueness of equilibrium.

In many cases, however, differentiability is an implication of incentive-compatibility. For a large class of signaling games and related settings, [Mailath \(1987\)](#) has shown that any incentive-compatible separating strategy of the informed agent must be differentiable and hence satisfy the standard differential equation. Unfortunately, the assumptions in [Mailath \(1987\)](#) rule out many important applications. In particular, as we describe below, they do not cover the models of [Leland and Pyle \(1977\)](#), [Glosten \(1989\)](#), and [DeMarzo and Duffie \(1999\)](#) that are at the core of modern theories of corporate finance and market microstructure.

In this note, we provide appropriate generalizations of [Mailath \(1987\)](#) to cover these models. Our results provide a foundation for the standard approach to problems of information transmission in finance, in which the issue of differentiability is often implicitly ignored. The new results can be grouped into three categories. First, we show that the original results extend to unbounded type spaces. This is important since many applications, in particular in finance, naturally involve unbounded type spaces, for example, when using normally distributed returns. Second, we provide sufficient conditions that apply locally instead of globally. These conditions can be used when the sufficient conditions do not hold globally (because, for example, a derivative vanishes somewhere), but do hold locally (because the derivative cannot vanish everywhere). We provide an example in which global differentiability can be shown by "patching together" local arguments. Third, we show that differentiability can obtain even in linear models, which are not covered by [Mailath \(1987\)](#). This extends the analysis to the many models

in corporate finance that use risk-neutrality, where the classic first-order conditions of asset pricing do not apply.

2 The Model

An informed agent knows the state of nature $\omega \in \Omega \subset \mathbb{R}$ and one or more uninformed agents react to the informed agent's action $x \in \mathcal{X} \subset \mathbb{R}$ on the basis of inferences drawn from x about ω . The sets Ω and \mathcal{X} are connected; the sets may be bounded or unbounded, we do not require them to be open or closed. For our purposes, this interaction can be summarized by the \mathcal{C}^2 function

$$\begin{aligned} V : \Omega^2 \times \mathcal{X} &\longrightarrow \mathbb{R}, \\ (\omega, \hat{\omega}, x) &\longmapsto V(\omega, \hat{\omega}, x), \end{aligned} \tag{1}$$

denoting the informed agent's payoff from taking action x when the true state of nature is ω and the uninformed agents believe it is $\hat{\omega}$.

As an example, consider the canonical signaling game (Spence (1973); Cho and Kreps (1987)). There is an informed agent who, knowing ω , chooses an action (or costly message) x , followed by an uninformed agent who observing x but not ω , chooses a response $r \in \mathcal{R} \subset \mathbb{R}$. The informed agent's payoff is given by $v(x, r, \omega)$ and the uninformed agent's payoff is given by $u(x, r, \omega)$. Given x and beliefs $\xi \in \Delta(\Omega)$, denote by $\rho(x, \xi)$ a best response for the uninformed player. In particular, if the uninformed agent has point belief $\hat{\omega}$ after observing x , $\rho(x, \hat{\omega})$ is a best response. The informed agent's payoff, given the uninformed agent's best response ρ , can then be written as

$$V(\omega, \hat{\omega}, x) \equiv v(x, \rho(x, \hat{\omega}), \omega),$$

which is of the form assumed in (1) if ρ is twice continuously differentiable.

Note, however, that the framework is more general than just the signaling model. In Section 3.2 below, for example, we apply our results to a screening model.

We study interactions in which the informed agent's information is fully revealed (as in separating equilibria in signalling games). This means that the informed agent's action is given by a *one-to-one* function $X : \Omega \rightarrow \mathcal{X}$, so that $\omega \neq \omega'$ implies $X(\omega) \neq X(\omega')$.¹ Furthermore, X must be *incentive-*

¹Environments with private information typically have many equilibria, not all of them separating. In signaling games, refinements in the spirit of those proposed by Kohlberg and Mertens (1986) and Cho and Kreps (1987) provide formal support for a focus on separation. While Kohlberg and Mertens's (1986) strategic stability has an abstract continuity motivation, the "intuitive" motivations for some of its implications seem less persuasive (Mailath, Okuno-Fujiwara, and Postlewaite, 1993).

compatible, which means that the informed agent finds it optimal to follow this strategy when she knows ω :

$$X(\omega) \in \arg \max_{x \in X(\Omega)} V(\omega, X^{-1}(x), x). \quad (\text{IC})$$

The following assumptions adopt Mailath's (1987) local concavity conditions (4) and (5) to our setting.²

Assumption 1 *The first-best contracting problem (the problem under full information),*

$$\max_{x \in \mathcal{X}} V(\omega, \omega, x)$$

has a unique solution for all $\omega \in \Omega$, denoted $X^{FB}(\omega)$, and

$$V_{33}(\omega, \omega, X^{FB}(\omega)) \equiv \left. \frac{\partial^2 V(\omega, \omega, x)}{\partial x^2} \right|_{x=X^{FB}(\omega)} < 0$$

for all $\omega \in \Omega$.

If \mathcal{X} is compact, the first-best may lie on the boundary of \mathcal{X} , in which case V_{33} is the appropriate one-sided derivative.

Assumption 2 *There exists $k > 0$ such that for all $(\omega, x) \in \Omega \times \mathcal{X}$,*

$$V_{33}(\omega, \omega, x) \geq 0 \Rightarrow |V_3(\omega, \omega, x)| > k.$$

Assumptions 1 and 2 are weaker than strict concavity but stronger than strict quasi-concavity of $V(\omega, \omega, \cdot)$.

The following theorem is the key result of Mailath (1987).

Theorem 1 (Mailath (1987)) *Let $\Omega = [\omega_1, \omega_2]$ and $\mathcal{X} = \mathbb{R}$ and let X be one-to-one and incentive-compatible. Suppose Assumptions 1 and 2 hold, and $V_{13}(\omega, \hat{\omega}, x) \neq 0$ and $V_2(\omega, \hat{\omega}, x) \neq 0$ for all $(\omega, \hat{\omega}, x) \in \Omega^2 \times \mathcal{X}$.*

1. *If $V_3(\omega, \hat{\omega}, X(\hat{\omega}))/V_2(\omega, \hat{\omega}, X(\hat{\omega}))$ is a strictly monotone function of ω for all $\hat{\omega}$, then X is differentiable in the interior of Ω , $\text{int}(\Omega)$.*

²Since Mailath (1987) took \mathbb{R} as the action space, while we allow for arbitrary real intervals, the assumption that the first order condition $V_3 = 0$ has a unique solution has been replaced with the requirement that the first-best contracting problem has a unique solution.

2. (a) If $X(\omega_1) = X^{FB}(\omega_1)$ and $V_2(\omega, \hat{\omega}, x) > 0$ for all $(\omega, \hat{\omega}, x) \in \Omega^2 \times \mathcal{X}$, then X is differentiable on Ω .
- (b) If $X(\omega_2) = X^{FB}(\omega_2)$ and $V_2(\omega, \hat{\omega}, x) < 0$ for all $(\omega, \hat{\omega}, x) \in \Omega^2 \times \mathcal{X}$, then X is differentiable on Ω .

If X is differentiable, then it satisfies the differential equation

$$X'(\omega) = -\frac{V_2(\omega, \omega, X(\omega))}{V_3(\omega, \omega, X(\omega))}. \quad (\text{DE})$$

The differential equation (DE) is a trivial consequence of the incentive constraint (IC), which yields the first-order condition $V_2 + X'V_3 = 0$, given differentiability.

The assumption that V_2 never equals zero, and so never changes sign (“belief monotonicity”) implies that the direction of belief manipulation the informed agent has an incentive to engage in is unambiguous: if $V_2 > 0$, she benefits from the uninformed side believing her to be of a higher type (respectively, of a lower type if $V_2 < 0$). The assumption that V_{13} never changes sign (“type monotonicity”) means that the informed agent’s marginal utility from x is monotone in her type. Neither assumption need be satisfied in standard examples, as the following section shows.

The condition that V_3/V_2 is a strictly monotone function of ω for all $(\hat{\omega}, X(\hat{\omega}))$ is a weak form of single crossing; we discuss the role of the single crossing property when we introduce Theorem 4.

For signaling games satisfying standard monotonicity properties, the initial value condition pinning down the value of X at either ω_1 or ω_2 in parts 2a and b of Theorem 1 is a simple consequence of sequential rationality:³ Suppose $V_2 > 0$. Then $\hat{\omega} = \omega_1$ is the worst belief the uninformed agents can have about the informed agent. It is then immediate that in any Nash equilibrium with X separating, if $X^{FB}(\omega_1) \in X(\Omega)$ then $X(\omega_1) = X^{FB}(\omega_1)$.⁴ On the other hand, if $X^{FB}(\omega_1) \notin X(\Omega)$, then in response to a deviation to the action $x = X^{FB}(\omega_1)$, sequential rationality requires the uninformed agents to choose a best reply to *some* belief, and $V_2 > 0$ again implies that ω_1 has a profitable deviation.

³For signaling games with finite type and action spaces, sequential rationality is formalized as sequential equilibrium (Kreps and Wilson, 1982), and with infinite type and action spaces, by various versions of perfect Bayes equilibrium.

⁴Suppose $X^{-1}(X^{FB}(\omega_1)) = \omega' \neq \omega_1$. Then,

$$V(\omega_1, \omega', X^{FB}(\omega_1)) > V(\omega_1, \omega_1, X^{FB}(\omega_1)) > V(\omega_1, \omega_1, X(\omega_1)),$$

and so X is not incentive compatible, a contradiction.

3 Three Examples

3.1 Equity Issues

The classic model of equity issues is due to [Leland and Pyle \(1977\)](#). It considers an owner of a firm who wants to raise funds on the stock market by selling her holdings. The uninformed side of the market is “the stock market”: a large group of equally informed and well-diversified investors. Investors are willing to invest if in expectation they earn the risk-free rate, normalized to 0.

The company is worth $\omega + \varepsilon$ in the future, where $\omega \in \Omega = [\omega_1, \infty)$ is a positive number and ε is a zero-mean random variable defined on an interval $[\underline{\varepsilon}, \bar{\varepsilon}]$. The expected value of the firm, therefore, is ω . The owner has personal wealth (outside the firm) of w_0 and is risk-averse, with an increasing, strictly concave, twice continuously differentiable money utility function U . The capital market is risk-neutral. The owner considers diversifying his risk by selling a fraction $1 - x$ of the firm in exchange for a payment of t by the capital market.

The owner’s utility from an allocation $(x, t) \in [0, 1] \times \mathbb{R}$ is

$$E_\varepsilon U(x(\omega + \varepsilon) + w_0 + t) \tag{2}$$

and that of the capital market (using risk-neutrality)

$$(1 - x)\omega - t. \tag{3}$$

Because the owner is risk-averse, the first-best is $X^{FB}(\omega) = 0$, i.e., to sell the firm completely, regardless of ω . The interaction between the two sides of the market is given by a signaling game in which the owner, knowing the value of ω , proposes an equity issue (x, t) which the stock market accepts or rejects.

As is well known, this game has a large number of equilibria. The literature usually considers equilibria with (i) maximum information transmission that (ii) leave zero expected profits to the market conditional on each type. Property (i) restricts attention to strategies $(X, T) : \Omega \rightarrow [0, 1] \times \mathbb{R}$ that are one-to-one (fully separating), while property (ii) implies transfers $T(\hat{\omega}) = (1 - x)\hat{\omega}$, where $\hat{\omega}$ is the inferred expected value of the firm. One can then ignore $t = T(\hat{\omega})$ in the analysis and denote a strategy of the informed player (the owner) by $X(\omega)$.

The payoff function V of the informed investor as defined in (1) is

$$V(\omega, \hat{\omega}, x) = E_\varepsilon U(w_0 + x(\omega + \varepsilon) + (1 - x)\hat{\omega}). \tag{4}$$

We have

$$V_2(\omega, \widehat{\omega}, x) = (1 - x)E_\varepsilon U'(w_0 + x(\omega + \varepsilon) + (1 - x)\widehat{\omega}),$$

and

$$V_{13}(\omega, \widehat{\omega}, x) = E_\varepsilon U'(w_0 + x(\omega + \varepsilon) + (1 - x)\widehat{\omega}) \\ + x(\omega - \widehat{\omega})E_\varepsilon U''(w_0 + x(\omega + \varepsilon) + (1 - x)\widehat{\omega}).$$

Simple examples show that in general V_{13} can be 0, violating type-monotonicity. Furthermore, Ω is not compact.

3.2 Market Microstructure

Consider a market for a risky asset in which risk neutral market makers provide liquidity to an informed trader who, depending on her private information, may wish to buy or sell the risky asset. Following [Glosten \(1989\)](#), we refer to the informed trader as the investor. Let $x \in \mathbb{R}$ denote the quantity of the risky asset traded by the investor, with $x > 0$ corresponding to a purchase and $x < 0$ to a sale. The corresponding monetary transfer from the investor to the market maker is denoted by $t \in \mathbb{R}$; if $t < 0$, $-t$ is the amount received by the investor. If as in standard market microstructure theory, p is the price of the asset, then $t = px$.

The following is [Mailath and Nöldeke's \(2008\)](#) generalization of [Glosten \(1989\)](#). The final value of the risky asset is $\nu = s + \varepsilon$. The investor privately observes s and her endowment θ of the risky asset before trade takes place. The random variables (s, θ) describing the investor's private information are uncorrelated and elliptically distributed ([Fang, Kotz, and Ng, 1990](#)) with variances $\sigma_s^2 > 0$ and $\sigma_\theta^2 > 0$. The random variable ε , realized after trade, is normally distributed with variance $\sigma_\varepsilon^2 > 0$ and independent of (s, θ) . The variables s , ε , and θ all have zero mean.

After a trade x resulting in a monetary transfer t , the investor's final wealth is $w = (x + \theta)(s + \varepsilon) - t$. (The risk-free rate and the investor's initial money holdings are assumed to be zero.) The investor has CARA preferences with risk aversion parameter $\gamma > 0$. As ε is normally distributed this yields, as usual, a quadratic representation of the investor's preferences over $(x, t) \in \mathbb{R}^2$ conditional on her private information. Defining

$$r \equiv \gamma\sigma_\varepsilon^2 > 0, \tag{5}$$

such a representation is given by

$$\mathcal{U}(x, t \mid s, \theta) = (s - r\theta)x - rx^2/2 - t.$$

While the private information of the investor is two-dimensional, her preferences depend on this information only through the one-dimensional variable $s - r\theta$, which reflects a blend of the investor's informational and hedging motives for trade. Setting

$$\omega \equiv E[\nu \mid s - r\theta] = E[s + \varepsilon \mid s - r\theta],$$

the linear conditional expectation property of elliptically distributed random variables ([Hardin, 1982](#)) implies

$$\omega = \frac{s - r\theta}{b}, \tag{6}$$

where

$$b \equiv \frac{\sigma_s^2 + r^2\sigma_\theta^2}{\sigma_s^2} > 1. \tag{7}$$

Conditional on ω , the investor's preferences over trade-transfer pairs are thus described by the utility function

$$\mathcal{U}(x, t \mid \omega) = b\omega x - rx^2/2 - t. \tag{8}$$

Market makers are risk neutral and maximize expected trading profits. It suffices to consider aggregate trading profits $t - \nu x$. Conditional on ω , expected aggregate trading profits are given by

$$\mathcal{V}(x, t \mid \omega) = t - \omega x. \tag{9}$$

The analysis of the model can be conducted in the reduced form environment, with the investor's private information summarized by her one-dimensional type ω and payoff functions given by (8) for the investor and (9) for market makers.

The above assumptions on the information structure and traders' preferences are as in [Glosten \(1989\)](#), with the important exception that the random variables (s, θ) describing the investor's private information are *not* required to be normally distributed. The assumption that these variables are elliptically distributed is enough to yield payoff functions (8) and (9) identical to those arising in [Glosten's \(1989\)](#) normal environment (and used in [Hellwig's \(1992\)](#) analysis of [Glosten's](#) competitive model).⁵ If ω is normally distributed, then the support Ω of its distribution is \mathbb{R} , but in general, Ω can be bounded (it must contain 0 as its midpoint).

⁵The distribution of ω is not completely arbitrary, as (6) determines ω as a function of the elliptically distributed random variables (s, θ) and the underlying parameters γ and σ_ε^2 . In particular, the distribution function of ω is symmetric and has finite variance (see [Mailath and Nöldeke \(2008\)](#) for details).

The strategic interaction between the two sides of the market is a screening game in which the market makers compete for the investor's trade. Hence, each market maker i offers a menu $\{(X_i(\omega), T_i(\omega))_{\omega \in \Omega} : X_i(\omega) \in \mathbb{R}, T_i(\omega) \in \mathbb{R}, \forall \omega \in \Omega\}$ of trading possibilities to the informed investor, and the investor then chooses one allocation $(X_i(\hat{\omega}), T_i(\hat{\omega}))$ from one menu.

We again consider outcomes with maximum information transmission, i.e., trading schedules that are separating with respect to ω . Competition between market makers implies that if there exists a separating equilibrium in the screening game, each market maker must make zero expected profits on each type. By (9), this means

$$T_i(\omega) = \omega X_i(\omega) \quad \text{for all } \omega \in \Omega \text{ and all } i. \quad (10)$$

Hence, the trading schedule pins down the pricing schedule.

By (8), the payoff function V of the informed investor as defined in (1) is then

$$V(\omega, \hat{\omega}, x) = (b\omega - \hat{\omega})x - rx^2/2. \quad (11)$$

We have

$$V_2(\omega, \hat{\omega}, x) = -x, \quad (12)$$

$$V_{13}(\omega, \hat{\omega}, x) = b, \quad (13)$$

$$\text{and} \quad \frac{d}{d\omega} \left\{ \frac{V_3(\omega, \hat{\omega}, x)}{V_2(\omega, \hat{\omega}, x)} \right\} = -\frac{b}{x}. \quad (14)$$

Equations (12) and (14) violate the assumptions of Theorem 1. Furthermore, while it is possible in the equity issue model of the previous subsection to restrict Ω arbitrarily to a compact interval, the case of an unbounded Ω is important in this case (arising, for example, when ω is normal).

3.3 Security Design

The fundamental question in corporate finance is how to allocate the cash flow generated by a firm's assets among its different providers of capital. DeMarzo and Duffie (1999) have argued that this problem should be analyzed in two steps. First, the firm's owners or managers design the security, and second the security is sold to investors. Since the second step may take place significantly later than the first, the firm may have obtained private information concerning the security's payoff once it sells the security. For this second step, DeMarzo and Duffie (1999) therefore consider the following game.

The security has an expected payoff $\omega \in \Omega$, where ω is private information of the firm and $\Omega \subset \mathbb{R}$ is a potentially unbounded interval with left endpoint ω_1 . The firm considers selling a quantity $x \in [0, 1]$ of the security to market investors. There are gains from trade because the firm discounts the security's cash flows at a higher rate than the market. Let $\delta < 1$ be the firm's discount rate relative to that of the market (which is normalized to 1). The firm and the market are both risk-neutral. If the firm sells the amount x of the security for a total of t , the firm's payoff is

$$t + (1 - x)\delta\omega \tag{15}$$

and the market investors' payoff is

$$x\omega - t. \tag{16}$$

Market investors are competitive and must make zero expected profits for each value of ω . Hence, if they believe the expected value of the security to be $\hat{\omega}$, they will pay $t = x\hat{\omega}$. Inserting this into (15) yields the payoff function V of the informed investor as defined in (1):

$$\begin{aligned} V(\omega, \hat{\omega}, x) &= x\hat{\omega} + (1 - x)\delta\omega \\ &= \delta\omega + (\hat{\omega} - \delta\omega)x. \end{aligned}$$

The informed agent's payoff function V is linear in x and therefore violates Assumption 1 of Theorem 1.

4 The Generalized Theorems

In this section, we provide two theorems that significantly expand the applicability of Theorem 1. Our first result is that incentive-compatibility implies differentiability in models with linear payoffs and compact choice sets, as in DeMarzo and Duffie (1999).

Assumption 3 *The set \mathcal{X} is compact and the function V is affine in $x \in \mathcal{X}$,*

$$V(\omega, \hat{\omega}, x) = A(\omega, \hat{\omega}) + B(\omega, \hat{\omega})x, \tag{17}$$

with $B(\omega, \omega) \neq 0$ for all $\omega \in \Omega$.

Under Assumption 3, $X^{FB}(\omega)$ is on the boundary of \mathcal{X} .

Theorem 2 *Let Ω and \mathcal{X} be intervals in \mathbb{R} and let X be one-to-one and incentive-compatible. Suppose Assumption 3 holds. Then X is differentiable at every $\omega \in \Omega$ and satisfies the linear differential equation*

$$B(\omega, \omega)X'(\omega) + B_2(\omega, \omega)X(\omega) = -A_2(\omega, \omega). \quad (18)$$

Theorem 2 is useful because many standard models in corporate finance, as in Industrial Organization, work with linear preferences, which often gives rise to valuations V of the form (17). The theorem is surprising because constrained optimization problems with linear objective functions often yield discontinuous solutions. Interestingly, the assumption that the action set \mathcal{X} is compact does not force the solution to lie on the boundary: the optimal X typically lies in the interior of \mathcal{X} . Instead, compactness is needed to prove that for any ω_0 and any sequence $\omega_n \rightarrow \omega_0$, $V(\omega_0, \omega_0, X(\omega_n)) \rightarrow V(\omega_0, \omega_0, X(\omega_0))$. This is a crucial insight that helps to establish the continuity of X , from which, in turn the differentiability of X can be deduced.

Instead of assuming compactness of \mathcal{X} , this insight can also be proved by our relaxed concavity assumptions 1 and 2.

Theorem 3 *Suppose Ω and \mathcal{X} are connected subsets of \mathbb{R} and X is one-to-one and incentive-compatible. Suppose Assumptions 1 and 2 hold.*

1. *For any $\omega \in \Omega$, if $X(\omega) = X^{FB}(\omega)$ then X is continuous at ω .*
2. *For any $\omega_0 \in \text{int}(\Omega)$, if $V(\omega_0, \omega_0, \cdot)$ is a monotone function in x , and if either $X(\omega_0) \neq X^{FB}(\omega_0)$ or $V_2(\omega_0, \omega_0, X^{FB}(\omega_0)) \neq 0$, then X is differentiable at ω_0 .*
3. *Suppose $V_{13}(\omega, \omega, x) \neq 0$ for all $(\omega, x) \in \Omega \times \mathcal{X}$, $V_2(\omega, \hat{\omega}, X(\hat{\omega})) \neq 0$ for all $\omega, \hat{\omega} \in \Omega$, and $V_3(\omega, \hat{\omega}, X(\hat{\omega}))/V_2(\omega, \hat{\omega}, X(\hat{\omega}))$ is a strictly monotone function of ω for all $\hat{\omega}$. Then X is differentiable on $\text{int}(\Omega)$.*
4. *Assume that $V_{13}(\omega, \omega, x) \neq 0$ and $V_{12}(\omega, \omega, x) \leq 0$ for all $(\omega, x) \in \Omega \times \mathcal{X}$. If $V_2(\omega, \omega, X(\omega)) > 0$ or if $V_2(\omega, \omega, X(\omega)) < 0$ for all ω in an open subset $\Omega_0 \subset \Omega$, then X is differentiable on Ω_0 .*
5. *Assume that $V_{13}(\omega, \omega, x) \neq 0$ for all $(\omega, x) \in \Omega \times \mathcal{X}$.*
 - (i) *Assume that $\Omega = [\omega_1, \omega_2]$ or $\Omega = [\omega_1, \infty)$ and that $X(\omega_1) = X^{FB}(\omega_1)$. If $V_2(\omega, \omega, X(\omega)) > 0$ for all $\omega \in \Omega$ then X is differentiable on Ω ,*

(ii) Assume that $\Omega = [\omega_1, \omega_2]$ or $\Omega = (-\infty, \omega_2]$ and that $X(\omega_2) = X^{FB}(\omega_2)$. If $V_2(\omega, \omega, X(\omega)) < 0$ for all $\omega \in \Omega$ then X is differentiable on Ω .

At all points of differentiability, X satisfies the differential equation (DE).

The proof of both theorems, which recycles Mailath’s original proof and adds a number of new elements, is in the appendix.

Theorem 3 generalizes Theorem 1 in several respects. First, the theorem does not assume compactness of Ω . The argument is not entirely trivial because Mailath’s proof uses uniform convergence (for which compactness is needed) and exploits the behavior of X on the boundary of Ω . Second, the assumptions on the partial derivatives need not hold for all $(\omega, \hat{\omega}, x) \in \Omega^2 \times \mathcal{X}$. Here, the necessary changes in Mailath’s proof are simple, but the new generality is useful because (i) the restriction to the diagonal $\hat{\omega} = \omega$ has bite and (ii) usually there is some a priori information about the graph of X that can be used.

Theorem 3.1 is a useful partial result that follows directly from Mailath’s proof. The assumption in this statement seems difficult to verify a priori, because one needs to know X , which one actually wants to characterize. However, since it is more difficult to show the continuity of X than deducing the differentiability from the continuity of X , this statement is useful to “fill possible holes” left by the other statements. Subsection 5.2 provides an example for this technique.

Theorem 3.5 clarifies the role of the boundary conditions in Theorem 1 and shows that only one boundary condition is necessary to obtain the result. This extends the validity of the theorem to the case of intervals that are either unbounded from below or from above. Theorem 3.3 is the same statement as in Theorem 1 but without the restrictions on Ω . The comparison of Theorem 3.3 and Theorem 3.5 therefore extends and clarifies Mailath’s (1987) observation that in order to prove differentiability one can use single crossing or a boundary condition.⁶

The two more substantial results are Theorem 3.2 and Theorem 3.4. They also show that in order to prove differentiability, neither single crossing nor a boundary condition are necessary. Theorem 3.2 is useful because

⁶However, note that these two are not the only alternatives, as Theorems 2 and 3 show. The single-crossing property is a standard condition on the indifference curves of the informed agent (we discuss it in more detail below).

the required monotonicity is often easy to verify, and is structurally novel because it is local (i.e., it only requires conditions at ω_0 to establish differentiability at ω_0). Unlike the statements in Theorem 1, Theorem 3.4 only requires assumptions on V_2 for ω in a subset $\Omega_0 \subset \Omega$. This is particularly useful if the regularity assumptions for V are known not to hold on the whole domain. The condition $V_{12} \leq 0$ (“manipulation monotonicity”) in this statement is mild and satisfied in all examples we know of. It requires that the informed agent’s gain from manipulating the uninformed beliefs upwards does not increase in her type.

Theorems 2 and 3 identify conditions under which incentive-compatibility implies differentiability. To complete our discussion we now briefly turn to the question under what conditions the converse is true, i.e. when differentiability implies incentive-compatibility. Under the assumption of Theorem 1, Mailath (1987, Theorem 3) showed that the converse holds if V satisfies the single-crossing property on the graph of X . The next theorem shows that this statement continues to be true under weaker assumptions. Moreover, the property is also *locally* necessary (the statement of Mailath (1987, Theorem 3) is incorrect).

The Spence (1973)-Mirrlees (1971) single-crossing property requires the agent’s marginal rate of substitution between her action (x) and that of the uninformed agents be appropriately monotone in her type. In our examples, the uninformed agents’ action is a monetary transfer, and as is typical, the action is monotone in beliefs about type. Consequently, in our reduced form model, the appropriate marginal rate of substitution is between $\hat{\omega}$ and x , that is, $V_3(\omega, \hat{\omega}, x)/V_2(\omega, \hat{\omega}, x)$. The adjective “appropriately” captures the requirement that, for example, in job market signaling, single crossing is implied by more (rather than less) able workers having a lower marginal cost of education. If less able workers have the lower marginal cost of education, then the marginal rate substitution between education and wage is an *increasing* function of ability. While monotonic, such a marginal rate of substitution precludes the existence of a separating equilibrium.

The single-crossing property imposes a uniform structure on the derivatives of V that implies global optimality from the first-order condition (which is essentially the differential condition (DE)). Conversely, the second-order condition for local optimality implied by incentive-compatibility is essentially the local single-crossing condition.

The following theorem on the role of single crossing is proved in the appendix. Since X' and V_2 do not change sign, (19) and (20) both imply that $V_3(\omega, \hat{\omega}, x)/V_2(\omega, \hat{\omega}, x)$ is monotonic in ω (with the signs of X' and V_2 jointly determining the appropriate monotonicity). Condition (19) implies

a global monotonicity, in that it holds everywhere on the graph of X , while (20) implies a local monotonicity, in that it only holds for the derivative evaluated at $\hat{\omega} = \omega$.

Theorem 4 *Assume that the one-to-one function X is continuous on Ω and satisfies the differential equation (DE) on the interior of Ω . Suppose $V_2(\omega, \hat{\omega}, X(\hat{\omega})) \neq 0$ for all $\omega, \hat{\omega} \in \Omega$.*

1. *If*

$$X'(\omega)V_2(\omega, \hat{\omega}, X(\hat{\omega}))\frac{d}{d\omega}\left\{\frac{V_3(\omega, \hat{\omega}, X(\hat{\omega}))}{V_2(\omega, \hat{\omega}, X(\hat{\omega}))}\right\} \geq 0 \quad (19)$$

for all $\omega, \hat{\omega} \in \Omega$, then X is incentive-compatible.

2. *If X is incentive-compatible, then*

$$X'(\omega)V_2(\omega, \omega, X(\omega))\frac{d}{d\omega}\left\{\frac{V_3(\omega, \hat{\omega}, X(\hat{\omega}))}{V_2(\omega, \hat{\omega}, X(\hat{\omega}))}\right\}\Bigg|_{\hat{\omega}=\omega} \geq 0 \quad (20)$$

for all $\omega \in \Omega$.

5 The Examples Revisited

5.1 Equity Issues

In equilibrium, the payoff of the informed party in the model of Section 3.1, (4), is strictly decreasing in x : under truth-telling ($\hat{\omega} = \omega$), investors pay the expected value of the firm, and any share the informed owner holds yields a mean-preserving spread around the mean. Since the informed owner is risk-averse, her utility is therefore decreasing in the size of her shareholdings. Since $V_2(\omega, \omega, 0) > 0$ for all ω , the conditions of Theorem 3.2 are satisfied.

Note that the conditions are not satisfied for $\omega = \omega_1$ (the left boundary), and indeed we have $X'(\omega_1) = \infty$.

5.2 Market Microstructure

From (11), the first-best in the model of Section 3.2 is given by

$$X^{FB}(\omega) = \frac{b-1}{r}\omega.$$

Under any competitive incentive-compatible separating price-quantity schedule, type $\omega = 0$ must get her first-best allocation, $x = 0$.⁷ Separation

⁷Since $V(0, 0, X(0)) \geq 0$ (type $\omega = 0$ has the option of choosing $x = 0$), and $V(0, 0, x) = -rx^2/2$, which is strictly negative if $x \neq 0$, we have $X(0) = 0$.

then implies that every other type must choose a non-zero quantity. Hence, $V_2(\omega, \omega, X(\omega)) \neq 0$ for all $\omega \neq 0$ by (12). Since $V_{12} \equiv 0$, Theorem 3.4 therefore implies that any incentive-compatible schedule X must be differentiable on the open sets $(0, \infty)$ and $(-\infty, 0)$.

By Theorem 3.1, the schedule X is continuous at $\omega = 0$. Calculating the derivatives of X on $(0, \infty)$ and $(-\infty, 0)$ from (DE) shows that the left-hand and right-hand derivative of X at $\omega = 0$ exist and are identical. Hence, X is differentiable on all of Ω .

One of the main insights in [Glosten \(1989\)](#) is his non-existence result. In particular, he shows that in the case $b \leq 2$, the differential equation (DE) for $\Omega = \mathbb{R}$ does not have a separating solution (see also [Hellwig \(1992\)](#)). His conclusion is that too much competition can be detrimental for market activity. This conclusion requires that every equilibrium trading schedule is differentiable, a result missing in [Glosten \(1989\)](#), but which is implied by Theorem 3.

It is worth noting that [Glosten's \(1989\)](#) conclusion also requires viewing separation as an implication of competition. Without an appeal to refinements (see footnote 1), there is no foundation for such a view.⁸ [Mailath and Nöldeke \(2008\)](#) argue that competitive pricing does not lead to market breakdown, even in the presence of extreme adverse selection (in the sense of unbounded supports of the private information).

5.3 Security Design

Since the firm's payoff function V is linear in x , if the firm's strategy X is incentive-compatible and one-to-one, Theorem 2 implies that it is differentiable for all $\omega > 0$ and satisfies the differential equation

$$(1 - \delta)\omega X'(\omega) + X(\omega) = 0. \quad (21)$$

It can be easily verified that (21) has the solution

$$X(\omega) = a\omega^{-\frac{1}{1-\delta}}, \quad (22)$$

where $a \geq 0$ is a constant of integration. Since $X(\omega) \in [0, 1]$ by construction, (22) implies that $\omega_1 = \inf \Omega$ and a must satisfy $\omega_1 \geq a^{1-\delta}$ for a solution to exist. In particular, if the interaction in the model is a signaling game (as

⁸[Gale \(1992, 1996\)](#) describes a Walrasian approach to competition in markets with adverse selection that is related to the ideas of [Kohlberg and Mertens \(1986\)](#) and which yields similar conclusions.

in Section 3 of [DeMarzo and Duffie \(1999\)](#)), then firm type ω_1 must obtain its most preferred allocation $x = 1$, and the constant of integration is

$$a = \omega_1^{\frac{1}{1-\delta}}.$$

Note that this implies that Ω must be bounded away from 0 for a non-trivial solution to exist.⁹

A Proofs of Theorems 2 and 3

In what follows, Ω and \mathcal{X} are connected subsets of \mathbb{R} , and X is a one-to-one function satisfying (IC).

Lemma A *If X is continuous at ω_0 and $V_3(\omega_0, \omega_0, X(\omega_0)) \neq 0$, then X is differentiable at ω_0 with derivative*

$$X'(\omega_0) = -\frac{V_2(\omega_0, \omega_0, X(\omega_0))}{V_3(\omega_0, \omega_0, X(\omega_0))}.$$

If ω_0 is on the boundary of Ω , then the derivative is the appropriate one-sided derivative. This is essentially Proposition 2 of [Mailath \(1987\)](#). Its proof requires no modification, since that proof only requires V be \mathcal{C}^2 (in which case, it also has bounded derivatives on any compact neighborhood).

Lemma B *Suppose either Assumptions 1 and 2 or Assumption 3 hold. If X is continuous at $\omega_0 \in \text{int}(\Omega)$ and either $V_2(\omega_0, \omega_0, X^{FB}(\omega_0)) \neq 0$ or $V_2(\omega_0, \omega_0, X(\omega_0)) \neq 0$, then $V_3(\omega_0, \omega_0, X(\omega_0)) \neq 0$ and Lemma A applies.*

Proof. Observe that $X(\omega_0)$ cannot be on the boundary of \mathcal{X} , since it is both continuous at ω_0 and one-to-one. Thus, if $X^{FB}(\omega_0)$ is on the boundary (as it is under Assumption 3), we immediately have $X(\omega_0) \neq X^{FB}(\omega_0)$ and so $V_3(\omega_0, \omega_0, X(\omega_0)) \neq 0$.

Suppose $X^{FB}(\omega_0) \in \text{int}(\mathcal{X})$ (and so it is Assumptions 1 and 2 that hold). Since X^{FB} is continuous, for η sufficiently small and for all $\omega \in [\omega_0 - \eta, \omega_0 + \eta]$, $X^{FB}(\omega) \in \text{int}(\mathcal{X})$ and so $V_3(\omega, \omega, X^{FB}(\omega)) = 0$.

Suppose $V_2(\omega_0, \omega_0, X^{FB}(\omega_0)) \neq 0$. Then, for $\eta > 0$ sufficiently small, $V_2(\omega, \omega, X^{FB}(\omega)) \neq 0$ on the same neighborhood of ω_0 . The argument showing that $X(\omega_0) \neq X^{FB}(\omega_0)$ is now identical to the proof of ([Mailath, 1987](#), Proposition 3, p. 1362).

⁹This result and solution (22) are in [DeMarzo and Duffie \(1999\)](#), which cites an earlier unpublished version of their paper for the complete proof.

Finally, suppose $V_2(\omega_0, \omega_0, X(\omega_0)) \neq 0$. If $X(\omega_0) = X^{FB}(\omega_0)$, then $V_2(\omega_0, \omega_0, X^{FB}(\omega_0)) \neq 0$, which by the previous paragraph implies $X(\omega_0) \neq X^{FB}(\omega_0)$, a contradiction. ■

Lemma C *Suppose either Assumptions 1 and 2 or Assumption 3 hold. For each non-empty compact interval $[\underline{\omega}, \bar{\omega}] \subset \Omega$, $X([\underline{\omega}, \bar{\omega}])$ is bounded.*

Proof. Under Assumption 3, \mathcal{X} is compact, and so we immediately have the result.

Suppose it is Assumptions 1 and 2 that hold. The continuity of V and the Maximum Theorem imply that the first-best X^{FB} is a continuous function on Ω .

Suppose that X is unbounded on $[\underline{\omega}, \bar{\omega}]$ and let $\omega_n \in [\underline{\omega}, \bar{\omega}]$, $n = 1, 2, \dots$ be a sequence such that $x_n = X(\omega_n) \rightarrow \infty$ (the case $x_n \rightarrow -\infty$ is handled analogously). We may assume that, by taking subsequences if necessary, the sequence ω_n converges to some $\omega_0 \in [\underline{\omega}, \bar{\omega}]$. There is $N \in \mathbb{N}$ such that $X(\omega_n) > X^{FB}(\omega_0)$ for all $n \geq N$. Assumptions 1 and 2 imply that $V(\omega_0, \omega_0, X(\omega_n)) \rightarrow -\infty$. By the continuity of V , we also have $V(\omega_n, \omega_n, X(\omega_n)) \rightarrow -\infty$. This, however, contradicts the incentive-compatibility of X . ■

Lemma D *Suppose either Assumptions 1 and 2 or Assumption 3 hold. If $\omega \rightarrow \omega_0$ then $V(\omega_0, \omega_0, X(\omega)) \rightarrow V(\omega_0, \omega_0, X(\omega_0))$.*

Proof. Fix a compact neighborhood N in Ω containing ω_0 . By Lemma C, $X(N)$ is bounded. Hence, V is uniformly continuous on $N^2 \times \text{cl}(X(N))$, where $\text{cl}(\cdot)$ denotes the closure.

Fix $\varepsilon > 0$. Uniform continuity implies that there is a $\delta_1 > 0$ with $\{\omega \in \Omega; |\omega - \omega_0| < \delta_1\} \subset N$ such that for all $x \in X(N)$,

$$|\omega - \omega_0| < \delta_1 \implies |V(\omega_0, \omega, x) - V(\omega_0, \omega_0, x)| < \varepsilon.$$

For these ω , incentive compatibility implies

$$V(\omega_0, \omega_0, X(\omega_0)) \geq V(\omega_0, \omega, X(\omega)) > V(\omega_0, \omega_0, X(\omega)) - \varepsilon. \quad (\text{A.1})$$

On the other hand, there is a $\delta_2 > 0$ with $\{\omega \in \Omega; |\omega - \omega_0| < \delta_2\} \subset N$ such that for all $x \in X(N)$,

$$\begin{aligned} |\omega - \omega_0| < \delta_2 \implies |V(\omega, \omega_0, x) - V(\omega_0, \omega_0, x)| &< \varepsilon/2 \\ \text{and } |V(\omega, \omega, x) - V(\omega_0, \omega_0, x)| &< \varepsilon/2. \end{aligned}$$

Hence, for these ω , incentive compatibility implies

$$\begin{aligned} V(\omega_0, \omega_0, X(\omega)) &> V(\omega, \omega, X(\omega)) - \frac{\varepsilon}{2} \\ &\geq V(\omega, \omega_0, X(\omega_0)) - \frac{\varepsilon}{2} > V(\omega_0, \omega_0, X(\omega_0)) - \varepsilon. \end{aligned} \quad (\text{A.2})$$

Therefore, for $\omega \in \Omega$ with $|\omega - \omega_0| < \min(\delta_1, \delta_2)$, we have

$$V(\omega_0, \omega_0, X(\omega_0)) - \varepsilon < V(\omega_0, \omega_0, X(\omega)) < V(\omega_0, \omega_0, X(\omega_0)) + \varepsilon,$$

where the first inequality is from (A.2) and the second (A.1).

Hence, $\lim_{\omega \rightarrow \omega_0} V(\omega_0, \omega_0, X(\omega)) = V(\omega_0, \omega_0, X(\omega_0))$. \blacksquare

Theorem 2. *Suppose Assumption 3 holds. Then X is differentiable at every $\omega \in \Omega$ and satisfies the linear differential equation*

$$B(\omega, \omega)X'(\omega) + B_2(\omega, \omega)X(\omega) = -A_2(\omega, \omega).$$

Proof. Fix $\omega_0 \in \Omega$ and a compact neighborhood in Ω containing ω_0 . Consider a sequence $\omega_n \rightarrow \omega_0$. By the compactness of \mathcal{X} , the sequence $X(\omega_n)$ has a convergent subsequence that converges to some $\hat{x} \in \mathcal{X}$. By Lemma D, $V(\omega_0, \omega_0, X(\omega_n)) \rightarrow V(\omega_0, \omega_0, X(\omega_0))$. By Assumption 3, $V(\omega_0, \omega_0, \cdot)$ is strictly monotone in x . The continuity of V therefore implies that $\hat{x} = X(\omega_0)$. Hence, X is continuous at ω_0 . From Assumption 3, and Lemma A, X is differentiable at ω_0 . The differential equation (18) is obtained by re-arranging the differential equation in Lemma A. \blacksquare

Theorem 3.1. *Suppose Assumptions 1 and 2 hold. For any $\omega \in \Omega$, if $X(\omega) = X^{FB}(\omega)$ then X is continuous at ω .*

Theorem 3.2. *Suppose Assumptions 1 and 2 hold. For any $\omega_0 \in \text{int}(\Omega)$, if $V(\omega_0, \omega_0, \cdot)$ is a monotone function in x , and if either $X(\omega_0) \neq X^{FB}(\omega_0)$ or $V_2(\omega_0, \omega_0, X^{FB}(\omega_0)) \neq 0$, then X is differentiable at ω_0 .*

Proofs of Theorem 3.1 and 3.2. Consider $\omega_0 \in \text{int}(\Omega)$ and fix a compact neighborhood N in Ω containing ω_0 . Consider a sequence $\omega_n \rightarrow \omega_0$ in N . By Lemma C, the sequence $X(\omega_n)$ has a convergent subsequence that converges to an $\hat{x} \in \text{cl}(X(N))$. By Lemma D, on that subsequence,

$$V(\omega_0, \omega_0, X(\omega_n)) \rightarrow V(\omega_0, \omega_0, X(\omega_0)). \quad (\text{A.3})$$

Assumptions 1 and 2 imply that $V(\omega_0, \omega_0, x)$ is strictly quasi-concave in x , with a unique maximum at $x = X^{FB}(\omega_0)$. Equation (A.3) then implies that X must be continuous at ω_0 when $X(\omega_0) = X^{FB}(\omega_0)$.

If $V(\omega, \omega, \cdot)$ is monotone, it is strictly monotone (and so one-to-one) because of Assumption 2. Hence, $\hat{x} = X(\omega_0)$, and so X is continuous at ω_0 . The second result then follows from Lemmas A and B. ■

Lemma E *Suppose Assumptions 1 and 2 hold. If X is discontinuous at some point ω_0 , then there are two possible actions $x' < x''$ such that for all sequences $\omega_n \rightarrow \omega_0$, the set $\{X(\omega_n)\}$ is bounded and every convergent subsequence of $\{X(\omega_n)\}$ converges to either x' or x'' . The type ω_0 is indifferent between the two limit actions, $X(\omega_0) = x'$ or x'' , and $x'^{FB}(\omega_0) < x''$.*

Proof. Suppose X is discontinuous at ω_0 and fix a compact neighborhood in Ω around ω_0 , $[\omega_0 - \eta, \omega_0 + \eta] \cap \Omega$. Consider two sequences, $\{\omega_n^-\}$ and $\{\omega_n^+\}$, in $[\omega_0 - \eta, \omega_0 + \eta] \cap \Omega$.

If the sequences $X(\omega_n^-)$ and $X(\omega_n^+)$ do not converge to $X(\omega_0)$ they have, by Lemma C, convergent subsequences that converge to x^- and x^+ , respectively, and we restrict attention to these subsequences.

Incentive-compatibility implies

$$\begin{aligned} V(\omega_n^-, \omega_n^-, X(\omega_n^-)) &\geq V(\omega_n^-, \omega_n^+, X(\omega_n^+)) \\ \text{and} \quad V(\omega_n^+, \omega_n^+, X(\omega_n^+)) &\geq V(\omega_n^+, \omega_n^-, X(\omega_n^-)). \end{aligned}$$

Taking limits and comparing the two inequalities gives

$$V(\omega_0, \omega_0, x^-) = V(\omega_0, \omega_0, x^+). \quad (\text{A.4})$$

A similar exercise with ω_0 replacing ω_n^- shows that

$$V(\omega_0, \omega_0, X(\omega_0)) = V(\omega_0, \omega_0, x^+).$$

The strict quasi-concavity of V (Assumptions 1 and 2) implies that the equation $V(\omega_0, \omega_0, x) = k$ can have at most two distinct solutions in x . Let x' and x'' denote these two solutions, with one of them equaling $X(\omega_0)$.

Equation (A.4), with the strict quasi-concavity, also implies that $X^{FB}(\omega_0)$ lies strictly between x' and x'' . ■

Lemma F *Suppose that Assumptions 1 and 2 hold, $V_{13}(\omega, \omega, x) \neq 0$ and $V_2(\omega, \omega, X^{FB}(\omega)) \neq 0$ for all $(\omega, x) \in \Omega^* \times \mathcal{X}$, where Ω^* is a connected subset of Ω . Then X can have at most one point of discontinuity ω_0 in Ω^* . At the discontinuity,*

1. X is continuous from either the left or the right,
2. the left-hand and the right-hand limits exist, and
3. the jump of X is of the same sign as V_{13} , i.e.,

$$\left(\lim_{\omega \searrow \omega_0} X(\omega) - \lim_{\omega \nearrow \omega_0} X(\omega) \right) \cdot V_{13} > 0. \quad (\text{A.5})$$

Proof. Define, for arbitrary $\omega, \hat{\omega} \in \Omega$ and $x \in \mathcal{X}$,

$$g(\omega, \hat{\omega}, x) \equiv V(\omega, \hat{\omega}, x) - V(\omega, \omega_0, X(\omega_0)).$$

Since V is \mathcal{C}^2 , g is \mathcal{C}^2 on $\Omega^2 \times \mathcal{X}$. Moreover,

$$g(\omega, \omega_0, X(\omega_0)) = g_1(\omega, \omega_0, X(\omega_0)) = 0, \quad \forall \omega. \quad (\text{A.6})$$

Incentive compatibility implies

$$g(\omega_0, \omega, X(\omega)) \leq 0 \quad (\text{A.7})$$

$$\text{and } g(\omega, \omega, X(\omega)) \geq 0. \quad (\text{A.8})$$

For any $\omega \in \Omega$ and $\lambda \in [0, 1]$, define

$$[\omega; \lambda]_1 \equiv (\lambda \omega_0 + (1 - \lambda)\omega, \omega, X(\omega))$$

and for any $\mu \in [0, 1]$, define

$$[\omega; \mu]_{23} \equiv (\omega_0, \mu \omega_0 + (1 - \mu)\omega, \mu X(\omega_0) + (1 - \mu)X(\omega)).$$

Expanding $g(\omega, \omega, X(\omega))$ around $(\omega_0, \omega, X(\omega))$ to the second order yields

$$g(\omega, \omega, X(\omega)) = g(\omega_0, \omega, X(\omega)) + g_1(\omega_0, \omega, X(\omega))(\omega - \omega_0) + \frac{1}{2}g_{11}([\omega; \lambda]_1)(\omega - \omega_0)^2$$

for some $\lambda \in [0, 1]$. Next, expanding $g_1(\omega_0, \omega, X(\omega))$ around $(\omega_0, \omega_0, X(\omega_0))$ to the first order yields

$$g_1(\omega_0, \omega, X(\omega)) = g_1(\omega_0, \omega_0, X(\omega_0)) + g_{12}([\omega; \mu]_{23})(\omega - \omega_0) + g_{13}([\omega; \mu]_{23})(X(\omega) - X(\omega_0))$$

for some $\mu \in [0, 1]$. Because the first term on the right-hand side is 0 by (A.6), combining these two expressions yields

$$\begin{aligned} g(\omega, \omega, X(\omega)) &= g(\omega_0, \omega, X(\omega)) \\ &\quad + (\omega - \omega_0) \left\{ \left[\frac{1}{2} g_{11}([\omega; \lambda]_1) + g_{12}([\omega; \mu]_{23}) \right] (\omega - \omega_0) \right. \\ &\quad \left. + g_{13}([\omega; \mu]_{23})(X(\omega) - X(\omega_0)) \right\}. \end{aligned} \quad (\text{A.9})$$

Expressions (A.7), (A.8), and (A.9) imply

$$\begin{aligned} 0 \geq g(\omega_0, \omega, X(\omega)) &\geq -(\omega - \omega_0) \left\{ \left[\frac{1}{2} g_{11}([\omega; \lambda]_1) + g_{12}([\omega; \mu]_{23}) \right] (\omega - \omega_0) \right. \\ &\quad \left. + g_{13}([\omega; \mu]_{23})(X(\omega) - X(\omega_0)) \right\} \end{aligned} \quad (\text{A.10})$$

for some $\lambda, \mu \in [0, 1]$.

Suppose X is discontinuous at $\omega_0 \in \Omega^*$. By Lemma E, there exists $x_0 \neq X(\omega_0)$ and a sequence $\{\omega_n\} \subset \Omega^*$ with $\omega_n \rightarrow \omega_0$ satisfying $X(\omega_n) \rightarrow x_0$. Without loss of generality, we may assume $\omega_n > \omega_0$ for all n , or $\omega_n < \omega_0$ for all n .

Suppose $\omega_n > \omega_0$ for all n . Focusing on the left-most and right-most term of the inequality chain (A.10), setting $\omega = \omega_n$, and dividing through by $\omega_n - \omega$ yields

$$\left[\frac{1}{2} g_{11}([\omega; \lambda]_1) + g_{12}([\omega; \mu]_{23}) \right] (\omega_n - \omega_0) + g_{13}([\omega; \mu]_{23})(X(\omega_n) - X(\omega_0)) \geq 0. \quad (\text{A.11})$$

Since $g_{13}([\omega; \mu]_{23}) = V_{13}([\omega; \mu]_{23})$, in the limit, this implies

$$(x_0 - X(\omega_0))V_{13}^0 > 0, \quad (\text{A.12})$$

where

$$V_{13}^0 = V_{13}(\omega_0, \omega_0, \mu X(\omega_0) + (1 - \mu)x_0)$$

and the inequality is strict because $x_0 \neq X(\omega_0)$ and $V_{13}(\omega, \omega, x) \neq 0$.

Suppose now that $\omega_n < \omega_0$ for all n . In that case, the inequality in (A.11) is reversed, and as is the *strict* inequality in (A.12). This implies that at a point of discontinuity, the value x_0 can only be the limiting value from one side. We now argue that x_0 is the limiting value for all such sequences.

Suppose $\omega_n \searrow \omega_0$ and $X(\omega_n) \rightarrow x_0 \neq X(\omega_0)$, and $V_{13}^0 > 0$ (the other cases are handled similarly). Then, from (A.12), $x_0 > X(\omega_0)$. Suppose there is another sequence $\tilde{\omega}_n \searrow \omega_0$ with $X(\tilde{\omega}_n) \rightarrow X(\omega_0)$. Since X^{FB} is continuous, for large n and small η , we have $X(\tilde{\omega}_n) < X^{FB}(\omega) < X(\omega_n)$ for all n and all $\omega \in [\omega_0, \omega_0 + \eta]$. Fix ω_n and $\tilde{\omega}_m$ with $\omega_0 < \omega_n < \tilde{\omega}_m < \omega_0 + \eta$. Theorem 3.1 and Lemma B imply that for all $\omega \in [\omega_n, \tilde{\omega}_m]$, $X(\omega) \neq X^{FB}(\omega)$.

Since for all $\omega \in [\omega_n, \tilde{\omega}_m]$, we have $X(\tilde{\omega}_m) < X^{FB}(\omega) < X(\omega_n)$, X is discontinuous at some $\omega \in [\omega_n, \tilde{\omega}_m]$ with some left limit (i.e., for some sequence $\omega_n^\dagger \nearrow \omega$) exceeding $X^{FB}(\omega)$ and some right limit being less than $X^{FB}(\omega)$. But this contradicts (A.12).

Hence, X is continuous from either the left or the right, and (A.5) holds.

Since X^{FB} is continuous and $X(\omega) \neq X^{FB}(\omega)$ (from Lemmas B and E), X can have at most one discontinuity on Ω^* . \blacksquare

Theorem 3.3. *Suppose Assumptions 1 and 2 hold, $V_{13}(\omega, \omega, x) \neq 0$ for all $(\omega, x) \in \Omega^* \times \mathcal{X}$, $V_2(\omega, \hat{\omega}, X(\hat{\omega})) \neq 0$ for all $\omega, \hat{\omega} \in \Omega^*$, and that $V_3(\omega, \hat{\omega}, X(\hat{\omega}))/V_2(\omega, \hat{\omega}, X(\hat{\omega}))$ is a strictly monotone function of $\omega \in \Omega^*$ for all $\hat{\omega} \in \Omega^*$, where Ω^* is a connected subset of Ω . Then X is differentiable on $\text{int}(\Omega^*)$.*

Proof. The result follows from Lemma B, once we have proved that X is continuous in the interior of Ω^* .

Suppose that X is discontinuous at $\omega_0 \in \text{int}(\Omega^*)$. Assume that $V_{13}(\omega, \omega, x) > 0$ (the other case is analogous). By Lemma F, X is continuous for all $\omega \neq \omega_0$ and so, by Lemma B, differentiable at all such $\omega \in \text{int}(\Omega^*)$ with derivative

$$X'(\omega) = -\frac{V_2(\omega, \omega, X(\omega))}{V_3(\omega, \omega, X(\omega))}.$$

Since by Lemmas E and F, $X(\omega)$ is strictly smaller than the first-best for $\omega < \omega_0$ and strictly greater for $\omega > \omega_0$, we have $(\omega - \omega_0)V_3(\omega, \omega, X(\omega)) < 0$ for $\omega \neq \omega_0$, and so $(\omega - \omega_0)X'(\omega)V_2(\omega, \omega, X(\omega)) > 0$ for $\omega \neq \omega_0$. Since $V_2(\omega, \omega, X(\omega))$ does not change sign on Ω^* by assumption, X' has one sign for $\omega < \omega_0$ and the other sign for $\omega > \omega_0$.

By assumption, $V_3(\omega, \hat{\omega}, X(\hat{\omega}))/V_2(\omega, \hat{\omega}, X(\hat{\omega}))$ is a strictly monotone function of $\omega \in \Omega^*$, and so

$$X'(\omega)V_2(\omega, \hat{\omega}, X(\hat{\omega}))\frac{\partial}{\partial \omega} \left[\frac{V_3(\omega, \hat{\omega}, X(\hat{\omega}))}{V_2(\omega, \hat{\omega}, X(\hat{\omega}))} \right] < 0 \quad (\text{A.13})$$

for ω either below or above ω_0 .

Suppose it is the former (the latter is handled similarly). Choose arbitrary $\omega' < \omega'' < \omega_0$ in $\text{int}(\Omega^*)$. Consider $V(\omega''^{-1}(x), x)$ as a function of x . By the differentiability of X and the Intermediate Value Theorem, there exists an $\bar{\omega} \in (\omega', \omega'')$ such that

$$\begin{aligned} V(\omega'', \omega', X(\omega')) &= V(\omega'', \omega'', X(\omega'')) - \left\{ V_2(\omega'', \bar{\omega}, X(\bar{\omega})) \frac{dX^{-1}(X(\bar{\omega}))}{dx} \right. \\ &\quad \left. + V_3(\omega'', \bar{\omega}, X(\bar{\omega})) \right\} (X(\omega'') - X(\omega')) \end{aligned}$$

$$\begin{aligned}
&= V(\omega'', \omega'', X(\omega'')) - \left\{ V_2(\omega'', \bar{\omega}, X(\bar{\omega})) \left(-\frac{V_3(\bar{\omega}, \bar{\omega}, X(\bar{\omega}))}{V_2(\bar{\omega}, \bar{\omega}, X(\bar{\omega}))} \right) \right. \\
&\quad \left. + V_3(\omega'', \bar{\omega}, X(\bar{\omega})) \right\} (X(\omega'') - X(\omega')) \\
&= V(\omega'', \omega'', X(\omega'')) - \left\{ \frac{V_3(\omega'', \bar{\omega}, X(\bar{\omega}))}{V_2(\omega'', \bar{\omega}, X(\bar{\omega}))} - \frac{V_3(\bar{\omega}, \bar{\omega}, X(\bar{\omega}))}{V_2(\bar{\omega}, \bar{\omega}, X(\bar{\omega}))} \right\} \\
&\quad \times V_2(\omega'', \bar{\omega}, X(\bar{\omega})) (X(\omega'') - X(\omega')) \\
&= V(\omega'', \omega'', X(\omega'')) - \int_{\bar{\omega}}^{\omega''} \frac{\partial}{\partial \omega} \left(\frac{V_3(\omega, \bar{\omega}, X(\bar{\omega}))}{V_2(\omega, \bar{\omega}, X(\bar{\omega}))} \right) d\omega \\
&\quad \times V_2(\omega'', \bar{\omega}, X(\bar{\omega})) (X(\omega'') - X(\omega')) \\
&> V(\omega'', \omega'', X(\omega'')).
\end{aligned}$$

The strict inequality (which follows from (A.13) and $\bar{\omega} < \omega''$) contradicts incentive compatibility. \blacksquare

Lemma G *Suppose Assumptions 1 and 2 hold. Let Ω_0 be an open subset of Ω on which X is differentiable. Assume that $V_2(\omega, \omega, X(\omega)) \neq 0$ for all $\omega \in \Omega_0$. Then*

$$V_{12}(\omega, \omega, X(\omega)) + X'(\omega)V_{13}(\omega, \omega, X(\omega)) \geq 0 \quad (\text{A.14})$$

for all $\omega \in \Omega_0$.

Proof. By Lemma A, $X'(\omega) \neq 0$ for all $\omega \in \Omega_0$, and so X^{-1} is differentiable for all $x \in X(\Omega_0)$. Since $X(\omega)$ must maximize $V(\omega, X^{-1}(x), x)$, the implied first order condition evaluated at $\omega = X^{-1}(x)$,

$$V_2(X^{-1}(x), X^{-1}(x), x) \frac{dX^{-1}(x)}{dx} + V_3(X^{-1}(x), X^{-1}(x), x) = 0,$$

must hold for all $x \in X(\Omega_0)$. Since the equation is an identity, the first derivative must also equal zero:

$$[V_{12} + V_{22}] \left(\frac{dX^{-1}}{dx} \right)^2 + [V_{13} + 2V_{23}] \frac{dX^{-1}}{dx} + V_2 \frac{d^2 X^{-1}}{dx^2} + V_{33} = 0, \quad (\text{A.15})$$

where all the partial derivatives of V are evaluated at $(X^{-1}(x), X^{-1}(x), x)$.

The second derivative of $V(\omega, X^{-1}(x), x)$ is

$$\frac{d}{dx^2} V(\omega, X^{-1}(x), x) = V_{22} \left(\frac{dX^{-1}}{dx} \right)^2 + 2V_{23} \frac{dX^{-1}}{dx} + V_2 \frac{d^2 X^{-1}}{dx^2} + V_{33}, \quad (\text{A.16})$$

where now all partial derivatives are evaluated at $(\omega, X^{-1}(x), x)$. Evaluating (A.16) at $\omega = X^{-1}(x)$, and substituting from (A.15), yields

$$\begin{aligned} \frac{d}{dx^2} V(X^{-1}(x), X^{-1}(x), x) &= -V_{12} \left(\frac{dX^{-1}}{dx} \right)^2 - V_{13} \frac{dX^{-1}}{dx} \\ &= - \left(\frac{dX^{-1}}{dx} \right)^2 (V_{12} + X'V_{13}). \end{aligned} \quad (\text{A.17})$$

Since $V(\omega, X^{-1}(x), x)$ must have a local maximum at $x = X(\omega)$, the right-hand side of (A.17) must be (weakly) negative for all $\omega \in \Omega_0$, which yields (A.14). \blacksquare

Theorem 3.4. *Suppose Assumptions 1 and 2 hold, $V_{13}(\omega, \omega, x) \neq 0$ and $V_{12}(\omega, \omega, x) \leq 0$ for all $(\omega, x) \in \Omega \times \mathcal{X}$. If $V_2(\omega, \omega, X(\omega)) > 0$ or if $V_2(\omega, \omega, X(\omega)) < 0$ for all ω in an open subset $\Omega_0 \subset \Omega$, then X is differentiable on Ω_0 .*

Proof. By Lemma F, X can have at most one discontinuity in Ω_0 , say ω_0 . By Lemma A, X is differentiable on $\Omega_0 \setminus \{\omega_0\}$, and by Lemma G, it satisfies (A.14) there.

Since $V_{12} \leq 0$ and $V_{13} \neq 0$, (A.14) immediately implies that X' must have the same sign as V_{13} for all $\omega \in \Omega_0 \setminus \{\omega_0\}$.

Recall that for all $\omega \in \Omega_0 \setminus \{\omega_0\}$,

$$\begin{aligned} X(\omega) > X^{FB}(\omega) &\Leftrightarrow V_3(\omega, \omega, X(\omega)) < 0 \\ &\Leftrightarrow X'(\omega)V_2(\omega, \omega, X(\omega)) > 0, \end{aligned} \quad (\text{A.18})$$

where the first equivalence follows from the definition of the first-best and the second from the form of the derivative of X derived in Lemma A. By Lemma E, we have both $X(\omega'^{FB}(\omega'))$ and $X(\omega''^{FB}(\omega''))$ for some $\omega', \omega'' \in \Omega_0$ (since ω_0 is a point of discontinuity). Since V_2 does not change sign on Ω_0 , X' does. But X' must have the same sign as V_{13} on $\Omega_0 \setminus \{\omega_0\}$, which does not change its sign by assumption. Contradiction. \blacksquare

Theorem 3.5. *Suppose Assumptions 1 and 2 hold and $V_{13}(\omega, \omega, x) \neq 0$ for all $(\omega, x) \in \Omega \times \mathcal{X}$.*

(i) *Assume that $\Omega = [\omega_1, \omega_2]$ or $\Omega = [\omega_1, \infty)$ and that $X(\omega_1) = X^{FB}(\omega_1)$. If $V_2(\omega, \omega, X(\omega)) > 0$ for all $\omega \in \Omega$ then X is differentiable on Ω ,*

(ii) *Assume that $\Omega = [\omega_1, \omega_2]$ or $\Omega = (-\infty, \omega_2]$ and that $X(\omega_2) = X^{FB}(\omega_2)$. If $V_2(\omega, \omega, X(\omega)) < 0$ for all $\omega \in \Omega$ then X is differentiable on Ω .*

Proof. Assume that $V_2(\omega, \omega, X(\omega)) > 0$ for all $\omega \geq \omega_1$ (the other case is handled similarly). Suppose that X is discontinuous at $\omega_0 > \omega_1$ (by Lemma F, there can be no other discontinuity). Denote $\Omega_0 = \{\omega \in \Omega; \omega > \omega_1, \omega \neq \omega_0\}$. By Lemma A, X is differentiable on Ω_0 , and by Lemma G, it satisfies (A.14) there.

Since $V_2(\omega^w, \omega^w, X(\omega^w)) \neq 0$, by assumption, the continuity of X at ω_1 and Lemma A imply that $|X'(\omega)| \rightarrow \infty$ for $\omega \rightarrow \omega_1$. Hence, (A.14) implies that X' must have the same sign as V_{13} for $\omega \in \Omega_0$ sufficiently close to ω_1 .

As in (A.18), we now have for all $\omega \in \Omega_0$

$$X(\omega) > X^{FB}(\omega) \Leftrightarrow X'(\omega) > 0.$$

But according to Lemma F, at the discontinuity, the jump of X has the same sign as V_{13} , which is impossible. ■

B Proof of Theorem 4

First note that it suffices to prove the result assuming Ω is open (since if Ω includes a boundary, and X is incentive compatible on the interior of Ω , then continuity implies X is incentive compatible on Ω).

1. (Global sufficiency of single crossing for IC.) Since X satisfies (DE), it satisfies the first order condition implied by (IC), and so satisfies (IC) if

$$\frac{d}{dx} V(\omega, X^{-1}(x), x) \cdot (x - X(\omega)) \leq 0 \quad \forall x \in X(\Omega), \omega \in \Omega. \quad (\text{B.19})$$

The derivative equals

$$\begin{aligned} & V_2(\omega, X^{-1}(x), x) \left(\frac{dX}{d\hat{\omega}} \Big|_{X^{-1}(x)} \right)^{-1} + V_3(\omega, X^{-1}(x), x) \\ &= V_2(\omega, X^{-1}(x), x) \frac{-V_3(X^{-1}(x), X^{-1}(x), x)}{V_2(X^{-1}(x), X^{-1}(x), x)} + V_3(\omega, X^{-1}(x), x) \\ &= V_2(\omega, X^{-1}(x), x) \left\{ \frac{V_3(\omega, X^{-1}(x), x)}{V_2(\omega, X^{-1}(x), x)} - \frac{V_3(X^{-1}(x), X^{-1}(x), x)}{V_2(X^{-1}(x), X^{-1}(x), x)} \right\}. \end{aligned}$$

If X is strictly increasing and $V_2(\omega, \omega, X(\omega)) > 0$ (the other possibilities handled *mutatis mutandis*), (B.19) is satisfied when $V_3(\omega, \hat{\omega}, X(\hat{\omega}))/V_2(\omega, \hat{\omega}, X(\hat{\omega}))$ is an increasing function of ω for all $\hat{\omega} \in \Omega$.

2. (Local necessity of single crossing for IC.) Suppose X satisfies (IC). The second order condition is

$$\left. \frac{d^2}{dx^2} V(\omega, X^{-1}(x), x) \right|_{x=X(\omega)} \leq 0,$$

which is, after substituting for $d^2 X^{-1}(x)/dx^2$,

$$-\frac{dX^{-1}}{dx} \cdot \left\{ V_{12}(\omega, \omega, X(\omega)) \frac{dX^{-1}}{dx} + V_{13}(\omega, \omega, X(\omega)) \right\} \leq 0,$$

i.e.,

$$-\frac{dX^{-1}}{dx} \cdot \left\{ V_{13}(\omega, \omega, X(\omega)) - V_{12}(\omega, \omega, X(\omega)) \frac{V_3(\omega, \omega, X(\omega))}{V_2(\omega, \omega, X(\omega))} \right\} \leq 0.$$

Multiplying both sides of this inequality by $-(X')^2$ yields an expression equivalent to (20).

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