Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://economics.sas.upenn.edu/pier

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"A Monetary Theory with Non-Degenerate Distributions"

by

Guido Menzio, Shouyong Shi and Hongfei Sun

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A Monetary Theory with Non-Degenerate Distributions*

Guido Menzio
University of Pennsylvania
(gmenzio@sas.upenn.edu)

Shouyong Shi
University of Toronto
(shouyong@chass.utoronto.ca)

Hongfei Sun
Queen's University
(hfsun@econ.queensu.ca)

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Abstract

Dispersion of money balances among individuals is the basis for a range of policies but it has been abstracted from in monetary theory for tractability reasons. In this paper, we fill in this gap by constructing a tractable search model of money with a non-degenerate distribution of money holdings. We assume search to be directed in the sense that buyers know the terms of trade before visiting particular sellers. Directed search makes the monetary steady state block recursive in the sense that individuals' policy functions, value functions and the market tightness function are all independent of the distribution of individuals over money balances, although the distribution affects the aggregate activity by itself. Block recursivity enables us to characterize the equilibrium analytically. By adapting lattice-theoretic techniques, we characterize individuals' policy and value functions, and show that these functions satisfy the standard conditions of optimization. We prove that a unique monetary steady state exists. Moreover, we provide conditions under which the steady-state distribution of buyers over money balances is non-degenerate and analyze the properties of this distribution.

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*Menzio: Department of Economics, University of Pennsylvania, 3718 Locust Walk Philadelphia, Pennsylvania 19104, USA. Shi: Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ontario, Canada, M5S 3G7; Sun: Department of Economics, Queen's University, 94 University Ave., Kingston, Ontario, Canada, K7L 3N6. We have received helpful comments from participants of seminars and conferences at the Society for Economic Dynamics (Istanbul, 2009 and Montreal, 2010), U. of Calgary (2010), Tsinghua Macro Workshop (Beijing, 2010), Singapore Management U. (2010), National Taiwan U. (2010), Chicago FED Conference on Money, Banking, Payments and Finance (Chicago, 2009 and 2010), Research and Money and Markets (Toronto, 2009), the Society for the Advancement of Economic Theory (Ischia, Italy, 2009) and the Texas Monetary Conference (Austin, 2009). Shi and Sun gratefully acknowledge financial support from the Social Sciences and Humanities Research Council of Canada. Shi also acknowledges financial support from the Bank of Canada Fellowship and the Canada Research Chair. The opinion expressed in the paper is our own and does not reflect the view of the Bank of Canada. All errors are ours.

1. Introduction

Money is unevenly distributed among individuals at any given point of time. Because this distribution implies dispersion in individuals' marginal value of money and consumption, the distribution has important implications for the efficiency of resource allocation and is the basis for a range of policies. For example, many central banks use open market operations and overnight markets to supply liquidity or channel liquidity from one set of individuals to another. Despite this importance of a non-degenerate distribution of money holdings, monetary theory has often abstracted from it, largely for tractability reasons. To fill in this gap between theory and policy, we construct a tractable model with a microfoundation of money and a non-degenerate distribution of money holdings. We prove that a unique monetary steady state exists and analyze its properties.

The microfoundation of money we refer to is the so-called search theory of money, pioneered by Kiyotaki and Wright (1989). This is a natural framework to use to study the role of the distribution of money holdings. It endogenously generates a positive value for fiat money, an object with no intrinsic value. The framework models exchange as a decentralized process in which each trade involves only a small group (usually two) of anonymous individuals who do not have a double coincidence of wants. In this environment, fiat money facilitates exchange. In addition, decentralized exchange naturally induces a non-degenerate distribution of buyers over money balances. Two individuals with the same amount of money may meet trading partners who differ in money holdings, tastes, and productivity, in which case they trade away different amounts of money. Thus, even if all individuals hold the same amount of money initially, the distribution of buyers over money balances can fan out as the exchange continues.

It has been a challenge to characterize an equilibrium with such a non-degenerate distribution while keeping the model non-trivial for macro analysis. The difficulty lies in the endogeneity and the potentially large dimensionality of the distribution. The distribution of money holdings is an aggregate state variable that can affect individuals' trading decisions in general. In turn, the trading decisions of all individuals together affect the evolution of the distribution. An equilibrium typically needs to determine individuals' decisions and the aggregate distribution simultaneously. This is a difficult task because the distribution can potentially have a large dimension. To avoid the difficulty, earlier search models restrict individuals to hold either zero or one unit of money (e.g., Shi, 1995, and Trejos and Wright, 1995). This restriction not only makes the distribution of buyers over money balances degenerate, but also makes the analysis of some policies contrived because it artificially ties the number of money holders to the money stock in the economy. In more recent attempts, Shi (1997) and Lagos and Wright (2005) offer tractable models where money and goods are fully divisible. However, Shi (1997) assumes that each household consists of a large number of members who share consumption and utility, and Lagos and Wright (2005) assume that individuals have quasi-linear preferences over a good which can be traded in a centralized market to immediately rebalance money holdings. Both assumptions make the distribution of money balances among the households degenerate.

In this paper, we construct a monetary search model where money distribution can be non-degenerate. The main deviation from the literature lies in the way we model search. The monetary search literature assumes search to be undirected in the sense that individuals do not know the terms of trade before they are matched. In contrast, we assume search to be directed in the sense that individuals know the terms of trade before a match, as in Peters (1991), Moen (1997), Acemoglu and Shimer (1999), and Burdett, Shi and Wright (2001). In particular, for each type of good, there is a continuum of submarkets, each of which specifies the terms of trade and a tightness (i.e., the ratio of trading posts to buyers). Buyers choose which submarket to enter and firms choose how many trading posts to create in each submarket. There is a cost of creating a trading post for a period, and the number of trading posts in each submarket is determined endogenously by free entry. Once inside a submarket, buyers and trading posts are brought into bilateral meetings through a frictional matching function that has constant returns to scale. The matching probability for a buyer or a trading post is a function of the tightness of the submarket. In equilibrium, the tightness in each submarket is consistent with buyers' choices on which submarket to enter and firms' choices on the creation of trading posts.

Directed search allows buyers to go directly to sellers who sell the goods they want. More importantly, directed search allow buyers with different money holdings to optimally sort into submarkets that differ in the terms of trade. Specifically, because the marginal value of money is lower to a buyer who has a relatively high money balance, such a buyer has a strong desire to spend a relatively large amount of money on consumption goods and to spend it sooner than later. To satisfy this desire, the buyer chooses to enter a submarket where he has a relatively high matching probability to trade a relatively large amount of money for a large quantity of goods. Firms cater to this desire by creating a relatively large number of trading posts per buyer in this submarket. Because buyers with different money holdings choose not to mix with each other, a buyer's optimal choices depend on the buyer's own money balance and the tightness of the submarket he will enter, but not on the distribution of individuals over money balances. Moreover, because each submarket is tailored to only one group of buyers with a particular money balance, the tightness of each submarket that ensures zero profit for a trading post does not depend on the distribution of money holdings. Precisely, individuals' policy functions, value functions and the market tightness function are all independent of the distribution of individuals over money holdings. We refer to this feature of the equilibrium as *block recursivity*.

Block recursivity makes the analytical characterization of the equilibrium tractable. Although the distribution of individuals affects the aggregate activity, it is not part of the state space in individuals' decision problems. As a result, we can characterize an individual's policy and value functions as functions of only the individual's own money balance. Having done so for each money balance separately, we can compute the net flows of individuals across money balances to obtain money distribution. In the equilibrium, an individual goes through a purchasing cycle. When the individual has no money, he works to obtain money and then becomes a buyer. Starting with a high money balance, a buyer enters a submarket where he has a high matching probability, spends a large amount of money and obtains a large quantity of goods. For the next trade, the buyer will go into a submarket where the matching probability is lower, the required spending is lower and the quantity of goods obtained in a trade is lower. The buyer will continue this pattern until he depletes his balance, at which point he will work again.

The analytical characterization of the equilibrium enables us to prove that a unique monetary steady state exists, to determine when the steady-state distribution of buyers over money balances is non-degenerate, and to analyze the properties of this distribution. The distribution is degenerate when individuals are sufficiently impatient. In this case, all buyers hold the same amount of money and spend the entire amount in one trade. Although this result provides a case to rationalize the behavioral pattern assumed in models with indivisible money (e.g., Shi, 1995, and Trejos and Wright, 1995), our model does not share a key result on policy analysis with those models. That is, a one-time change in the money stock affects real activities in models with indivisible money, but it is neutral in the steady state in our model regardless of whether the distribution is degenerate or not.

The steady-state distribution of buyers over money balances is non-degenerate if individuals are sufficiently patient, if the utility function of consumption is sufficiently concave, if the disutility function of labor supply is not very convex, and if the cost of creating a trading post is low. In subsection 4.3 we will explain intuitively why these conditions are needed for money distribution to be non-degenerate. Moreover, the distribution has a particular shape. Starting from the highest money balance in the equilibrium, buyers go through a sequence of trades before running out of money. The frequency function of the distribution of buyers is a decreasing function of money holdings in this sequence, because the buyers who hold a high balance trade relatively quickly and exit from that balance.

A large part of this paper is devoted to the analysis of a buyer's decision problem. This analysis is necessary here because it establishes the properties of the policy and value functions that are needed for block recursivity. The analysis is of independent interest because it provides a set of analytical tools to overcome some difficulties in the use of dynamic programming. The difficulties are that a buyer's objective function is not concave and that a buyer's value function cannot be assumed to be differentiable a priori. These difficulties prevent us from using the standard approach in dynamic programming (e.g., Stokey et al, 1989) to analyze the policy and value functions. To overcome these difficulties, we adapt lattice-theoretic techniques (see Topkis, 1998) to prove that a buyer's policy functions are monotone functions of the buyer's money balance. Using this result, we prove further that the optimal choices obey the first-order conditions, the value functions are differentiable and the envelope conditions hold. By validating these standard conditions, we make the model easy to use. In addition, this procedure of analyzing a dynamic programming problem is intuitive and the techniques used are likely to apply to a variety of dynamic models that involve both discrete and continuous choices.

Our paper is related to the literature on directed search cited earlier, most of which studies non-monetary economies. In this literature, Shi (2009) studies a block recursive equilibrium, which is explored further by Menzio and Shi (2008, 2010) and Gonzalez and Shi (2010). In particular, our use of lattice-theoretic techniques in dynamic programming with non-concave objective functions has similarities to that in Gonzalez and Shi (2010). The monetary issues in our paper are obviously different from the issues in labor search. Also, a monetary equilibrium is more challenging to characterize than a non-monetary labor equilibrium. First, in a monetary model, an individual's gain from a match depends not only on how the match surplus is split, but also on how all individuals in the economy value money. The equilibrium must determine this value of money. Second, money balance is a state variable in an individual's decision problem and it can be accumulated or decumulated over time through trade. Third, a buyer's objective function is not supermodular, which prevents a straightforward application of lattice-theoretic techniques. We overcome this difficulty by decomposing a buyer's decision problem into several steps and applying lattice-theoretic techniques in each step.

In the money literature, Corbae et al (2003) assume search to be directed, but they focus on the formation of trading coalitions and assume that money and goods are indivisible. Also, Rocheteau and Wright (2005) check the robustness of their model to the use of directed search, and Galenianos and Kircher (2008) and Julien et al (2008) examine directed search with auctions. These papers do not formulate a block recursive equilibrium. Moreover, money distribution in these papers is either degenerate or temporary which does not have important wealth effects.[1] In the money literature with undirected search, Green and Zhou (1998) take the first step to characterize the distribution of money holdings. They restrict goods to be indivisible and money to be in discrete units. By making goods divisible, Zhu (2005) studies a sequence of economies with discrete money and characterizes the limit where the size of discreteness goes to zero. Finally, some authors have numerically computed a monetary equilibrium with a non-degenerate distribution (e.g., Molico, 2006, and Chiu and Molico, 2008).

2. A Monetary Economy with Directed Search

2.1. The model environment

There are I types of individuals and I types of perishable goods indexed by $i \in \{1, 2, ..., I\}$, where $I \geq 3$. Each type i consists of a continuum of individuals with measure one who are specialized in the consumption of good i and the production of good $i+1$ (modulo I). The preferences of a type i individual are represented by the utility function $\sum_{t=0}^{\infty} \beta^t [U(q_t) - h(\ell_t)]$, where $\beta \in (0, 1)$ is the discount factor, $U : \mathbb{R}_+ \to \mathbb{R}$ is the utility of consumption of good i, and $h : [0, 1] \to \mathbb{R}$ is the disutility of labor. We assume that U is strictly increasing, strictly concave and twice continuously differentiable, with the boundary properties: $U(0) = 0$, $U'(\infty) = 0$, and $U'(0)$ is sufficiently large. Similarly, we assume that h is strictly increasing, strictly convex and twice continuously differentiable, with the boundary properties: $h(0) = 0$ and $h'(1) = \infty$.

The economy is also populated by I types of firms. Each type i consists of a large number of firms that are specialized in the production and distribution of good i. A type i firm operates a technology of constant returns to scale that transforms each unit of labor supplied by individuals of type $i-1$ (modulo I) into one unit of good i.[2] Moreover, a type i firm can open a trading post in the market for good i using $k > 0$ units of labor supplied by individuals of type $i-1$ (modulo I). Firms are owned by the individuals through a balanced mutual fund.

In addition to consumption goods, there is an object called fiat money which is intrinsically worthless, perfectly divisible and costlessly storable. In this paper, we focus on the case in which the supply of fiat money per capita, M, is constant over time. To simplify the notation, we choose labor, instead of goods or money, as the numeraire in this model.

[1] As in Lagos and Wright (2005), the three papers assume that each decentralized market is followed by a centralized market where preferences are quasi-linear over a homogeneous good. As a result, any non-degenerate distribution of money holdings induced by the decentralized market becomes degenerate immediately in the ensuing centralized market.

[2] The assumption that the cost of production is linear is made without loss of generality. Because the disutility function of labor supply, $h(.)$, is assumed to be strictly convex, the disutility of producing goods is strictly convex in the quantity of production.

In every period, a labor market and a product market open. Firms can participate in both markets in the same period. In contrast, individuals can participate in either the labor market or the product market. That is, in a given period, individuals must choose whether to become workers or buyers. Before making this choice, individuals can play a fair lottery. Even though individuals are risk averse, a lottery can be desirable because the value function without the lottery can be non-concave at particular money balances. One cause of non-concavity is the discrete nature of the decision on which market to enter. Another cause is the tradeoff between the matching probability and the surplus of trade in the product market, to be described later.

The labor market is centralized and frictionless. Each firm chooses how much labor to demand taking as given the nominal wage rate. Similarly, each worker chooses how much labor to supply taking as given the nominal wage rate. In equilibrium, the nominal wage rate, ωM, equates total demand for labor by all firms to the supply of labor by all workers. We will simply refer to ω as the nominal wage rate. Workers are paid in money instead of goods because they do not want to consume the good produced by the firm in which they work and because goods are perishable between periods. Moreover, a firm cannot pay its employees with an IOU because firms are better off exiting the market than honoring their IOUs.

The product market is decentralized and characterized by search frictions. Buyers and trading posts meet in pairs and there is no record keeping of their actions once they exit a trade. More specifically, the market for each type i good is organized in a continuum of submarkets indexed by the terms of trade $(x, q) \in \mathbb{R}_+ \times \mathbb{R}_+$, to be explained below. Each buyer chooses which submarket to visit in order to find a seller and each firm chooses how many trading posts to open in each submarket in order to meet some buyers. The buyers who visit a submarket and the trading posts in that submarket are brought into contact by a frictional matching process. When a buyer chooses which submarket to visit and a firm chooses how many trading posts to create in a submarket, they take into account the fact that matching probabilities vary with the terms of trade across the submarkets. Hence, the search process is directed as in Moen (1997), Acemoglu and Shimer (1999), Burdett et al (2001) and Delacroix and Shi (2006).

It is clear that type i buyers will choose to participate only in the submarkets where trading posts are created by type i firms. A buyer in submarket (x, q) finds a trading post with probability $b = \lambda(\theta(x, q))$. The function $\lambda : \mathbb{R}_+ \rightarrow [0, 1]$ is a strictly increasing function with boundary conditions $\lambda(0) = 0$ and $\lambda(\infty) = 1$. The function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the ratio of trading posts to buyers in submarket (x, q) which we refer to as the *tightness* of the submarket. Similarly, a trading post located in submarket (x, q) is visited by a buyer with probability $s = \rho(\theta(x, q))$, where $\rho : \mathbb{R}_+ \rightarrow [0, 1]$ is a strictly decreasing function such that $\rho(\theta) = \lambda(\theta)/\theta$, $\rho(0) = 1$ and $\rho(\infty) = 0$.

Since b and s are both functions of θ, we can express a trading post's matching probability as a function of a buyer's matching probability; that is, $s = \mu(b) \equiv \rho(\lambda^{-1}(b))$. Clearly, $\mu(b)$ is a decreasing function. We assume that $1/\mu(b)$ is strictly convex in b.

When a buyer and a seller meet in submarket (x, q), they exchange q units of the consumption good for $x\omega M$ units of fiat money.[3] The buyer must pay the seller with money because neither barter nor credit is feasible. The buyer cannot pay the seller with goods because goods are perishable and there is no double coincidence of wants in goods between the buyer and the seller. Moreover, the buyer cannot pay the seller with an IOU because individuals are anonymous; once they exit a trade, they can renege on their IOUs without fear of retribution. Thus, the amount of money that a buyer can spend in a trade is bounded above by the balance he carries into the trade. Note that x is the traded amount of money measured in units of labor, which we refer to as the *real balance* traded in a match.

2.2. An individual's decisions

Let $V(m)$ denote the lifetime utility of an individual who starts a period with $m\omega M$ units of money, where m is the individual's real balance (in units of labor). We refer to V as the individual's *ex-ante value function*, since it is measured before the individual chooses whether to play a lottery and whether to be a worker or a buyer in the period. Let $B(m)$ denote the lifetime utility of an individual who enters the product market with the real balance m. Similarly, let $W(m)$ denote the lifetime utility of an individual that enters the labor market with the real balance m. We will refer to B as the *buyer's value function* and W as the *worker's value function*.

A worker chooses labor supply, ℓ, where the disutility of labor is $h(\ell)$. The wage income is ℓ units of real balances. In addition to the wage, the individual also owns a diversified portfolio of the firms. However, the return to this ownership is zero since all firms earn zero profit in the equilibrium. Thus, a worker who enters the labor market with a real balance m will have a real balance $m + \ell$ at the end of the period. The discounted value of this balance is $\beta V(m + \ell)$. The value function of the worker, $W(m)$, obeys:

$$W(m) = \max_{\ell \in [0,1]} \left[\beta V(m + \ell) - h(\ell)\right]. \tag{2.1}$$

Denote the optimal choice of ℓ as $\ell^*(m)$ and the implied real balance at the end of the period as $y^*(m) = m + \ell^*(m)$. We refer to $\ell^*(.)$ and $y^*(.)$ as a worker's policy functions.

[3] The price of goods in a submarket alone is an inadequate description of a submarket because a buyer may not spend all the money in a trade.

A buyer chooses which submarket (x, q) to enter. Once in submarket (x, q), the buyer will meet a trading post with probability $\lambda(\theta(x, q))$. In the match, the buyer will trade away a real balance x for q units of goods. The lifetime utility will be $U(q) + \beta V(m - x)$, which consists of the utility of consumption and the discounted value of the residual balance $(m - x)$. With probability $1 - \lambda(\theta(x, q))$, the buyer will not have a match and will hold onto the real balance m which will yield $\beta V(m)$ as the lifetime utility. Because the buyer's choice of x is feasible if and only if $x \in [0, m]$, the value function of the buyer, $B(m)$, obeys:

$$\begin{aligned} B(m) = \quad & \max_{(x,q)} \{\lambda(\theta(x, q)) [U(q) + \beta V(m - x)] + [1 - \lambda(\theta(x, q))] \beta V(m)\} \\ \text{s.t.} \quad & x \in [0, m], \quad q \geq 0. \end{aligned} \tag{2.2}$$

The buyer's optimal choices are represented by the policy functions $(x^*(m), q^*(m))$.

An individual chooses whether to be a worker or a buyer in the period. The value function induced by this choice is:

$$\tilde{V}(m) = \max\{W(m), B(m)\}. \tag{2.3}$$

Notice that $\tilde{V}$ may not be concave over some intervals of the real balance, even when W and B are concave functions. Thus, there is a potential gain to the individual from playing fair lotteries before making the above choice on whether to be a worker or a buyer. Denote a lottery as $(z_j, \pi_j)_{j=1,2}$. With probability π_1 the low prize of the lottery, z_1, is realized, in which case the individual's lifetime utility is $\tilde{V}(z_1)$. With probability π_2 the high prize of the lottery, z_2, is realized, in which case the individual's lifetime utility is $\tilde{V}(z_2)$. Thus, the individual's ex ante value function induced by the lottery choice is:

$$V(m) = \max_{(z_1, z_2, \pi_1, \pi_2)} \left[\pi_1 \tilde{V}(z_1) + \pi_2 \tilde{V}(z_2)\right] \tag{2.4}$$

$$\begin{aligned} \text{s.t.} \quad & \pi_1 z_1 + \pi_2 z_2 = m, \quad \pi_1 + \pi_2 = 1, \quad z_2 \geq z_1, \\ & \pi_j \in [0, 1] \text{ and } z_j \geq 0 \text{ for } j = 1, 2. \end{aligned}$$

Let $(z_j^*(m), \pi_j^*(m))_{j=1,2}$ denote the individual's optimal choice of a lottery.[4]

2.3. A firm's decisions

A firm chooses how many trading posts to create in each submarket and how much labor to employ. The firm's demand for labor is equal to the sum of labor required for producing goods and creating trading posts. For the decision on the creation of trading posts, consider submarket

[4] For any given m, we choose $(z_1(m), z_2(m))$ as the tighest lottery at m to simplify the analysis. That is, $z_1(m)$ is the largest prize smaller than or equal to m, and $z_2(m)$ is the smallest prize greater than or equal to m.

(x, q). The cost of creating a trading post is k units of labor. A trading post in submarket (x, q) will be visited by a buyer with probability $\rho(\theta(x, q))$, in which case the firm uses q units of labor to produce q units of goods and exchanges them for a real balance x. Thus, the expected benefit of creating a trading post in submarket (x, q) is $\rho(\theta(x, q))(x - q)$ units of labor. If $\rho(\theta(x, q))(x - q) < k$, it is optimal for the firm not to create any trading post in submarket (x, q). If $\rho(\theta(x, q))(x - q) > k$, it is optimal for the firm to create infinitely many trading posts in submarket (x, q). If $\rho(\theta(x, q))(x - q) = k$, the firm is indifferent between creating different numbers of trading posts in submarket (x, q).

Notice that the case $\rho(\theta(x, q))(x - q) > k$ never occurs, because the case implies $\theta(x, q) = \infty$ and, hence, $\rho(\theta(x, q)) = 0$, which violates the condition for the case. Thus, in any submarket (x, q) that is visited by a positive number of buyers, the tightness $\theta(x, q)$ is consistent with the firm's incentive to create trading posts if and only if

$$\rho(\theta(x, q))(x - q) \leq k \quad \text{and} \quad \theta(x, q) \geq 0, \tag{2.5}$$

where the two inequalities hold with complementary slackness. In any submarket (x, q) that is not visited by buyers, the tightness can be arbitrary if k is greater than $\rho(\theta(x, q))(x - q)$. However, following Shi (2009), Menzio and Shi (2008, 2010) and Gonzalez and Shi (2010), we restrict attention to equilibria in which (2.5) also holds for such submarkets.[5] Note that (2.5) implies that the firm earns zero profit.

2.4. Equilibrium definition and block recursivity

We define a monetary steady state as follows:

Definition 2.1. *A* monetary steady state *consists of value functions, (V, W, B), policy functions, $(\ell^*, x^*, q^*, z^*, \pi^*)$, market tightness function θ, a wage rate ω, and a distribution of individuals over real balances G that satisfy the following requirements:*
(i) W satisfies (2.1) with ℓ^* as the associated policy function;
(ii) B satisfies (2.2) with (x^*, q^*) as the associated policy functions;
(iii) V satisfies (2.4) with (z^*, π^*) as the associated policy functions;
(iv) θ satisfies (2.5) for all $(x, q) \in \mathbb{R}^2_+$;

[5] This restriction on the beliefs out of the equilibrium "completes" the market in the following sense: A submarket is inactive only if, given that some buyers are present in the submarket, the expected benefit to a lone trading post in the submarket is still lower than the cost of the trading post. This restriction can be justified by a "trembling-hand" argument that a small measure of buyers appear in every submarket exogenously. Similar restrictions are common in the literature on directed search, e.g., Moen (1997) and Acemoglu and Shimer (1999).

(v) G is the ergodic distribution generated by $(\ell^*, x^*, q^*, z^*, \pi^*, \theta)$;
(vi) ω is such that $\omega < \infty$ and $\int m \, dG(m) = 1/\omega$.

Requirements (i)-(iv) are explained by previous subsections. Requirement (v) asks the distribution of individuals over real balances to be stationary and consistent with the flows of individuals induced by optimal choices. Requirement (vi) asks that money should have a positive value and that all money should be held by the individuals. Specifically, the sum of real balances across individuals is the integral of m according to the distribution G. The total real balance available in the economy is $1/\omega$, which is the nominal balance M divided by the monetary wage rate ωM. Notice that we did not specify the labor market clearing condition in the above definition, because such a condition is implied by requirement (vi) in a closed economy.

Equilibrium objects and requirements in Definition 2.1 can be grouped into two blocks. The first block consists of the value functions, the policy functions and the market tightness function, which are determined by requirements (i) - (iv). The second block consists of the distribution of individuals over money balances and the wage rate, which are determined by requirements (v) and (vi). The second block depends on the objects in the first block, but the first block is self-enclosed and not affected by the second block. That is, the value functions, the policy functions and the market tightness function are independent of the distribution and the wage rate. We refer to this property of the equilibrium as *block recursivity*, following the usage in recent literature on labor search (Shi, 2009, Menzio and Shi, 2008, 2010, and Gonzalez and Shi, 2010). Clearly, even when an equilibrium is block recursive, the distribution still affects the aggregate activity.

Block recursivity is an attractive property of our model because it allows us to solve for equilibrium value functions, policy functions and the market tightness function without having to solve for the entire distribution of individuals over money balances. After obtaining these objects in the first block, we can compute the distribution of individuals over money balances by simply equating the flows of individuals into and out of each level of money balance. In contrast, when the steady state is not block recursive, the distribution is an aggregate state variable that appears in individuals' policy and value functions. In this case, one must compute the objects in the two blocks simultaneously and, since the distribution is endogenous and potentially has a large dimension, the computation of an equilibrium is complicated. In fact, it is to circumvent this complexity that monetary models have imposed assumptions on the model environment to make the distribution degenerate (e.g., Shi, 1997, Lagos and Wright, 2005). With block recursivity, the steady state is tractable even when the distribution of real balances is non-degenerate.

It is easy to understand why the equilibrium in our model is block recursive. As formulated in subsection 2.2, individuals' value and policy functions satisfy three functional equations that are independent of the distribution G and the wage rate ω. These decision problems are related to the general equilibrium of the economy only through the market tightness function θ. In particular, the tightness function provides all the relevant information needed for a buyer to optimally choose which submarket to visit. In making this decision, the buyer faces a tradeoff between the terms of trade in a submarket (x, q) and the matching probability in the submarket, $\lambda(\theta(x, q))$. Because the matching technology has constant returns to scale, the buyer's matching probability in a submarket is only a function of the tightness in the submarket. If the market tightness function is independent of G and ω, then so are the buyer's optimal choices and value function. The market tightness function is indeed independent of G and ω. In each submarket (x, q), the tightness $\theta(x, q)$ must be consistent with a firm's incentive to create trading posts. If a firm chooses to create a trading post in submarket (x, q), the firm's net profit from the trading post must be zero; that is, the expected benefit from the trading post must be equal to the cost of the trading post. The cost of creating a trading post is a constant, k. The expected benefit is the firm's gain from a trade, $(x - q)$, multiplied by the post's matching probability, $\rho(\theta)$. Thus, the zero-profit condition pins down the tightness of each submarket as a function of the terms of trade in the submarket, independently of G and ω.

The assumption of directed search is necessary for the steady state to be block recursive. To see why, consider an alternative environment of the model in which search is random in the sense that buyers cannot direct their search toward sellers who offer particular terms of trade. If the terms of trade are posted before a meeting takes place, whether they generate a non-negative surplus to a randomly met buyer depends on money holdings of the particular buyer. In this case, the probability that a meeting will result in trade depends on the distribution of buyers over money balances. If the terms of trade are instead bargained after a meeting takes place, they will depend on money holdings of the buyer in the match. In this case, the seller's surplus from a trade will depend on the distribution of buyers over money balances. In both cases, the distribution of individuals over money balances, G, affects individuals' value functions and a firm's expected benefit of a trading post. Because the tightness of the market is such that the expected benefit of a trading post is equal to the cost of creating the trading post, the tightness is also a function of the distribution G when search is undirected.

3. Equilibrium Policy and Value Functions

In this section we establish existence, uniqueness and other features of equilibrium value and policy functions. A center piece of this analysis is subsection 3.2 on a buyer's value and policy functions. In particular, we prove that a buyer's policy functions $(x^*(m), q^*(m))$ are monotonically increasing, which implies that buyers choose to sort themselves out according to money holdings. That is, a buyer with more money chooses to search in a submarket where the buyer can spend a larger balance and get a higher quantity of goods. In such a submarket the buyer also has a higher matching probability. Sorting leads to a stylized pattern of purchases over time by a buyer and a straightforward characterization of the equilibrium in section 4.

Monotonicity of policy functions is also critical for us to prove that the standard conditions of optimization, such as the first-order conditions and the envelope conditions, hold in our model. The characterization of a buyer's problem is technically challenging because the problem is not well-behaved. In fact, a buyer's objective function is not concave in the choice and state variables jointly. For this reason, we cannot use standard arguments (e.g., Stokey et al, 1989) to establish monotonicity of the policy functions and differentiability of the value function and, in turn, to establish the validity of the envelope and first-order conditions. Instead, we develop an alterative set of arguments that first prove monotonicity of the policy functions, then differentiability of the value function and finally the validity of the first-order and envelope conditions. These arguments are of independent interest because they are likely to apply to a variety of dynamic models that involve both discrete and continuous choices.

A map of the analysis in this section is as follows. First, we assume that individuals' money holdings are bounded above by $\bar{m} < \infty$, an assumption we will validate later in Theorem 3.5. Let $\mathcal{C}[0, \bar{m}]$ denote the set of continuous and increasing functions on $[0, \bar{m}]$, and let $\mathcal{V}[0, \bar{m}]$ denote the subset of $\mathcal{C}[0, \bar{m}]$ that contains all concave functions. Taking an arbitrary ex ante value function $V \in \mathcal{V}[0, \bar{m}]$, we use subsection 3.1 to characterize a worker's problem. Second, with the same function $V \in \mathcal{V}[0, \bar{m}]$, we use subsection 3.2 to characterize a buyer's problem. Third, in subsection 3.3, we characterize an individual's lottery choice and obtain an update of the ex ante value function, denoted as TV. We prove that T is a monotone contraction mapping on $\mathcal{V}[0, \bar{m}]$, and so there is a unique fixed point for the ex ante value function. Finally, we verify that individuals' money holdings are indeed bounded above by $\bar{m} < \infty$.

3.1. A worker's value and policy functions

Let $\bar{m}$ be a sufficiently large upper bound on individuals' money holdings and V any arbitrary function in $\mathcal{V}[0, \bar{m}]$. Given V, the worker's problem, (2.1), generates the worker's value function $W(m)$, the policy function of labor supply $\ell^*(m)$, and the policy function of the end-of-period balance $y^*(m) = m + \ell^*(m)$. We have the following lemma (see Appendix A for a proof):

Lemma 3.1. *For any $m \in [0, \bar{m}]$ and $V \in \mathcal{V}[0, \bar{m}]$, the following properties hold:*
(i) $W \in \mathcal{V}[0, \bar{m}]$; i.e., W is continuous, increasing and concave on $[0, \bar{m}]$;
(ii) $\ell^*(m)$ is unique, continuous and decreasing in m, and $y^*(m)$ is unique, continuous and strictly increasing in m;
(iii) For all m such that $\ell^*(m) > 0$, $W'(m)$ and $V'(y^*(m))$ exist and satisfy:

$$W'(m) = \beta V'(m + \ell^*(m)) = h'(\ell^*(m)). \tag{3.1}$$

The first equality is the envelope condition and the second equality the first-order condition.

In part (i) of Lemma 3.1, the value function of a worker is continuous and increasing in the worker's money holdings because the ex ante value function has these properties. A worker's value function is also concave because the ex ante value function is concave and the disutility function of labor supply is convex, which make the worker's objective function concave jointly in the choice ℓ and the state variable m. Part (ii) of Lemma 3.1 states existence, uniqueness and monotonicity of a worker's policy functions. These properties are intuitive. By supplying higher labor, a worker obtains a higher balance which increases the ex ante value function next period. Since the ex ante value function is concave, the marginal benefit of labor supply is decreasing. In contrast, the marginal disutility of labor supply is strictly increasing. Thus, for any given balance, a worker's optimal labor supply is unique. Such uniqueness implies that the policy function of labor supply is continuous in the worker's money holdings. Moreover, since the gain from working is smaller when a worker already has a relatively high balance, the policy function of labor supply is decreasing in the worker's money holdings. Similarly, a worker's policy function of the end-of-period money holdings is unique and continuous. This function is strictly increasing in m because a higher balance has a strictly positive marginal benefit to a worker.

Part (iii) of Lemma 3.1 states that if a worker's optimal labor supply is strictly positive, then the worker's value and policy functions satisfy the envelope condition and the first-order condition. Notice that the choice $\ell = 1$ is never optimal, because the marginal disutility of labor supply at this choice is $h'(1) = \infty$. Hence, a worker's optimal labor supply is interior if it is strictly positive. An interior choice is a common requirement for the first-order and the envelope conditions to apply, and the requirement is not binding in the equilibrium.[6] Let us draw attention to the fact that part (iii) uses the derivative $V'(y^*(m))$. Although we have not assumed that V is differentiable everywhere, we have assumed that V is concave. Concavity of V implies that V is differentiable almost everywhere, and the one-sided derivatives of V exist (see Royden, 1988, pp113-114). Part (iii) of Lemma 3.1 implies that a worker's optimal labor supply always generates an end-of-period balance $y^*(m)$ at which the ex ante value function is differentiable.

To establish Lemma 3.1 and especially part (iii), we augment the standard approach in dynamic programming (see Stokey et al, 1989, p85). To do so, we transform a worker's problem (2.1) into one where the choice is the end-of-period balance y instead of labor supply:

$$W(m) = \max_{y \geq m} \left[\beta V(y) - h(y-m)\right]. \tag{3.2}$$

The standard approach in dynamic programming is applied as follows. First, with any concave V, the objective function in (3.2) is concave in (y, m) jointly. This feature ensures not only that the optimal choice $y^*(m)$ is unique for each m, but also that $W(m)$ is concave. Second, with concavity of W and the objective function, the result in Benveniste and Scheinkman (1979) applies here. That is, for any balance m at which the optimal choice is interior (i.e., $y^*(m) > m$), the derivative $W'(m)$ exists and satisfies the envelope condition, $W'(m) = h'(y^*(m) - m)$. Third, rewriting this envelope condition as $W'(m) = h'(\ell^*(m))$, we use concavity of W and convexity of h to deduce that the policy function $\ell^*(m)$ is decreasing.

The derivative $W'(m)$ exists because the marginal disutility of labor, $h'(\ell)$, is continuous and strictly increasing. To elaborate, suppose that $W'(m)$ does not exist at a particular m where the optimal choice $\ell^*(m)$ is interior. Since W is concave, then the marginal value of money balance to a worker is strictly greater on the left-hand side of m than on the right-hand side of m. This outcome is inconsistent with a worker's choice of the end-of-period balance. To a worker, the marginal benefit of having a higher balance before going to work is that the worker can reduce the hours of work needed to achieve any given end-of-period balance. This benefit is captured by $h'(y^*(m) - m)$, which is continuous in m. Thus, $W'(m)$ must exist.

We augment the standard approach above with a proof that $V'(y^*(m))$ exists and satisfies (3.1). The proof is a generalized envelope argument which compares two ways of calculating the marginal value of money to a worker. One way to calculate this marginal value of money is $W'(m)$. For an alternative way, let us go back to the original formulation of a worker's problem, (2.1),

[6] If an individual's balance is so high that optimal labor supply is zero at such a balance, then it is optimal for the individual to choose to enter the goods market as a buyer rather than the labor market as a worker.

where the marginal value of money to a worker comes from directly affecting the end-of-period balance, $m+\ell$. Because the objective function in (2.1) is concave in (ℓ, m) jointly, we can derive a generalized version of the envelope theorem. That is, the left-hand derivative, $\beta V'(y^{*-}(m))$, is equal to $W'(m^-)$, and the right-hand derivative, $\beta V'(y^{*+}(m))$, is equal to $W'(m^+)$. Because $W'(m)$ exists, then the derivative $\beta V'(y^*(m))$ must exist and be equal to $W'(m)$. This is the envelope condition of W given by the first equality in (3.1). The first-order condition of $\ell^*(m)$ given by the second equality in (3.1) comes from substituting $W'(m) = h'(\ell^*(m))$.

The above lemma holds for all $m \geq 0$. Of particular interest is the case $m = 0$. For a worker with $m = 0$, denote the optimal end-of-period balance as $\hat{m} = y^*(0) = \ell^*(0)$. This worker's value function is $W(0) = \beta V(\hat{m}) - h(\hat{m})$. Lemma 3.1 implies that

$$V'(\hat{m}) = \frac{1}{\beta} h'(\hat{m}) = \frac{1}{\beta} W'(0). \tag{3.3}$$

3.2. A buyer's value and policy functions

We now analyze a buyer's problem (2.2), given any arbitrary ex ante value function $V \in \mathcal{V}[0, \bar{m}]$. In subsection 3.2.1, we reformulate the buyer's problem, describe the difficulty in analyzing the problem, outline our approach, and present the main results in Theorem 3.2. In subsections 3.2.2 and 3.2.3, we establish two lemmas which together constitute a proof of Theorem 3.2.

3.2.1. Descriptions of the difficulty, our approach and main results

For convenience, we express a buyer's choices as (x, b) instead of (x, q), where b is the buyer's matching probability in a submarket, and express q as a function of (x, b). Recall that $b = \lambda(\theta(x, q))$, that a trading post's matching probability is $s = \rho(\theta(x, q))$, and that $s = \mu(b) \equiv \rho(\lambda^{-1}(b))$. Thus, the market tightness condition (2.5) can be equivalently written as

$$s = \mu(b) = \begin{cases} \frac{k}{x-q}, & \text{if } k \leq x - q \\ 1, & \text{otherwise.} \end{cases} \tag{3.4}$$

In any submarket with $x - q \leq k$, the tightness is 0, and a buyer's matching probability is $b = \mu^{-1}(1) = 0$. In any submarket with $x - q > k$, the tightness is strictly positive, and a buyer's matching probability is $b = \mu^{-1}(\frac{k}{x-q}) > 0$. Thus, in any submarket (x, q) with positive tightness, we can express the quantity of goods traded in a match as

$$q = Q(x, b) \equiv x - \frac{k}{\mu(b)}. \tag{3.5}$$

Note that if a buyer has a balance $m \leq k$, the only submarkets that the buyer can afford to visit have $x - q \leq m \leq k$ and, hence, have zero tightness. For such a buyer, the optimal choice is $b^*(m) = 0$, and the value function is $B(m) = \beta V(m)$.

Let us focus on the non-trivial case $m > k$. In this case, the buyer's problem (2.2) can be transformed into the following one in which the choices are (x, b):

$$\begin{aligned} B(m) \quad &= \max_{(x,b)} \{\beta V(m) + b\left[u(x,b) + \beta V(m-x) - \beta V(m)\right]\} \\ \text{s.t.} \qquad & x \in [0,m], \quad b \in [0,1], \end{aligned} \tag{3.6}$$

where $u(x,b) = U(Q(x,b))$.[7] Let $(x^*(m), b^*(m))$ denote the buyer's policy functions of (x, b) and let $\phi(m)$ denote the policy function of the residual balance $(x - m)$. Then,

$$q^*(m) \equiv Q\left(x^*(m), b^*(m)\right), \quad \phi(m) \equiv m - x^*(m). \tag{3.7}$$

The objective function in (3.6) is not concave jointly in the choices (x, b) and the state variable m. The objective function involves the product of the buyer's trading probability, b, and the buyer's surplus of trade. Even if these terms are concave separately, the product of the two may not be concave in (x, b, m) jointly. The lack of concavity presents a major difficulty in using the standard approach in dynamic programming to analyze policy and value functions, because the approach starts with the requirement that the objective function be concave jointly in the choice and state variables (see Stokey et al, 1989, and the analysis of a worker's problem in subsection 3.1). Attempts to make a buyer's objective function concave entail additional restrictions on the endogenous function V that are difficult to be verified as the outcomes of (2.4).

To analyze a buyer's problem, we use lattice-theoretic techniques (see Topkis, 1998). The procedure almost reverses the steps of the standard approach. First, we establish monotonicity of the policy functions using lattice-theoretic techniques. Second, using monotonicity of the policy functions, we prove that the value functions $B(m)$ and $V(m)$ are differentiable along the equilibrium path, i.e., at money balances induced by optimal choices. This result allows us to characterize the policy functions with the first-order conditions and envelope conditions. Finally, we prove that the ex ante value function is differentiable at all money balances. This procedure is natural in the sense that it first establishes a basic property of functions, i.e., monotonicity, and then progresses to a stronger property – differentiability.[8]

[7] Note that for $q \geq 0$, the buyer's choices must satisfy $x \geq k/\mu(b)$. However, there is no need to add this constraint to the problem (3.6) because it is not binding in any realized trade. For any choices (x, b) such that $x < k/\mu(b)$ and $x > 0$, the quantity of goods is $q < 0$ and the utility of consumption is $u(x, b) < U(0) = 0$. In this case, the buyer's surplus from trade is $u(x, b) + \beta V(m - x) - \beta V(m) < 0$. The buyer can avoid this loss by choosing $b = 0$.

[8] There are other approaches that establish differentiability of the value function in the presence of a non-concave

Recall that $\mathcal{C}[0,\bar{m}]$ denotes the set of continuous and increasing functions on $[0,\bar{m}]$, and $\mathcal{V}[0,\bar{m}]$ denotes the subset of $\mathcal{C}[0,\bar{m}]$ that contains all concave functions. The following theorem states the main result of our procedure:

Theorem 3.2. *Take any arbitrary $V \in \mathcal{V}[0,\bar{m}]$. Then, $B \in \mathcal{C}[0,\bar{m}]$. If $m \leq k$, then $b^*(m) = 0$ and $B(m) = \beta V(m)$; if $m > k$, then $B(m)$ satisfies (3.6). Consider any $m \in [k,\bar{m}]$ such that $b^*(m) > 0$. The results (i)-(iii) below hold:*
(i) For each m, the optimal choices $(x^*(m), b^*(m))$ and the implied quantities $(q^*(m), \phi(m))$ are unique. The policy functions $x^*(m)$, $b^*(m)$, $q^*(m)$ and $\phi(m)$ are continuous and increasing.
(ii) The optimal choice $b^*(m)$ satisfies the first-order condition:

$$u(x,b) + b\,u_2(x,b) = \beta\left[V(m) - V(m-x)\right]. \tag{3.8}$$

For all m such that $\phi(m) > 0$, $\phi(m)$ satisfies the first-order condition:[9]

$$V'(\phi(m)) = \frac{1}{\beta} u_1\left(x^*(m), b^*(m)\right). \tag{3.9}$$

(iii) $B'(m)$ exists if and only if $V'(m)$ exists, and B is strictly increasing.
Consider any $m < \bar{m}$ such that $b^*(m) > 0$. If $B(m) = V(m)$ and if there exists a neighborhood O surrounding m such that $B(m') \leq V(m')$ for all $m' \in O$, then (iv) and (v) below hold. These two parts also hold for $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$:
(iv) The derivatives $B'(m)$ and $V'(m)$ exist and satisfy:

$$V'(m) = \frac{b^*(m)}{1 - \beta\left[1 - b^*(m)\right]} u_1\left(x^*(m), b^*(m)\right) = B'(m). \tag{3.10}$$

(v) If $\phi(m) > 0$, then b^* and ϕ are strictly increasing at m, and V is strictly concave at $\phi(m)$, with $V'(\phi(m)) > V'(m)$.

Parts (ii)-(iv) of this theorem assure that one can use the standard apparatus of optimization to analyze a buyer's optimal decisions and value function. We will establish Lemmas 3.3 and 3.4 which together prove Theorem 3.2. A reader who is eager to see the implications of the above theorem may want to go directly to subsection 3.3.

objective function. However, these approaches do not prove monotonicity of the policy functions. Moreover, they are not applicable in our model. Specifically, these approaches assume the objective function to be equi-differentiable (Milgrom and Segal, 2002) or differentiable with respect to the state variable (Clausen and Strub, 2010). In our model, the objective function in (2.2) contains both $V(m)$ and $V(m-x)$, where x is a choice and m a state variable. For this objective function to satisfy either of the aforementioned assumptions, the value function V must be differentiable, which is a result to be proven.

[9] If $\phi(m) = 0$, then (3.9) is replaced with $V'(0) \leq \frac{1}{\beta} u_1\left(m, b^*(m)\right)$.

3.2.2. A buyer's policy functions and monotonicity

To apply lattice-theoretic techniques (Topkis, 1998) to (3.6), we investigate whether the objective function in (3.6) is supermodular in the choice variables (x, b) and the state variable m, i.e., whether the objective function has increasing differences in (x, b), (x, m) and (b, m).[10] When the objective function has increasing differences in two variables, there is complementarity between the two variables which intuitively leads to a monotonic relationship between the two variables. As a preliminary step toward using the techniques and developing the intuition, we examine the properties of the functions $Q(x, b)$ and $u(x, b)$. The function $Q(x, b)$, defined in (3.5), determines the quantity of goods sold to a buyer who has a matching probability b and spends a real balance x in the trade in a submarket with positive tightness. For all (x, b) such that $Q(x, b) > 0$, it is easy to verify that the function Q has the following properties:

$$Q_1(x, b) > 0,\ Q_2(x, b) < 0,\ Q(x, b) \text{ is (weakly) concave, and } Q_{12} = 0. \tag{3.11}$$

It is intuitive that Q strictly increases in x and strictly decreases in b. For any given matching probability, the more a buyer is willing to pay, the higher the quantity of good he can obtain. For any given payment, however, a buyer must accept a relatively low quantity of goods in order to increase the matching probability. This is because the cost of production must be relatively low in order to induce firms to set up a large number of trading posts needed to increase the matching probability for a buyer.

It is also intuitive that Q is (weakly) concave in (x, b) jointly and its cross partial derivative with respect to x and b is zero. The function Q is strictly concave in b because increasing the number of trading posts has a diminishing marginal effect in increasing a buyer's matching probability. In order to increase a buyer's matching probability further, the additional number of trading posts created for a buyer must increase, and a firm must be compensated for creating the additional posts with an increasingly larger reduction in the quantity of goods traded for a given x. Moreover, because the amount of labor needed to produce any quantity of goods is assumed

[10] Consider a set $S = \{s = (s_1, s_2, ..., s_n) : s_i \in A_i\}$, where $A_i \subseteq \mathbb{R}$ for each i. For any s and s' in S, define the binary relation "$\geq$" by the requirement that $s \geq s'$ if and only if $s_i \geq s'_i$ for all $i \in \{1, 2, ..., n\}$. Let S be a partially ordered set with the relation "$\geq$". For s and s' in S, define $s \vee s' = \{(s''_1, s''_2, ..., s''_n) : s''_i = \max\{s_i, s'_i\}, \text{ all } i\}$ and $s \wedge s' = \{(s''_1, s''_2, ..., s''_n) : s''_i = \min\{s_i, s'_i\}, \text{ all } i\}$. The set S is called a lattice if $s \vee s' \in S$ and $s \wedge s' \in S$ for all s and s' in S. A function $f(s)$ defined on S is supermodular if $f(s) + f(s') \leq f(s \vee s') + f(s \wedge s')$ for all s and s' in S, and strictly supermodular if the inequality is strict for all $s \neq s'$. The function $f(s)$ has (strictly) increasing differences in the pair (s_i, s_j) if the difference, $f(s_1, ..., s_i, ..., s_j, ...s_n) - f(s_1, ..., s_i, ..., s'_j, ..., s_n)$, is (strictly) increasing in s_i for all $s_j > s'_j$ in A_j. If A_i is a lattice for each $i \in \{1, 2, ..., n\}$ and S has the direct product topology, then $f(s)$ is (strictly) supermodular if and only if f has (strictly) increasing differences in (s_i, s_j) for all i and j in $\{1, 2, ..., n\}$ with $i \neq j$. In problem (3.6), $s = (x, b, m)$. Since each of the three variables in s belongs in an interval (which is a lattice), the direct product of these intervals forms a lattice S. Thus, the objective function in (3.6) is supermodular in (x, b, m) if and only if it has increasing differences in the three pairs of variables.

to be a linear function of the quantity, Q is linear in x and separable in (x, b). As a result, Q is weakly concave in (x, b) and $Q_{12} = 0$.[11]

The function $Q(x, b)$ is used in the objective function in (3.6) to express the utility of consumption as $u(x, b) = U(Q(x, b))$. It is easy to verify that for all (x, b) such that $Q(x, b) > 0$,

$$u_1(x, b) > 0,\ u_2(x, b) < 0,\ u(x, b) \text{ is strictly concave, and } u_{12} > 0. \tag{3.12}$$

The first-order properties of u directly come from the first-order properties of Q and the fact that U is strictly increasing. The second-order properties of u are stronger than those of Q because the utility function U is strictly concave. In particular, the property $u_{12} > 0$ says that $u(x, b)$ is strictly supermodular. This property is intuitive. Consider $u_1(x, b)$, the marginal increase in utility caused by an increase in spending. In a submarket where the buyer's matching probability is relatively high, the quantity of goods that the buyer obtains in a trade with any given spending is relatively low, because a firm must be compensated for creating a large number of trading posts to deliver the high matching probability for the buyer. At such low consumption, an increase in spending can increase the utility of consumption by a relatively large amount. Thus, $u_1(x, b)$ is higher in a submarket with a higher b than in a submarket with a lower b.

Despite strict supermodularity of $u(x, b)$, a buyer's objective function in (3.6) is not supermodular in (x, b). Thus, we cannot apply Topkis' (1998) theorems directly to the buyer's problem. To get around this problem, we write the objective function in (3.6) as $\beta V(m) + b\ R(x, b, m)$, where R is the buyer's surplus from trade defined as follows:

$$R(x, b, m) = u(x, b) + \beta V(m - x) - \beta V(m). \tag{3.13}$$

We decompose the buyer's problem into two steps. In the first step, we fix b and characterize the optimal choice of x. For any given (b, m), the optimal choice of x maximizes $R(x, b, m)$. Denote

$$\tilde{x}(b, m) = \arg\max_{x \in [0, m]} R(x, b, m), \quad \tilde{R}(b, m) = R(\tilde{x}(b, m), b, m). \tag{3.14}$$

In the second step, we characterize the optimal choice of b as

$$b^*(m) = \arg\max_{b \in [0,1]}\ b\ \tilde{R}(b, m). \tag{3.15}$$

Although Topkis' theorems are not applicable directly to a buyer's problem, we show that they are applicable in each step above. In particular, to apply Topkis' theorems in the second step requires only that $b\ \tilde{R}(b, m)$ be supermodular in (b, m), i.e., that this function be supermodular

[11] If the amount of labor needed to produce any quantity of goods is assumed to be a strictly convex function of the quantity, then Q is strictly concave in (x, b) and $Q_{12} > 0$. These features of Q will strengthen our results.

in (b, m) when spending optimally depends on (b, m) as $x = \tilde{x}(b, m)$. This requirement is weaker than the requirement that $b\ R(x, b, m)$ be supermodular in all (x, b, m). The following lemma states the result of this two-step procedure (see Appendix B for a proof):

Lemma 3.3. *For any $V \in \mathcal{V}[0, \bar{m}]$, $B(m) \in \mathcal{C}[0, \bar{m}]$. If $m \leq k$, then $b^*(m) = 0$ and $B(m) = \beta V(m)$; if $m > k$, $B(m)$ solves (3.6). Moreover, for all $m \in [k, \bar{m}]$ such that $b^*(m) > 0$, the policy functions are monotone as stated in part (i) of Theorem 3.2.*

The function $B(m)$ is continuous and increasing because the ex ante value function $V(m)$ has these features for all $m \leq \bar{m}$. As explained in subsection 3.2.1, B has two segments. If $m \leq k$, then $b^*(m) = 0$ and $B(m) = \beta V(m)$; if $m > k$, $B(m)$ solves (3.6).

As stated in part (i) of Theorem 3.2, optimal choices $(x^*(m), b^*(m))$ are unique for any given balance m, and so the policy functions are continuous. We prove this result by establishing the feature that the logarithmic transformation of the part to be maximized in the buyer's problem, $bR(x, b, m)$, is strictly concave in (x, b). This feature is intuitive. Because the matching function has diminishing marginal returns to the number of trading posts, a buyer's matching probability is strictly concave in the tightness of the submarket. This implies that the marginal cost of increasing a buyer's matching probability is increasing, in the sense that the buyer must either spend an increasingly larger amount of money to purchase a given quantity of goods or obtain an increasingly smaller quantity of goods for any given spending. Thus, for any given balance, a buyer finds a unique pair of (x, b) that offers the best trade-off between the quantity of goods traded and the probability of the trade. That is, given his balance, a buyer chooses a unique submarket to enter, rather than being indifferent between different submarkets.

Part (i) of Theorem 3.2 also states that the policy functions $x^*(m)$ and $b^*(m)$ are increasing for all m such that $b^*(m) > 0$. This feature arises intuitively from the assumption that the ex ante value function V is concave. After a trade, a buyer's residual balance is valued with V next period. Concavity of V implies that for the same spending, a buyer with a higher balance will have a lower marginal value of the residual balance than will a buyer with a lower balance. This motivates the buyer with a higher balance to enter a submarket where he has a higher matching probability and spend more money to increase consumption.

Let us explain how we implement the two-step procedure to turn the above intuition into a formal proof. We first establish that a buyer's surplus $R(x, b, m)$ defined by (3.13) is supermodular in (x, b, m); that is, R has increasing differences in (x, b), (b, m) and (x, m). The function R has strictly increasing differences in (x, b) because $u(x, b)$ has this property, as explained earlier. The function R has (weakly) increasing differences in (b, m) because b and m are separable in R. To see why R has increasing differences in (x, m), let us fix b at any arbitrary level in $[0, 1]$ and consider any $x_2 > x_1$ and $m_2 > m_1 \geq k$, with $x_i \in [0, m_i]$ for $i = 1, 2$. For any given balance m, an increase in spending from x_1 to x_2 increases the buyer's surplus by

$$R(x_2, b, m) - R(x_1, b, m) = [u(x_2, b) - u(x_1, b)] - \beta [V(m - x_1) - V(m - x_2)].$$

This amount consists of the increase in the utility of consumption resulted from higher spending and the reduction in the value function next period resulted from a lower residual balance after trade. Note that for any given b, the increase in the utility of consumption is independent of the buyer's balance. Thus, relative to a buyer with a lower balance m_1, a buyer with a higher balance m_2 can obtain the following additional surplus from increasing spending from x_1 to x_2:

$$\begin{aligned} & [R(x_2, b, m_2) - R(x_1, b, m_2)] - [R(x_2, b, m_1) - R(x_1, b, m_1)] \\ = \; & \beta [V(m_1 - x_1) - V(m_1 - x_2)] - \beta [V(m_2 - x_1) - V(m_2 - x_2)]. \end{aligned}$$

This additional surplus is positive because V is concave, i.e., because a buyer with a higher balance has a lower marginal value of money than does a buyer with a lower balance. Thus, R has increasing differences in (x, m).

Next, notice that the feasibility set in the maximization problem in (3.14) is increasing in m and independent of b. With a supermodular objective function, a feasibility set increasing in m and a unique optimal choice $\tilde{x}$, we conclude that $\tilde{x}(b, m)$ is increasing in (b, m) and that $\tilde{R}(b, m)$ is increasing and supermodular in (b, m) (see Topkis 1998, pp.70-76). Proceeding to the second step and the problem in (3.15), we use these features of $\tilde{x}(b, m)$ and $\tilde{R}(b, m)$ to verify that $b\tilde{R}(b, m)$ is supermodular in (b, m). Together with uniqueness of the optimal choice b^*, this feature implies that $b^*(m)$ and the maximized function, $b^*(m)\tilde{R}(b^*(m), m)$, are increasing in m.

Finally, by changing the choices from (x, b) to (x, q) and to $(m - x, b)$, in turn, we use the two-step procedure above to prove that $q^*(m)$ and $\phi(m)$ are increasing. Intuitively, consumption is a normal good in the current period and in the future. When a buyer has more money, it is optimal for him to increase spending in the current period and also to keep a higher residual balance to increase spending in the future.

In summary, buyers sort themselves into different submarkets according to their balance. A buyer with more money chooses to enter a submarket where he will have a higher matching probability and once he is matched in the submarket, he will spend a larger amount of money, buy a larger quantity of goods, and exit the trade with a higher balance.

3.2.3. First-order conditions, envelope theorems and value functions

The remaining parts of Theorem 3.2, (ii)-(v), describe the first-order conditions, the envelope condition and additional properties of the value functions. They are restated in the following lemma and proven in Appendix C:

Lemma 3.4. *Consider any $m \in [k, \bar{m}]$ such that $b^*(m) > 0$. For any $V \in \mathcal{V}[0, \bar{m}]$, parts (ii) and (iii) of Theorem 3.2 hold. For any $m < \bar{m}$ such that $b^*(m) > 0$, if $B(m) = V(m)$ and if there exists a neighborhood O surrounding m such that $B(m') \leq V(m')$ for all $m' \in O$, then parts (iv) and (v) of Theorem 3.2 hold. These two parts also hold for $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$.*

Part (ii) of Theorem 3.2 states that optimal choices b^* and x^* satisfy the first-order conditions. In the first-order condition of b^*, (3.8), the left-hand side is the marginal benefit of increasing b, represented by the increase in expected utility of consumption resulting from a higher matching probability. The right-hand side of (3.8) is the buyer's opportunity cost of a trade represented by the reduction in the future value function resulting from a lower future balance. Thus, the first-order condition of b requires intuitively that the optimal choice b^* equates the marginal benefit and the marginal cost of changing b. Similarly, the first-order condition of x^*, (3.9), requires that the marginal cost of increasing spending, represented by the marginal value of the residual balance ϕ, should be equal to the marginal utility of consumption brought about by higher spending.

The first-order condition of b^* holds regardless of whether the ex ante value function V is differentiable. This is because the choice b does not appear in V, which implies that the buyer's objective function in (3.6) is differentiable with respect to b for any given (x, m). In contrast, the choice x appears in V through the residual balance. Thus, the first-order condition of the optimal choice x^*, (3.9), requires the derivative $V'(\phi(m))$ to exist. That is, it is optimal for a buyer to choose spending in such a way that steers the residual balance away from any level at which V is not differentiable. Similar to the existence of $V'(y^*(m))$ in Lemma 3.1, this result comes from a generalized envelope argument that compares two ways of calculating the marginal value of money to a buyer. Consider the buyer's gain from increasing the balance to m from slightly below m. When choosing (x, b), the increase in the balance yields that gain $b^*(m)\beta V'(\phi^-(m))$ $+(1-b^*(m))\beta V'(m^-)$, where $\phi^-(m) = m^- - x^*(m)$. Alternatively, the buyer can choose $(m-x, a)$, where $a = m - x + q$. Let $(\phi(m), a^*(m))$ be such optimal choices and $(x^*(m), b^*(m))$ the implied choices of (x, b). The marginal value of money on the left-side of m is $b^*(m)u_1(x^*(m), b^*(m))$ $+(1 - b^*(m))\beta V'(m^-)$. The two ways of computing the marginal value of money to a buyer yield $\beta V'(\phi^-(m)) = u_1(x^*(m), b^*(m))$. Since a similar equation holds for $\beta V'(\phi^+(m))$, then βV must be differentiable at $\phi(m)$ and the derivative is equal to $u_1(x^*(m), b^*(m))$.

Part (iii) of Theorem 3.2 ties differentiability of B to that of V and states that B is strictly increasing. To obtain these results, we use the fact that a concave function has both left-hand and right-hand derivatives (see Royden, 1988, pp113-114). Because V is concave, then $V'(m^-)$ and $V'(m^+)$ exist for all m, with $V'(m^-) \geq V'(m^+)$. Using the functional equation, (3.6), and the definitions of one-sided derivatives, we can obtain the following generalized envelope conditions:

$$B'\left(m^+\right) = b^*(m)u_1\left(x^*(m), b^*(m)\right) + \beta\left(1 - b^*(m)\right)V'\left(m^+\right) \tag{3.16}$$

$$B'\left(m^-\right) = b^*(m)u_1\left(x^*(m), b^*(m)\right) + \beta\left(1 - b^*(m)\right)V'\left(m^-\right). \tag{3.17}$$

From these conditions it is evident that $B'(m)$ exists if and only if $V'(m)$ exists. Moreover, since $V'(m^-) \geq V'(m^+) \geq 0$ and $u_1 > 0$, the above conditions imply that $B'(m^-) \geq B'(m^+) > 0$ if $b^*(m) > 0$; that is, B is strictly increasing for all m such that $b^*(m) > 0$.

Part (iv) of Theorem 3.2 is the envelope condition of a buyer's problem. It is valid for any $m < \bar{m}$ that satisfies $B(m) = V(m)$ and is surrounded by a neighborhood O such that $B(m') \leq V(m')$ for all $m' \in O$. Note that such a neighborhood O always exists if V is the equilibrium ex ante value function, but it is required here since Theorem 3.2 states the results under any arbitrary function V. Part (iv) states that B and V are differentiable for any m that satisfies the hypotheses and the common derivative is equal to the discounted marginal utility of consumption, given by the expression in the middle of (3.10). It is apparent that if V is differentiable at a balance m, then (3.16) and (3.17) imply that $B'(m)$ and $V'(m)$ are both equal to the expression in the middle of (3.10). Thus, it suffices to explain why V is differentiable at m with the described properties. Because $B(m') \leq V(m')$ for all m' in the neighborhood O and $B(m) = V(m)$, continuity of the two functions implies $B'(m^+) \leq V'(m^+)$ and $B'(m^-) \geq V'(m^-)$. Substituting $B'(m^+) \leq V'(m^+)$ into (3.16), we conclude that $V'(m^+)$ is greater than or equal to the expression in the middle of (3.10). Similarly, substituting $B'(m^-) \geq V'(m^-)$ into (3.17), we conclude that $V'(m^-)$ is less than or equal to the expression in the middle of (3.10) and, hence, less than $V'(m^+)$. However, concavity of V requires $V'(m^-) \geq V'(m^+)$. Therefore, $V'(m)$ must exist and be equal to the expression in the middle of (3.10).

At $m = \bar{m}$, part (iv) of Theorem 3.2 holds under the additional condition $B'(\bar{m}) = V'(\bar{m})$. So far, we have imposed $\bar{m}$ exogenously as an upper bound on individuals' money holdings and have not characterized B and V on the right-hand side of $\bar{m}$. Although we will later prove that the finite upper bound $\bar{m}$ exists and satisfies $B'(\bar{m}) = V'(\bar{m})$, we want to prevent the circularity in the argument by not presuming here that the neighborhood O exists for $m = \bar{m}$.

Part (v) of Theorem 3.2 describes additional properties at a balance m that satisfies the above hypotheses and induces a strictly positive residual balance. First, note that under the hypothesis $B(m) = V(m)$, a buyer with the balance m does not have the need to use a lottery. This implies that the marginal value of money is strictly decreasing at m and so, for any given trading probability, the buyer's surplus of trade is strictly increasing in m locally. If such a buyer has additional money, he prefers to enter a submarket with a strictly higher trading probability and spend more in order to capture the higher surplus of trade. That is, $b^*(m)$ and $x^*(m)$ are strictly increasing at such m. Second, since future consumption is a normal good, it is optimal for the buyer to keep part of this additional money as the residual balance. That is, $\phi(m)$ is also strictly increasing at such m. Third, the ex ante value function must be strictly concave at $\phi(m) > 0$: if V is linear at $\phi(m)$, the buyer should have spent more because the marginal cost of doing so is locally constant.

3.3. Lotteries and the ex ante value function

We have characterized a worker's decisions and a buyer's decisions, taking the ex ante value function as any arbitrary $V \in \mathcal{V}[0, \bar{m}]$, i.e., any continuous, increasing and concave function on $[0, \bar{m}]$. Part (i) of Lemma 3.1 states that a worker's problem (2.1) defines a mapping $T_W : \mathcal{V}[0, \bar{m}] \to \mathcal{V}[0, \bar{m}]$ that maps an ex ante value function $V \in \mathcal{V}[0, \bar{m}]$ into a worker's value function $W \in \mathcal{V}[0, \bar{m}]$. Similarly, Theorem 3.2 implies that a buyer's problem (2.2) defines a mapping $T_B : \mathcal{V}[0, \bar{m}] \to \mathcal{C}[0, \bar{m}]$ that maps an ex ante value function $V \in \mathcal{V}[0, \bar{m}]$ into a buyer's value function $B \in \mathcal{C}[0, \bar{m}]$. In the equilibrium, the ex ante value function must satisfy (2.4). If we substitute $W = T_W V$ and $B = T_B V$ into (2.3) to obtain $\tilde{V}$, then the right-hand side of (2.4) is a mapping on V. Denote this mapping as T and write (2.4) as $V(m) = TV(m)$. The ex ante value function in the equilibrium is a fixed point of T. We need to show that T maps $\mathcal{V}[0, \bar{m}]$ into $\mathcal{V}[0, \bar{m}]$ and that it has a fixed point. Moreover, we need to verify that there indeed exists a finite upper bound $\bar{m}$ on individuals' money holdings in the equilibrium.

The functional equation (2.4) involves a maximization problem over the choice of lotteries. This choice is necessary for the ex ante value function to be concave. We have used concavity of any arbitrary function V to ensure that an individual's optimal choices are unique. Also, concavity of V is important for a buyer's policy functions to be monotone, since we have used it to ensure various components of a buyer's objective functions to be supermodular. To preserve these properties of the optimal choices, it is necessary that T maps the elements in $\mathcal{V}[0, \bar{m}]$, which are concave functions, back into the elements in $\mathcal{V}[0, \bar{m}]$.

There are two causes of why $\tilde{V}$ defined by (2.3) is not concave at all money balances. The first exists when the balance m satisfies $B(m) > W(m)$ and when B is not concave at m. An individual with such a balance prefers to be a buyer, and so $\tilde{V}(m) = B(m)$. If B is not concave near m, the individual wants to use a lottery to convexify the feasibility set of values. The second cause of non-concavity of $\tilde{V}$ occurs when money balance is low. Specifically, when $m < k$, the value of going to the market as a worker is higher than the value of being a buyer, because $W(m) > \beta V(m) = B(m)$ for such m. Since $B(m') > W(m')$ for sufficiently large m', then B crosses W from below as m increases. This feature implies that $\tilde{V}$ is strictly convex in a neighborhood of the balance at which B crosses W. Thus, to an individual with a sufficiently low balance, there is a gain from playing a lottery. In Figure 1 we depict these two lotteries: the lottery for low money holdings makes V the dashed line connecting points A and C, and a lottery at higher money holdings makes V the dashed line connecting points D and E.

Let us refer to the lottery for $m \in [0, m_0]$ as the *lottery for low money holdings*. The low prize of this lottery is $z_1^* = 0$, the high prize is $z_2^* = m_0$, and the probability of winning the high prize is $\pi_2^*(m) = m/m_0$. The high prize is determined as

$$m_0 = \arg\max_{z \geq m} \left[\frac{m}{z} \tilde{V}(z) + \left(1 - \frac{m}{z}\right) \tilde{V}(0) \right]. \tag{3.18}$$

It is clear that m_0 is independent of the individual's balance m, provided $m \leq m_0$.

The following theorem states existence, uniqueness and other properties of the equilibrium ex ante value function as well as the properties of the upper bound $\bar{m}$ (see Appendix D for a proof):

Theorem 3.5. *(i) T is a self-map on $\mathcal{V}[0, \bar{m}]$ and has a unique fixed point V.*
(ii) $V(m) > W(m) > 0$ for all $m > 0$; $V(0) = W(0) > 0$, and $W(m) \geq B(m)$ for all $m \in [0, k]$.
(iii) There exists $m_0 \in (k, \bar{m}]$ such that an individual with $m < m_0$ will play the lottery with the prize m_0, which satisfies $V(m_0) = B(m_0)$, $b^*(m_0) > 0$ and $\phi(m_0) = 0$. Moreover, if $m_0 < \bar{m}$, then (3.10) holds for $m = m_0$, and $V'(m_0) = B'(m_0) > 0$.
(iv) $V'(m) > 0$ exists for all $m \in [0, \bar{m})$; $B'(m)$ exists for all $m \in [k, \bar{m})$ such that $b^*(m) > 0$.
(v) There exists $\bar{m} < \infty$ such that individuals' money balances satisfy $m \leq \bar{m}$. Moreover, $\bar{m}$ satisfies $\bar{m} = \hat{z}_2 = z_2^*(\hat{m})$, $B(\bar{m}) = V(\bar{m})$ and $B'(\bar{m}) = V'(\bar{m})$.

For part (i), we verify that the mappings on V defined by a worker's problem, (2.1), and a buyer's problem, (2.2), are monotone and feature discounting with the factor β. As a result, T is a monotone contraction mapping that maps continuous, increasing and concave functions into continuous and increasing functions. In addition, since TV is generated by the optimal choice of a two-point lottery, it is a concave function (see Lemma F.1, Menzio and Shi, 2010). Thus, T is a monotone contraction mapping on $\mathcal{V}[0, \bar{m}]$ and has a unique fixed point.

Part (ii) of Theorem 3.5 compares V, W and B, which can be seen from Figure 1. Part (iii) formally characterizes the lottery for low money holdings. In particular, after winning the prize m_0, an individual strictly prefers to be a buyer and he spends all of the balance m_0 in one trade. Part (iv) states that the ex ante value function is differentiable and strictly increasing for all $m < \bar{m}$ and a buyer's value function is differentiable at all balances at which the buyer has a positive matching probability. This result on differentiability of V is stronger than the one in Theorem 3.2 which stated only that V is differentiable at money balances generated by optimal choices and at $m < \bar{m}$ such that $B(m) = V(m)$. The strong result on differentiability is not surprising, given the results we have obtained so far. The ex ante value function is clearly differentiable at any $m \in [0, m_0)$, because it is a straight line in this region. If $m_0 < \bar{m}$, then V is also differentiable at m_0, because the high prize of the lottery is interior and $B(m_0) = V(m_0)$. Furthermore, V is differentiable in $(m_0, \bar{m})$: If V were not differentiable at some $m \in (m_0, \bar{m})$, then V would be strictly concave at m and hence $V(m) = B(m)$, in which case part (iv) of Theorem 3.2 would imply the contradiction that V is differentiable at m.

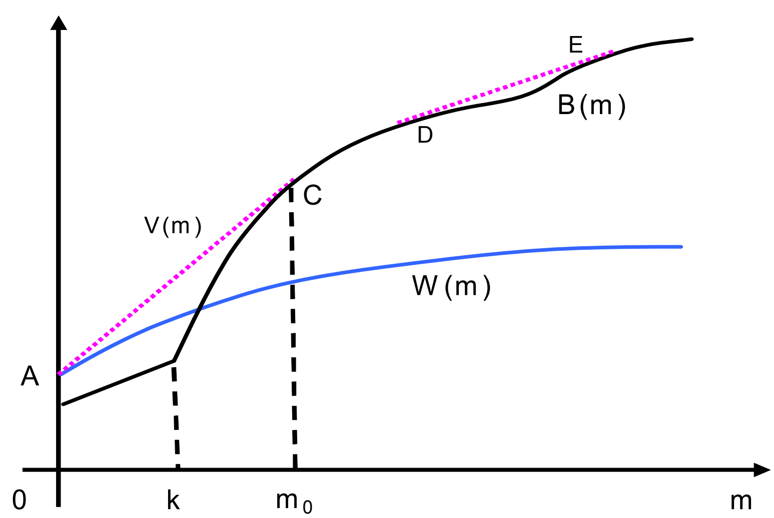

Figure 1. Lotteries and the ex ante value function

Finally, part (v) of Theorem 3.5 states that individuals' money holdings are endogenously bounded above by $\bar{m} < \infty$; in addition, $\bar{m}$ is equal to the high prize of the lottery played by an individual with the balance $\hat{m}$. Moreover, $\bar{m}$ has the properties $B(\bar{m}) = V(\bar{m})$ and $B'(\bar{m}) = V'(\bar{m})$, and so we can eliminate the qualifications "if $m < \bar{m}$" and "if $m_0 < \bar{m}$" in various parts of Theorems 3.2 and 3.5. It is intuitive that individuals' money holdings are finite. Because the marginal utility of consumption is diminishing, the marginal value of money is diminishing. In contrast, the marginal cost of obtaining money, in terms of the disutility of labor, is strictly increasing. Thus, if an individual has a sufficiently large balance, he strictly prefers spending some of rather than working to accumulate even more money. This force puts an endogenous upper bound on individuals' money holdings prior to the play of lotteries. Moreover, since the marginal value of money to a buyer is diminishing, it is bounded above. This implies that if an individual plays a lottery with any finite amount of money, the high prize of the optimal lottery must be finite. Thus, individuals' money holdings after the play of lotteries are also finite. Because $\bar{m}$ is equal to the high prize of the optimal lottery that may be played by an individual whose balance is $\hat{m} < \infty$, the construction of the lottery yields $B(\bar{m}) = V(\bar{m})$ and $B'(\bar{m}) = V'(\bar{m})$.

4. Monetary Equilibrium

In this section we characterize the spending pattern, prove existence and uniqueness of the monetary steady state, and examine the steady-state distribution of real balances.

4.1. Equilibrium pattern of spending

Let us begin with some features of optimal choices established in section 3. First, a worker with no money supplies $\ell^*(0)$ units of labor and obtains $\hat{m}$ units of money. Second, an individual with a balance m may play a lottery before going to the market, where the low prize of the lottery is $z_1^*(m)$ and the high prize is $z_2^*(m)$. If $m < m_0$, then the individual will certainly play a lottery with $z_1^*(m) = 0$ and $z_2^*(m) = m_0$. Third, buyers sort into different submarkets according to their money balances. A buyer with a balance m $(\geq m_0)$ chooses to enter the submarket where he has a matching probability $b^*(m)$ and, after being matched, he spends an amount $x^*(m)$ units of money, buys $q^*(m)$ units of goods, and exits the trade with a residual balance $\phi(m) = m - x^*(m)$. The functions $b^*(m)$, $x^*(m)$, $q^*(m)$ and $\phi(m)$ are all increasing in m.

Denote $\phi^0(m) = m$ and $\phi^{i+1}(m) = \phi(\phi^i(m))$ for $i = 0, 1, 2, ...$ For any arbitrary balance $m \geq m_0$, let $n(m)$ be the number of purchases that a buyer with m can make before his balance falls below m_0, i.e., $\phi^{n(m)-1}(m) \geq m_0 > \phi^{n(m)}(m)$. Also, denote $\hat{n} = n(\hat{m})$, $\hat{z}_j = z_j^*(\hat{m})$ and $\hat{n}_j = n(\hat{z}_j)$, where $j \in \{1, 2\}$. We prove the following lemma in Appendix E:

Lemma 4.1. *(i) If $\hat{m} < m_0$, then $\hat{z}_1 = 0$, $\hat{z}_2 = m_0$, and $\hat{n}_2 = 1$;*
(ii) The only lottery that is possibly played in the steady state is the lottery at $\hat{m}$, with $\hat{z}_1$ and $\hat{z}_2$ as the prizes, and this lottery is indeed played if $B(\hat{m}) < V(\hat{m})$;
(iii) If $\hat{m} \geq m_0$, then the following properties hold for $j \in \{1, 2\}$: (a) $b^*(\phi^{i-1}(\hat{z}_j)) > 0$ for all $i = 1, 2, ..., \hat{n}_j$; (b) $\phi^i(\hat{z}_j) \geq m_0$, $V(\phi^i(\hat{z}_j)) = B(\phi^i(\hat{z}_j))$ and V is strictly concave at $\phi^i(\hat{z}_j)$ for all $i = 1, 2, ..., \hat{n}_j - 1$; (c) $\phi^{\hat{n}_j}(\hat{z}_j) = 0$.

Part (i) of Lemma 4.1 is implied by part (iii) of Theorem 3.5 for $m = \hat{m}$. If $\hat{m} < m_0$, a worker who has worked to obtain the balance $\hat{m}$ will play the lottery with the prize m_0 next period and, if he wins the lottery, he will spend the entire amount m_0 in one trade. We will provide a precise condition for $\hat{m} < m_0$ later in Theorem 4.3. At this point, let us relate the case to convexity of the disutility function of labor supply, $h(\ell)$. A sufficiently convex disutility function of labor supply means that the marginal disutility increases rapidly with labor supply. In this case it is optimal for a worker to work a small amount of time in a period, which leads to $\hat{m} < m_0$. Note that regardless of how convex h is, it is not optimal for a worker to work for consecutive periods unless the worker does not win the prize m_0 of the lottery. The use of the lottery with the prize m_0 is a better way to smooth the cost of labor than working for consecutive periods.

Even when $\hat{m} > m_0$, an individual holding $\hat{m}$ chooses to play a lottery if $B(\hat{m}) < V(\hat{m})$, as stated in part (ii) of Lemma 4.1.[12] Recall from subsection 3.2 that a buyer's value function may be convex at some $\hat{m} > m_0$. If $\hat{m}$ lies near such a convex region of B, then it is beneficial for an individual to play a lottery before going to the market. In contrast to the case $\hat{m} < m_0$, an individual with $\hat{m} > m_0$ will always go to the market as a buyer after playing a lottery, regardless of whether the high or the low prize is realized. Both prizes are greater than or equal to m_0 if $\hat{m} > m_0$. Part (ii) of Lemma 4.1 states further that the only possible lottery played in the steady state is the one at $\hat{m}$. If $\hat{m} < m_0$, the statement is obviously true. If $\hat{m} \geq m_0$, the statement is implied by part (iii) of Lemma 4.1, which we explain below.

Part (iii) of Lemma 4.1 describes a stylized purchasing cycle by a buyer who enters the market with a balance $\hat{z}_j$, which is a prize of the lottery at $\hat{m}$. Such a buyer will trade with positive probability every period until running out of money (part (a)), his value function will be strictly concave at the residual balance if this balance is strictly positive (part (b)), and in the last trade in the cycle, he will spend all of his money instead of leaving a small amount to play the lottery for low money holdings (part (c)). Because of parts (b) and (c), the buyer has no need for a lottery at any residual balance resulted from trade. Moreover, as the buyer's balance diminishes with each trade, the buyer goes through a sequence of submarkets where the trading probability is increasingly lower, the required spending in a trade is increasingly lower, the quantity of goods traded is increasingly lower, and the residual balance after trade is increasingly lower.

Part (iii) of Lemma 4.1 comes from repeated applications of parts (iv) and (v) of Theorem 3.2. To see this, note first that a buyer's value function at either prize $\hat{z}_j$ of the lottery played at $\hat{m} \geq m_0$ satisfies $B(\hat{z}_j) = V(\hat{z}_j)$ and $B'(\hat{z}_j) = V'(\hat{z}_j)$. Since $\hat{z}_j \geq m_0$, then $b^*(\hat{z}_j) \geq b^*(m_0) > 0$. If it is optimal for the buyer with $\hat{z}_j$ to drive the residual balance to $\phi(\hat{z}_j) = 0$, then the individual

[12] The case where a lottery is not played at $\hat{m} > m_0$ can be treated as a degenerate lottery at $\hat{m}$.

will become a worker next period. If it is optimal for the buyer with $\hat{z}_j$ to keep a strictly positive residual balance $\phi(\hat{z}_j) > 0$, then all the hypotheses in part (v) of Theorem 3.2 are satisfied with $m = \hat{z}_j$. In this case, the ex ante value function is strictly concave at $\phi(\hat{z}_j)$. Strict concavity of V at $\phi(\hat{z}_j)$ implies that no lottery is needed at $\phi(\hat{z}_j)$, and so B is equal to V at $\phi(\hat{z}_j)$. Then, $m = \phi(\hat{z}_j)$ satisfies the hypotheses in part (iv) of Theorem 3.2, which implies that B and V are differentiable at $\phi(\hat{z}_j)$ and $B'(\phi(\hat{z}_j)) = V'(\phi(\hat{z}_j))$. Moreover, strict concavity of V at $\phi(\hat{z}_j)$ implies that $\phi(\hat{z}_j) \geq m_0$, because $\phi(\hat{z}_j) < m_0$ would imply a contradiction that V is linear near $\phi(\hat{z}_j)$. With the balance $\phi(\hat{z}_j) \geq m_0$, the buyer will trade with a strictly positive probability and the outcome of the trade is either $\phi^2(\hat{z}_j) = 0$ or $\phi^2(\hat{z}_j) > 0$. If $\phi^2(\hat{z}_j) = 0$, the round of purchases ends. If $\phi^2(\hat{z}_j) > 0$, we can repeat the above argument to conclude that V is strictly concave at $\phi^2(\hat{z}_j)$, the two value functions B and V are equal to each other at $\phi(\hat{z}_j)$, and their derivatives at $\phi^2(\hat{z}_j)$ are given by (3.10) with $m = \phi^2(\hat{z}_j)$. In this case, the buyer has no need for a lottery at $\phi^2(\hat{z}_j)$ and the residual balance satisfies $\phi^2(\hat{z}_j) \geq m_0$. This pattern continues until the $\hat{n}_j$-th trade, in which the buyer spends all the money.

The purchasing cycle above has some similarity to that in the inventory model of money (see Baumol, 1952, and Tobin, 1956). However, our model has the following features that are absent in the inventory model. First, our model has a microfoundation for money. Second, there are matching frictions, which imply that a buyer does not always have a match. Thus, the number of periods which a buyer spends in a purchasing cycle is larger than the number of purchases. Third, the trading probability and the terms of trade are endogenous. As each purchase reduces a buyer's balance, the buyer chooses to spend a longer time to get the next trade and, when he gets the next trade, he spends less money and obtains a smaller quantity of goods.

4.2. Equilibrium distribution of real balances

Let $G(m)$ be the measure of individuals holding a balance less than or equal to m immediately after the outcomes of the lotteries are realized in a period. From the previous subsection we know that if $\hat{m} < m_0$, the individuals in the market are either buyers with the balance m_0 or workers with no money. In this case, the support of G is $\{m_0, 0\}$. We also know that if $\hat{m} \geq m_0$, a worker who obtains the balance $\hat{m}$ may play a lottery next period before going to the market as a buyer. Depending on the realization of the lottery, $\hat{z}_j$, the individual will go through a purchasing cycle in which the individual's money holdings will be characterized by the sequence $\{\phi^i(\hat{z}_j)\}_{i=0}^{\hat{n}_j-1}$. Thus, if $\hat{m} \geq m_0$, the support of G is $\{\phi^i(\hat{z}_1)\}_{i=0}^{\hat{n}_1-1} \cup \{\phi^i(\hat{z}_2)\}_{i=0}^{\hat{n}_2-1} \cup \{0\}$. Since the distribution has a discrete support, we denote the corresponding frequency function as g.

It is straightforward to calculate the steady-state distribution of real balances. In the steady state, the measure of individuals who hold each balance in the support of G should be constant over time. If $\hat{m} \geq m_0$ (i.e., $\hat{n}_2 \geq 1$), we can express this requirement as follows:

$$0 = g(0)\,\hat{\pi}_j - b^*(\hat{z}_j)g(\hat{z}_j), \qquad j = 1, 2; \tag{4.1}$$

$$\begin{gathered} 0 = b^*(\phi^{i-1}(\hat{z}_j))g(\phi^{i-1}(\hat{z}_j)) - b^*(\phi^i(\hat{z}_j))g(\phi^i(\hat{z}_j)) \\ \text{for } 1 \leq i \leq \hat{n}_j - 1 \text{ and } j = 1, 2; \end{gathered} \tag{4.2}$$

$$g(0) = \sum_{j=1,2} b^*(\phi^{\hat{n}_j - 1}(\hat{z}_j))g(\phi^{\hat{n}_j-1}(\hat{z}_j)), \tag{4.3}$$

where $\hat{\pi}_j = \pi_j^*(\hat{m})$ for $j \in \{1, 2\}$. Equation (4.1) sets the change in the measure of individuals who hold the balance $\hat{z}_j$ to zero. The flow of individuals into the balance $\hat{z}_j$ consists of those workers in the current period who will win the prize $\hat{z}_j$ of the lottery at $\hat{m}$ next period. The size of this inflow is $g(0)\hat{\pi}_j$. The outflow of individuals from the balance $\hat{z}_j$ consists of the buyers with the balance $\hat{z}_j$ who successfully trade in the current period. The size of this outflow is $b^*(\hat{z}_j)g(\hat{z}_j)$. Similarly, (4.2) sets the change in the measure of individuals who hold the balance $\phi^i(\hat{z}_j)$ to zero, where $i \in \{1, 2, ..., \hat{n}_j - 1\}$. The inflow of individuals into the balance $\phi^i(\hat{z}_j)$ consists of the buyers with the balance $\phi^{i-1}(\hat{z}_j)$ who successfully trade in the current period, and the outflow consists of the buyers with the balance $\phi^i(\hat{z}_j)$ who successfully trade in the current period. Finally, (4.3) sets the change in the measure of individuals who hold no money to zero. In any period, the individuals who have no money are the workers. Since every worker obtains a balance $\hat{m}$ by working for one period, the size of the outflow from the group is $g(0)$. The inflow comes from the buyers who are in the last period of their purchasing cycle and who successfully trade in the current period, as given by the right-hand side of (4.3).

Equations (4.1) – (4.3) solve for the steady-state distribution as

$$\left.\begin{aligned} & g(\phi^i(\hat{z}_j)) = \tfrac{g(0)\hat{\pi}_j}{b^*(\phi^i(\hat{z}_j))} \text{ for } j = 1, 2, \text{ and } 0 \leq i \leq \hat{n}_j - 1; \\ & g(0) = \left[1 + \textstyle\sum_{j=1,2} \sum_{i=0}^{\hat{n}_j - 1} \tfrac{\hat{\pi}_j}{b^*(\phi^i(\hat{z}_j))}\right]^{-1}. \end{aligned}\right\} \tag{4.4}$$

The formula (4.4) is also valid for the case $\hat{m} < m_0$. In this case, $\hat{z}_1 = 0$ and $\hat{n}_2 = 1$ in (4.4), and so the steady-state distribution is $g(m_0) = 1 - g(0)$ and $g(0) = b^*(m_0)/[b^*(m_0) + \pi_2^*(m_0)]$.

4.3. Existence and uniqueness of a monetary steady state

In section 3, we have characterized individuals' policy and value functions, which are independent of the nominal wage rate ω. The market tightness function θ is solved by (2.5), which is independent of ω. Moreover, given the policy functions, (4.4) solves the steady-state distribution of real balances independently of ω. Thus, for a monetary steady state to exist, it suffices to solve for ω by requirement (vi) of Definition 2.1. This requirement can be rewritten as

$$\omega = \left[\sum_{j=1,2} \sum_{i=0}^{\hat{n}_j - 1} \phi^i(\hat{z}_j) g(\phi^i(\hat{z}_j)) \right]^{-1}. \tag{4.5}$$

Because all of the elements on the right-hand side of (4.5) are independent of ω, the formula determines a unique, finite value of ω in the steady state. We summarize this result and other properties of the steady state in the following theorem (see Appendix F for a proof):

Theorem 4.2. *A unique monetary steady state exists and is block recursive. Money is neutral in the steady state. The distribution of buyers over money balances is degenerate if $\beta \leq \beta_0$, where $\beta_0 > 0$ is defined in Appendix F. On the other hand, if β is sufficiently close to one, the distribution of buyers over money balances is non-degenerate if and only if*

$$m_c > m_0(m_c) = q_0(m_c) + \frac{k}{\mu(b_0(m_c))}, \tag{4.6}$$

where m_c, $q_0(m)$, and $b_0(m)$ are defined in Appendix F. Moreover, the frequency function of the distribution satisfies $g(\phi^i(\hat{z}_j)) > g(\phi^{i-1}(\hat{z}_j))$ for all $i = 1, 2, ..., \hat{n}_j - 1$ and $j = 1, 2$.

Money is neutral in the sense that a one-time change in the money stock has no effect on real variables in the steady state. This is intuitive in our model. Money holdings, m, the quantity of money traded in a match, x^*, and the residual balance after a trade, $\phi(m)$, are all measured in terms of labor. These real quantities are given by the policy functions that are independent of the money stock. Similarly, a one-time change in the money stock does not affect the quantity of goods traded, labor supply and the distribution of individuals.

The distribution of buyers over money balances may or may not be degenerate in the steady state. If individuals are sufficiently impatient in the sense $\beta \leq \beta_0$, then $\hat{m} \leq m_0$ and the distribution of buyers over money balances is degenerate. In this case, all buyers in the market hold the same amount of money, m_0. Moreover, a buyer spends all the money whenever he has a match and an individual alternates stochastically between being a buyer with a balance m_0 and a worker with no money. Thus, when $\beta \leq \beta_0$, our model endogenously generates the patterns that are assumed in earlier models with indivisible money (e.g., Shi, 1995, and Trejos and Wright, 1995). However, our model does not share the result of these models that a one-time change in the money stock affects real activities. Instead, money is neutral in the steady state here.

It is intuitive that the distribution of buyers over money balances is degenerate when individuals are sufficiently impatient.[13] Consider a buyer with the highest equilibrium balance, $\hat{z}_2$. The buyer can spend this balance in one trade or spread it over several periods in a sequence of purchases. If the buyer spends the entire balance in one trade, he consumes a large amount of goods in the period. The upside of doing so is that the utility of current consumption is not discounted. The downside relative to spreading consumption over several periods is that the marginal utility of large consumption is low. When the buyer is sufficiently impatient, the upside of spending all the money at once outweighs the downside. In fact, if $\beta \leq \beta_0$, the highest equilibrium balance is $\hat{z}_2 = m_0$. In this case, all buyers in the market hold the same balance m_0.

Thus, for money distribution to be non-degenerate among buyers, a necessary condition is that individuals are patient. However, high patience is not sufficient for a non-degenerate distribution; even in the limit $\beta \to 1$, the additional condition (4.6) is needed. In (4.6), m_c is the solution for $\hat{m}$ under the supposition that $\hat{z}_2 \leq m_0$. When (4.6) is satisfied, the supposition is contradicted, in which case the equilibrium must have $\hat{m} > m_0$. That is, (4.6) is a necessary condition for the distribution of buyers over money balances to be non-degenerate. In the limit $\beta \to 1$, the condition is also sufficient for the distribution to be non-degenerate.

The condition (4.6) is complicated because m_c, $q_0(m)$ and $b_0(m)$ are defined implicitly through some equations (see Appendix F). To illustrate the elements involved, consider:

Example: $U(q) = \frac{(q+0.1)^{1-\sigma}-(0.1)^{1-\sigma}}{1-\sigma}$, $h(\ell) = 10[1-(1-\ell)^\eta]$, and $\mu(b) = 1-b$. A higher value of σ indicates stronger concavity of U, and a higher value of η (< 1) indicates less convexity of h. For any given (σ, η), we denote $K(\sigma, \eta)$ as the critical level of k such that (4.6) is satisfied if and only if $k < K(\sigma, \eta)$. Figure 2.1 depicts $K(\sigma, 0.5)$ for $\sigma \in [1.1, 3]$ and Figure 2.2 depicts $K(2, \eta)$ for $\eta \in [0.1, 0.9]$. The function $K(\sigma, 0.5)$ is increasing in σ. This means that when the utility function of consumption is more concave, (4.6) is more easily satisfied for any given (k, η). Also, $K(2, \eta)$ is increasing in η. This means that as the disutility function of labor supply becomes less convex, (4.6) is more easily satisfied for any given (k, σ).

The above example indicates some elements, in addition to high patience, that are intuitively important for a buyer to choose to run down the balance in several periods rather than in one period. First, the cost of creating a trading post cannot be too high. If a trading post is very costly to create, the number of trading posts in each submarket is small and, hence, the matching probability is very low for a buyer. It is optimal to spend all the money in one trade in this

[13]The critical level β_0 depends on other features of the model environment. In particular, β_0 increases in the degree of convexity of the disutility function of labor supply. Thus, consistent with an earlier explanation, the case $\hat{m} < m_0$ is more likely to occur if the disutility function of labor supply is more convex.

case because, if the buyer keeps any residual balance, it will be difficult to find a match in the future to spend it. Second, the utility function of consumption needs to be sufficiently concave. Intuitively, a more concave utility function increases a buyer's incentive to smooth consumption over time by making a round of relatively small purchases rather than one large purchase. Third, the disutility of labor supply cannot be very convex. As explained earlier, if the marginal cost of labor supply increases very quickly, the optimal choice is to work for a relatively small amount of money in a period, spend it all in a trade, and work again, rather than work in one period for a large amount of money and make a sequence of purchases.[14]

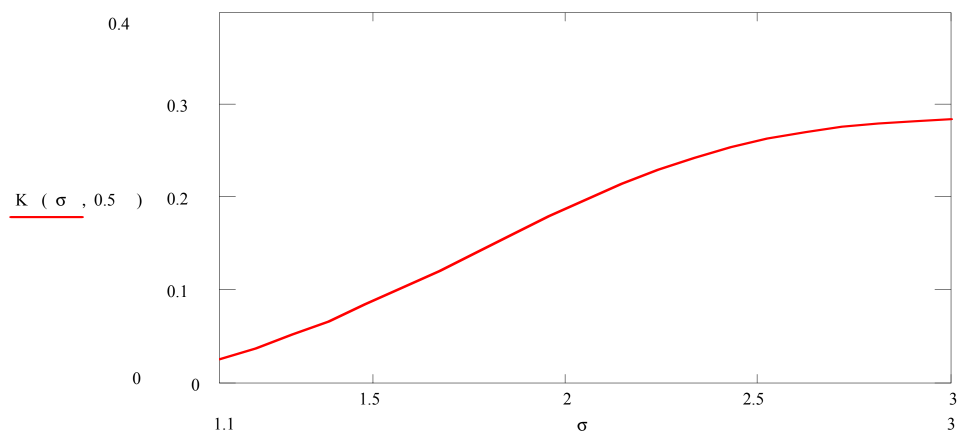

Figure 2.1

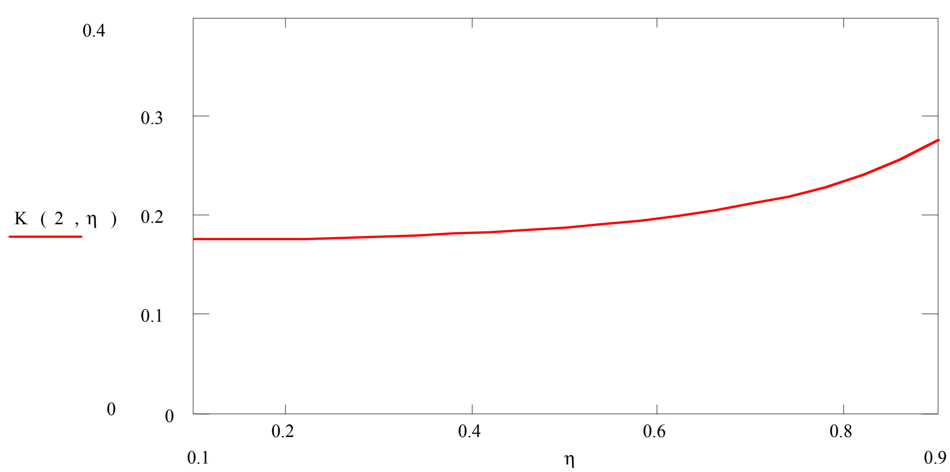

Figure 2.2

[14]Another element for a non-degenerate distribution is that $\mu(b)$ should not increase very quickly with b. If $\mu(b)$ increases very quickly with b, the amount of money required for obtaining any given quantity of goods increases quickly with b. In this case, the benefit of acquiring a large balance and going through a sequence of purchases is small relative to the cost of labor supply, and so a buyer will make only one purchase before working again.

Theorem 4.2 also describes the shape of the steady-state distribution. To explain this shape, consider first the case where an individual with the balance $\hat{m}$ has no need for a lottery. Then, the measure of buyers increases as their money holdings strictly decrease in the purchasing cycle; i.e., the equilibrium frequency function g is a strictly decreasing function of money holdings among buyers. This is an intuitive consequence of buyers' optimal choices described in Theorem 3.2. Because buyers who hold a relatively high balance choose to trade with a relatively high probability, they exit quickly from the high balance into the lower level of money balance and, hence, a relatively small number of buyers are left holding a high balance in the steady state.[15] Next, consider the case where an individual with the balance $\hat{m}$ has the need for a lottery. From each prize of the lottery, $\hat{z}_j$ $(j = 1, 2)$, a buyer's balance in a purchasing cycle goes through the sequence $\{\phi^i(\hat{z}_j)\}_{i=0}^{\hat{n}_j - 1}$. The above feature of the distribution of real balances holds true for each of these two sequences. That is, for each $j \in \{1, 2\}$, the measure of buyers holding $\phi^i(\hat{z}_j)$ increases with i and, hence, decreases with $\phi^i(\hat{z}_j)$ for all $i \in \{0, 1, ..., \hat{n}_j - 1\}$. However, with a non-degenerate lottery at $\hat{m}$, the overall frequency function of money holdings is not necessarily monotone. For example, $g(\phi^i(\hat{z}_1))$ may be greater than, less than or equal to $g(\phi^i(\hat{z}_2))$ for a particular i, and the result of the comparison between the two may vary over i.

A non-degenerate distribution of real balances has a wealth effect in the sense that a transfer of money between two sets of buyers who hold different balances affects the sum of the values of these buyers. Recall that a buyer's marginal value of money increases strictly as the buyer's balance decreases with each purchase. That is, $V'(\phi^i(\hat{z}_j)) > V'(\phi^{i-1}(\hat{z}_j))$ for all $i = 0, 1, ..., \hat{n}_j$ and $j = 1, 2$ (see part (v) of Theorem 3.2). A transfer of money from a buyer with a relatively high balance to a buyer with a relatively low balance reduces the gap between the two buyers' marginal values of money. This transfer increases the sum of the values of the two buyers.

5. Concluding Remarks

In this paper, we have constructed and analyzed a tractable search model of money where the distribution of money holdings can be non-degenerate. We modeled search as a directed process in the sense that buyers know the terms of trade before visiting particular sellers. We showed that the monetary steady state is block recursive in the sense that individuals' policy functions, value functions and the market tightness function are all independent of the distribution of individuals over money balances, although the distribution affects the aggregate activity by itself. Using lattice-theoretic techniques, we characterized individuals' policy and value functions, and showed

[15]This result generalizes the result of Green and Zhou (1998) from an economy where goods are indivisible and money are in discrete units to an economy with fully divisible money and goods.

that these functions satisfy the standard conditions of optimization. We proved that a unique monetary steady state exists, provided conditions under which the steady-state distribution of buyers over money balances is non-degenerate, and analyzed the properties of this distribution.

We hope that our model provides a new starting point for examining both the long-run and short-run effects of policies. Although the monetary steady state is block recursive, the distribution of real balances does matter for the aggregate real activity and welfare. Policies that permanently redistribute the purchasing power between individuals with different real balances, such as inflation, affect the steady-state real activity and welfare. Even if a policy does not have long-run effects, such as a one-time injection of money, it can still affect the real activity and welfare in the short run. The reason is that individuals' decisions on how much money to spend and to work for depend on the rate of return to money, which can be expressed as ω_t/ω_{t+1}. By affecting this rate of return in the short run, a one-time injection of money can affect individuals' decisions temporarily. Along this line, our model is natural to use for examining how the so-called liquidity effect of monetary policy (see Lucas, 1990) depends on the way in which money is injected. For example, the short-run effect of a lump-sum injection is likely to be different from that of a proportional injection, and the short-run effect of an injection of money to buyers is likely to be different from that of an injection to firms.

Our model of directed search will be useful for simplifying this analysis of a dynamic equilibrium, as well as the steady state. The above discussion suggests that individuals' value and policy functions outside the steady state in our model do not depend on the distribution G directly; rather, they depend on G in the short run only through the rate of return to money, ω_t/ω_{t+1}. A common practice of many central banks is to specify a path of the nominal interest rate. Such a policy determines the path of the rate of return to money, ω_t/ω_{t+1}. Given this path, individuals' decisions and the market tightness are independent of the distribution G outside the steady state as well as in the steady state. Hence, in this case, our analysis can be modified in a straightforward manner to study dynamics. Furthermore, even if the path of the nominal interest rate is determined endogenously in the equilibrium rather than being specified by monetary policy, directed search still simplifies the task of computing a dynamic equilibrium. Because the distribution can affect individuals' decisions and market tightness only through a one-dimensional variable, ω_t/ω_{t+1}, the dynamic equilibrium of our model can be solved using the approximation technique of Krusell and Smith (1998). In contrast, under random search, the individuals' value and policy functions depend directly on G and, hence, the equilibrium of the model requires more computationally intensive techniques (see e.g. Molico, 2006, and Chiu and Molico, 2008).

Appendix

A. Proof of Lemma 3.1

Take any $V \in \mathcal{V}[0, \bar{m}]$ as the ex ante value function appearing in a worker's maximization problem, (2.1). The objective function in (2.1) is continuous, bounded on $[0, \bar{m}]$ and increasing in m. Then, the Theorem of the Maximum implies $W \in \mathcal{C}[0, \bar{m}]$, i.e., a continuous and increasing function on $[0, \bar{m}]$. Because the objective function $[\beta V(m+\ell) - h(\ell)]$ is strictly concave in (ℓ, m) jointly, its maximized value, $W(m)$, is concave in m, and the optimal choice ℓ^* is unique. With uniqueness, the Theorem of the Maximum implies that the policy function $\ell^*(m)$ is continuous (see Stokey et al, 1989, p62). The choice $\ell = 1$ can never be optimal under the assumption $h'(1) = \infty$. It may be possible that the optimal choice is $\ell^*(m) = 0$ when m is sufficiently high. In this case, it is evident that $\ell^*(m) = 0$ is (weakly) increasing in m and $y^*(m) = m$ is strictly increasing in m.

The remainder of this proof focuses on the case where $\ell^*(m) > 0$. In this case, $y^*(m) = m + \ell^*(m) > m$. Reformulate a worker's problem as (3.2), where the choice is the end-of-period balance $y = m+\ell$. The objective function in (3.2) is strictly concave in (y, m) jointly and $h(y-m)$ is continuously differentiable in (y, m). Thus, the result in Benveniste and Scheinkman (1979) applies (see also Stokey et al, 1989, p85). That is, for all m such that the optimal choice $y^*(m)$ is interior, $W(m)$ is differentiable and the derivative satisfies:

$$W'(m) = h'(y^*(m) - m) = h'(\ell^*(m)).$$

In addition, using concavity of W and strict convexity of h, we can deduce from the equation $W'(m) = h'(\ell^*(m))$ that $\ell^*(m)$ is decreasing in m.

Return to the original maximization problem of a worker, (2.1). Consider any $m \in [0, \bar{m}]$ such that $\ell^*(m) > 0$. Because $\ell^*(m)$ is continuous, there exists $\varepsilon_0 > 0$ such that $\ell^*(m \pm \varepsilon) > 0$ for all $\varepsilon \in [0, \varepsilon_0]$. Moreover, we can choose sufficiently small ε_0 so that for any $\varepsilon \in [0, \varepsilon_0]$, the choice $\ell^*(m - \varepsilon)$ is feasible to a worker who holds a balance m and the choice $\ell^*(m)$ is feasible to a worker who holds a balance $m - \varepsilon$. Then, for any $\varepsilon \in [0, \varepsilon_0]$, the optimality of ℓ^* implies:

$$\begin{aligned} W(m) &= F(\ell^*(m), m) \geq F(\ell^*(m-\varepsilon), m), \\ W(m-\varepsilon) &= F(\ell^*(m-\varepsilon), m-\varepsilon) \geq F(\ell^*(m), m-\varepsilon), \end{aligned}$$

where $F(\ell, m)$ temporarily denotes the objective function in (2.1). Hence,

$$\frac{F(\ell^*(m-\varepsilon), m) - F(\ell^*(m-\varepsilon), m-\varepsilon)}{\varepsilon} \leq \frac{W(m) - W(m-\varepsilon)}{\varepsilon} \leq \frac{F(\ell^*(m), m) - F(\ell^*(m), m-\varepsilon)}{\varepsilon}.$$

Since $W'(m)$ exists, taking the limit $\varepsilon \searrow 0$ on the above relations yields $\beta V'(y^{*-}(m)) = W'(m)$, where $y^{*-}(m) = m^- + \ell^*(m)$. Note that the one-sided derivatives $V'(y^{*-})$ and $V'(y^{*+})$ exist because V is a concave function (see Royden, 1988, pp113-114). Similarly, we can prove that $\beta V'(y^{*+}(m)) = W'(m)$, where $y^{*+}(m) = m^+ + \ell^*(m)$. Therefore, V is differentiable at $y^*(m)$ and the derivative satisfies $\beta V'(y^*(m)) = W'(m)$, which is the first equality in (3.1). Substituting $W'(m) = h'(\ell^*(m))$ yields the second equality in (3.1).

Finally, since V' is decreasing and h' is strictly increasing, the first-order condition $\beta V'(y^*(m)) = h'(y^*(m) - m)$ implies that $y^*(m)$ is strictly increasing. **QED**

B. Proof of Lemma 3.3

Take any $V \in \mathcal{V}[0, \bar{m}]$ as the ex ante value function appearing in a buyer's maximization problem, (2.2). Applying the Theorem of the Maximum to the problem, we conclude that B is continuous on $[0, \bar{m}]$ and that a solution to the buyer's maximization problem exists (see Stokey et al, 1989, p62). Since V is increasing, the objective function in (2.2) is increasing in m. Since the feasibility set in the maximization problem is also increasing in m, then B is increasing, i.e., $B \in \mathcal{C}[0, \bar{m}]$. As explained earlier in subsection 3.2.1, B has two segments. If $m \leq k$, then $b^*(m) = 0$ and $B(m) = \beta V(m)$; if $m > k$, $B(m)$ solves (3.6).

If $b^* = 0$, the choice of x is irrelevant for the buyer because a trade does not take place. For the remainder of the proof, we focus on the case where $b^*(m) > 0$. Temporarily denote $F(x, b, m) = b\, R(x, b, m)$, where the surplus function R is defined in (3.13). Optimal choices (x^*, b^*) maximize $F(x, b, m)$.

(1) A buyer's optimal choices are unique and the policy functions are continuous.

If $R(x, b^*, m) < 0$, the optimal choice is $b^* = 0$; if $R(x, b^*, m) = 0$, then the choice $b^* = 0$ is not dominated by other choices of b. Because we focus on $b^* > 0$, it suffices to examine a buyer's optimal choices when $R(x, b, m) > 0$. With $b > 0$ and $R(x, b, m) > 0$, we can transform a buyer's maximization problem as

$$B(m) = \beta V(m) + \exp\left\{\max_{x,b} \left[\ln b + \ln R(x, b, m)\right]\right\}.$$

The function $(\ln b)$ is concave. Recall that $u(x, b)$ is strictly concave in (x, b) jointly. Since V is concave, then $V(m - x)$ is concave in x. Thus, $R(x, b, m)$ defined in (3.13) is strictly concave in (x, b) jointly. Since the logarithmic function is strictly increasing and strictly concave, the function $[\ln b + \ln R(x, b, m)]$ is strictly concave in (x, b) jointly. The Theorem of the Maximum implies that a buyer's optimal choices (x^*, b^*) are unique for each m and the policy functions $(x^*(m), b^*(m))$ are continuous. So are the policy functions $q^*(m)$ and $\phi(m)$.

(2) Monotonicity of the policy functions $x^*(m)$ and $b^*(m)$.

Consider any $m \in [k, \bar{m}]$ such that $b^*(m) > 0$. We solve a buyer's maximization problem in two steps: first, the optimal choice of x solves the problem in (3.14) for any given (b, m); second, the optimal choice of b solves the problem in (3.15).

Take the first step. For any given (b, m), the optimal choice of x maximizes $R(x, b, m)$ and is denoted $\tilde{x}(b, m)$ as in (3.14). Because $u(x, b)$ is strictly concave in x and V is concave, $R(x, b, m)$ is strictly concave in x, which implies that a unique $\tilde{x}$ exists for any given (b, m). We prove that $R(x, b, m)$ is supermodular. Since the choice set of x, $[0, m]$, is increasing in m and independent of b, supermodularity of $R(x, b, m)$ and uniqueness of $\tilde{x}$ imply that the maximizer $\tilde{x}(b, m)$ is an increasing function of (b, m) (see Topkis, 1998, p76) and that the maximized value of R is supermodular in (b, m) (see Topkis, 1998, p70).

To prove that $R(x, b, m)$ is supermodular, note that the feasibility set of (x, b, m) is $\{(x, b, m) : 0 \leq x \leq m,\ 0 \leq b \leq 1,\ k \leq m \leq \bar{m}\}$. This set is a sublattice in $\mathbb{R}^3_+$ with the usual relation "$\geq$". It suffices to prove that R has increasing differences in the three pairs, (b, m), (x, b) and (x, m) (see Topkis, 1998, p45). Take arbitrary m_1, m_2, x_1, x_2, b_1 and b_2 from the feasibility set, with $m_2 > m_1$, $x_2 > x_1$, and $b_2 > b_1$. Because R is separable in b and m, it is clear that R has (weakly) increasing differences in (b, m). For the differences in (x, b), compute:

$$R(x_2, b, m) - R(x_1, b, m) = [u(x_2, b) - u(x_1, b)] + \beta [V(m - x_2) - V(m - x_1)].$$

Since $u(x, b)$ is strictly supermodular in (x, b), we have:

$$\begin{aligned} & [R(x_2, b_2, m) - R(x_1, b_2, m)] - [R(x_2, b_1, m) - R(x_1, b_1, m)] \\ = \; & [u(x_2, b_2) - u(x_1, b_2)] - [u(x_2, b_1) - u(x_1, b_1)] > 0. \end{aligned}$$

That is, R has strictly increasing differences in (x, b). For the differences in (x, m), we have:

$$\begin{aligned} & [R(x_2, b, m_2) - R(x_1, b, m_2)] - [R(x_2, b, m_1) - R(x_1, b, m_1)] \\ = \; & \beta [V(m_1 - x_1) - V(m_1 - x_2)] - \beta [V(m_2 - x_1) - V(m_2 - x_2)] \geq 0. \end{aligned}$$

The inequality follows from concavity of V (see Royden, 1988, p113) and the facts that $m_1 - x_1 < m_2 - x_1$, $m_1 - x_2 < m_2 - x_2$, and $(m_1 - x_1) - (m_1 - x_2) = (m_2 - x_1) - (m_2 - x_2) = x_2 - x_1 > 0$. Thus, $R(x, b, m)$ has increasing differences in (x, m).

Denote $\tilde{R}(b, m) = R(\tilde{x}(b, m), b, m)$ as in (3.14). From the above proof, $\tilde{R}(b, m)$ is supermodular in (b, m). Because $R(x, b, m)$ strictly decreases in b for any given (x, m), then $\tilde{R}(b, m)$ is strictly decreasing in b. To examine the dependence of $\tilde{R}(b, m)$ on m, take arbitrary m_1 and m_2 in $[k, \bar{m}]$, with $m_2 \geq m_1$. We have:

$$R(x, b, m_2) - R(x, b, m_1) = \beta [V(m_1) - V(m_1 - x)] - \beta [V(m_2) - V(m_2 - x)] \geq 0,$$

where the inequality follows from concavity of V. Since the above result holds for all (x, b), then

$$\tilde{R}(b, m_1) = R(\tilde{x}(b, m_1), b, m_1) \leq R(\tilde{x}(b, m_1), b, m_2) \leq R(\tilde{x}(b, m_2), b, m_2) = \tilde{R}(b, m_2).$$

Note that for the second inequality we have used the fact that $\tilde{x}(b, m_1)$ is feasible in the problem $\max_{x \leq m_2} R(x, b, m_2)$. Thus, $\tilde{R}(b, m)$ increases in m.

Now let us take the second step, i.e., to characterize the optimal choice of b. Denote the optimal choice of b as $b^*(m) = \arg\max_{b \in [0,1]} f(b, m)$, where

$$f(b, m) = F(\tilde{x}(b, m), b, m) = b\tilde{R}(b, m).$$

We show that f is supermodular in (b, m). Take arbitrary $b_1, b_2 \in [0, 1]$, with $b_2 > b_1$, and arbitrary $m_1, m_2 \in [k, \bar{m}]$, with $m_2 > m_1$. Compute:

$$\begin{aligned} & [f(b_2, m_2) - f(b_1, m_2)] - [f(b_2, m_1) - f(b_1, m_1)] \\ = \; & b_2 \left[\tilde{R}(b_2, m_2) - \tilde{R}(b_1, m_2) + \tilde{R}(b_1, m_1) - \tilde{R}(b_2, m_1)\right] \\ & + (b_2 - b_1)\left[\tilde{R}(b_1, m_2) - \tilde{R}(b_1, m_1)\right]. \end{aligned}$$

Because $\tilde{R}(b, m)$ is supermodular in (b, m), the first difference on the right-hand side is positive. Because $\tilde{R}(b, m)$ is increasing in m, the second difference on the right-hand side is also positive. Thus, $f(b, m)$ is supermodular in (b, m) on $[0, 1] \times [k, \bar{m}]$. Note also that the choice set for b, $[0, 1]$, is independent of m and that the optimal choice b^* is unique. Thus, $b^*(m)$ is increasing in m (see Topkis, 1998, p76). Since $\tilde{x}(b, m)$ is increasing in (b, m), the optimal choice of x, given by $x^*(m) = \tilde{x}(b^*(m), m)$, is increasing in m.

(3) $q^*(m)$ is an increasing function.

Denote $a = m - x + q$ and use (q, a) as a buyer's choices. Using (3.4), we can express:

$$m - x = a - q, \quad b = \mu^{-1}\left(\frac{k}{m - a}\right).$$

Because $b \geq 0$, the relevant domain of a is $[0, m - k]$. The relevant domain of q is $[0, a]$. A buyer chooses $(q, a) \in [0, a] \times [0, m - k]$ to solve:

$$\max_{(q,a)} \mu^{-1}\left(\frac{k}{m - a}\right)[U(q) + \beta V(a - q) - \beta V(m)].$$

We can divide this problem into two steps: first solve q for any given (a, m) and then solve a.

For any given (a, m), the optimal choice of q, denoted as $\tilde{q}(a)$, solves:

$$J(a) \equiv \max_{0 \leq q \leq a} [U(q) + \beta V(a - q)].$$

Note that q and J do not depend on m for any given a. It is easy to see that the objective function above is supermodular in (q, a). Since the choice set, $[0, a]$, is increasing in a and $\tilde{q}$ is unique, then $\tilde{q}(a)$ and $J(a)$ increase in a (see Topkis, 1998, p76 and p70).

The optimal choice of a is $a^*(m) = \arg\max_{0\le a\le m-k} \Delta(a,m)$, where

$$\Delta(a,m) = \mu^{-1}\left(\frac{k}{m-a}\right)[J(a) - \beta V(m)].$$

Note that if $J(a) < \beta V(m)$, the buyer can choose $a = m - k$ to obtain $\Delta = 0$. Thus, focus on the case where $J(a) \ge \beta V(m)$. Since $\mu(b)$ is strictly decreasing in b and $1/\mu(b)$ is strictly convex in b, the function $\mu^{-1}(\frac{k}{m-a})$ strictly increases in m, strictly decreases in a, and is strictly supermodular in (a,m). Thus, for arbitrary $a_2 > a_1$ and $m_2 > m_1 \ge k$, we have:

$$\begin{aligned}
& \Delta(a_2,m_2) - \Delta(a_1,m_2) - \Delta(a_2,m_1) + \Delta(a_1,m_1) \\
= & \left[\mu^{-1}\left(\tfrac{k}{m_2-a_2}\right) - \mu^{-1}\left(\tfrac{k}{m_1-a_2}\right)\right][J(a_2) - J(a_1)] \\
& + \left[\mu^{-1}\left(\tfrac{k}{m_2-a_1}\right) - \mu^{-1}\left(\tfrac{k}{m_2-a_2}\right)\right][\beta V(m_2) - \beta V(m_1)] \\
& + \left[\mu^{-1}\left(\tfrac{k}{m_2-a_2}\right) - \mu^{-1}\left(\tfrac{k}{m_2-a_1}\right) - \mu^{-1}\left(\tfrac{k}{m_1-a_2}\right) + \mu^{-1}\left(\tfrac{k}{m_1-a_1}\right)\right][J(a_1) - \beta V(m_1)].
\end{aligned}$$

The first term on the right-hand side is positive because $J(a)$ increases in a and $\mu^{-1}(\frac{k}{m-a})$ increases in m. The second term on the right-hand side is positive because $\mu^{-1}(\frac{k}{m-a})$ decreases in a and $V(m)$ increases in m. The third term on the right-hand side is strictly positive because $\mu^{-1}(\frac{k}{m-a})$ is strictly supermodular in (a,m). Therefore, $\Delta(a,m)$ is strictly supermodular. Since the choice set $[0, m-k]$ is also increasing in m, the solution $a^*(m)$ increases in m (see Topkis, 1998, p76). Since $\tilde{q}(a)$ increases in a, then $q^*(m) = \tilde{q}(a^*(m))$ increases in m.

(4) $\phi(m)$ is an increasing function.

We reformulate a buyer's problem by letting the choices be (ϕ, a), where a is defined as $a = \phi + q$. From the definition of a and (3.4), we can express

$$q = a - \phi, \quad b = \mu^{-1}\left(\frac{k}{m-a}\right).$$

The relevant domain of ϕ is $[0, \min\{m, a\}]$, and of a is $[0, m-k]$. A buyer solves:

$$\max_{(\phi,a)} \mu^{-1}\left(\frac{k}{m-a}\right)[U(a-\phi) + \beta V(\phi) - \beta V(m)]. \tag{B.1}$$

As in the above formulation where the choices are (q,a), we can divide the maximization problem into two steps. First, for any given a, the optimal choice of ϕ solves:

$$J(a) = \max_{\phi\ge 0}[U(a-\phi) + \beta V(\phi)]. \tag{B.2}$$

Note that we have written the constraint on ϕ as $\phi \ge 0$, instead of $\phi \in [0, \min\{m, a\}]$. The optimal choice satisfies $\phi < m$, because $\phi = m$ implies $x = 0$ which is not optimal (in the case with $b > 0$). Also, $\phi < a$ under the assumption that $U'(0)$ is sufficiently large. Denote the solution for ϕ as $\tilde{\phi}(a)$. Second, the optimal choice of a solves

$$B(m) - \beta V(m) = \max_{0 \le a \le m-k} \mu^{-1}\left(\frac{k}{m-a}\right)[J(a) - \beta V(m)]. \tag{B.3}$$

Similar to the procedure used in the above formulation of the problem where the choices are (q, a), we can prove that $a^*(m)$ increases in m and, hence, $\phi^*(m)$ increases in m. **QED**

C. Proof of Lemma 3.4

Take any $V \in \mathcal{V}[0, \bar{m}]$ as the ex ante value function appearing in a buyer's problem and consider any arbitrary $m \in [k, \bar{m}]$ such that $b^*(m) > 0$. Parts (1) - (4) below establish Lemma 3.4.

(1) The one-sided derivatives of B satisfy (3.16) and (3.17). $B'(m)$ exists if and only if $V'(m)$ exists. Moreover, $B(m)$ is strictly increasing.

Consider the formulation of a buyer's problem, (B.1), where the choices are ϕ and $a = \phi + q$. Let a and a' be arbitrary levels in $[0, m-k]$. Note that the constraint on the choice ϕ is $\phi \ge 0$, which does not depend on a. Thus, the choice $\tilde{\phi}(a)$ is feasible in the maximization problem with a' and the choice $\tilde{\phi}(a')$ is feasible in the maximization problem with a. Using a proof similar to the one in Appendix A that established the existence of $V'(y^*(m))$, we can prove that $J'(a^-)$ and $J'(a^+)$ both exist and are equal to

$$J'(a) = U'(\tilde{q}(a)) > 0, \tag{C.1}$$

where $\tilde{\phi}$ and $\tilde{q}(a) \equiv a - \tilde{\phi}(a)$ are given in part (4) of the above proof of Lemma 3.3.

Next, we prove that the objective function in (B.3) is strictly concave in a and derive the first-order condition of a. Recall that $\tilde{q}(a)$ is an increasing function, as shown in the above proof of Lemma 3.3. This result and (C.1) together imply that $J'(a)$ is decreasing, i.e., that $J(a)$ is concave. Because $J(a)$ is increasing and concave, and $\mu^{-1}(\frac{k}{m-a})$ is strictly decreasing and strictly concave in a, it can be verified that the objective function in (B.3) is strictly concave in a. Strict concavity of the objective function implies that the optimal choice of a is unique. Also, because the objective function is differentiable in a, the optimal choice of a is given by the first-order condition. Deriving the first-order condition, substituting $J'(a)$ from (C.1), and substituting $\mu^{-1}(\frac{k}{m-a^*}) = b^*(m)$, we obtain:

$$J(a^*) - \beta V(m) + U'(\tilde{q}(a^*))\frac{k\mu' b^*(m)}{\mu^2} \le 0 \quad \text{and} \quad a^* \le m-k, \tag{C.2}$$

where the two inequalities hold with complementary slackness.

Now we derive (3.16) and (3.17), which clearly imply that $B'(m)$ exists if and only if $V'(m)$ exists. Note that $b^*(m) > 0$ implies $a^*(m) < m - k$. Because $a^*(m) < m - k$ and $a^*(m)$ is continuous, there exists $\varepsilon > 0$ such that $a^*(m + \varepsilon) < m - k$ and $a^*(m) < m - \varepsilon - k$. Consider the neighborhood $O(m) = (m - \varepsilon, m + \varepsilon)$. For any $m' \in O(m)$, the choice $a^*(m')$ is feasible in the problem where the balance is m, and the choice $a^*(m)$ is feasible in the problem where the balance is m'. Applying to (B.3) a proof similar to Appendix A that established the existence of $V'(y^*(m))$, we can derive the formulas of $B'(m^+)$ and $B'(m^-)$ for any m such that $b^*(m) > 0$. These formulas and the first-order condition of a^*, (C.2), together yield:

$$\begin{aligned} B'(m^+) &= b^*(m)\left[J'(a^*) - \beta V'(m^+)\right] + \beta V'(m^+), \\ B'(m^-) &= b^*(m)\left[J'(a^*) - \beta V'(m^-)\right] + \beta V'(m^-). \end{aligned}$$

Again, we have used the fact that a concave function has one-sided derivatives. Substituting $J'(a^*)$ from (C.1) and $U'(q^*) = u_1(x^*, b^*)$ into the above equations, we obtain (3.16) and (3.17).

Finally, we prove that B is strictly increasing. Since V is concave and increasing, $V'(m^-) \geq V'(m^+) \geq 0$. Since $b^* \leq 1$ and $J'(a^*(m)) > 0$, the above equations for $B'(m^+)$ and $B'(m^-)$ imply that $B'(m^-) \geq B'(m^+) \geq b^*(m)J'(a^*) > 0$, where we have used the hypothesis $b^*(m) > 0$. Therefore, $B(m)$ is strictly increasing if $b^*(m) > 0$.

(2) The optimal choice b^* satisfies the first-order condition (3.8). If $\phi(m) > 0$, then $V'(\phi(m))$ exists, and the optimal choice x^*satisfies the first-order condition (3.9).

For any given (x, m), the objective function in a buyer's problem (3.6) is differentiable with respect to b. Thus, if the optimal choice b^* is interior, it satisfies the first-order condition (3.8). Now consider the optimal choice x^* and assume $\phi(m) > 0$ (i.e., $x^*(m) < m$). Since $x^*(m) < m$, a procedure similar to the derivation of $J'(a)$ in part (1) above but applied to (3.6) yields:

$$\begin{aligned} B'(m^+) &= \beta\left[b^*(m)V'(\phi^+(m)) + (1 - b^*(m))V'(m^+)\right] \\ B'(m^-) &= \beta\left[b^*(m)V'(\phi^-(m)) + (1 - b^*(m))V'(m^-)\right], \end{aligned}$$

where $\phi^+(m) = m^+ - x^*(m)$ and $\phi^-(m) = m^- - x^*(m)$. Comparing these equations with (3.16) and (3.17) yields that $V'(\phi^+(m)) = V'(\phi^-(m)) = V'(\phi)$ which is given by (3.9).

(3) For any $m \in [k, \bar{m})$ such that $b^*(m) > 0$, if $B(m) = V(m)$ and if there exists a neighborhood $O \ni m$ such that $B(m') \leq V(m')$ for all $m' \in O$, then $B'(m)$ and $V'(m)$ exist and satisfy (3.10) in part (iv) of Theorem 3.2. Also, (3.10) holds for $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$.

Take any $m \in [k, \bar{m})$ such that $b^*(m) > 0$, $B(m) = V(m)$ and the neighborhood O described above exists. Because $B(m') \leq V(m')$ for all $m' \in O(m)$, continuity of B and V implies that $B'(m^-) \geq V'(m^-)$ and $B'(m^+) \leq V'(m^+)$. Substituting $V'(m^-) \leq B'(m^-)$ into the right-hand side of (3.17), we get:

$$V'\left(m^-\right) \leq B'(m^-) \leq \frac{b^*(m)}{1-\beta\left[1-b^*(m)\right]} u_1\left(x^*(m), b^*(m)\right). \tag{C.3}$$

Similarly, substituting $V'(m^+) \geq B'(m^+)$ into the right-hand side of (3.16), we get:

$$V'\left(m^+\right) \geq B'(m^+) \geq \frac{b^*(m)}{1-\beta\left[1-b^*(m)\right]} u_1\left(x^*(m), b^*(m)\right). \tag{C.4}$$

On the other hand, concavity of V implies $V'\left(m^-\right) \geq V'\left(m^+\right)$. Thus, $V'\left(m^-\right) = B'(m^-) = B'(m^+) = V'\left(m^+\right)$. Moreover, $V'(m)$ and $B'(m)$ satisfy (3.10).

If $m = \bar{m}$, it is still true that $B'(m^-) \geq V'(m^-)$, and so (C.3) also holds at $m = \bar{m}$. However, since we cannot presume $V'(\bar{m}^+) \geq B'(\bar{m}^+)$, we cannot conclude that (C.4) holds at this point. If $B'(\bar{m}) = V'(\bar{m})$, however, (3.16) and (3.17) imply that (3.10) holds at $m = \bar{m}$.

(4) Consider any $m \in [k, \bar{m})$ such that $b^*(m) > 0$ and $\phi(m) > 0$. If $B(m) = V(m)$ and if there exists a neighborhood $O \ni m$ such that $B(m') \leq V(m')$ for all $m' \in O$, then b^* and ϕ are strictly increasing at m and V is strictly concave at $\phi(m)$, with $V\left(\phi(m)\right) = B\left(\phi(m)\right)$ and $V'(\phi(m)) > V'(m)$. These properties also hold for $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$.

Take any arbitrary $m_1 \in [k, \bar{m}]$ such that $b^*(m_1) > 0$, $\phi(m_1) > 0$, $B(m_1) = V(m_1)$ and the neighborhood O described above exists. If $m_1 = \bar{m}$, then add the assumption $B'(m_1) = V'(m_1)$. Shorten the notation $\phi(m_1)$ as ϕ_1 and $b^*(m_1)$ as b_1^*. As a preliminary step, we prove $V'(\phi_1) > V'(m_1)$ so that V must be strictly concave in some sections of $[\phi_1, m_1]$. By the construction of m_1, $V'(m_1)$ satisfies (3.10) and $V'(\phi_1)$ satisfies (3.9). Subtracting the two relations yields:

$$V'\left(\phi_1\right) - V'\left(m_1\right) = \frac{1-\beta}{\beta\left[1-\beta\left(1-b_1^*\right)\right]} u_1(x_1^*, b_1^*) > 0.$$

Next, we prove that $b^*(.)$ is strictly increasing at m_1. Let m_2 be sufficiently close to m_1 so that $\phi(m_2) > 0$ and $b^*(m_2) > 0$ (which is feasible because $\phi(m)$ and $b(m)$ are continuous functions). Shorten the notation $(x^*(m_i), b^*(m_i), \phi(m_i))$ to (x_i^*, b_i^*, ϕ_i), where $x_i^* = m_i - \phi_i$ and $i = 1, 2$. Since the proofs of strict monotonicity of $b^*(m_1)$ at m_1 are similar in the cases $m_2 > m_1$ and $m_2 < m_1$, let us consider only the case $m_2 > m_1$. By Lemma 3.3, $x_2^* \geq x_1^*$, $b_2^* \geq b_1^*$ and $\phi_2 \geq \phi_1$. We prove that $b_2^* > b_1^*$. Because $b_i^* > 0$, the first-order condition for b, (3.8), yields:

$$u(x_i^*, b_i^*) + \beta[V(\phi_i) - V(m_i)] + b_i^* u_2(x_i^*, b_i^*) = 0.$$

Subtracting these conditions for $i = 1, 2$, and re-organizing, we get:

$$\begin{aligned} & u(x_1^*, b_1^*) - u(x_1^*, b_2^*) - b_2^* u_2(x_2^*, b_2^*) + b_1^* u_2(x_1^*, b_1^*) \\ = & \left\{u(x_2^*, b_2^*) + \beta[V(m_2 - x_2^*) - V(m_2)] - u(x_1^*, b_2^*) - \beta[V(m_2 - x_1^*) - V(m_2)]\right\} \\ & +\beta\left\{V(m_1) - V(m_1 - x_1^*) + V(m_2 - x_1^*) - V(m_2)\right\}. \end{aligned}$$

The term in the first pair of braces on the right-hand side is equal to $[R(x_2^*, b_2^*, m_2) - R(x_1^*, b_2^*, m_2)]$, where R is defined by (3.13). This term is non-negative because x_2^* maximizes $R(x, b_2^*, m_2)$ and x_1^* is a feasible choice of x in this maximization problem (as $x_1^* \leq x_2^*$). The term in the second pair of braces is strictly positive because V is strictly concave in some sections of $[\phi_1, m_1] \subset [\phi_1, m_2]$, as proven earlier. Thus, the left-hand side of the above equation must be strictly positive. Noting that $u_2(x_2^*, b_2^*) \geq u_2(x_1^*, b_2^*)$ (because $u_{12} > 0$ and $x_2^* \geq x_1^*$), we get:

$$u(x_1^*, b_1^*) - u(x_1^*, b_2^*) - b_2^* u_2(x_2^*, b_2^*) + b_1^* u_2(x_1^*, b_1^*) > 0.$$

The left-hand side of this inequality is a strictly increasing function of b_2^*, and it is equal to 0 when $b_2^* = b_1^*$. Thus, the inequality implies $b_2^* > b_1^*$.

Let us complete the proof of part (4). Since (3.10) and (3.9) hold at $m = m_1$, we can combine the two relations to obtain:

$$V'(\phi_1) = V'(m_1) \left[\frac{1-\beta}{\beta b^*(m_1)} + 1 \right].$$

Because $b^*(m_1)$ is strictly increasing at m_1 and V is concave, the right-hand side above is strictly decreasing in m_1. Thus, the above equation requires V to be strictly concave at $\phi(m_1)$ and ϕ to be strictly increasing at m_1. **QED**

D. Proof of Theorem 3.5

We prove each part of Theorem 3.5 in turn.

Part (i). Let us express the functional equation (2.1) in a worker's problem as $W(m) = T_W V(m)$ for $m \in [0, \bar{m}]$, and express the functional equation (2.2) in a buyer's problem as $B(m) = T_B V(m)$ for $m \in [0, \bar{m}]$. Substituting these expressions into (2.3) to obtain $\tilde{V}$, we can rewrite (2.4) as $V(m) = TV(m)$, where

$$TV(m) \equiv \max_{(z_1, z_2, \pi_1, \pi_2)} [\pi_1 \max\{T_W V(z_1), T_B V(z_1)\} + \pi_2 \max\{T_W V(z_2), T_B V(z_2)\}] \tag{D.1}$$

$$\text{s.t.} \quad \begin{array}{l} \pi_1 z_1 + \pi_2 z_2 = m, \quad \pi_1 + \pi_2 = 1, \quad z_2 \geq z_1, \\ \pi_j \in [0,1] \text{ and } z_j \geq 0 \text{ for } j = 1, 2. \end{array}$$

Lemma 3.1 proves that T_W maps $\mathcal{V}[0, \bar{m}]$ into $\mathcal{V}[0, \bar{m}]$; i.e., T_W maps the set of continuous, increasing and concave functions on $[0, \bar{m}]$ into itself. Theorem 3.2 proves that T_B maps $\mathcal{V}[0, \bar{m}]$ into $\mathcal{C}[0, \bar{m}]$ (but not necessarily into $\mathcal{V}[0, \bar{m}]$). Thus, the objective function in (D.1) is a continuous function of z_1 and z_2. Also, the objective function is increasing in π_1 and π_2, and the feasibility set in the above problem is increasing in m. These features of the maximization problem above imply that T maps $\mathcal{V}[0,\bar{m}]$ into $\mathcal{C}[0,\bar{m}]$. Moreover, since the function $\max\{T_W V(z), T_B V(z)\}$ is continuous in z on a closed interval $[0,\bar{m}] \ni z$, the lottery in (2.4) makes $TV(m)$ a concave function (see Appendix F in Menzio and Shi, 2010, for a proof). Thus, T is a self-map on $\mathcal{V}[0,\bar{m}]$.

It is evident from (2.1) and (2.2) that T_W and T_B are monotone mappings, and so T is a monotone mapping. It is also easy to verify that T_W and T_B feature discounting with a factor $\beta \in (0,1)$. Thus, T features discounting with a factor β. Hence, T satisfies Blackwell's sufficient conditions for a monotone contraction mapping, which imply that T has a unique fixed point $V \in \mathcal{V}[0,\bar{m}]$ (see Stokey et al, 1989).

Part (ii). For a worker with any balance m, the choice of working zero hours yields the value $\beta V(m)$. Because this choice is always feasible, $W(m) \geq \beta V(m)$ for all m. For a buyer who holds $m \leq k$, the value is $B(m) = \beta V(m) \leq W(m)$. It is clear that $V(0) = \tilde{V}(0) = W(0)$. Also, $V(0) \geq 0$, because an individual without money can always choose not to trade. To prove $V(0) > 0$, suppose $V(0) = 0$, to the contrary. In this case, $0 = V(0) \geq W(0) \geq \beta V(0) = 0$, and so $W(0) = V(0) = 0$. Using the definition of $W(0)$, we have $\beta V(\hat{m}) - h(\hat{m}) = 0$. Since this equation has a unique solution and since $\hat{m} = 0$ satisfies the equation, then $\hat{m} = 0$. Recall that $\hat{m} = \ell^*(0)$ is the optimal labor supply of an individual without money and that the policy function $\ell^*(m)$ is decreasing in m. Thus, $\hat{m} = 0$ implies that $\ell^*(m) = 0$ for all $m \geq 0$. In this case, no individual will work for money, and so a monetary equilibrium does not exist. Therefore, for a monetary equilibrium to exist, it must be the case that $V(m) \geq W(m) \geq W(0) = V(0) > 0$ for all m.

We now prove that $V(m) > W(m)$ for all $m > 0$. For all $m > 0$ such that the constraint $y^* \geq m$ is binding for a worker, (3.2) yields $W(m) = \beta V(m) < V(m)$. Now consider $m > 0$ such that the constraint $y^* \geq m$ is not binding for a worker. Contrary to the result in this part, suppose $V(\tilde{m}) = W(\tilde{m})$ for some $\tilde{m} > 0$ such that $y^*(\tilde{m}) > \tilde{m}$. Since $\beta V'(y^*(\tilde{m})) = W'(\tilde{m})$ by (3.1) in Lemma 3.1, then $V'(y^*(\tilde{m})) > 0$, and concavity of V implies $V'(\tilde{m}^-) > 0$. In this case,

$$V'\left(\tilde{m}^-\right) \leq W'(\tilde{m}) = \beta V'(y^*(\tilde{m})) \leq \beta V'(\tilde{m}^-) < V'(\tilde{m}^-).$$

The first inequality follows from the hypothesis $V(\tilde{m}) = W(\tilde{m})$ and the fact $V(m) \geq W(m)$ for all $m < \tilde{m}$. The equality is from (3.1). The second inequality follows from concavity of V, and the last inequality from $V'(\tilde{m}^-) > 0$. Since the above result is a contradiction, we conclude that $V(m) > W(m)$ for all $m > 0$.

Part (iii). We prove first that there is some $m' \in (0,\infty)$ such that $B(m') > W(m')$. Suppose, to the contrary, that $B(m) \leq W(m)$ for all $m \in (0,\infty)$. Then, $\tilde{V}(m) = W(m)$ for all m. Since $W(m)$ is concave (see Lemma 3.1), $\tilde{V}(.)$ is concave in this case, and so $V(m) = \tilde{V}(m) = W(m)$

for all m. In this case, (3.2) yields

$$V(m) = \max_{y \geq m} [\beta V(y) - h(y-m)], \quad \text{all } m > 0.$$

If $y^*(m) = m$, the above equation yields $V(m) = 0$, which contradicts part (ii) above. If $y^*(m) > m$, (3.1) in Lemma 3.1 implies that W is differentiable at m, with $W'(m) = \beta V'(y^*(m)) > 0$. Since $W(m) = V(m)$ for all $m > 0$ in this case, $V'(m) = W'(m) = \beta V'(y^*(m)) \leq \beta V'(m)$. This implies $V'(m) = 0 = V'(y^*(m))$, which contradicts $V'(y^*(m)) > 0$.

Next, we prove that there exists $m_0 \in (k, \bar{m}]$ with $V(m_0) = B(m_0)$ such that an individual with $m < m_0$ will play the lottery with the prize m_0. For an individual with a balance $m \in (0, k)$, the lottery with $z_1 = 0$ and $z_2 = m'$ yields a value higher than $\tilde{V}(m)$, where m' is described above. Thus, these individuals will participate in lotteries. However, m' may not necessarily be the optimal prize of the lottery for these individuals. The optimal prize is m_0, defined by (3.18). Clearly, $m_0 > k > 0$, $V(m_0) = \tilde{V}(m_0) = B(m_0)$, and $V(m) \geq \tilde{V}(m)$ for all $m \in [0, m_0]$.

Now we prove that $b^*(m_0) > 0$ and $\phi(m_0) = 0$. Suppose $b^*(m_0) = 0$ to the contrary, and so $B(m_0) = \beta V(m_0)$. Since $V(m_0) = B(m_0)$, as shown above, then $V(m_0) = 0$, which contradicts the above result in part (ii) that $V(m) > 0$ for all $m \geq 0$. Thus, it must be true that $b^*(m_0) > 0$. Since $V(m_0) = B(m_0)$, (C.3) holds for $m = m_0$ which, together with $b^*(m_0) > 0$, implies $V'(m_0^-) < u_1(x^*(m_0), b^*(m_0))/\beta$. Since $V(m)$ is linear for $m \in [0, m_0]$, then $V'(\phi(m_0)) = V'(m_0^-) < u_1(x^*(m_0), b^*(m_0))/\beta$. If $\phi(m_0) > 0$, then (3.9) holds for $m = m_0$, which yields the contradiction that $V'(\phi(m_0)) = u_1(x^*(m_0), b^*(m_0))/\beta$. Thus, it must be true that $\phi(m_0) = 0$.

Finally, since $V(m_0) = B(m_0)$ and $b^*(m_0) > 0$, m_0 satisfies the hypotheses in part (iv) of Theorem 3.2 if $m_0 < \bar{m}$. Thus, if $m_0 < \bar{m}$, then (3.10) holds for $m = m_0$, which implies $V'(m_0) = B'(m_0) > 0$.

Part (iv). We first prove that $V'(m)$ exists for all $m \in [0, \bar{m})$ and $B'(m)$ exists for all $m \in [k, \bar{m})$ such that $b^*(m) > 0$. If $V'(m)$ exists for all $m \in [0, \bar{m})$, then part (iii) of Theorem 3.2 implies that $B'(m)$ exists for all $m \in [k, \bar{m})$ such that $b^*(m) > 0$. To prove that $V'(m)$ exists for all $m \in [0, \bar{m})$, note that the lottery with the prize m_0 implies that $V'(m)$ exists for all $m \in [0, m_0)$. If $m_0 = \bar{m}$, then $V'(m)$ exists for all $m \in [0, \bar{m})$. If $m_0 < \bar{m}$, then $V'(m_0)$ also exists, as shown in part (iii) above. What remains to be proven is that $V'(m)$ exists for all $m \in (m_0, \bar{m})$. Suppose to the contrary that $V'(\tilde{m})$ does not exist for some $\tilde{m} \in (m_0, \bar{m})$. In this case, $V'(\tilde{m}^-) > V'(\tilde{m}^+)$, and so V is strictly concave at $\tilde{m}$. Because $V(m) > W(m)$ for all $m > 0$, as proven in part (ii) above, we must have $V(\tilde{m}) = B(\tilde{m})$. Also, $b^*(\tilde{m}) \geq b^*(m_0) > 0$. Thus, the hypotheses in part (iv) of Theorem 3.2 are true for $m = \tilde{m}$, and so $V'(\tilde{m})$ exists. This contradicts the supposition that $V'(\tilde{m})$ does not exist.

Next, we prove that $V'(m) > 0$ for all $m \in [0, \bar{m})$. For all $m \in [0, m_0)$, $V(m)$ is linear and $V'(m) = V'(m_0^-) > 0$. If $m_0 = \bar{m}$, then $V'(m) > 0$ for all $m \in [0, \bar{m})$. If $m_0 < \bar{m}$, then $V'(m_0) = B'(m_0) > 0$, as proven in part (iii) above. We need to prove $V'(m) > 0$ for all $m \in [m_0, \bar{m})$. Consider any $m > m_0$. Since $b^*(m_0) > 0$ and $b^*(m)$ is an increasing function (see part (i) of Theorem 3.2), then $b^*(m) > 0$, which further implies that $B(m)$ is strictly increasing (see part (iii) of Theorem 3.2). Because $\tilde{V}(m) = B(m)$ for all $m \geq m_0$, then $\tilde{V}(m)$ is strictly increasing over such m. Recall that $V(m)$ is constructed with lotteries over $\tilde{V}(m)$. If $V(m_1) = V(m_2)$ for some $m_2 > m_1 > m_0$, contrary to the claimed result, then $V(m)$ must be constant for all $m \in [m_1, m_2]$. Extend this interval to $[m_1', m_2']$, with $m_1' \leq m_1$ and $m_2' \geq m_2$, so that $V(m_1') = \tilde{V}(m_1')$ and $V(m_2') = \tilde{V}(m_2')$. Then, $\tilde{V}(m_2') = V(m_2) = V(m_1) = \tilde{V}(m_1')$, which contradicts strict monotonicity of $\tilde{V}$.

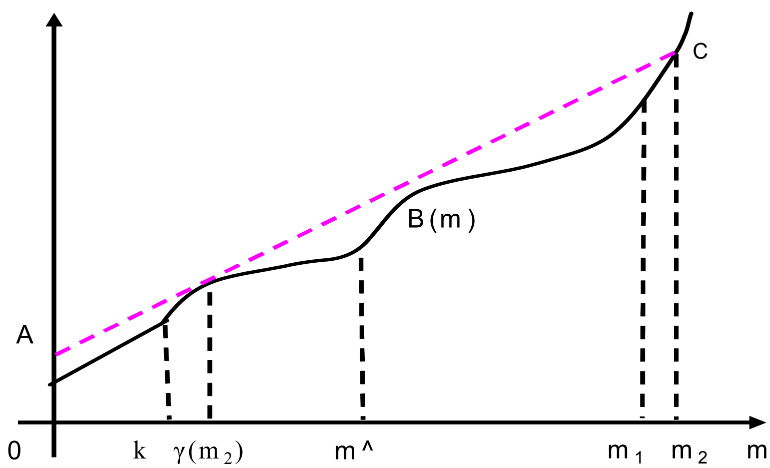

Figure 3.1

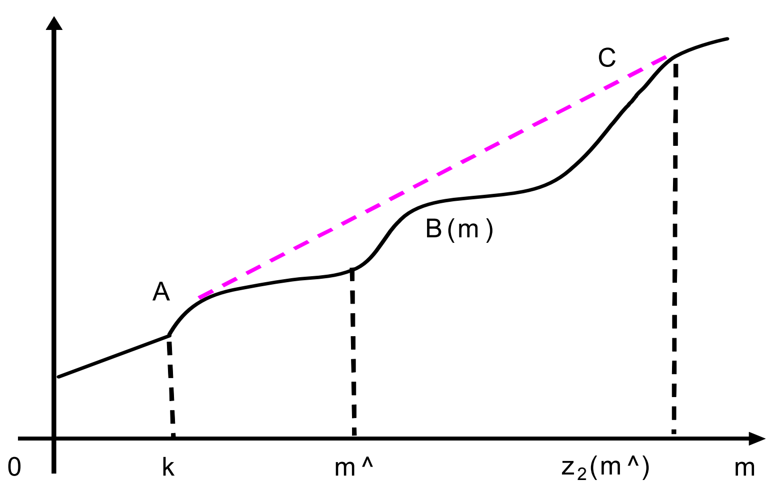

Figure 3.2

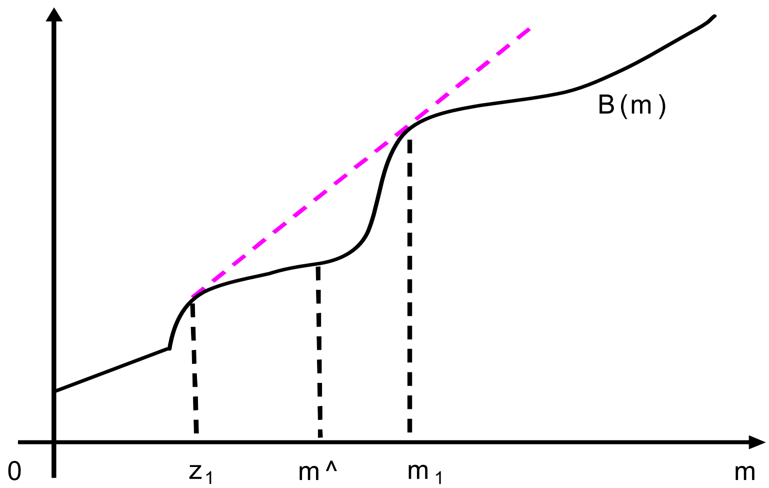

Figure 3.3

Part (v). For each exogenous upper bound on individuals' money holdings, the policy and value functions are characterized by the results in section 3 up to part (iv) of the current theorem. Now we allow the upper bound to vary and prove that individuals' money holdings in the equilibrium are indeed bounded above by $\bar{m}$ that satisfies part (v) of the current theorem. Note first that the balance obtained by a worker is $\hat{m} = \ell^*(0) \leq 1$, which is clearly bounded above. If $B(\hat{m}) = V(\hat{m})$ and $B'(\hat{m}) = V'(\hat{m})$, then $z_2^*(\hat{m}) = \hat{z}_2 = \hat{m}$ in which case we can set $\bar{m} = \hat{m}$, which satisfies the properties stated in this part. If $B(\hat{m}) < V(\hat{m})$ and $z_2^*(\hat{m}) < \infty$, then we can set $\bar{m} = z_2^*(\hat{m})$, which satisfies the properties stated in this part. Thus, it suffices to show that an equilibrium does not have the "unbounded case" where $B(\hat{m}) < V(\hat{m})$ and $z_2^*(\hat{m}) = \infty$. To begin, note that the unbounded case occurs if and only if there exists a finite $m_1 > \hat{m}$ such that the following two conditions are satisfied:

(A) $B(m)$ is strictly increasing and strictly convex for all $m \geq m_1$;

(B) for every $m_2 \geq m_1$, there exists $z_1 < \hat{m}$ such that, for all $m \in (z_1, m_2)$, $B(m)$ lies strictly below the line segment connecting $B(z_1)$ and $B(m_2)$.

Figure 3.1 depicts this unbounded case. If (A) is violated, as depicted in Figure 3.2, then there must exist a finite number $m_1 > \hat{m}$ such that $B(m)$ is concave for all $m \geq m_1$. In this case, the high prize of the lottery at $\hat{m}$ is $z_2^*(\hat{m}) < \infty$ and so $\bar{m} = z_2^*(\hat{m})$ has the properties stated in this part of Theorem 3.5. If (B) is violated, as depicted in Figure 3.3, then there must exist a finite $m_1 > \hat{m}$ and an associated $z_1 < \hat{m}$ such that the low prize of the lottery at $\hat{m}$ is z_1 and the high prize is m_1 and that, for all $m > m_1$, the function $B(m)$ lies strictly below the extension of the line connecting $B(z_1)$ and $B(m_1)$. In this case, $\bar{m} = m_1$ satisfies the properties in this part of

Theorem 3.5. Note that in the case depicted in Figure 3.3, $B(m)$ can still be strictly increasing and strictly convex for sufficiently large m, but such a section of B is irrelevant in the equilibrium because an individual's balance never goes above m_1.

On the way to derive a contradiction, suppose that there exists a finite $m_1 > \hat{m}$ such that (A) and (B) above are satisfied, as depicted in Figure 3.1. Consider any arbitrary $m_2 \geq m_1$. When individuals' money holdings are exogenously capped by m_2, the lottery at $\hat{m}$ is well-defined, with m_2 as the high prize, and all characterizations of the policy and value functions that we have obtained so far (including parts (i)-(iv) of the current Theorem 3.5) remain valid with $\bar{m} = m_2$. However, since $B'(m_2) > V'(m_2)$ in this case, we have $B'(\bar{m}) > V'(\bar{m})$. Denote the low prize of the lottery at $\hat{m}$ as $z_1^*(\hat{m}) = \gamma(m_2)$ so as to emphasize its dependence on the exogenous upper bound m_2. Without loss of generality, assume that $\hat{m} \geq m_0$, i.e., $B(\gamma(m_2)) = V(\gamma(m_2))$. (If $B(\gamma(m_2)) < V(\gamma(m_2))$, then $\gamma(m_2) = 0$, in which case the proof is still valid after replacing $V(\gamma(m_2))$ below with $V(0)$.) Denote

$$\alpha(m_2) = \frac{B(m_2) - V(\gamma(m_2))}{m_2 - \gamma(m_2)},$$

$$\hat{V}(m) = B(m_2) - \alpha(m_2)(m_2 - m), \quad m \in [0, m_2].$$

Here, $\alpha(m_2)$ is the slope of the line connecting $B(\gamma(m_2))$ and $B(m_2)$, and $\hat{V}(m)$ is the extension of this line to the domain $[0, m_2]$ (depicted in Figure 3.1 by the dashed line from point A to point C). It is clear that $V(m) \leq \hat{V}(m)$ for $m \in [0, m_2]$, with equality if $m \in [\gamma(m_2), m_2]$. We have:

$$\begin{aligned}
B(m_2) &= \max_{b\in[0,1], x\in[0,m_2]} \{b\left[u(x,b) + \beta V(m_2 - x)\right] + (1-b)\beta V(m_2)\} \\
&\leq \max_{b\in[0,1], x\in[0,m_2]} \left\{b\left[u(x,b) + \beta \hat{V}(m_2 - x)\right] + (1-b)\beta \hat{V}(m_2)\right\} \\
&= \max_{b\in[0,1], x\in[0,m_2]} \left\{b\left[u(x,b) - \beta\alpha(m_2)x\right] + \beta \hat{V}(m_2)\right\} \\
&\leq D(\alpha(m_2)) + \beta\hat{V}(m_2), \\
\text{where} \quad & D(\alpha(m_2)) \equiv \max_{b\in[0,1], q\geq 0} b\left\{U(q) - \beta\alpha(m_2)\left[q + \tfrac{k}{\mu(b)}\right]\right\}.
\end{aligned}$$

The first inequality follows from the fact that $V(m) \leq \hat{V}(m)$ for all $m \in [0, m_2]$, and the ensuing equality comes from the linearity of $\hat{V}$. The second inequality comes from ignoring the constraint $x \leq m_2$, substituting $x = q + \frac{k}{\mu(b)}$, and solving the maximization problem with (q, b) as the choices. By construction, $B(m_2) = V(m_2) = \hat{V}(m_2)$, and so the above result implies $B(m_2) \leq \frac{D(\alpha(m_2))}{1-\beta}$.

The notation $D(\alpha(m_2))$ emphasizes the fact that D depends on m_2 only through $\alpha(m_2)$. Applying the envelope condition to the maximization problem that defines $D(\alpha)$ above, we have $D'(\alpha) < 0$. Because $\gamma(m_2)$ is the low prize of the lottery at $\hat{m}$, we have $V'(\gamma(m_2)^+) \leq \alpha(m_2) \leq V'(\gamma(m_2)^-)$. Since m_1 satisfies (A), we can verify that $\gamma(m_2)$ decreases in m_2 for all $m_2 \geq m_1$.

Concavity of V implies that $\alpha(m_2)$ increases in m_2. Thus, for all $m_2 \geq m_1$, we have $\alpha(m_2) \geq \alpha(m_1) > 0$ and $D(\alpha(m_2)) \leq D(\alpha(m_1)) < \infty$. Therefore, $B(m_2) \leq D(\alpha(m_1))/(1-\beta) < \infty$ for all $m_2 \geq m_1$. This result contradicts the supposition that $B(m)$ is a strictly increasing and convex function for all $m_2 \geq m_1$. **QED**

E. Proof of Lemma 4.1

Part (i) of the lemma is implied by part (iii) of Theorem 3.5, with $m = \hat{m}$. Part (ii) of the lemma is obvious if $\hat{m} < m_0$ and, if $\hat{m} \geq m_0$, it is implied by part (iii) of the lemma. In particular, since part (iii) implies that $B(\phi^i(\hat{z}_j)) = V(\phi^j(\hat{z}_j))$, $B'(\phi^i(\hat{z}_j)) = V'(\phi^i(\hat{z}_j))$ and V is strictly concave at $\phi^i(\hat{z}_j)$ for all i in the set $\{0, 1, 2, ..., \hat{n}_j - 1\}$, then $\phi^i(\hat{z}_j) \geq m_0$ and a buyer with the balance $\phi^i(\hat{z}_j)$ has no need for a lottery for any i in the aforementioned set. Thus, the only lottery possibly played in the steady state is the one at $\hat{m}$.

We use induction to prove parts (a) and (b) of part (iii) of the lemma. Assume $\hat{m} \geq m_0$, as is required in part (iii), and take $\hat{z}_j$ as either prize of the lottery at $\hat{m}$. Start with $i = 1$. Because $\hat{m} \geq m_0$, then $\hat{z}_j \geq m_0$, and so $b^*(\hat{z}_j) \geq b^*(m_0) > 0$, where the strict inequality comes from part (iii) of Theorem 3.5. Thus, part (a) holds true for $i = 1$. Moreover, by the construction of the lottery at $\hat{m}$, $B(\hat{z}_j) = V(\hat{z}_j)$ and $B'(\hat{z}_1) = V'(\hat{z}_1)$. Also, since $\hat{z}_2 = \bar{m}$, part (v) of Theorem 3.5 implies $B'(\hat{z}_2) = V'(\hat{z}_2)$. Thus, $m = \hat{z}_j$ satisfies the hypotheses in part (v) of Theorem 3.2 which implies that, if $\phi(\hat{z}_j) > 0$, then V is strictly concave at $\phi(\hat{z}_j)$. Strict concavity of V at $\phi(\hat{z}_j)$ implies $V(\phi(\hat{z}_j)) = B(\phi(\hat{z}_j))$: if $B(\phi(\hat{z}_j)) < V(\phi(\hat{z}_j))$, V around $\phi(\hat{z}_j)$ would be a linear segment generated by the lottery in (2.4), which would contradict strict concavity of V at $\phi(\hat{z}_j)$. Thus, $m = \phi(\hat{z}_j)$ satisfies the hypotheses in part (iv) of Theorem 3.2 which implies $V'(\phi(\hat{z}_j)) = B'(\phi(\hat{z}_j))$. Moreover, strict concavity of V at $\phi(\hat{z}_j)$ implies that $\phi(\hat{z}_j) \geq m_0$, because V is linear for all $m < m_0$. Thus, parts (b) holds true for $i = 1$ if $\phi(\hat{z}_j) > 0$. If $\phi(\hat{z}_j) = 0$, on the other hand, part (b) is vacuous.

Suppose that parts (a) and (b) hold for an arbitrary $i \in \{1, 2, ..., \hat{n}_j - 1\}$, we prove that they hold for $i+1$ and, by induction, they hold for all $i \in \{1, 2, ..., \hat{n}_j - 1\}$. Because $\phi^i(\hat{z}_j) \geq m_0$ by the supposition, $b^*(\phi^i(\hat{z}_j)) \geq b^*(m_0) > 0$, and so part (a) holds for $i+1$. If $i = \hat{n}_j - 1$, then part (b) is vacuous for $i+1$. If $i < \hat{n}_j - 1$, then $\phi^{i+1}(\hat{z}_j) > 0$. Since $V(\phi^i(\hat{z}_j)) = B(\phi^i(\hat{z}_j))$ and V is strictly concave at $\phi^i(\hat{z}_j)$, by the supposition, then $m = \phi^i(\hat{z}_j)$ satisfies the hypotheses in part (v) of Theorem 3.2 which implies that V is strictly concave at $\phi^{i+1}(\hat{z}_j)$. Strict concavity of V at $\phi^{i+1}(\hat{z}_j)$ implies $V(\phi^{i+1}(\hat{z}_j)) = B(\phi^{i+1}(\hat{z}_j))$ and $\phi^{i+1}(\hat{z}_j) \geq m_0$. Thus, $m = \phi^{i+1}(\hat{z}_j)$ satisfies the hypotheses in part (iv) of Theorem 3.2 which implies $V(\phi^{i+1}(\hat{z}_j)) = B(\phi^{i+1}(\hat{z}_j))$. Hence, part (b) holds for $i+1$.

If $i = \hat{n}_j$, part (a) follows from the same proof as above, and part (b) is vacuous.

Finally, suppose $\phi^{\hat{n}_j}(\hat{z}_j) > 0$, contrary to part (c). Because part (b) holds for $i = \hat{n}_j - 1$, then $m = \phi^{\hat{n}_j - 1}(\hat{z}_j)$ satisfies all the hypotheses in part (v) of Theorem 3.2 which implies that V is strictly concave at $\phi^{\hat{n}_j}(\hat{z}_j)$. A contradiction. **QED**

F. Proof of Theorem 4.2

The text preceding the theorem has established that a unique monetary steady state exists, the steady state is block recursive, and the frequency function g is independent of ω. These results imply that money is neutral in the steady state. Turn to the result that from either $\hat{z}_j$ ($j = 1, 2$), the frequency function, $g(\phi^i(\hat{z}_j))$, is decreasing in $\phi^i(\hat{z}_j)$. To prove this result, note that $\phi^i(\hat{z}_j) = \phi^{i-1}(\hat{z}_j) - x^*(\phi^{i-1}(\hat{z}_j)) < \phi^{i-1}(\hat{z}_j)$ for all $1 \le i \le \hat{n}_j$ and $j = 1, 2$. By part (iii) of Theorem 3.5, $b^*(m_0) > 0$. For each $j \in \{1, 2\}$, Lemma 4.1 implies that $\phi^i(\hat{z}_j) \ge m_0$ and $b^*(\phi^i(\hat{z}_j)) > 0$ for all $1 \le i \le \hat{n}_j - 1$. Thus, for all $1 \le i \le \hat{n}_j - 1$, $\phi^i(\hat{z}_j)$ satisfies part (v) of Theorem 3.2, which implies that $b^*(.)$ is strictly increasing at $\phi^i(\hat{z}_j)$ for each i and j. With this feature, (4.4) implies that $g(\phi^i(\hat{z}_j)) > g(\phi^{i-1}(\hat{z}_j))$ for all $i = 1, 2, ..., \hat{n}_j - 1$ and $j = 1, 2$.

Next, we prove that there exists $\beta_0 > 0$ such that if $\beta \le \beta_0$, then $\hat{m} < m_0$ and $\phi(\hat{z}_2) = 0$. Let us shorten the notation $b^*(m_0)$ to b_0 and $q^*(m_0)$ to q_0. Define $\bar{q}(\beta)$ and $\underline{q}$ by

$$\frac{U(\underline{q})}{U'(\underline{q})} - \underline{q} = k, \quad U'(\bar{q}(\beta)) = \frac{1}{\beta} h'(\bar{q}(\beta)) + k). \tag{F.1}$$

We then define β_0 as

$$\beta_0 = \max_{\beta \in [0,1]} \left\{ \beta : \bar{q}(\beta) \le \underline{q} \right\}. \tag{F.2}$$

For any $\beta \in (0, 1]$, the assumptions on U and h imply that $\bar{q}(\beta)$ and $\underline{q}$ are well defined. In particular, the assumptions on U imply that $[\frac{U(q)}{U'(q)} - q]$ is a strictly increasing function of q whose value at $q = 0$ is 0. Moreover, $\bar{q}(\beta)$ and $\underline{q}$ have the following features:
(a) $\bar{q}'(\beta) > 0$ and $\lim_{\beta \to 0} \bar{q}(\beta) = 0 < \underline{q}$: These follow from the assumptions on U and h.
(b) $\underline{q} < q_0$: To verify this feature, note that $V'(0) = V'(m_0)$. Since $V'(m_0)$ satisfies part (v) of Theorem 3.2 with $m = m_0$, we have:

$$V'(0) = \frac{b_0 U'(q_0)}{1 - \beta + \beta b_0}, \tag{F.3}$$

where we have substituted $u_1(m_0, b_0) = U'(q_0)$. Also, the lottery in (3.18) implies $V(m) = V(0) + mV'(0)$ for all $m \in [0, m_0]$. Substituting $V(m_0)$ from this result and $V'(0)$ from (F.3) into the Bellman equation for $B(m_0)$ $(= V(m_0))$, we obtain:

$$b_0 \left[U(q_0) - m_0 U'(q_0) \right] = (1 - \beta) V(0). \tag{F.4}$$

Here, we have substituted $u_1(m_0, b_0) = U'(q_0)$ and $u(m_0, b_0) = U(q_0)$. Because $V(0) > 0$ by part (ii) of Theorem 3.5 and $b_0 > 0$, (F.4) implies $U(q_0) > m_0 U'(q_0)$. Because $m_0 > q_0 + k$ (as $\mu(b_0) < 1$), this result further implies $\frac{U(q_0)}{U'(q_0)} - q_0 > k = \frac{U(\underline{q})}{U'(\underline{q})} - \underline{q}$, which is equivalent to $\underline{q} < q_0$.
(c) $\bar{q}(\beta) > q^*(\hat{m})$ for all $\beta \in (0, 1]$: By the definition of $\bar{q}(\beta)$ in (F.1), $\bar{q}(\beta) > q^*(\hat{m})$ if and only if $U'(q^*(\hat{m})) > \frac{1}{\beta} h'(q^*(\hat{m})) + k)$. The latter relation is verified as follows:

$$U'(q^*(\hat{m})) \geq U'(q^*(\hat{z}_2)) > V'(\hat{z}_2) = V'(\hat{m}) = \frac{1}{\beta} h'(\hat{m}) > \frac{1}{\beta} h'(q^*(\hat{m})) + k).$$

The first inequality comes from the fact $q^*(\hat{m}) \leq q^*(\hat{z}_2)$. To obtain the second inequality, we apply (3.10) for $m = \hat{z}_2$, which yields $V'(\hat{z}_2) = \frac{b^*(\hat{z}_2) U'(q^*(\hat{z}_2))}{1 - \beta + \beta b^*(\hat{z}_2)}$. The first equality above comes from the fact that V is linear between $\hat{m}$ and $\hat{z}_2$, and the second equality above from the definition of $\hat{m}$. The last inequality comes from $\hat{m} = q^*(\hat{m}) + \frac{k}{\mu(b^*(\hat{m}))}$ and $\mu(b^*(\hat{m})) < 1$.

Feature (a) implies that the set $\{\beta \in [0, 1] : \bar{q}(\beta) \leq \underline{q}\}$ is non-empty and that $\beta_0 > 0$ is well-defined. Moreover, $\bar{q}(\beta) \leq \underline{q}$ for all $\beta \leq \beta_0$. Using features (b) and (c), we conclude that if $\beta \leq \beta_0$, then $q^*(\hat{m}) < \bar{q}(\beta) \leq \underline{q} < q_0$. Recall that $q^*(m)$ is an increasing function. Thus, if $\beta \leq \beta_0$ then $\hat{m} < m_0$, in which case $\phi(\hat{z}_2) = \phi(m_0) = 0$.

As a preliminary step toward finding a condition for $\phi(\hat{z}_2) > 0$, we consider the limit $\beta \to 1$ and characterize the optimal choices in more detail. This exercise is guided by the above result that $\phi(\hat{z}_2) = 0$ if β is small. Note that although $\lim_{\beta \to 1} V(m) = \infty$, the limit of $(1 - \beta)V(m)$ is strictly positive and finite for all $m \in [0, \infty)$. Also, the limit of $[V(m) - V(0)]$ is finite for all $m < \infty$. We characterize in detail the optimal choices of a buyer with the balance m_0 in the limit $\beta \to 1$. First, taking the limit $\beta \to 1$ on (F.3) and (F.4) yields:

$$V'(0) = U'(q_0), \tag{F.5}$$

$$b_0 \left[U(q_0) - m_0 U'(q_0) \right] = \lim_{\beta \to 1} \left[(1 - \beta) V(0) \right]. \tag{F.6}$$

Second, since $u_2 = u_1 k \mu' / \mu^2$, the first-order condition of b_0 (see (3.8)) yields:

$$\frac{U(q_0)}{U'(q_0)} - m_0 + \frac{k \mu'(b_0) b_0}{[\mu(b_0)]^2} = 0, \tag{F.7}$$

where we have used (F.5) for $V'(0)$. Substituting $b_0 = \mu^{-1}(\frac{k}{m_0 - q_0})$ into (F.7), we can prove that $q_0 = q^*(m_0)$ is a strictly increasing function of m_0.

We are now ready to prove that $\phi(\hat{z}_2) > 0$ in the limit $\beta \to 1$ if and only if $m_0 < \hat{z}_2$. The "only if" part of this statement is apparent, because $m_0 \geq \hat{z}_2$ implies $\phi(\hat{z}_2) = \phi(m_0) = 0$. To prove the "if" part of the statement, we verify that $\phi(\hat{z}_2) = 0$ implies $m_0 \geq \hat{z}_2$ in the limit $\beta \to 1$.

Suppose $\phi(\hat{z}_2) = 0$. Using part (ii) of Theorem 3.2, we deduce that $\beta V'(0) \leq U'(q^*(\hat{z}_2))$. Taking the limit $\beta \to 1$ and using (F.5), we write this condition as $q_0 \geq q^*(\hat{z}_2)$. Because $q^*(m)$ is strictly increasing at $m = m_0$, then $m_0 \geq \hat{z}_2$.

The above procedure leads to the conclusion that when β is sufficiently close to one, $\phi(\hat{z}_2) > 0$ if and only if $m_0 < \hat{z}_2$. To characterize the condition $m_0 < \hat{z}_2$ explicitly, we suppose that the opposite, $m_0 \geq \hat{z}_2$, is true. After solving q_0 from (F.8) as $q_0(\hat{m})$ and b_0 from (F.9) as $b_0(\hat{m})$, we will solve the number $\hat{m}$ from (F.10) as m_c. Because the supposition $m_0 \geq \hat{z}_2$ implies $\hat{m} \leq m_0$, the supposition leads to a contradiction if $\hat{m} = m_c$ satisfies $\hat{m} > m_0$, i.e., if (4.6) holds. Therefore, if (4.6) holds, then $m_0 < \hat{z}_2$ and $\phi(\hat{z}_2) > 0$ for β sufficiently close to one.

To carry out the procedure described above, we suppose $m_0 \geq \hat{z}_2$ and consider the limit $\beta \to 1$. Since $m_0 \geq \hat{m}$ in this case, the lottery for low money holdings implies $V'(0) = V'(\hat{m})$. Because the definition of $\hat{m}$ in the limit $\beta \to 1$ implies $V'(\hat{m}) = h'(\hat{m})$, then $V'(0) = h'(\hat{m})$. Substituting this result into (F.5), we solve $q_0 = q_0(\hat{m})$ where

$$q_0(\hat{m}) \equiv U'^{-1}(h'(\hat{m})). \tag{F.8}$$

Substituting $m_0 = q_0 + \frac{k}{\mu(b_0)}$ and $q_0 = q_0(\hat{m})$ into (F.7) yields:

$$\frac{k}{\mu(b_0)} - \frac{k\mu'(b_0)b_0}{[\mu(b_0)]^2} = \frac{U(q_0(\hat{m}))}{h'(\hat{m})} - q_0(\hat{m}). \tag{F.9}$$

Since $\mu'(b) < 0$ and $1/\mu(b)$ is strictly convex in b, the left-hand side of (F.9) is strictly increasing in b_0. Thus, for any given $\hat{m}$, (F.9) solves for a unique b_0. Denote this solution as $b_0(\hat{m})$.

Moreover, since $\hat{m} \leq m_0$ under the supposition $m_0 \geq \hat{z}_2$, the lottery for low money holdings implies that $V(\hat{m})$ is linear in $\hat{m}$ and the slope of the line is $V'(\hat{m}) = h'(\hat{m})$ in the limit $\beta \to 1$. That is, $V(\hat{m}) - V(0) = \hat{m}h'(\hat{m})$. On the other hand, in the limit $\beta \to 1$, a worker's Bellman equation yields $V(\hat{m}) - V(0) = h(\hat{m}) + \lim_{\beta\to 1}[(1-\beta)V(0)]$. Thus, $\lim_{\beta\to 1}[(1-\beta)V(0)] = \hat{m}h'(\hat{m}) - h(\hat{m})$. Substituting this result and $b_0 = b_0(\hat{m})$, we rewrite (F.6) as

$$-\frac{k\mu'(b_0)(b_0)^2}{[\mu(b_0)]^2}\bigg|_{b_0=b_0(\hat{m})} = \hat{m} - \frac{h(\hat{m})}{h'(\hat{m})}. \tag{F.10}$$

The right-hand side of (F.10) is a strictly increasing function of $\hat{m}$. From (F.8) and (F.9), we can verify that $q_0'(\hat{m}) < 0$, $b_0'(\hat{m}) < 0$, $q_0(0) = \infty$, $b_0(0) = 1$, $q_0(\infty) = 0$ and $\mu(b_0(\infty)) > 0$. With these properties and the maintained assumptions on the function $\mu(b)$, we can verify that the left-hand side of (F.10) is a strictly decreasing function of $\hat{m}$ and that there is a unique solution to (F.10) for the number $\hat{m}$. This solution, denoted as m_c, is the one used in (4.6). **QED**

References

[1] Acemoglu, D. and R. Shimer (1999): Efficient Unemployment Insurance, Journal of Political Economy 107, 893-928.

[2] Baumol, W.J. (1952): The Transactions Demand for Cash: An Inventory Theoretic Approach, Quarterly Journal of Economics 66, 545-556.

[3] Benveniste, L.M. and J.A. Scheinkman (1979): On the Differentiability of the Value Function in Dynamic Models of Economics, Econometrica 47, 727-732.

[4] Burdett, K., Shi, S. and R. Wright (2001): Pricing and Matching with Frictions, Journal of Political Economy 109, 1060-1085.

[5] Chiu, J. and M. Molico, (2008), Uncertainty, Inflation, and Welfare, Working Papers 08-13, Bank of Canada.

[6] Clausen, A. and C. Strub (2010): Envelope Theorems for Non-Smooth and Non-Concave Optimization, Manuscript, University of St. Gallen.

[7] Corbae, D., Temzelides, T. and R. Wright (2003): Directed Matching and Monetary Exchange, Econometrica 71, 731-756.

[8] Delacroix, A. and S. Shi (2006): Directed Search On the Job and the Wage Ladder, International Economic Review 47, 651-699.

[9] Galenianos, M. and P. Kircher (2008): A Model of Money with Multilateral Matching, Journal of Monetary Economics 55, 1054-1066.

[10] Gonzalez, F.M. and S. Shi (2010): An Equilibrium Theory of Learning, Search and Wages, Econometrica 78, 509-537.

[11] Green, E. and R. Zhou (1998): A Rudimentary Model of Search with Divisible Money and Prices, Journal of Economic Theory 81, 252-271.

[12] Julien, B., Kennes, J. and I. King (2008): Bidding for Money, Journal of Economic Theory 142, 196-217.

[13] Kiyotaki, N. and R. Wright (1989): On Money as a Medium of Exchange, Journal of Political Economy 97, 927-954.

[14] Krusell, P. and A. Smith (1998): Income and Wealth Heterogeneity in the Macroeconomy, Journal of Political Economy 106, 867-896.

[15] Lagos, R. and R. Wright (2005): A Unified Framework for Monetary Theory and Policy Analysis, Journal of Political Economy 113, 463-484.

[16] Lucas, R.E., Jr. (1990): Liquidity and Interest Rates, Journal of Economic Theory 50, 237-264.

[17] Menzio, G. and S. Shi (2008): Efficient Search on the Job and the Business Cycle, Working Paper 327, University of Toronto.

[18] Menzio, G. and S. Shi (2010): Block Recursive Equilibria for Stochastic Models of Search on the Job, Journal of Economic Theory 145, 1453-1494.

[19] Milgrom, P. and I. Segal (2002): Envelope Theorems for Arbitrary Choice Sets, Econometrica 70, 583-601.

[20] Moen, E. (1997): Competitive Search Equilibrium, Journal of Political Economy 105, 694–723.

[21] Molico, M. (2006), The Distribution of Money and Prices in Search Equilibrium, International Economic Review 47, 701-722.

[22] Peters, M. (1991): Ex Ante Price Offers in Matching Games: Non-Steady State, Econometrica 59, 1425-1454.

[23] Rocheteau, G. and R. Wright (2005): Money in Competitive Equilibrium, in Search Equilibrium, and in Competitive Search Equilibrium, Econometrica 73, 175-202.

[24] Royden, H.L. (1988): Real Analysis. New York: Macmillan.

[25] Shi, S. (1995): Money and Prices: A Model of Search and Bargaining, Journal of Economic Theory 67, 467-496.

[26] Shi, S. (1997): A Divisible Search Model of Fiat Money, Econometrica 65, 75-102.

[27] Shi, S. (2009): Directed Search for Equilibrium Wage-Tenure Contracts, Econometrica 77, 561-584..

[28] Stokey, N., R.E. Lucas, Jr., with E.C. Prescott (1989): Recursive Methods in Economic Dynamics. Cambridge, Massachusetts: Harvard University Press.

[29] Tobin, J. (1956): The Interest-Elasticity of Transactions Demand for Cash, Review of Economics and Statistics 38, 784-793.

[30] Topkis, D.M. (1998): Supermodularity and Complementary. New Jersey: Princeton University Press.

[31] Trejos, A. and R. Wright (1995): Search, Bargaining, Money and Prices, Journal of Political Economy 103, 118-141.

[32] Zhu, T. (2005): Existence of a Monetary Steady State in a Matching Model: Divisible Money, Journal of Economic Theory 123, 135-160.