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"Inference of Signs of Interaction Effects in Simultaneous Games with Incomplete Information"

by

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Inference of Signs of Interaction Effects in Simultaneous Games with Incomplete Information*

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Inference of Signs of Interaction Effects in Simultaneous Games with Incomplete Information

Abstract

This paper studies the inference of interaction effects, i.e., the impacts of players' actions on each other's payoffs, in discrete simultaneous games with incomplete information. We propose an easily implementable test for the signs of state-dependent interaction effects that does not require parametric specifications of players' payoffs, the distributions of their private signals or the equilibrium selection mechanism. The test relies on the commonly invoked assumption that players' private signals are independent conditional on observed states. The procedure is valid in the presence of multiple equilibria, and, as a by-product of our approach, we propose a formal test for multiple equilibria in the data-generating process. We provide Monte Carlo evidence of the test's good performance in finite samples. We also implement the test to infer the direction of interaction effects in couples' joint retirement decisions using data from the Health and Retirement Study.

JEL Codes: C01, C72

1 Introduction

Strategic interaction effects occur when a player's action choice affects not only his or her own payoff but also those of other players. In simultaneous discrete games of incomplete information, each person has a private signal about his or her payoff, while the joint distribution of such private signals is common knowledge among all players. In a Bayesian Nash equilibrium (BNE), individuals act to maximize their expected payoffs given their knowledge of these distributions as well as the payoff structure. Such models have found applications in a wide array of empirical contexts where players are uncertain about their competitors' payoffs given their own information. These include airing commercials at radio stations (Sweeting (2008)) and peer effects in recommendations by financial analysts (Bajari, Hong, Krainer, and Nekipelov (2009)).

Earlier works have studied the identification and estimation of these games using a wide spectrum of restrictions. These include (but are not limited to) the independence of private information variables from observable covariates, parametric specification of relevant distributions or utility functions, or constraints on the set of Bayesian Nash equilibria.[1] In comparison, in this paper we focus on inference of the signs of interaction effects, which are allowed to be individual-specific and state-dependent, under a minimal set of nonparametric restrictions on private signals and payoff structures. Our choice of focus is motivated by two considerations. First, signs of interaction effects alone have important policy implications. For example, consider the context of couples' joint retirement decisions as studied by Banks, Blundell, and Casanova Rivas (2007) and Casanova Rivas (2009). If spouses enjoy retirement more when their partner is retired as well (i.e., interaction effects are positive), then

[1]Wan and Xu (2010), for example, consider the set of monotone, threshold-crossing Bayesian Nash equilibria. Recent work by Grieco (2010) studies a class of games with flexible information structures that also subsume games with complete information where players know each other's payoffs for sure. In a similar spirit, Navarro and Takahashi (2009) suggest a test for the information structure that, among other things, relies on a degenerate equilibrium selection rule and independence of residuals and observed covariates.

any exogenous change in retirement provisions that force the wife to delay her retirement should at the same time also dampen the husband's incentive to retire. Second, while point identification and estimation of the full structure of such games inevitably hinge on stochastic restrictions on private signal distributions (such as perfect knowledge or independence) and parametric specifications of payoffs (such as index specification), inference on signs of interaction effects can be done under minimal nonparametric restrictions on the structure. Such inference is valid even in the presence of multiple equilibria and does not invoke any assumptions on the equilibrium selection mechanism in the data-generating process. This feature of our procedure is particularly notable, since almost all previous work has relied on stringent assumptions about equilibrium selection or multiplicity to attain identification (e.g., the single-equilibrium assumption in Bajari, Hong, Krainer, and Nekipelov (2009) and Tang (2009), equilibrium uniqueness in Seim (2006) or Aradillas-Lopez (2009), or the parametric specification of equilibrium selection mechanism as in Sweeting (2008) or other work).[2]

We first show how the existence of multiple equilibria in the data can be exploited to infer *the signs* of strategic interactions. If private signals are independent from each other conditional on observed covariates, the joint probability of a profile of actions must be the product of marginal probabilities for individual actions in any single equilibrium. When multiple equilibria exist in the data, the choice probabilities observed are mixtures of those implied in each single equilibrium. We show in Section 3 that signs of correlations between players' actions are determined by signs of the strategic interaction effects. As a byproduct, the correlations also allow us to identify the existence of multiple equilibria in the data (see below). The assumption of conditional independence of private information is commonly maintained in the literature on estimation and inference in statistic games with incomplete

[2]As indicated in Berry and Tamer (2007), another possibility is to resort to partial identification. Examples of such a strategy *under complete information* are Beresteanu, Molchanov, and Molinari (2009), Ciliberto and Tamer (2009), Galichon and Henry (2009) and earlier references cited in Berry and Tamer (2007).

information (see, for example, Seim (2006), Aradillas-Lopez (2009), Berry and Tamer (2007), Bajari, Hong, Krainer, and Nekipelov (2009), Bajari, Hahn, Hong, and Ridder (2009), Brock and Durlauf (2007), Sweeting (2008) and Tang (2009)).[3] The assumption is also common in the literature on the estimation of dynamic games with incomplete information.

We then generalize these arguments for identifying the signs of interaction effects so as to allow for the possibility that there is only a unique equilibrium for a given state in the data. This is done by exploiting an exclusion restriction on the states. The idea relies on the following simple intuition. Suppose that for some player i, there exists a subvector of state variables that affect other players' payoffs or private signals but not his or her own. Then the correlation between actions chosen by i and others across different realizations of such "excluded" states must be solely determined by the direction of others' interaction effects on i's payoffs, provided private signals are independent given observed states. Such exclusion restrictions on state variables arise naturally in many applications. For example, in a static entry-and-exit game between two firms, it might be plausible to assume that some idiosyncratic factors affecting *Firm A*'s costs (such as geographic location) may not enter *Firm B*'s profits or private information directly.

Another contribution of this paper is to introduce formal tests for the presence of multiple equilibria in the data-generating process. Such tests arise as a natural by-product of the logic underlying our inference of the signs of interaction effects. Note that the test for multiple equilibria is of practical importance in structural empirical research. When the number of players within a game is large, the conditional choice probabilities *within a particular equilibrium* may be consistently estimated from average choices in each game (see,

[3] A recent exception is Wan and Xu (2010), who nevertheless rely on further restrictions on the solution concept and the environment to obtain identification. To handle the correlation in private types, they rely on monotone, threshold-crossing equilibrium strategies and impose restrictions on the magnitude of the strategic interaction parameters (see Assumption A in their paper). In a subsection, Aradillas-Lopez (2009) also suggests an estimation procedure to handle cases in which the assumption is violated, but relies on the assumption that a single equilibrium is played in the data.

for example, Brock and Durlauf (2007), p.58). Nevertheless, when the number of players in each game is small (as is typically the case in the empirical games literature), the conditional choice probabilities will not be reliably estimated within individual games. It is then necessary to pool data across games in which the same equilibrium is played so as to estimate the choice probabilities using more data. In this case, testing for multiple equilibria is of interest in its own right. Besides, most of the known methods for semiparametric estimation of incomplete information games (without explicitly specifying an equilibrium selection rule) has relied on the existence of a single equilibrium in the data (e.g. Aradillas-Lopez (2009), Bajari, Hong, Krainer, and Nekipelov (2009) and Tang (2009)).[4] Hence it is *imperative* to devise a formal test for the assumption of unique equilibrium in the data-generating process. The test we propose in this paper exploits the observation (also mentioned in the parametric model by Sweeting (2008)) that if private signals are i.i.d. across individuals, players' actions must be independent in a single equilibrium but correlated when there are multiple equilibria.

An innovation of our test for multiple equilibria is to use a stepwise multiple testing procedure to infer whether *each individual player* has different strategies across the multiple equilibria in the data-generating process. This is particularly interesting for structural estimation of games involving more than three players, in which a subset of players may stick to the same strategy across multiple equilibria. Semiparametric methods based on the assumption of a unique equilibrium can still be applied to consistently estimate payoff parameters for those players who do not switch between strategies in multiple equilibria. Hence, it is useful to infer the *identity* of such players from observed distributions of actions.

For a parametric model with constant, state-independent interaction effects, Sweeting (2008) proposed a procedure to check for multiple equilibria in the data by calculating the percentage of pairs of players whose actions are correlated. In comparison, we develop

[4]For an illustration of how misspecification of the equilibrium selection rule can affect inference in a complete information game with a small number of players, see Honoré and de Paula (forthcoming).

stronger, new results by extending this intuition in a more general context where individual-specific interaction effects may depend on the states in unrestricted ways. Besides, our test also addresses two subtle issues not noted in Sweeting (2008).[5] First, our test for multiple BNE in the data is based on testing whether each individual's action is correlated with the *total number* of competitors choosing the same action. Therefore, our test has power under alternatives in which multiple BNE exist in the data with only a very small number of players switching strategies across the multiple equilibria.[6] Second, we apply a multiple testing procedure proposed by Romano and Wolf (2005) to test the joint null hypothesis that the equilibrium in the data is unique (i.e., that none of the N players has switched between strategies in the data). Within $N-1$ steps, the procedure leads to a decision to reject or not to reject the joint null. And if the joint null is rejected, the procedure infers the exact identities of players who have switched between strategies in the data. This test is known to effectively control the family-wise error rate, or the probability of rejecting at least one of the true null hypotheses.

The paper proceeds as follows. We present our basic model and empirical characterization in the next section. In Section 3, we present the main results on the identification of the sign of interaction effects. Section 4 outlines general testing procedures for inference. Monte Carlo experiments and an application to joint retirement are presented in Sections 5 and 6. Section 7 concludes.

[5]The procedure used in Sweeting (2008) did not prescribe a decision rule to draw conclusions about the existence of multiple BNE.

[6]In such a case, the number of players whose actions are correlated *pairwise* is very small. For example, if only 3 players out of a group of 30 actively switch between strategies across multiple equilibria in the data, then the percentage of pairs with correlated actions is only $\binom{3}{2}/\binom{30}{2} < 0.0069$. The procedure proposed in Sweeting (2008) would fail to reject the null of a unique equilibrium in the data in this case.

2 The Model and Empirical Context

We consider a simultaneous discrete game with incomplete information involving N players. Each player i chooses an action D_i from two alternatives $\{1,0\}$. A vector of states $X \in \mathbb{R}^K$ is common knowledge among all players. A vector of private information (or "types") $\epsilon \equiv (\epsilon_i)_{i=1}^N \in \mathbb{R}^N$ is such that ϵ_i is only observed by player i. (Throughout the paper we will use upper case letters for random variables and lower case for their realized values. We use Ω_R to denote the support of any generic random vector R.) Conditional on a given state $X = x$, private information ϵ is jointly distributed according to the CDF $F_{\epsilon|X}(.|x)$. The payoff for player i from choosing action 1 is $U_{1i}(X, \epsilon_i) \equiv u_i(X) + (\sum_{j \neq i} D_j)\delta_i(X) - \epsilon_i$, while the return from the other action $U_{0i}(X, \epsilon_i)$ is normalized to 0. Intuitively, $u_i(X)$ specifies a base return from action 1 for player i. Meanwhile $\delta_i(X)$ captures interaction effects on i's payoff due to another player j who chooses 1. (This specification subsumes that of Sweeting (2008) in the context of binary choices, since it allows the interaction effects to depend on states X and u_i, δ_i to take general forms.) The return functions $(u_i, \delta_i)_{i=1}^N$ and the distribution (though not the realization) of private information $F_{\epsilon|X}$ are common knowledge among all players. We maintain the following major identifying restrictions on $F_{\epsilon|X}$ throughout the paper.

Assumption 1 *Conditional on any $x \in \Omega_X$, ϵ_i is independent of $(\epsilon_j)_{j \neq i}$ for all i and has continuous, positive densities over the support $\Omega_{\epsilon_i|X=x}$.*

Assumption 1 allows X to be correlated with private information of the players, as may be desirable in empirical applications. This is a common assumption in the econometric literature dealing with incomplete information (dynamic and simultaneous) games. A pure strategy for player i in this Bayesian game is a mapping $s_i : \Omega_{X,\epsilon_i} \to \{0,1\}$. Letting $S_i(X, \epsilon_i)$ denote an equilibrium strategy for player i, equilibrium behavior prescribes:

$$S_i(X, \epsilon_i) = \begin{cases} 1, & \text{if } u_i(X) + \delta_i(X) \sum_{j \neq i} \mathbb{E}\left[S_j(X, \epsilon_j)|X, \epsilon_i\right] - \epsilon_i \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

and because of Assumption 1, $\mathbb{E}[S_j(X,\epsilon_j)|X,\epsilon_i] = \mathbb{E}[S_j(X,\epsilon_j)|X] \equiv p_j(x)$.[7] Hence, under this assumption, a Bayesian Nash equilibrium (BNE) in pure strategies (given state x) can be characterized by a profile of choice probabilities $p(x) \equiv [p_1(x), \ldots, p_N(x)]$ such that for all $x \in \Omega_X$,

$$p_i(x) = F_{\epsilon_i|X}\left(u_i(x) + \delta_i(x)\sum_{j\neq i} p_j(x)\right) \textit{ for all } i = 1, 2, ., N \tag{1}$$

where $p_i(x)$ is player i's probability of choosing action 1 conditional on the state x and $F_{\epsilon_i|X}$ is the marginal distribution of ϵ_i conditional on X. Let $\mathcal{L}_{x,\theta}$ denote the set of BNE (as characterized by solutions in p to (1)) for a given x and structure $\theta \equiv \{(u_i,\delta_i)_{i=1,2}, F_{\epsilon|X}\}$. The existence of pure-strategy BNE for any given x follows from Brouwer's Fixed Point Theorem and the continuity of $F_{\epsilon_i|X}$ under Assumption 1. In general there may be multiple BNE, depending on the specifications of $F_{\epsilon|X}$, u_i and δ_i.

We assume that econometricians have access to a large cross-section of games between N players. In each game, they observe choices of actions by all players and realized states x, but do not observe $(\epsilon_i)_{i=1}^N$ or know the form of $(u_i,\delta_i)_{i=1}^N$ and $F_{\epsilon|X}$. Our analysis posits (i) that the structure $(u_i,\delta_i)_{i=1}^N$ and $F_{\epsilon|X}$ are fixed across all games observed, and (ii) that the choice data observed is generated by players following the pure strategies prescribed by BNE. Econometricians are interested in learning (at least some features of) the structure $(u_i,\delta_i)_{i=1}^N$ and $F_{\epsilon|X}$ from the observable joint distribution of X and $(D_i)_{i=1}^N$.

Suppose the choices observed in the data are known to be generated from a single BNE in the data-generating process for all $x \in \Omega_X$. This may arise either because the solution to (1) is unique or because the equilibrium selection in the data-generating process is degenerate in one of the multiple solutions. Then (1) offers a link between observable conditional choice patterns and structural elements $(u_i,\delta_i)_{i=1}^N, F_{\epsilon|X}$. Estimation can be done

[7]Given Assumption 1, other players' private types can be easily included in the utility function. If $U_{1i}(X,\epsilon_i) \equiv u_i(X) + f_i(X,\epsilon_{-i}) + (\sum_{j\neq i} D_j)\delta_i(X) - \epsilon_i$, Assumption 1 implies that $\mathbb{E}[f_i(X,\epsilon_{-i})|X,\epsilon_i] = \mathbb{E}[f_i(X,\epsilon_{-i})|X]$ and one can simply focus on $\hat{U}_{1i}(X,\epsilon_i) \equiv \hat{u}_i(X) + (\sum_{j\neq i} D_j)\delta_i(X) - \epsilon_i$ where $\hat{u}_i(X) \equiv u_i(X) + \mathbb{E}[f_i(X,\epsilon_{-i})|X]$.

under various restrictions on u, δ and $F_{\epsilon|X}$ (see Aradillas-Lopez (2009), Berry and Tamer (2007), Bajari, Hong, Krainer, and Nekipelov (2009) and Tang (2009) for more details).

This link may nonetheless break down when there are multiple equilibria in the data-generating process. To see this, let $\Lambda_{x,\theta}$ be an equilibrium selection mechanism (i.e. a distribution over $\mathcal{L}_{x,\theta}$) in the data-generating process that may depend on x and θ. That Λ depends on x captures the idea that, as stated in Myerson (1991) (pp. 371-2), "...in a game with multiple equilibria, anything that tends to focus the players' attention on one particular equilibrium, *in a way that is commonly recognized*, tends to make this the equilibrium that the players will expect and thus actually implement. The focal equilibrium could be determined by any of a wide range of possible factors, including environmental factors and cultural traditions (which fall beyond the scope of analysis in mathematical game theory), special mathematical properties of the various equilibria, and preplay statements made by the players or an outside arbitrator..."

Since X is commonly observed by all players in our model, we allow it to affect the equilibrium selection mechanism accordingly. For any x such that $\mathcal{L}_{x,\theta}$ is not a singleton, the conditional choice probability observed in the data is a mixture of the conditional choice probabilities implied by each pure-strategy BNE in $\mathcal{L}_{x,\theta}$. That is, $p_i^*(x) = \int_{\mathcal{L}_{x,\theta}} p_i^l d\Lambda_{x,\theta}(p^l)$, where $p_i^*(x)$ is the actual marginal probability that i chooses 1 conditional on x observed from data, and $p^l \equiv (p_i^l)_{i=1}^N$ is a generic element in the set of possible BNE $\mathcal{L}_{x,\theta}$, with l indexing the equilibria in $\mathcal{L}_{x,\theta}$ and p_i^l, the marginal probability for i to choose 1 given x and θ implied in equilibrium l. While the fixed point characterization (1) holds for every single BNE $p^l \in \mathcal{L}_{x,\theta}$ by definition, it does not necessarily hold for the vector of mixture marginals $p^* \equiv (p_i^*)_{i=1}^N$ observed. Because the data will provide information on the mixtures of equilibria, not on the individual equilibria themselves, there will be limits to what can be learned about the structure from the data without imposing additional assumptions. This point is illustrated in the appendix using results from the literature on identifiability (or lack thereof) in mixture models.

Researchers have taken different approaches to deal with the issue of multiple equilibria in empirical works. Such strategies include (a) the use of a parametric equilibrium selection rule; (b) the assumption that only one equilibrium is played in all games; (c) sufficient conditions for uniqueness of the equilibrium; and/or (d) partial identification and estimation of the identified set. Each of these strategies (which can also be combined) has some limitations. We are interested in constructing a robust way to test for the existence of multiple equilibria and to recover the sign of interactions under weak stochastic restrictions on the distribution of private information.

3 Identifying Signs of Interaction Effects

In this section, we show how to detect the presence of multiple BNE in the data observed and identify signs of interaction effects $\delta_i(x)$ for any i under a given state x. The sign reveals the nature of strategic incentives among players. Compared with earlier works, our sign identification has several innovations and contributions. First, our test does not invoke any parametric restrictions on players' preferences or distributions of private information. Second, it allows the strategic incentives (as captured by the sign of δ_i) to be a function of states x. Third, our approach is robust to the presence of multiple BNE. If in fact the existence of multiple BNE at first precludes complete identification of the structure, it makes possible the identification of the sign of interaction effects in contrast to states where equilibrium is unique. This intriguing possibility is outlined, for example, in Manski (1993)[8], and clearly observed in Sweeting (2008).

We first show how to detect the existence of multiple BNE in the data using observed distributions. Define

$$\gamma_i^l(x) \equiv \mathbb{E}_l \left(\sum\nolimits_{j \neq i} D_j | X = x \right) = \sum\nolimits_{j \neq i} p_j^l(x)$$

[8] "The prospects for identification may improve if $f(\cdot, \cdot)$ is non-linear in a manner that generates multiple social equilibria" (p. 539).

where $\mathbb{E}_l$ denotes the expectation with respect to the distribution of $(D_i)_{i=1}^N$ induced in the equilibrium $p^l \in \mathcal{L}_{x,\theta}$. Define $sign(a)$ to be 1 if $a > 0$, -1 if $a < 0$ and 0 if $a = 0$. For any player $i \in \{1, ., N\}$, let $\tilde{\gamma}_i^*(x)$ denote the conditional expectation of the product $D_i(\sum_{j \neq i} D_j)$ given x observed from the data-generating process. That is, $\tilde{\gamma}_i^*(x) \equiv \int_{p^l \in \mathcal{L}_{x,\theta}} p_i^l(x)\gamma_i^l(x) d\Lambda_{x,\theta}(p^l)$, where θ denotes the true structure, and $\Lambda_{x,\theta}$ denotes the equilibrium-selection mechanism in the data generating process. Let $p_i^*(x)$ be the actual probability that i chooses 1 given x observed in the data (i.e. $p_i^*(x) = \int_{p^l \in \mathcal{L}_{x,\theta}} p_i^l(x) d\Lambda_{x,\theta}(p^l)$), and let $\gamma_i^*(x) \equiv \sum_{j \neq i} p_j^*(x)$. $\mathcal{L}_{x,\theta}^+$ denotes the subset of $\mathcal{L}_{x,\theta}$ that occurs in the data-generating process with positive probability (i.e., $\mathcal{L}_{x,\theta}^+ \equiv \{p^l : \Lambda_{x,\theta}(p^l) > 0\}$). We say multiple BNE exist in the data-generating process whenever $\mathcal{L}_{x,\theta}^+$ is not a singleton.

Proposition 1 *Suppose Assumption 1 holds. (i) For any given x, multiple BNE exist in the data-generating process if and only if $\tilde{\gamma}_i^*(x) \neq p_i^*(x)\gamma_i^*(x)$ at least for some i; (ii) For all i and x such that $\tilde{\gamma}_i^*(x) \neq p_i^*(x)\gamma_i^*(x)$,*

$$sign\left(\tilde{\gamma}_i^*(x) - p_i^*(x)\gamma_i^*(x)\right) = sign(\delta_i(x))$$

Proof. Under Assumption 1, D_i must be independent of $\sum_{j \neq i} D_j$ conditional on x in every single BNE p^l in $\mathcal{L}_{x,\theta}$.

(Sufficiency of (i)) Suppose there is a unique BNE in the data-generating process. That is, $\mathcal{L}_{x,\theta}^+$ is a singleton $\{p^l\}$. Then $p_i^*(x) = p_i^l(x)$, $\gamma_i^*(x) = \sum_{j \neq i} p_j^l(x)$ and $\tilde{\gamma}_i^*(x) = p_i^l(x) \sum_{j \neq i} p_j^l(x)$ for all i in state x. Hence $\tilde{\gamma}_i^*(x) = p_i^*(x)\gamma_i^*(x)$ for all i.

(Necessity of (i)) Suppose $\mathcal{L}_{x,\theta}^+$ is not a singleton in state x. Then there exists at least some i and $p^l, p^k \in \mathcal{L}_{x,\theta}^+$ such that $p_i^l \neq p_i^k$.[9] Also note that for a player such as i, $\delta_i(x)$ must necessarily be non-zero. By definition,

$$\begin{aligned} & \tilde{\gamma}_i^*(x) - p_i^*(x)\gamma_i^*(x) \\ = & \int_{p^l \in \mathcal{L}_{x,\theta}^+} p_i^l(x)\gamma_i^l(x) d\Lambda_{x,\theta} - \int_{p^l \in \mathcal{L}_{x,\theta}^+} p_i^l(x) d\Lambda_{x,\theta} \int_{p^l \in \mathcal{L}_{x,\theta}^+} \gamma_i^l(x) d\Lambda_{x,\theta} \end{aligned} \quad (2)$$

[9] The choice of l, k in general is specific to i. We suppress this dependence for notational ease.

Suppose $\delta_i(x) > 0$. The equilibrium characterization in (1) implies that there exists a strictly increasing function g_i such that $\gamma_i^l(x) = g_i(p_i^l(x)) \equiv \left(F^{-1}_{\epsilon_i|X}(p_i^l(x)) - u_i(x)\right)/\delta_i(x)$ for each single p^l in $\mathcal{L}_{x,\theta}$.[10] Thus for x given, (2) can be written as

$$\tilde{\gamma}_i^*(x) - p_i^*(x)\gamma_i^*(x) = \int_0^1 g_i(z) z d\tilde{\Lambda}_{i,x,\theta}(z) - \int_0^1 z d\tilde{\Lambda}_{i,x,\theta}(z) \int_0^1 g_i(z) d\tilde{\Lambda}_{i,x,\theta}(z)$$

where $z \equiv p_i^l(x)$ and $\tilde{\Lambda}_{i,x,\theta}$ is a distribution of $p_i^l(x)$ induced by the equilibrium selection mechanism $\Lambda_{x,\theta}$ defined on $\mathcal{L}_{x,\theta}$. Thus (2) takes the simple form of the covariance of a random variable z and a strictly increasing function of itself:

$$\begin{aligned}
\text{cov}(Z, g_i(Z)) &= \mathbb{E}\left[(Z - \mathbb{E}(Z))(g_i(Z) - \mathbb{E}(g_i(Z)))\right] \\
&= \mathbb{E}\left[(Z - \mathbb{E}(Z))(g_i(Z) - g_i(\mathbb{E}(Z)))\right] \\
&\quad + \mathbb{E}\left[(Z - \mathbb{E}(Z))(g_i(\mathbb{E}(Z)) - \mathbb{E}(g_i(Z)))\right] \\
&= \mathbb{E}\left[(Z - \mathbb{E}(Z))(g_i(Z) - g_i(\mathbb{E}(Z)))\right]
\end{aligned}$$

Because g_i is strictly increasing in $[0,1]$ for given x, we have $z_1 > z_2 \Rightarrow g_i(z_1) > g_i(z_2)$. Consequently, $(z - \mathbb{E}(Z))(g_i(z) - g_i(\mathbb{E}(Z))) > 0$ for any $z \neq \mathbb{E}(Z)$, and the covariance is strictly positive, provided the distribution $\tilde{\Lambda}_{i,x,\theta}$ is not degenerate on $\mathcal{L}^+_{x,\theta}$. Hence $\tilde{\gamma}_i^*(x) - p_i^*(x)\gamma_i^*(x) > 0$ if multiple BNE exist in the data-generating process in state x. The case with $\delta_i(x) < 0$ is proved by symmetric arguments. The proof of (ii) is already included in the proof of (i) above. ■

In applications, one plausible scenario is that in which $\delta_i(x) = \delta_i$ for any $x \in \Omega_x$. In this case, the sign of δ_i is identified as long as multiple equilibria exist on a set of x with positive probability. If ($\mathbb{P}$-almost) no x induces multiple equilibria in the data, the conditional choice probabilities will factor for ($\mathbb{P}$-almost) every x. Consequently, $\tilde{\gamma}_i^*(x) = p_i^*(x)\gamma_i^*(x)$, ($\mathbb{P}$-a.e.) $\Rightarrow \mathbb{E}_X(\tilde{\gamma}_i^*(X)) = \mathbb{E}_X(p_i^*(X)\gamma_i^*(X))$. On the other hand, it is not difficult to verify that when $\delta_i(x) = \delta_i$ for any $x \in \Omega_X$ and there are multiple equilibria in the data-generating process, $sign(\delta_i) = sign\left(\mathbb{E}_X(\tilde{\gamma}_i^*(X) - p_i^*(X)\gamma_i^*(X))\right)$.[11]

[10] The form of g_i may depend on θ and x in general. We suppress this dependence for notational ease.

[11] One could use this result to devise a directional test as in Section 4.3 to do inference on interaction signs

We exploit part (i) in Proposition 1 to devise a test for multiple BNE in the data-generating process in the Section 4. Part (ii) of the proposition suggests that $\tilde{\gamma}_i^*(x) - p_i^*(x)\gamma_i^*(x) > 0$ only if $\delta_i(x) > 0$ *and* there exist multiple BNE in state x in the data-generating process. However, the reverse of this statement only holds when $N = 2$. This is because when $N \geq 3$, there can exist i and x such that multiple BNE exist at x in the data-generating process and $\delta_i(x) \neq 0$, but $p_i^l = p_i^*$ for all p_i^l in $\mathcal{L}_{x,\theta}^+$ so that $\tilde{\gamma}_i^*(x) = p_i^*(x)\gamma_i^*(x)$. The following example illustrates this point.

Example 1 Consider a simple 3-by-2 game involving three players. Suppress the dependence on x for notational ease. Let $u_1 = 0.5$, $u_2 = u_3 = 0.3611$, $\delta_i = -1$ and $\epsilon_i \sim \mathcal{N}(0.10, 0.25^2)$ for all i. Then there exist (at least) two distinct BNE:

$$\begin{aligned} p^1 \text{ with } p_1^1 &= 0.0611;\ p_2^1 = 0.7756;\ p_3^1 = 0.0107 \\ p^2 \text{ with } p_1^2 &= 0.0611;\ p_2^2 = 0.0107;\ p_3^2 = 0.7756 \end{aligned} \tag{3}$$

Player 1 chooses alternative 1 with the same probability in both BNE, while both 2 and 3 play different strategies across the two BNE (p^1 and p^2) with $p_i^1 \neq p_i^2$ for $i = 2, 3$. ||

Therefore, part (ii) does not guarantee the identification of signs of $\delta_i(x)$ for *all* i and x in general, due to the need to distinguish players who do or do not incur the same probability for choosing 1 across different equilibria. Let $\iota(x) \equiv \{i : p_i^l(x) = \bar{p}_i(x)$ for some $\bar{p}_i$ in all $p^l \in \mathcal{L}_{x,\theta}^+\}$. Obviously $\bar{p}_i(x) = p_i^*(x)$ for all $i \in \iota(x)$. Let $\iota^c(x)$ denote the set of players not in $\iota(x)$. When there is a unique BNE at x in the data-generating process (i.e., $\mathcal{L}_{x,\theta}^+$ is a singleton), all players in the game belong to $\iota(x)$. When $\mathcal{L}_{x,\theta}^+$ is not a singleton, $\iota(x)$ consists only of players who choose 1 with the same probability in *all* $p^l \in \mathcal{L}_{x,\theta}^+$. By construction, $\tilde{\gamma}_i^*(x) = p_i^*(x)\gamma_i^*(x)$ if and only if $i \in \iota(x)$. A corollary to Proposition 1 is that for i and x such that $i \in \iota^c(x)$, the sign of $\delta_i(x)$ is directly identified as the sign of $\tilde{\gamma}_i^*(x) - p_i^*(x)\gamma_i^*(x)$. Additional restrictions are needed in order to identify $sign(\delta_i(x))$ for $i \in \iota(x)$. For any

and multiple equilibria. We focus on the more general case of state-dependent interaction effects.

given x and i, define the equivalence class as $\Upsilon_i(x) \equiv \{x' : u_i(x) = u_i(x'),\ \delta_i(x) = \delta_i(x'),\ F_{\epsilon_i|X=x} = F_{\epsilon_i|X=x'}\}$. (There is no loss of generality in introducing this notation, since we allow the possibility that $\Upsilon_i(x)$ is a singleton consisting of x only.)

In empirical applications, the equivalence class $\Upsilon_i(x)$ is often a non-singleton set that can happen with positive probability for all i, x. For example, consider entry-exit games involving N firms. The state variables X may include a vector X_0 that consists of market- or sector-wide factors affecting the demand for goods produced. The vector X may also include a group of mutually exclusive vectors $(X_i)_{i=1}^N$ with X_i capturing observable firm-specific factors that affect only i's profitability but not its rivals (e.g., X_i may include labor costs or local regulations pertaining to the geographic location of firm i). The vector of private information $(\epsilon_i)_{i=1}^N$ may well capture all other firm-specific factors (such as idiosyncratic costs) affecting profitability and unobservable to opponent firms and econometricians. In situations in which, given one's own states, the rivals' states (such as their labor costs) have no bearing on one's own profitability, then ϵ_i is independent of X_{-i} given X_0, X_i. In such an environment, $\Upsilon_i(x) = \{\tilde{x} : (\tilde{x}_0, \tilde{x}_i) = (x_0, x_i)\}$ where $x = (x_0, x_i, x_{-i})$.

Assumption 2 *For all i and x s.t. $\delta_i(x) \neq 0$, $\exists \omega_i(x) \subset \Upsilon_i(x)$ with positive probability such that $\forall x' \in \omega_i(x)$, either (i) $i \in \iota^c(x) \cup \iota^c(x')$ or (ii) $i \in \iota(x) \cap \iota(x')$ and $p_i^*(x) \neq p_i^*(x')$.*

More intuitively, Assumption 2 is satisfied as long as there is enough variation in the equivalence class $\Upsilon_i(x)$ to induce changes in $p_i^l(x')$ in equilibria that happen with positive probability in the data. If $\Upsilon_i(x)$ is a singleton (for i), this will happen when x induces multiple equilibria and the $p_i(x)$ is not the same in all equilibria. For non-singleton $\Upsilon_i(x)$, Assumption 2 may hold even when the equilibrium is unique or when $p_i(x)$ is the same across all equilibria. In some empirical contexts, researchers know a priori which i, x satisfy $\delta_i(x) \neq 0$. (For example, when interaction effects are known to be increasing or decreasing in the number of players choosing the same action together, then $\delta_i(x)$ are non-zero for all i, x.) In such cases, Assumption 2 can be checked directly using observable distributions as Proposition 2 suggests below. The following example illustrates in detail how more primitive conditions can lead to Assumption 2 in simple examples.

Example 2 Consider a 2-by-2 game with states $X \equiv (X_0, X_1, X_2)$ where Assumption 1 holds and $u_i, \delta_i, F_{\epsilon_i|X}$ depend only on $X_{0i} \equiv (X_0, X_i)$ and not on the other state variables. The probability for choosing action 1 in equilibrium is given by

$$\begin{bmatrix} p_1(x) \\ p_2(x) \end{bmatrix} = \begin{bmatrix} F_{\epsilon_1|x_{01}}(u_1(x_{01}) + \delta_1(x_{01})p_2(x)) \\ F_{\epsilon_2|x_{02}}(u_2(x_{02}) + \delta_2(x_{02})p_1(x)) \end{bmatrix} \tag{4}$$

Then $\Upsilon_i(x) \equiv \{x' : x'_{0i} = x_{0i}\}$. Within this framework, Assumption 2 can be satisfied in a given state $x \equiv (x_0, x_1, x_2)$ under several different specifications. First, consider the following specification:

For $i = 1, 2$, $\exists \omega_i(x) \subset \Upsilon_i(x)$ with positive probability such that $\forall x' \in \omega_i(x)$,

$$F_{\epsilon_j|x_{0j}}(u_j(x_{0j}) + \delta_j(x_{0j})t) \neq F_{\epsilon_j|x'_{0j}}(u_j(x'_{0j}) + \delta_j(x'_{0j})t) \tag{5}$$

for all $t \in [0, 1]$ where $j \neq i$ is the identity of the other player. (For example, this inequality can hold if $\delta_j(x_{0j}) = \delta_j(x'_{0j})$, $F_{\epsilon_j|x_{0j}} = F_{\epsilon_j|x'_{0j}}$ but $u_j(x_{0j}) \neq u_j(x'_{0j})$ for all $x' \in \omega_i(x)$.)

The event "$1 \in \iota(x) \cap \iota(x')$ and $p_1^*(x) = p_1^*(x')$" can never happen for any $x' \in \omega_1(x)$ whenever $\delta_1(x) \neq 0$. Suppose it does. Then $p_1^l(z) = p_1^*(z)$ in all $p^l \in \mathcal{L}_{z,\theta}^+$ for $z \in \{x, x'\}$. Therefore (4) implies $p_2^l(z) = p_2^*(z)$ in all $p^l \in \mathcal{L}_{z,\theta}^+$ for $z \in \{x, x'\}$ and $2 \in \iota(x) \cap \iota(x')$. Then $p_1^*(x) = p_1^*(x')$ and the inequality (5) above suggests $p_2^*(x) \neq p_2^*(x')$ for all $x' \in \omega_1(x)$. This in turn implies $p_1^*(x) \neq p_1^*(x')$ in (4) by definition of $x' \in \omega_1(x) \subset \Upsilon_1(x)$ whenever $\delta_1(x) \neq 0$. This contradicts the supposition that $p_1^*(x) = p_1^*(x')$. Hence Assumption 2 holds for $i = 1$ if $\delta_1(x) \neq 0$. Symmetric arguments prove the case with $i = 2$. $\|$

Note that when $\delta_i(x) = 0$ for either $i = 1$ or 2, $i \in \iota(x) \cap \iota(x')$ and $p_i^*(x) = p_i^*(x')$ for all $x' \in \Upsilon_i(x)$. This nevertheless does not violate Assumption 2, which covers only the case with $\delta_i(x) \neq 0$.

Now we resume our discussion of identification of the sign of $\delta_i(x)$. Define

$$\Psi_i(x) \equiv E\left(D_{i,g}\sum\nolimits_{j\neq i} D_{j,g}|X_g \in \Upsilon_i(x)\right) - E(D_{i,g}|X_g \in \Upsilon_i(x))E\left(\sum\nolimits_{j\neq i} D_{j,g}|X_g \in \Upsilon_i(x)\right)$$

where g indexes the independent games observed in the data, and $D_{i,g}$ is the decision made by i in game g.

Proposition 2 *Under Assumption 1 and Assumption 2, sign*$(\delta_i(x)) =$ *sign*$(\Psi_i(x))$ *for all* i, x.

Proof. Let $\Lambda_{x,\theta}$ denote the equilibrium selection mechanism defined over $\mathcal{L}^+_{x,\theta}$ under state x and let $\Lambda^*_{x,\theta}$ denote $\int_{\Upsilon_i(x)} \Lambda_{x',\theta} dF(x'|X' \in \Upsilon_i(x))$ (where $F(.|X \in S)$ is the distribution of X conditional on $X \in S$). Note that this is a distribution over the augmented support $\mathcal{L}^*_{x,\theta} \equiv \cup_{x'\in\Upsilon_i(x)}\mathcal{L}^+_{x',\theta}$ Let $\Lambda^*_{i,x,\theta}$ denote the distribution of $p^l_i(x)$ induced by $\Lambda^*_{x,\theta}$. Then (1) and *Assumption 2* imply that there exists an increasing (or decreasing) function g_i s.t. $p^l_i(x') = g_i(\gamma^l_i(x'))$ for all $p^l \in \mathcal{L}^+_{x',\theta}$ and all $x' \in \Upsilon_i(x)$ if $\delta_i(x') > 0$ (or < 0 respectively). Specifically, $g_i(t) \equiv F_{\epsilon_i|X=x'}(u_i(x') + \delta_i(x')t)$ for $t \in [0, 1]$. Note that this function is fixed for all $x' \in \Upsilon_i(x)$ due to Assumption 2. Also since $\omega_i(x)$ happens with a positive probability under Assumption 2, the distribution $\Lambda^*_{i,x,\theta}$ is non-degenerate. Then note

$$\begin{aligned}
\Psi_i(x) &= \int_{p^l\in\mathcal{L}^*_{x,\theta}} p^l_i(x)\gamma^l_i(x)d\Lambda^*_{x,\theta} - \int_{p^l\in\mathcal{L}^*_{x,\theta}} p^l_i(x)d\Lambda^*_{x,\theta}\int_{p^l\in\mathcal{L}^*_{x,\theta}} \gamma^l_i(x)d\Lambda^*_{x,\theta} \\
&= \int_0^1 g_i(z)zd\Lambda^*_{i,x,\theta}(z) - \int_0^1 zd\Lambda^*_{i,x,\theta}(z)\int_0^1 g_i(z)d\Lambda^*_{i,x,\theta}(z)
\end{aligned}$$

where $z \equiv p^l_i(x)$. Then the same argument as in Proposition 1 shows $\Psi_i(x) > 0$ (or < 0) whenever $\delta_i(x) > 0$ (or < 0) for all i, x. When $\delta_i(x) = 0$, $p^l_i(x')$ must be the same for all $p^l \in \mathcal{L}_{x',\theta}$ and all $x' \in \Upsilon_i(x)$, and the distribution $\Lambda^*_{i,x,\theta}$ is degenerate. Hence $\Psi_i(x) = 0$. This completes the proof. ■

If Assumption 2 fails for some i and x while Assumption 1 still holds, then the identification of the signs of $\delta_i(x)$ are affected only in the following sense. When $\Psi_i(x) = 0$, we cannot make decisions about the sign of $\delta_i(x)$. This is because either $\delta_i(x) = 0$ or the failure of Assumption 2 for i and x can result in $\Psi_i(x) = 0$. However, note on the other hand, if $\Psi_i(x) > 0$ (or $\Psi_i(x) < 0$), we can still identify the sign of $\delta_i(x) > 0$ (or $\delta_i(x) < 0$ respectively) regardless of Assumption 2.

4 Testing Multiple BNE and Interaction Signs

4.1 A Wald Test for Multiple BNE in the Data-Generating Process

Below, we propose a test for the presence of multiple equilibria in the data in an empirical context where researchers observe states and players' decisions from a large cross-section of independent games (indexed by $g = 1, ., G$), each defined by the same structural elements $(u_i, \delta_i, F_{\epsilon_{i|X}})_{i \leq N}$. Semiparametric estimation of games with incomplete information typically refrains from parametric assumptions on primitives or the equilibrium selection mechanism at the cost of assuming that the data observed are rationalized by the same BNE (see Aradillas-Lopez (2009), Bajari, Hong, Krainer, and Nekipelov (2009) and Tang (2009)). The applicability of these semiparametric approaches naturally hinges on validity of the "single equilibrium" assumption.[12] The procedures below formally test the existence of multiple BNE in the data observed under the maintained assumption of conditionally independent private signals.

The null hypothesis that "multiple BNE exist in the data for state x" is equivalently formulated as follows, based on Proposition 1:

$$\begin{aligned} H_0 &: \Delta_i(x) = 0 \ \forall i \leq N \qquad (6) \\ H_1 &: \exists\ i\ s.t.\ \Delta_i(x) \neq 0 \end{aligned}$$

[12]As mentioned in the introduction, "social interaction" models do not rely on this assumption but require the number of agents in each game to be large so that *within equilibrium* choice probabilities can be consistently estimated from average choices in each game.

where $\Delta_i(x) \equiv \tilde{\gamma}_i^*(x) - p_i^*(x)\gamma_i^*(x) = \sum_{j \neq i} \{E[D_i D_j|x] - E(D_i|x)E(D_j|x)\}$. Throughout this and the next section, we focus on a simple case where X contains only discrete coordinates. Then the sample analogs of expectations conditional on x are simple sample averages across games with $X = x$. We suppress x for notational ease when there is no ambiguity.

Let g index games observed in the data. For any subset $I \subset \{1,.,N\}$, let $D_{I,g} \equiv \Pi_{i \in I} D_{i,g}$ and $\mu_I \equiv E(D_{I,g}1(X_g = x))$. Let $\mu_0 \equiv \Pr(X_g = x)$. Let μ denote a $\tilde{N} \equiv \left(N + \binom{N}{2} + 1\right)$-vector consisting of μ_0, μ_i and μ_{ij} for all individual i and all pairs $i \neq j$. For example, with $N = 3$, $\mu \equiv (\mu_0, \mu_1, \mu_2, \mu_3, \mu_{12}, \mu_{13}, \mu_{23})'$. Define:

$$
\begin{aligned}
\overline{\mu}_i &\equiv (G)^{-1} \sum_g D_{i,g} 1(X_g = x) \; ; \; \overline{\mu}_{ij} \equiv (G)^{-1} \sum_g D_{ij,g} 1(X_g = x) \\
\overline{\mu}_0 &\equiv (G)^{-1} \sum_g 1(X_g = x) \; ; \; \overline{\mu}_G \equiv (\overline{\mu}_0 ., \overline{\mu}_i, ., \overline{\mu}_{ij}, .)'
\end{aligned}
$$

where $\overline{\mu}_G$ is the vector of sample analogs for μ. By the multivariate central limit theorem, $G^{1/2}(\overline{\mu}_G - \mu) \xrightarrow{d} N(\mathbf{0}_{\tilde{N}}, \mathbf{\Sigma})$ where $\mathbf{0}_{\tilde{N}}$ is a $\tilde{N}$-vector of zeros and Σ is the corresponding variance-covariance matrix.

Note that $\Delta_i(x) \equiv \sum_{j \neq i} \left(\frac{\mu_{ij}}{\mu_0} - \frac{\mu_i \mu_j}{\mu_0 \mu_0} \right)$. Let $\mathbf{T}_G$ be a N-vector with its i-th coordinate defined as

$$T_{G,i} = \hat{\Delta}_i(x) \equiv \sum_{j \neq i} \left(\frac{\overline{\mu_{ij}}}{\overline{\mu_0}} - \frac{\overline{\mu_i}}{\overline{\mu_0}} \frac{\overline{\mu_j}}{\overline{\mu_0}} \right)$$

Let V denote a N-by-$\tilde{N}$ matrix, with its i-th row V_i defined by the following table (where $\mu_{(m)}, V_{i,(m)}$ denote the m-th coordinates of two $\tilde{N}$-vectors μ and V_i respectively, and $j, k \neq i$),

$\mu_{(m)}$	μ_0	μ_i	μ_j	μ_{ij}	μ_{jk}
$V_{i,(m)}$	$\sum_{j \neq i}(-\frac{\mu_{ij}}{\mu_0^2} + \frac{2\mu_i \mu_j}{\mu_0^3})$	$-\sum_{j \neq i} \frac{\mu_j}{\mu_0^2}$	$-\frac{\mu_i}{\mu_0^2}$	$\frac{1}{\mu_0}$	0

Then the Delta Method implies

$$G^{1/2}(\mathbf{T}_G - \mathbf{\Delta}) \xrightarrow{d} N(\mathbf{0}_N, \mathbf{V\Sigma V}')$$

where $\mathbf{\Delta} \equiv (\Delta_i)_{i=1}^N$. Let $\hat{\mathbf{\Sigma}}, \hat{\mathbf{V}}$ be estimates for $\mathbf{\Sigma}, \mathbf{V}$ respectively, constructed by replacing μ_0, μ_I with non-parametric estimates

$$\hat{\mu}_0 = G^{-1} \sum_{g=1}^G 1(X_g = x) \qquad \hat{\mu}_I = G^{-1} \sum_{g=1}^G [\Pi_{i \in I} D_{i,g} 1(X_g = x)]$$

Proposition 3 *Suppose the data have G independent games with the same underlying structure. Then*

$$G(\mathbf{T}_G - \mathbf{\Delta})'(\hat{\mathbf{V}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}')^{-1}(\mathbf{T}_G - \mathbf{\Delta}) \xrightarrow{d} \chi^2_{df=N}$$

Under the null hypothesis, $\mathbf{\Delta} = \mathbf{0}_N$ and the chi-squared distribution can be used to obtain critical values for the test statistic $G\mathbf{T}'_G(\hat{\mathbf{V}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}')^{-1}\mathbf{T}_G$.

4.2 Inference of Players Who Switch Between Multiple Strategies

In addition to testing the joint hypothesis that $\mathbf{\Delta} = \mathbf{0}_N$ in (6), one would also be interested in finding out the identities of the set of players i in $\iota^c(x)$, i.e., all those who actively mix strategies across multiple BNE (indexed by $p_i^l(x)$). As mentioned in the introduction, this is interesting in particular for games with more than three players, where a subset of players may stick to the same strategy across multiple equilibria. Available semiparametric methods based on the assumption of unique equilibrium can still be applied to consistently estimate payoff parameters for those players who follow the same strategies across all equilibria in the game. Therefore it is useful to infer the *identity* of these players from the distributions of actions observed.

To accommodate this possibility, we resort to the statistical literature on multiple comparisons (for a recent survey, see Lehmann and Romano (2005)). This literature considers decision strategies that aggregate the individual tests for each i conditional on x:

$$\begin{aligned} H_i^0 &: \Delta_i(x) = 0 \\ H_i^1 &: \Delta_i(x) \neq 0 \end{aligned}$$

Given individual test statistics for each of the $i \leq N$ hypotheses, our objective is to define a decision rule that controls the *family-wise error rate*, or the probability of rejecting at least one of the true null hypotheses. More formally:

$$\text{FWE}_P = \text{Prob}_P\{\text{Reject at least one } H_i^0 : \Delta_i(x) = 0 \text{ where } i \in \mathbf{I}_0(P)\}$$

where the subscript P indicates the data-generating process and $\mathbf{I}_0(P) \subset \{1, \ldots, N\}$ is the set of indices i of true null hypotheses under P. A multiple testing procedure asymptotically controls the FWE_P at the rate α if $\limsup_{G \to +\infty} \mathrm{FWE}_P \leq \alpha$ for any P.

Well-known methods that asymptotically control for the family-wise error rate include the Bonferroni and the Holm's method. Both methods can be described in terms of the p-values for each of the individual hypotheses (indexed by i) above. We denote these p-values by $\hat{p}_{G,i}$. The Bonferroni method at level α rejects i if $\hat{p}_{G,i} \leq \alpha/N$. The Holm's procedure, which is less conservative than the Bonferroni method, follows a stepwise strategy. (For notational convenience, we suppress the dependence of the hypotheses and test statistics on x.) The Holm's procedure starts by ordering the p-values in ascending order: $\hat{p}_{G,(1)} \leq \hat{p}_{G,(2)} \leq \cdots \leq \hat{p}_{G,(N)}$. Let $H^0_{j_k} : \Delta_{j_k} = 0$ denote the single hypothesis corresponding to the k-th smallest p-value (i.e. $\hat{p}_{G,j_k} = \hat{p}_{G,(k)}$). Holm's stepwise method proceeds as follows. In the first step, compare $\hat{p}_{G,(1)}$ with α/N. If $\hat{p}_{G,(1)} \geq \alpha/N$, then accept all individual hypotheses and the procedure ends. Otherwise, reject the individual null hypothesis $H^0_{j_1} : \Delta_{j_1} = 0$ and move on to the second step. In the second step, the remaining $N-1$ hypotheses are all accepted if $\hat{p}_{G,(2)} \geq \alpha/(N-1)$. Otherwise reject $H^0_{j_2} : \Delta_{j_2} = 0$ and continue to the next step. More generally, compare $\hat{p}_{G,(k)}$ with $\alpha/(N-k+1)$ in the k-th step. Accept all remaining $N-(k-1)$ hypotheses if $\hat{p}_{G,(k)} \geq \alpha/(N-k+1)$. Otherwise, reject $H^0_{j_k}$ and move on to the next step. Continue doing so until all remaining hypotheses are accepted, or all hypotheses are rejected one by one in N steps.

Though less conservative than the Bonferroni method, the Holm's procedure can still be improved upon if one takes into account the dependence between the individual test statistics. To achieve this, we follow recent contributions by van der Laan, Dudoit, and Pollard (2004) and Romano and Wolf (2005).[13] Ordering the test statistics in descending order, we let $T_{G,(1)} \geq T_{G,(2)} \geq \cdots \geq T_{G,(N)}$. In the k-th step, a critical level c_k is obtained and

[13]The following description closely follows the presentation in Romano and Wolf (2005). For similar strategies controlling generalizations of the family-wise error rate, see Romano and Shaikh (2006). A recent application of such generalizations is Moon and Perron (2009).

those hypotheses with $T_{G,\cdot} \geq c_k$ are rejected. Let R_k be the number of hypotheses rejected after the first $k-1$ steps (i.e. the number of hypotheses not rejected at the beginning of the k-th step). As before, let $H^0_{i_k}$ denote the hypothesis whose test statistic is the k-th largest (i.e. $T_{G,i_k} = T_{G,(k)}$). Ideally, we want to obtain c_1 such that:

$$c_1 \equiv c_1(1-\alpha, P) = \inf \left\{ y : \text{Prob}_P \left\{ \max_{1 \leq j \leq N} T_{G,(j)} - \Delta_{i_j} \leq y \right\} \geq 1 - \alpha \right\}$$

where all statements are implicitly conditional on $X = x$. Subsequently, c_k is defined as

$$c_k \equiv c_k(1-\alpha, P) = \inf \left\{ y : \text{Prob}_P \left\{ \max_{R_k+1 \leq j \leq N} T_{G,(j)} - \Delta_{i_j} \leq y \right\} \geq 1 - \alpha \right\}$$

(also conditional on $X = x$). As pointed out in the references cited, because P is unknown in practice, we replace P by an estimate $\hat{P}_G$ and define

$$\hat{c}_k \equiv c_k(1-\alpha, \hat{P}_G) = \inf \left\{ y : \text{Prob}_{\hat{P}_G} \left\{ \max_{R_k+1 \leq j \leq N} T^*_{G,(j)} - \Delta^*_{i_j} \leq y \right\} \geq 1 - \alpha \right\} \quad (7)$$

where we follow Romano and Wolf (2005) and use $T^*_{G,(j)}$ and $\Delta^*_{i_j}$ to highlight that the sampling distribution of the test statistics is under $\hat{P}_G$ (not P). The following algorithm summarizes the stepwise multiple testing procedure we adopt from Romano and Wolf (2005).

Algorithm 1 (Basic Non-studentized Step-down Procedure)

Step 1. Relabel the hypotheses in descending order of the test statistics $T_{G,i}$. Let $H^0_{i_k}$ denote the individual null hypothesis whose test statistic is the k-th largest.

Step 2. Set $k = 1$ and $R_1 = 0$.

Step 3. For $R_k + 1 \leq s \leq N$, if $T_{G,(s)} - \hat{c}_k > 0$, then reject the individual null $H^0_{i_s}$.

Step 4. If no (further) null hypotheses are rejected, then stop. Otherwise, let R_{k+1} denote the total number of hypotheses rejected so far (i.e. R_k plus the number of hypotheses rejected in the k-th step), and set $k = k + 1$. Then return to Step 3 above.

In addition to estimating $\hat{c}_k$ via bootstrap, we also consider an alternative approach that uses the fact that the test statistics have a normal limiting distribution with a consistently estimable variance-covariance matrix.[14] We summarize the two approaches for estimating $\hat{c}_k$ in the following algorithms.

Algorithm 2.1 (Computing $\hat{c}_k$ Using Bootstrap)

Step 1. Let i_k and R_k be defined as in Algorithm 1 above.

Step 2. Generate B bootstrap data sets (typically with $B \geq 1000$).

Step 3. From each bootstrap data set (indexed by b), compute the vector of test statistics $\left(T^{*,b}_{G,1}, ., T^{*,b}_{G,N}\right)$.

Step 4. For $1 \leq b \leq B$, compute $\max^{*,b}_{G,k} = \max_{R_k+1 \leq s \leq N}(T^{*,b}_{G,i_s} - T_{G,i_s})$.

Step 5. Then compute $\hat{c}_k$ as the $(1-\alpha)$-th empirical quantile of the B values $\{\max^{*,b}_{G,k}\}_{b \leq B}$.

Algorithm 2.2 (Computing $\hat{c}_k$ Using Parametric Simulations)

Step 1. Estimate the covariance matrix of the vector of test statistics that corresponds to hypotheses which are not rejected after the first $k-1$ steps, i.e. $(T_{G,(R_k+1)}, T_{G,(R_k+2)}, ., T_{G,(N)})$. *Denote the estimate by $\hat{\Sigma}_k$.*

Step 2. Simulate a data set of M observations $\{v_m\}_{m=1}^{M}$ from the $(N-R_k)$-dimensional multivariate normal distribution with parameters $(0_{N-R_k}, \hat{\Sigma}_k)$, where 0_k is a k-vector of zeros.

Step 3. Then $\hat{c}_k$ is computed as the $(1-\alpha)$-th empirical quantile of the maximum coordinates of v_m in the simulated data.[15]

We also use a studentized version of the multiple testing method as in Romano and Wolf (2005). Let $\hat{\sigma}_{G,k}$ denote the estimates for the standard deviation of the test statistic $T_{G,k}$. To do so, we need an analogue of (7):

$$\hat{d}_k \equiv d_k(1-\alpha, \hat{P}_G) \equiv \inf\left\{y : \mathrm{Prob}_{\hat{P}_G}\left\{\max_{R_k+1 \leq j \leq N}(T^*_{G,(j)} - T_{G,i_j})/\hat{\sigma}^*_{G,i_j} \leq y\right\} \geq 1-\alpha\right\} \quad (8)$$

where $\hat{\sigma}^*_{G,i}$ are the estimates for standard deviations of $T_{G,i}$ computed from bootstrap samples. The studentized stepwise procedure is summarized in the following algorithm. As before, let R_k denote the total number of hypotheses that are not rejected in the first $k-1$ steps.

[14] See footnote 21 in Romano and Wolf (2005).

[15] M can be large relative to the number of bootstrap samples B in Algorithm 2.1.

Algorithm 3 (Studentized Step-down Procedure)

Step 1. Relabel the individual hypotheses in descending order of studentized test statistics $Z_{G,i} \equiv T_{G,i}/\hat{\sigma}_{G,i}$*, where* $\hat{\sigma}_{G,i}$ *are estimates for standard deviation of* $T_{G,i}$*.*

Step 2. Set $k = 1$ *and* $R_1 = 0$*.*

Step 3. For $R_k + 1 \le s \le S$*, if* $Z_{G,i_s} > \hat{d}_j$*, then reject the individual null* $H^0_{i_s}$*.*

Step 4. If no further individual null hypotheses are rejected, stop. Otherwise, let R_{k+1} *denote the total number of hypotheses rejected so far and set* $k = k+1$*. Then return to Step 3 above.*

The critical values for the studentized stepwise method $\hat{d}_k$ are computed by an algorithm similar to Algorithm 2.1 where standard errors $\left(\hat{\sigma}^{*,b}_{G,1}, ., \hat{\sigma}^{*,b}_{G,N}\right)$ are also computed in Step 3 and $\max^{*,b}_{G,k} \equiv \max_{R_k+1 \le s \le N}(T^{*,b}_{G,i_s} - T_{G,i_s})/\hat{\sigma}^{*,b}_{G,i_s}$ in Step 4.

In Section 5, we report the performance of three tests based on stepwise multiple testing procedures: (a) the non-studentized test with $\hat{c}_k$ computed from parametric simulations; (b) the non-studentized test with $\hat{c}_k$ computed via bootstrap; and (c) the studentized test with $\hat{d}_k$ computed via bootstrap. Because our setting corresponds to the smooth function model with i.i.d. data (Scenario 3.1 in Romano and Wolf (2005)), both strategies yield consistent tests that asymptotically control the family-wise error rate at level α. This would obtain from a slight modification in Theorem 3.1 in Romano and Wolf (2005) to accommodate two-sided hypotheses as indicated in Section 5 of that paper.

4.3 Inference on Signs of Interaction Effects

This section proposes a simple test for the sign of interaction effects for a player i in a given state x. It relies on the characterization in Proposition 2 and will hold when x induces multiple equilibria and choice probabilities vary across equilibria or when there are excluded regressors as discussed in Section 3. To fix ideas, we focus on the simple case with discrete X where any x in the support can happen with strictly positive probabilities. For any i, x, define

$$\hat{\Psi}_i(x) \equiv \hat{\psi}^{-1}_{i,x,1}\hat{\psi}_{i,x,2} - \hat{\psi}^{-2}_{i,x,1}\hat{\psi}_{i,x,3}\hat{\psi}_{i,x,4}$$

where

$$\hat{\psi}_{i,x,1} \equiv G^{-1}\sum_g 1\{x_g \in \Upsilon_i(x)\} \; ; \; \hat{\psi}_{i,x,2} \equiv G^{-1}\sum_g \left(D_{i,g}\left(\sum_{j\neq i} D_{j,g}\right) 1\{x_g \in \Upsilon_i(x)\}\right),$$

$$\hat{\psi}_{i,x,3} \equiv G^{-1}\sum_g \left(D_{i,g}1\{x_g \in \Upsilon_i(x)\}\right) \; ; \; \hat{\psi}_{i,x,4} \equiv G^{-1}\sum_g \left(\left(\sum_{j\neq i} D_{j,g}\right) 1\{x_g \in \Upsilon_i(x)\}\right)$$

When $\Upsilon_i(x) = \{x\}$, $\hat{\Psi}_i$ coincides with $T_{G,i}$ introduced in subsection 4.1. In this sense, $\hat{\Psi}_i$ generalizes $T_{G,i}$ for a non-singleton $\Upsilon_i(x)$. For notational ease, we drop the subscript i, x from the estimators when there is no ambiguity. Define

$$\hat{\mathbf{\Sigma}} \equiv \begin{bmatrix} \hat{\psi}_1(1-\hat{\psi}_1) & \hat{\psi}_2(1-\hat{\psi}_1) & \hat{\psi}_3(1-\hat{\psi}_1) & \hat{\psi}_4(1-\hat{\psi}_1) \\ \hat{\psi}_2(1-\hat{\psi}_1) & \hat{\psi}_5-\hat{\psi}_2^2 & \hat{\psi}_2(1-\hat{\psi}_3) & \hat{\psi}_5-\hat{\psi}_2\hat{\psi}_4 \\ \hat{\psi}_3(1-\hat{\psi}_1) & \hat{\psi}_2(1-\hat{\psi}_3) & \hat{\psi}_3-\hat{\psi}_3^2 & \hat{\psi}_2-\hat{\psi}_3\hat{\psi}_4 \\ \hat{\psi}_4(1-\hat{\psi}_1) & \hat{\psi}_5-\hat{\psi}_2\hat{\psi}_4 & \hat{\psi}_2-\hat{\psi}_3\hat{\psi}_4 & \hat{\psi}_6-\hat{\psi}_4^2 \end{bmatrix}$$

with

$$\hat{\psi}_5 \equiv G^{-1}\sum_g \left(D_{i,g}\left(\sum_{j\neq i} D_{j,g}\right)^2 1\{x_g \in \Upsilon_i(x)\}\right)$$

$$\hat{\psi}_6 \equiv G^{-1}\sum_g \left(\left(\sum_{j\neq i} D_{j,g}\right)^2 1\{x_g \in \Upsilon_i(x)\}\right)$$

Also define

$$\hat{\mathbf{V}} \equiv \left[-\hat{\psi}_2\hat{\psi}_1^{-2} + 2\hat{\psi}_3\hat{\psi}_4\hat{\psi}_1^{-3} \qquad \hat{\psi}_1^{-1} \qquad -\hat{\psi}_1^{-2}\hat{\psi}_4 \qquad -\hat{\psi}_1^{-2}\hat{\psi}_3 \right]$$

where $\hat{\mathbf{\Sigma}}$ and $\hat{\mathbf{V}}$ are analogous to the objects defined in subsection 4.1. Using the Delta Method and Slutsky's Theorem it is straightforward to verify that

$$\left(\hat{\mathbf{V}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}'/G\right)^{-1/2}\left(\hat{\Psi}_i(x) - \Psi_i(x)\right) \xrightarrow{d} \mathcal{N}(0,1)$$

Testing the sign of $\delta_i(x)$ amounts to testing the following three hypotheses:

$$H_1 : \Psi_i(x) > 0 \qquad H_2 : \Psi_i(x) = 0 \qquad H_3 : \Psi_i(x) < 0$$

Using the test statistic $\sqrt{G}(\hat{\mathbf{V}}\hat{\boldsymbol{\Sigma}}\hat{\mathbf{V}}')^{-1/2}\hat{\Psi}_i(x)$, we can choose critical regions at the two tails, each resulting in the rejection of H_2 in favor of either H_1 or H_3.[16] Proofs of consistency and asymptotic levels of the test should readily follow.

5 Monte Carlo Simulations

In this section we explore Monte Carlo experiments to illustrate the strategy presented in the previous section. The first design reproduces Example 1 and displays multiple equilibria. We use it to analyze the inference procedure on the existence of multiple equilibria and on the interaction signs when more than one equilibrium exists. Design 2 displays only one equilibrium and we use it to illustrate our procedure when multiple equilibria are absent but an excluded variable exists.

Design 1 We study the finite sample performance of the tests for multiple equilibria in Section 4 using a simple design of a 3-by-2 game in Example 1. The design is conditional on some state x and this dependence is suppressed for notational convenience. For some fixed state, let the players' baseline payoffs be $u_1 = 0.5$ and $u_2 = u_3 = 0.3611$, respectively, and let $\delta_i = -1$ and $\epsilon_i \sim \mathcal{N}(\mu = 0.1, \sigma^2 = 0.25^2)$ for all i. Let λ denote the probability with which the first Bayesian Nash equilibrium in (3) shows up in the data-generating process. We experiment with $\lambda = 0.1$, 0.25 or 0.5 and sample sizes $G = 1000$ or 3000.

For any (λ, G), we simulate a data set of players' binary decisions by letting

$$D_{i,g} = 1\left\{u_i - W_g\left(\sum_{j\neq i} p_j^1\right) - (1 - W_g)\left(\sum_{j\neq i} p_j^2\right) - \epsilon_{i,g} \geq 0\right\}$$

where in each game $g \leq G$, W_g is simulated from a Bernoulli distribution with success probability λ, $\epsilon_{i,g}$ from $\mathcal{N}(0.1, 0.25^2)$ and p^ls are propensity-scores in the two Bayesian Nash equilibria. For each (λ, G), we simulate $S = 1000$ data sets. For each data set, we employ the stepwise multiple testing procedure as described in Section 4.2, and make a decision to

[16] This is a *directional* hypothesis test. For a recent survey, see Shaffer (2006).

reject or not to reject the null hypothesis that there is a unique equilibrium in the data-generating process. We experiment with three different approaches for choosing the critical level $\hat{c}_k$ in Section 4.2: (i) simulation using estimated covariance matrix of T_G; (ii) bootstrap; and (iii) studentized bootstrap (Algorithms 3.2 and 4.2 in Romano and Wolf (2005)). For meaningful comparison between these three approaches, we use the same number of simulated multivariate normal vectors in (i) as the number of bootstrap samples drawn in (ii) and (iii) (which is denoted by B). We experiment with $B = 1000, 2000$. In Table 1 below, we report the probability of making a wrong decision (i.e., rejecting H_0 for $i = 1$ or not rejecting H_0 for $i = 2$ or 3) calculated from the $S = 1000$ simulated data sets in columns RP $1, 2, 3$.

Table 2 presents the tests of interaction signs for each of the three players. Since player 1 has the same conditional choice probabilities in the two equilibria, the test withholds judgment for most of the simulations. It detects a negative sign for the other two players.

Design 2 In this design, we consider a 3-player-by-2-action game where Assumption 2 is satisfied. The baseline payoff for player i is $u_i(x_i) = 1 + x_i$ where $x_1 \in \{-1, 2\}$ and $x_2 \in \{-1/2, 3/2\}$ and $x_3 \in \{-1, 3\}$. The state-dependent interaction effect for i is $\delta_i(x_i) = \delta x_i$ where δ is a parameter that controls the scale of the interaction effect. The private information ϵ_i is uniformly distributed over $(-c_i, c_i)$, where $c_i = 2(1 + x_i + |\delta x_i|)$.[17] Table 3 lists the marginal choice probabilities, or propensity scores, $p_i(x) \equiv \Pr(i \text{ chooses } 1|x)$ in the unique Bayesian Nash equilibria for each state $x \equiv (x_1, x_2, x_3)$. It is easy to verify that the Bayesian Nash equilibrium is unique for all x from Table 3, since all ϵ_i is uniformly distributed and all propensity scores are strictly between 0 and 1.

In Design 2, strategic interaction effects are state-dependent and individual-specific. For player 1, states in the first four rows in Table 3 form an equivalent class, while the other four rows form another equivalent class. We simulate $S = 1000$ samples, each with sample size $G = 5000$. For each of these samples, we calculate the test statistics T_G^* as defined in

[17] The parameter c_i is chosen this way to ensure there is a unique Bayesian Nash equilibrium under each state.

Section 4 and apply the following decision rule. If $T_G^* < -1.64$, then reject H_2 (no interaction effect) in favor of H_3 (negative interaction effect). If $T_G^* > 1.64$, then reject H_2 in favor of H_1 (positive interaction effect). Otherwise, do not reject H_2. Table 4 below summarizes the finite sample performance of our test. The two entries $[q_1, q_3]$ in the brackets report percentages of tests in $S = 1000$ simulations where H_2 is rejected in favor of H_1 (i.e., q_1) and the percentage of rejections in favor of H_3 (i.e. q_3), respectively. Recall that the sign of interaction effects for $\delta_i(x_i)$ is the same as the sign of x_i in our design as $\delta > 0$.

6 Empirical Illustration

As an application for the methodology outlined in the previous sections, we investigate the strategic behavior of couples over retirement decisions. A majority of retirees are married and many studies indicate that a significant proportion of individuals retire within a year of their spouse. Among the articles documenting the joint retirement of couples (and data sets employed) one could cite Hurd (1990) (New Beneficiary Survey), Blau (1998) (Retirement History Study), Gustman and Steinmeier (1992) (National Longitudinal Survey of Mature Women), Michaud (2004) (Health and Retirement Study) and Banks, Blundell, and Casanova Rivas (2007) (English Longitudinal Study of Ageing). Even though this is especially the case for couples closer in age, the distribution of differences in retirement timing between partners typically displays a spike at zero, regardless of the age difference (see, for instance, Figure 7 and Table 3 in Casanova Rivas (2009)). Following our framework, let δ_H denote the effect of the wife's retirement on the husband and δ_W denote the effect of the husband's retirement on the wife. For a two-player game, it is not hard to show that multiple equilibria can occur only when $\text{sign}(\delta_H) = \text{sign}(\delta_W)$.

We use the Health and Retirement Study for this analysis. The HRS is a panel data set, representative of non-institutionalized individuals and their spouses. There are currently eight available waves, covering every two years from 1992 to 2006. The study originally started with a cohort of individuals born between 1931 and 1941 (otherwise known as the HRS cohort). Soon after, the study added individuals in other cohorts. The HRS cohort has nevertheless been the most commonly studied, not only because more waves of information are available, but also because the cohort has been more frequently linked to other databases. We use couples in which at least one partner belongs to this cohort (i.e., born in the 1930s). Table 5 presents summary statistics on retirement for this cohort during the eight waves available.

For evidence of informational asymmetries within the couple regarding joint retirement, we present a two-way table with answers by husbands and wives to whether he or she expects to retire with his or her spouse.[18] We view the fact that a substantial proportion of couples showed opposing predictions as evidence that they are not fully aware of the other half's preference for retirement. If they had complete information, they would not have formed contradicting expectations about the possibility of joint retirement. Using a different survey focussing on elicited perceptions of spouse and own satisfaction within the marriage, Friedberg and Stern (2009) also show evidence of information asymmetries within marriage.

Because there are only two players, the statistic $G^{1/2}(\hat{\mathbf{V}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}')^{-1/2}\mathbf{T}_G$, calculated as prescribed in the first subsection of Section 4 is asymptotically standard normal under the null of no multiplicity and can be used to infer the existence of multiple equilibria and, in that case, the sign of the interaction effects. The results conditioning on age differences between husband and wife (husband's age - wife's age) are reported in Table 7. The test statistic is computed as prescribed in the previous sections and positive (negative) values of the statistic correspond to positive (negative) values of both δ_H and δ_W. According to the results in Section 4, under the null of a unique equilibrium played in the data, the test statistic we use below should follow a standard normal distribution. We find evidence of coordination motives between husband and wife for couples in which wives are at most two years older than the husband across all eight waves of the HRS data. Evidence for couples

[18]This variable is available for Wave 1 only.

in which the wife is much older than the husband is weaker. Similar findings obtain with the other cohorts in the HRS and conditioning variables.

Since the decision to retire may depend on more than the difference in age, we also perform the analysis above for a set of variables typically used in this literature. We focus on the fourth wave of the survey (1998), when respondents are in their early sixties. Following the literature, in addition to age difference, we condition on household wealth (tercile) and whether at least one member is in poor health. Table 8 presents more detailed information for these variables in 1998 (Wave 4).

Table 9 presents the test statistics with the conditioning variables mentioned above. The statistics are inconclusive for couples in which at least one of the partners is in poor health but tend to confirm our previous results otherwise.

Of course, some caveats apply. First, we assume that the decision problem is static. This would be reasonable when agents are impatient and/or when moving in and out of retirement is relatively costless so that individual choices can be treated as a succession of static decisions. Second, another explanation for the coincidence in retirement decisions (aside from taste interactions) is that husband and wife receive correlated (unobservable or omitted) shocks, driving them to retirement at similar times. This is outside the scope of the model we analyze, since Assumption 1 would then be violated.

7 Conclusion

In this paper we have shown how a condition typically employed in the analysis of simultaneous games of incomplete information leads to a simple and easily implementable test for the signs of interaction effects as well as the existence of multiple equilibria in the data-generating process. Inference of the signs of state-dependent and individual-specific interaction effects can be done under minimal assumptions that require only the conditional independence of private information, and the existence of state variables satisfying appropriate exclusion restrictions. Besides, given that many of the suggested methods for estimating and making inferences in such environments rely on the assumption that only one equilibrium is played in the data, this finding is relevant for the implementation of these techniques. With discrete covariates, such inference is implementable using well-known results in the multiple testing literature. When a continuous covariate is included, the testing procedure should account for the boundaries between regions with a different number of equilibria. Finally, the conditional independence assumption is also widely used in the dynamic games of incomplete information. In those settings, optimal decision rules involve not only equilibrium beliefs but continuation value functions that may also change across equilibria. Though a detailed analysis is deferred to future research, we speculate that our results generalize to such games under certain additional assumptions. In particular, the characterization of optimal policy rules in that context suggests that the existence of a unique equilibrium in the data can still be detected by the lack of correlation in actions across players of a given game as presented in the current paper.[19]

[19] With two actions, the optimal policy for a specific equilibrium would prescribe a decision rule like $S_i(X, \epsilon_i) = \mathbf{1}[u_i(X) + \delta_i(X)\sum_{j\neq i} p_j(X) + \beta\mathbb{E}(V_i(X', \epsilon_i'; p_1, \ldots, p_N)|X, \epsilon_i) - \epsilon_i \geq 0]$ where β is a discount factor, primed variables refer to the following period and $V_i(\cdot)$ is a continuation value defined by a Bellman equation where we make explicit the dependence on equilibrium choice probabilities $(p_1, \ldots, p_N)$ (see for example display (8) in Aguirregabiria and Mira (2007)). If the equilibrium is unique and Assumption 1 holds, conditional choice probabilities will factor as in the static case.

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Appendix:

Let θ denote the structure $(u_i, \delta_i)_{i=1}^N$ and $F_{\epsilon|X}$, and let $\mathcal{L}_{x,\theta}$ denote the choice probabilities profiles corresponding to BNE for a given x and parameter θ. That is, $\mathcal{L}_{x,\theta} \equiv \{p \in [0,1]^N : p$ solves (1) for θ and the given $x\}$. We let $\Lambda_{x,\theta}$ be an equilibrium selection mechanism. The following proposition illustrates the limits of what can be learned about the structure from the mixture data without imposing additional assumptions. Let $\#A$ denote the cardinality of set A and define $g : [0,1]^N \longrightarrow [0,1]^N$ as

$$g(p(x); x, \theta) \equiv \left(p_i(x) - F_{\epsilon_i|X}(u_i(x) + \delta_i(x) \sum_{j \neq i} p_j(x)) \right)_{i=1,\ldots,N}. \tag{9}$$

Proposition 4 *Assume*

$$det \left(\frac{\partial g(p(x); x, \theta)}{\partial p(x)} \right) \neq 0$$

Then the structure is not identified if $\#\mathcal{L}_{x,\theta} > \frac{2^N - 2}{N}$.

Proof. We first show that, for given x, the number of equilibria is finite. An equilibrium vector $p(x)$ is a fixed point to the mapping depicted on display (1). Equivalently, we represent it as a solution to the following equation:

$$g(p(x); x, \theta) = 0.$$

Notice that $\{0,1\} \cap F_{\epsilon_i|x}(\mathbb{R}) = \emptyset$ for any i, given the full support of ϵ_i. Consequently, for a solution vector, $p_i(x) \notin \{0,1\}$ and $p(x) \in (0,1)^N$. Since

$$\det \left(\frac{\partial g(p(x); x, \theta)}{\partial p(x)} \right) \neq 0$$

the Implicit Function Theorem directly implies that the set of fixed points to (9) is discrete (i.e. its elements are isolated points: each element is contained in a neighborhood with no other solutions to the system). Infinitesimal changes in $p(x)$ will imply a displacement of $g(\cdot; x, \theta)$ from zero, so local perturbations in $p(x)$ cannot be solutions to the system of equations. Since $p(x) \in [0,1]^N$, the set of solutions is a bounded subset of $\mathbb{R}^N$. In $\mathbb{R}^N$, every bounded infinite subset has a limit point (i.e., an element for which every neighborhood contains another element in the set) (Theorem 2.42 in Rudin (1976)). Consequently, a discrete set, having no limit points, cannot be both bounded and infinite. Being bounded and discrete, the set of solutions is finite.

In this case, the observed joint distribution of equilibrium actions is a finite mixture. Given Assumption 1, the cumulative distribution function for the observed actions is given by

$$\Phi(y_1, \ldots, y_N; x, \theta) = \sum_{\mathcal{L}_{x,\theta}} \Lambda_{x,\theta}(p^l(x)) \Pi_{i \in \{1,\ldots,N\}} (1 - p_i^l(x))^{1-y_i}$$

For a given x, the problem of retrieving this cdf and mixing probabilities from observed data is analyzed by Hall, Neeman, Pakyari, and Elmore (2005). In that paper, the authors show that the choice and mixing probabilities ($p_i^l(x)$ and $\Lambda_{x,\theta}$) cannot be obtained from observation of $\Phi(y_1, \ldots, y_N; x, \theta)$ if $\#\mathcal{L}_{x,\theta} > \frac{2^N-2}{N}$. Consequently, it is necessary for identifiability of the relevant probabilities that $\#\mathcal{L}_{x,\theta} \leq \frac{2^N-2}{N}$. Finally, if the equilibrium-specific choice probabilities cannot be identified, the utility function and the distribution of private components cannot be identified either (or else one could obtain the equilibrium specific choice probabilities and use those to obtain the mixing distribution from the data).

■

The condition that $\det\left(\frac{\partial g(p(x);x,\theta)}{\partial p(x)}\right) \neq 0$ is likely to be satisfied. With two players, for example, this determinant equals

$$1 - \delta_1(x)\delta_2(x) f_{\epsilon_1|X}(u_1(x) + \delta_1(x)p_2(x)) f_{\epsilon_2|X}(u_2(x) + \delta_2(x)p_1(x)).$$

Also when there are two players, the bound on the number of equilibria implies that, without further assumptions, the existence of more than one equilibrium precludes identification.

Table 1: Finite Sample Performance: Tests for Multiple Equilibria

(*Target probability for FWE:* $\alpha = 0.10$)

		$B = 1000$			$B = 2000$		
G	λ	$RP1$	$RP2$	$RP3$	$RP1$	$RP2$	$RP3$
1000	0.50	0.101	0.101	0.095	0.112	0.109	0.111
	0.25	0.093	0.094	0.085	0.094	0.096	0.089
	0.10	0.107	0.107	0.102	0.114	0.119	0.112
3000	0.50	0.108	0.109	0.105	0.087	0.089	0.083
	0.25	0.096	0.097	0.094	0.102	0.105	0.103
	0.10	0.093	0.090	0.092	0.111	0.107	0.108

NOTE: Design 1: Number of simulations $S = 1000$. G is the sample size. λ specifies the probability that an equilibrium is chosen. *RP1*, *2* and *3* are rejection frequencies following three tests respectively: (1) the non-studentized test with $\hat{c}_k$ from parametric simulations; (2) the non-studentized test with $\hat{c}_k$ computed via bootstrap; and (3) the studentized test with $\hat{d}_k$ computed via bootstrap.

Table 2: Finite Sample Performance: Test of Signs of Interaction Effects

Brackets include $[q_1, q_3]$.

G	λ	$i = 1$	$i = 2$	$i = 3$
1000	0.50	[0.036, 0.076]	[1.000,0.000]	[1.000,0.000]
	0.25	[0.035, 0.072]	[1.000,0.000]	[1.000,0.000]
	0.10	[0.040, 0.072]	[1.000,0.000]	[1.000,0.000]
3000	0.50	[0.054, 0.067]	[1.000,0.000]	[1.000,0.000]
	0.25	[0.048, 0.048]	[1.000,0.000]	[1.000,0.000]
	0.10	[0.049, 0.053]	[1.000,0.000]	[1.000,0.000]

NOTE: Design 1: S is 1000. G is the sample size. λ is the equilibrium selection probability. q_1 is the frequency of rejection of H_2 in favor of H_1. q_3 is the frequency of rejection of H_2 in favor of H_3.

Table 3: Propensity Scores in Bayesian Nash Equilibria

(p_1, p_2, p_3 in brackets)

x_1	x_2	x_3	$\delta = 0.8$	$\delta = 0.9$	$\delta = 1$
-1	$-1/2$	-1	$[0.3233, 0.5603, 0.3233]$	$[0.3060, 0.5561, 0.3060]$	$[0.2895, 0.5526, 0.2895]$
-1	$-1/2$	3	$[0.2523, 0.5288, 0.7098]$	$[0.2223, 0.5196, 0.7144]$	$[0.1927, 0.5111, 0.7183]$
-1	$3/2$	-1	$[0.2998, 0.7012, 0.2998]$	$[0.2790, 0.7033, 0.2790]$	$[0.2590, 0.7048, 0.2590]$
-1	$3/2$	3	$[0.2101, 0.7262, 0.7231]$	$[0.1710, 0.7323, 0.7300]$	$[0.1316, 0.7376, 0.7360]$
2	$-1/2$	-1	$[0.7124, 0.5286, 0.2518]$	$[0.7167, 0.5194, 0.2219]$	$[0.7203, 0.5109, 0.1922]$
2	$-1/2$	3	$[0.7479, 0.4754, 0.7477]$	$[0.7593, 0.4541, 0.7599]$	$[0.7704, 0.4322, 0.7717]$
2	$3/2$	-1	$[0.7249, 0.7263, 0.2098]$	$[0.7313, 0.7324, 0.1707]$	$[0.7369, 0.7376, 0.1314]$
2	$3/2$	3	$[0.7738, 0.7724, 0.7754]$	$[0.7927, 0.7903, 0.7955]$	$[0.8126, 0.8090, 0.8166]$

Table 4: Finite Sample Performance: Test of Signs of Interaction Effects

(No. of simulations: $S = 1000$. Brackets include $[q_1, q_3]$.)

	$G = 5000$			$G = 10000$		
	$\delta = 0.8$	$\delta = 0.9$	$\delta = 1.0$	$\delta = 0.8$	$\delta = 0.9$	$\delta = 1.0$
$X_1 = -1$	$[0.000, 0.469]$	$[0.001, 0.628]$	$[0.000, 0.854]$	$[0.000, 0.717]$	$[0.000, 0.890]$	$[0.000, 0.986]$
$X_2 = -1/2$	$[0.003, 0.359]$	$[0.000, 0.520]$	$[0.000, 0.714]$	$[0.000, 0.577]$	$[0.000, 0.790]$	$[0.000, 0.925]$
$X_3 = -1$	$[0.000, 0.483]$	$[0.000, 0.643]$	$[0.000, 0.834]$	$[0.000, 0.702]$	$[0.000, 0.888]$	$[0.000, 0.986]$
$X_1 = 2$	$[0.323, 0.004]$	$[0.459, 0.000]$	$[0.667, 0.000]$	$[0.484, 0.000]$	$[0.736, 0.000]$	$[0.910, 0.000]$
$X_2 = 3/2$	$[0.400, 0.000]$	$[0.617, 0.000]$	$[0.817, 0.000]$	$[0.665, 0.000]$	$[0.867, 0.000]$	$[0.979, 0.000]$
$X_3 = 3$	$[0.300, 0.004]$	$[0.496, 0.000]$	$[0.735, 0.000]$	$[0.545, 0.000]$	$[0.764, 0.000]$	$[0.930, 0.000]$

NOTE: q_1 is the frequency of rejection of H_2 in favor of H_1. q_3 is the frequency of rejection of H_2 in favor of H_3.

Table 5: Summary Statistics (HRS Cohort)

Variable	Wave 1 (1992)		Wave 2 (1994)		Wave 3 (1996)		Wave 4 (1998)	
	Mean	N	Mean	N	Mean	N	Mean	N
Joint Ret	0.055	4,754	0.102	4,260	0.139	4,057	0.165	3,834
Only Husband Ret	0.150	0.166	2.13	4,260	0.193	4,057	0.229	3,834
Only Wife Ret	0.064	4,754	0.086	4,260	0.096	4,057	0.097	3,834

Variable	Wave 5 (2000)		Wave 6 (2002)		Wave 7 (2004)		Wave 8 (2006)	
	Mean	N	Mean	N	Mean	N	Mean	N
Joint Ret	0.202	3,589	0.256	3,405	0.325	3,185	0.394	2,920
Only Husband Ret	0.259	3,589	0.265	3,405	0.265	3,185	0.259	2,920
Only Wife Ret	0.102	3,589	0.113	3,405	0.096	3,185	0.132	2,920

NOTE: The sample includes couples from the Health and Retirement Study with at least one partner in the HRS cohort, i.e., born between 1931 and 1941.

Table 6: Joint Retirement Expectations

	Wife	
Husband	Yes	No
Yes	492	146
No	136	349

NOTE: Available for Wave 1 only.

Table 7: Test of Multiplicity and Interaction Signs

	Wave 1		Wave 2		Wave 3		Wave 4	
Age Dif (Yrs)	Test Stat	N	Test Stat	N	Test Stat	N	Test Stat	N
< -5	1.78	155	0.96	145	1.22	137	0.84	123
-5 to -2	1.77	175	2.13	160	1.58	146	2.40	137
-2 to 2	4.38	1,536	6.30	1,407	5.80	1,316	4.94	1,245
2 to 5	4.35	1,305	5.51	1,196	5.05	1,114	4.18	1,038
> 5	5.33	1,870	4.83	1,645	5.73	1,546	5.01	1,468

	Wave 5		Wave 6		Wave 7		Wave 8	
Age Dif (Yrs)	Test Stat	N	Test Stat	N	Test Stat	N	Test Stat	N
< -5	1.12	117	0.65	109	0.94	103	0.79	87
-5 to -2	1.23	132	1.67	123	1.03	120	0.66	104
-2 to 2	4.83	1,167	3.91	1,105	3.43	1051	1.87	995
2 to 5	3.39	955	3.47	901	2.48	845	1.74	786
> 5	1.30	1,324	0.92	1,256	2.96	1,166	2.60	1,050

NOTE: The sample includes couples from the Health and Retirement Study with at least one partner in the HRS Cohort, i.e., born between 1931 and 1941. Age difference is husband's age minus wife's age. The test statistic is $G^{1/2}(\hat{\mathbf{V}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}')^{-1/2}\mathbf{T}_G$.

Table 8: Wave 4 (1998)

Variable	N	Mean
At least one with college or above	4,011	0.306
Total Household Wealth (Dollars)	4,011	378,591.2
Age (Husband) (Yrs.)	3,919	62.8
Age (Wife) (Yrs.)	3,926	58.8
Poor Health (Husband)	3,919	0.271
Poor Health (Wife)	3,926	0.240

NOTE: The sample includes couples from the Health and Retirement Study with at least one partner in the HRS cohort, i.e., born between 1931 and 1941.

Table 9: Test of Multiplicity and Interaction Signs

			Poor Health (At Least One)			
			No		Yes	
Age Diff	Total Wealth Tercile	College	Test Stat	N	Test Stat	N
$[-2\text{yrs.}, 2\text{yrs.}]$	1	Yes	1.64	145	1.05	205
		No	1.51	69	1.02	17
	2	Yes	2.53	203	1.27	87
		No	2.00	95	0.82	29
	3	Yes	2.42	156	1.14	56
		No	1.91	162	0.38	21
$(2\text{yrs.}, 5\text{yrs}]$	1	Yes	2.07	149	1.75	209
		No	1.67	35	-0.08	20
	2	Yes	1.40	152	1.52	96
		No	0.98	74	0.74	14
	3	Yes	1.13	106	1.34	46
		No	1.23	115	0.8	22
$> 5\text{yrs.}$	1	Yes	2.6	223	1.22	337
		No	1.74	75	0.83	59
	2	Yes	2.55	156	0.88	98
		No	2.28	113	0.62	39
	3	Yes	1.64	113	0.74	55
		No	3.10	159	1.33	41

NOTE: The sample includes couples from the Health and Retirement Study with at least one partner in the HRS cohort, i.e. born between 1931 and 1941. Age difference is husband's age minus wife's age. Tercile 1 is the one with highest total wealth. The test statistic is $G^{1/2}(\hat{\mathbf{V}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}')^{-1/2}\mathbf{T}_G$.