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"Competitive Equilibrium with Incomplete Financial Markets"

by

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Competitive Equilibrium with Incomplete Financial Markets*

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Abstract

This is my classic paper, written in early 1984, concerning existence and optimality in general financial equilibrium with incomplete markets for nominal assets, just now being published in a special issue of the Journal of Mathematical Economics.

I. **Introduction**

This paper presents a fairly general theory of competitive equilibrium when, in effect, trading across contingent commodities markets is constrained. The novel feature of my formulation is its representation of such an incomplete market structure as a purely financial phenomenon; trading is constrained simply because there are insufficient contingent claims instruments to provide households with all potentially desirable credit arrangements. Put succinctly, then, my investigation pursues the implications of "incomplete" markets following Arrow's [1] rather than Debreu's [3, Chapter 7] lead.

In the next section I describe the basic model (which is unfortunately, if unavoidably, quite notation intensive). The most crucial characteristic of this model is the restriction on rates of return to various financial instruments imposed by simple arbitrage

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considerations. Relying heavily on this restriction, in Sections III and IV I establish fundamental results concerning existence and optimality. Regarding existence, my main contribution is a theorem whose hypotheses are comparable in generality to those employed in standard Walrasian theory. In particular, in contrast to the (Debreuvian) approach taken in the deservedly influential paper of Radner [10], my (Arrovian) approach yields equilibrium without the artifice of introducing arbitrary (and, should I also say, self-contradictory?) bounds on presumed competitive behavior. On the other hand, with respect to optimality, my sole contribution is a refinement of the clever characterization due to Grossman [6] (and erected on foundations provided earlier by the insightful studies of Diamond [4] and Hart [7]). This patently severe qualification to the two basic theorems of welfare emphasizes that, with incomplete financial markets, competitive allocation is Pareto optimal only relative to a very narrow class of transaction-constrained perturbations.

Finally, in the concluding section I briefly discuss extensions of two different sorts – first, generalizations involving (for example) weaker hypotheses on household characteristics (in the spirit, say, of McKenzie [8]), and second, generalizations involving (for example) broader specification of financial instruments (in the spirit, say, of Hart [7] or, better, Gale [5]). This last analysis suggests one effective treatment for the inherent difficulty associated with endogenous rates of return (a difficulty which was also skillfully diagnosed in Hart's fine piece).

II. **The Model**

To avoid obscuring my presentation with inessential technical detail, I consider pure distribution within the simplest possible intertemporal context, where there are only two periods, today and tomorrow, denoted by the superscripts 0 and 1, respectively. There is a priori uncertainty about which of σ possible states of the world will prevail tomorrow; these are denoted by the superscripts $s \in S = \{1, 2, ..., \sigma\}$. In today's spot market, households can buy and sell – within the limits of their predetermined endowment income – either goods, which they consume, or bonds, which they hold until redeemed next period. Then, in tomorrow's spot market, they can again buy and sell goods – but now within the limits of their realized endowment cum portfolio income – which they also consume. Since all transactions take place on spot markets, both prices (for goods and bonds) and returns (on bonds) are denominated in the unit of account, say, dollars.

Before making this general framework more specific, I should first emphasize that there are only two differences between Arrow's and my version of contingent claims markets: First, there are typically fewer available (types of) bonds than potential states, and second, each bond normally yields returns in several states. Thus, in the limiting case, there may be only a single available bond which yields returns in every state, for example, money (or, more precisely, inside money).

1. **Economic Objects**

There are κ physical commodities or **goods**, denoted by the superscript $c \in C = \{1, 2, ..., \kappa\}$. Quantities of goods are represented by x^{0c} and x^{1sc}, or $x^0 = (x^{01}, x^{02}, \ldots, x^{0\kappa})$ and $x^{1s} = (x^{1s1}, x^{1s2}, \ldots, x^{1s\kappa})$, or $x = (x^0, x^{11}, x^{12}, \ldots, x^{1\sigma})$. Similarly, spot prices of goods are represented by p^{0c} and p^{1sc}, or $p^0 = (p^{01}, p^{02}, \ldots, p^{0\kappa})$ and $p^{1s} = (p^{1s1}, p^{1s2}, \ldots, p^{1s\kappa})$, or $\boldsymbol{p} = (p^0, p^{11}, p^{12}, \ldots, p^{1\sigma})$. There are also ι financial instruments or **bonds**, denoted by the superscript $i \in I = \{1, 2, ..., \iota\}$. Quantities of bonds are represented by β^i or $\beta = (\beta^1, \beta^2, \ldots, \beta^\iota)$, their spot prices (in today's spot market) by ψ^i or $\boldsymbol{\psi} = (\psi^1, \psi^2, \ldots, \psi^\iota)$.

The i^{th} bond is a commitment to pay (in tomorrow's spot market) P^{is} dollars if state s occurs. I will assume throughout the paper that there are no more bonds than states,

$$\iota \leqq \sigma \tag{1}$$

and, for the most part, that returns are both nonnegative and nontrivial,

$$P^i = (P^{i1}, P^{i2}, \ldots, P^{i\sigma}) > 0 \text{ for } i \in I.^1 \tag{2}$$

While my analysis therefore also covers the polar case where there may be complete markets ($\iota = \sigma$), I will be mainly concerned with the less well understood case where there aren't ($\iota < \sigma$).

2. **Household Behavior**

There are η active agents or **households**, denoted by the subscript $h \in H = \{1, 2, ..., \eta\}$. Each household has an endowment of goods, represented by the vector $\omega_h = (\omega_h^0, \omega_h^{11}, \omega_h^{12}, \ldots, \omega_h^{1\sigma})$, and preferences over goods, represented by the function $u_h : \mathbb{R}_+^{\kappa(\sigma+1)} \to \mathbb{R}$. For the bulk of my analysis I will assume that endowments are strictly positive, i.e., that

$$\omega_h \gg 0 \text{ for } h \in H, \tag{3}$$

[1] I adopt the generally accepted convention that, for any pair of vectors, say, $y' \in \mathbb{R}^n$ and $y'' \in \mathbb{R}^n, y' \geqq y''$ means $y^{j\prime} \geqq y^{j\prime\prime}$ for every j, $y' > y''$ means $y^{j\prime} \geqq y^{j\prime\prime}$ for every j, with strict inequality for some j, while $y' \gg y''$ means $y^{j\prime} > y^{j\prime\prime}$ for every j.

and that preferences are continuous, weakly monotone in each of the potential consumption bundles x_h^0 and x_h^{1s} for $s \in S$, and quasi-concave, i.e., in particular, that

$$x_h' \geqq x_h'' \geqq 0 \text{ [resp. } x_h' \geqq x_h'' \geqq 0 \text{ and}$$
$$x_h^{0\prime} \gg x_h^{0\prime\prime} \text{ or } x_h^{1s\prime} \gg x_h^{1s\prime\prime} \text{ for some } s] \text{ implies} \tag{4}$$
$$u_h(x_h') \geqq u_h(x_h'') \text{ [resp. } u_h(x_h') > u_h(x_h'')] \text{ for } h \in H.$$

Given spot prices $(\boldsymbol{p}, \boldsymbol{\psi})$, each household chooses its consumption plan x_h and portfolio holdings β_h so as to maximize its utility subject to (spot market) budget constraints, i.e., so that (x_h, β_h) is an optimal solution to the problem

$$\begin{aligned} &\text{maximize} && u_h(x_h') \\ &\text{subject to} && p^0 x_h^{0\prime} + \sum_i \psi^i \beta_h^{i\prime} \leqq p^0 \omega_h^0, \\ & && p^{1s} x_h^{1s\prime} \leqq p^{1s} \omega_h^{1s} + \sum_i P^{is} \beta_h^{i\prime} \text{ for } s \in S, \\ &\text{and} && x_h' \geqq 0 \end{aligned} \tag{5}$$

for $h \in H$. Notice especially that each household is thus free to buy or sell – within the limits of its budget constraints, of course – any quantity of bonds; if $\beta_h^i > 0$, then it is a creditor, while if $\beta_h^i < 0$, it is a debtor (with regard to the particular bond i).

3. Arbitrage Possibilities

The key to my analysis is the fact that the assumptions (2) and (4) together with the behavior described in (5) entail a severe restriction on the configuration of bond prices and their returns (which may prevail in competitive equilibrium). This can be most clearly seen by proceeding in several steps.

(i) Bond prices are strictly positive,

$$\boldsymbol{\psi} \gg 0. \tag{6}$$

Otherwise, if $\psi^i \leqq 0$ for some i, every household would want to buy unbounded amounts of this bond today, and thereby attain the prospect of unlimited consumption in some state s (for which $P^{is} > 0$) tomorrow.

From this first observation it follows that, by defining

$$b_h^i \equiv \psi^i \beta_h^i = \text{ dollars invested in bond } i \text{ by household } h$$

for $h \in H$, $i \in I$ and

$$r^{is} \equiv P^{is}/\psi^i = \text{ rate of return to bond } i \text{ in state } s$$

for $s \in S$, $i \in I$, we can streamline the description of household behavior by reformulating (5) in terms of choosing (x^h, b^h) to be an optimal solution to the problem

$$\begin{aligned} &\text{maximize} && u_h(x_h') \\ &\text{subject to} && p^0 x_h^{0\prime} + \sum_i b_h^{i\prime} \leqq p^0 \omega_h^0, \\ & && p^{1s} x_h^{1s\prime} \leqq p^{1s} \omega_h^{1s} + \sum_i r^{is} b_h^{i\prime} \text{ for } s \in S, \\ &\text{and} && x_h' \geqq 0 \end{aligned} \tag{7}$$

for $h \in H$. Note also that, in switching from returns to rates of return (given (6)), (2) becomes

$$r^i > 0 \text{ for } i \in I. \tag{8}$$

The important advantage of this reformulation stems from a second basic observation.

(ii) There are **"no fast bucks,"**

$$\{b : b \in \mathbb{R}^\iota \text{ and } (-\sum_i b^i, \sum_i r^i b^i) > 0\} = \emptyset. \tag{9}$$

Otherwise, if $(-\sum_i b^{i\prime}, \sum_i r^i b^{i\prime}) > 0$ for some b', every household would want to buy an unbounded portfolio (in the same proportions as b') today, and thereby attain unlimited consumption today (when $\sum_i b^{i\prime} < 0$) or the prospect of unlimited consumption in some state s tomorrow (when $\sum_i r^{is} b^{i\prime} > 0$).

(9) amounts to the analogue for finance of the familiar "no free lunch" postulate for production (explaining my choice of terminology). For my purposes it has an extremely useful restatement, whose full proof is fairly lengthy and therefore deferred to the Appendix.

Lemma 1. *There are "no fast bucks", condition (9), if and only if there are some positive values for wealth in each state, $\psi^s > 0$ for $s \in S$, such that the value of each bond's returns equals its price, $\sum_s \psi^s P^{is} = \psi^i$ or, equivalently,*

$$\sum_s \psi^s r^{is} = 1 \text{ for } i \in I. \tag{10}$$

These values can be interpreted as the prices associated with σ Arrow securities – where the return from the s^{th} such security is one dollar if state s occurs, and nothing otherwise, and where, for example, the set of returns generated by varying a portfolio consisting of one dollar's worth of these securities contains the set generated by varying one dollar's worth of the original bonds. In other words, when $\iota < \sigma$, Lemma 1 means that investing in the original bonds is tantamount to investing (subject to some additional linear constraints) in Arrow-Debreu securities. (Precisely, in dollar terms, a particular portfolio of original bonds b' is equivalent to the portfolio of Arrow-Debreu securities $a' = (\psi^1 \sum_i r^{i1} b^{i\prime}, \psi^2 \sum_i r^{i2} b^{i\prime}, \ldots, \psi^\sigma \sum_i r^{i\sigma} b^{i\prime})$.) A fortiori, when $\iota = \sigma$ – and the rates of return r^i for $i \in I$ are linearly independent – the two investment opportunities are identical.[2]

One final observation permits adopting an especially convenient normalization for the spot market prices of goods and rates of return on bonds.

(iii) Goods prices are nonnegative and nontrivial in each spot market,

$$p^0 > 0 \text{ and } p^{1s} > 0 \text{ for } s \in S. \tag{11}$$

Otherwise, for example, if $p^{0c} < 0$ for some c or $p^0 = 0$, every household would want to attain unlimited consumption today.

By virtue of the fact that each budget constraint in (7) is unaffected upon being multiplied through by a positive constant, (10) and (11) enable me to concentrate attention on prices and rates of return satisfying

$$\boldsymbol{p} \in P = \{\boldsymbol{p} : \boldsymbol{p} \geqq 0 \text{ and } \sum_c (p^{0c} + \sum_s p^{1sc}) = 1\}$$

[2]Lemma 1 can therefore be viewed as an extension of the characterization in Cass-Stiglitz [2] (Appendix I, pp. 149-153). As far as I can now recollect, this general sort of result – as well as its general method of proof – were first suggested to me by Steve Ross during the early 1970s.

and

$$r \in R = \{r : r^i \geqq 0 \text{ and } \sum_s r^{is} = 1 \text{ for } i \in I\},$$

respectively, where it will be very useful in the sequel to represent the whole array of rates of return by the $(\iota \times \sigma)$ – dimensional matrix

$$r = \begin{pmatrix} r^1 \\ r^2 \\ \vdots \\ r^\iota \end{pmatrix} = \begin{pmatrix} \ddots & & \\ & r^{is} & \\ & & \ddots \end{pmatrix}.$$

(Formally, given $\boldsymbol{p}$ satisfying (11) and r satisfying both (8) and (9) – and hence also (10) for some $\psi^s > 0$ for $s \in S$ – we can transform into $\boldsymbol{p}' \in P$ and $r' \in R$ using the formulas

$$p^{0\prime} = p^0 / \sum_c (p^{0c} + \sum_s \psi^s p^{1sc}) \text{ and } p^{1s\prime} = \psi^s p^{1s} / \sum_c (p^{0c} + \sum_s \psi^s p^{1sc}) \text{ for } s \in S$$

and

$$r^{is\prime} = \psi^s r^{is} \text{ for } s \in S, i \in I,$$

respectively. This maneuver is completed by transforming (x_h, b_h) into (x_h, b_h') where

$$b_h' = b_h / \sum_c (p^{0c} + \sum_s \psi^s p^{1sc}),$$

for $h \in H$.)

An attentive reader may have noticed that in going from the formulation (5) to (7), and then from the parameters $(\boldsymbol{p}, r)$ to $(\boldsymbol{p}', r') \in P \times R$, we have lost track of some of the original structure of the model. In particular, it might appear as if we need to account for which subset of R corresponds to the original specification of returns. This turns out, however, not to be a relevant consideration; both my existence results and my optimality characterization are stated in terms of an arbitrary element of R – and, from the preceding argument, we now know that any configuration of bond prices and their returns which may prevail in competitive equilibrium must yield some $r \in R$.[3]

[3] It is perhaps worth remarking too that, given the maintained assumption (2), there **always** exist positive bond prices such that rates of return lie in the subset R; simply pick **any** Arrow-Debreu security prices $\psi^s > 0$ for $s \in S$, and then define corresponding bond prices

$$\psi^i = \sum_s \psi^s P^{is} > 0 \text{ for } i \in I.$$

4. **Financial Equilibrium**

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_\eta)$ represent a goods allocation, and $\boldsymbol{b} = (b_1, b_2, \ldots, b_\eta)$ represent bond holdings (after today's trading). Then, on the one hand, an allocation $\boldsymbol{x}$ is **feasible** if it is nonnegative, $\boldsymbol{x} \geqq 0$, and – assuming free disposal of unwanted goods – materials balance,

$$\sum_h x_h^{0c} \leqq \sum_h \omega_h^{0c} \text{ for } c \in C \tag{12}$$

and

$$\sum_h x_h^{1sc} \leqq \sum_h \omega_h^{1sc} \text{ for } c \in C, s \in S.$$

On the other hand, it is **Pareto optimal** if there is no other feasible allocation $\boldsymbol{x}'$ which dominates $\boldsymbol{x}$ in terms of utility,

$$u_h(x'_h) \geqq u_h(x_h) \text{ for } h \in H, \text{ with strict inequality for some } h. \tag{13}$$

Finally, given permissible rates of return $r \in R$, a price-allocation-holding triplet $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b})$ is a **financial equilibrium** if, at the normalized prices $\boldsymbol{p} \in P$, each household maximizes, (x_h, b_h) is an optimal solution to the problem (7) for $h \in H$, and each market clears,

$$\sum_h x_h^{0c} \leqq \sum_h \omega_h^{0c}, \text{ with equality if } p^{0c} > 0, \text{ for } c \in C, \tag{14}$$

$$\sum_h b_h^i = 0 \text{ for } i \in I,$$

and

$$\sum_h x_h^{1sc} \leqq \sum_h \omega_h^{1sc}, \text{ with equality if } p^{1sc} > 0, \text{ for } c \in C, s \in S.$$

In analyzing the properties of financial equilibrium it is possible (and all but indispensable) to restrict attention to rates of return which are linearly independent,

$$\begin{aligned} r \in R^\iota \quad &= \{r: \quad r \in R \text{ and rank } r = \iota\} \\ &= \{r: \quad r \in R \text{ and if } b \in \mathbb{R}^\iota \text{ and } \sum_i r^i b^i = 0, \text{ then } b = 0\} \\ &= \{r: \quad r \in R \text{ and if } b \in \mathbb{R}^\iota \text{ and } (-\sum_i b^i, \sum_i r^i b^i) = 0, \text{ then } b = 0\} \\ &= \{r: \quad r \in R \text{ and if } b \in \mathbb{R}^\iota \text{ and } (-\sum_i b^i, \sum_i r^i b^i) \geqq 0, \text{ then } b = 0\}. \end{aligned} \tag{15}$$

The last two identities in (15) follow directly from (9) and Lemma 1, and, in particular, the former provides explicit justification for my asserting that such an additional requirement involves no real loss of generality; otherwise, if $(-\sum_i b^{i\prime}, \sum_i r^i b^{i\prime}) = 0$ for some $b' \neq 0$, households would be indifferent between having access to the original set of bonds and to any maximal subset yielding independent rates of return. (For elaboration of these points, see Cass-Stiglitz [2], especially pp. 149-150.) The usefulness of imposing (15) comes (in part) from the easily verified result that together with weak monotonicity (4) – which itself implies that, at an optimal solution, each budget constraint in (7) holds with equality – it implies that, in a financial equilibrium, market clearing (14) is equivalent to materials balance (12).[4] In other words, by assuming that rates of return are linearly independent, we can essentially ignore the bond markets per se. But, as we shall see shortly, (15) also affords several other substantial benefits.

III. Existence of Financial Equilibrium

1. The Existence Theorem

One of the two main results in this paper is the sine qua non for any model of competitive equilibrium, an existence proof. My basic argument takes a well-known direction (that pioneered by Arrow and Debreu [see, especially, Debreu 3, Chapter 5]). However, it deviates a bit from the usual route by deliberately concealing the role of bond prices in order to avoid difficulties with demand irregularities. For this reason I present it in some (but, I hope, not excessive) detail.

Theorem 1. *Given the maintained assumptions about households' endowments and preferences, there is a financial equilibrium* $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b})$ *corresponding to every return structure* $r \in R^\iota$.

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[4] Obviously, (14) implies (12). On the other side, (7) and (12) imply

$$\sum_c p^{0c} \sum_h (x_h^{0c} - \omega_h^{0c}) = \sum_h p^0 (x_h^0 - \omega_h^0) = -\sum_h \sum_i b_h^i = -\sum_i (\sum_h b_h^i) \leqq 0$$

and

$$\sum_c p^{1sc} \sum_h (x_h^{1sc} - \omega_h^{1sc}) = \sum_h p^{1s} (x_h^{1s} - \omega_h^{1s}) = \sum_h \sum_i r^{is} b_h^i = -\sum_i r^{is} (\sum_h b_h^i) \leqq 0$$

for $s \in S$. These inequalities in turn imply (from the last expression in (15) with $b = -\sum_h b_h$) market clearing for bonds, and hence (again from (12)), market clearing for goods.

Proof of Theorem 1. Pick a bound

$$\sum_h \omega_h << \bar{x} << \infty$$

and define the sets

$$P^\varepsilon = \{\boldsymbol{p} : \boldsymbol{p} \in P \text{ and } \sum_c p^{0c} \geqq \min[\varepsilon, 1/(\sigma+1)], \sum_c p^{1sc} \geqq \min[\varepsilon, 1/(\sigma+1)] \text{ for } s \in S\}$$

for $\varepsilon > 0$, and

$$X = \{\boldsymbol{x} : 0 \leqq x_h \leqq \bar{x} \text{ for } h \in H\}.$$

Now consider the equilibrium model obtained from the original model when the optimal solutions to (7) are replaced by the optimal solutions to the problem

$$\begin{aligned} &\text{maximize} && u_1(x_1') \\ &\text{subject to} && \boldsymbol{p}x_1' \leqq \boldsymbol{p}\omega_1 \\ &\text{and} && 0 \leqq x_1' \leqq \bar{x} \end{aligned} \tag{16}$$

together with the identity $b_1 = -\sum_{h>1} b_h$ for $h = 1$, and by the optimal solutions to the problem

$$\begin{aligned} &\text{maximize} && u_h(x_h') \\ &\text{subject to} && p^0 x_h^{0\prime} + \sum_i b_h^{i\prime} \leqq p^0 \omega_h^0, \\ & && p^{1s} x_h^{1s\prime} \leqq p^{1s}\omega_h^{1s} + \sum_i r^{is} b_h^i, \\ &\text{and} && 0 \leqq x_h' \leqq \bar{x} \end{aligned} \tag{17}$$

for $h > 1$. Then my proof involves establishing the following series of claims.

(i) The set of optimal solutions to (16), say, $x_1(\boldsymbol{p})$, is well-defined (i.e., nonempty and compact-valued), convex-valued and upper semi-continuous for $\boldsymbol{p} \in P$. Moreover, (constrained) consumption demand satisfies (the corresponding weak form of) Walras' law,

$$\boldsymbol{p}x_1 \leqq \boldsymbol{p}\omega_1, \text{ with equality if } x_1 << \bar{x}, \text{ for } x_1 \in x_1(\boldsymbol{p}). \tag{18}$$

Finally, if $x_1 \in x_1(\boldsymbol{p})$ and $x_1 << \bar{x}$, then x_1 is also an optimal solution to the problem

$$\begin{aligned} &\text{maximize} \quad u_1(x_1') \\ &\text{subject to} \quad \boldsymbol{p}x_1' \leqq \boldsymbol{p}\omega_1 \\ &\text{and} \qquad\quad x_1' \geqq 0. \end{aligned} \tag{19}$$

The set of optimal solutions to (17), say, $(x_h(\boldsymbol{p}, r),\, b_h(\boldsymbol{p}, r))$, is equally well-behaved for $(\boldsymbol{p}, r) \in P^{\varepsilon} \times R^{\iota}$, for every $\varepsilon > 0$.

Given my assumptions on ω_h and u_h, these results are well-established for the problems (16) and (19). For the problems (17) and (7), they follow from exactly parallel sorts of reasoning, once it is established that the constraint set in (17) is compact – as I will demonstrate in the Appendix.

Lemma 2a: *The correspondence $(x_h(\mathbf{p}, r), b_h(\boldsymbol{p}, r))$ is well-defined, convex-valued, and upper semi-continuous on $P^{\varepsilon} \times R^{\iota}$, for $\varepsilon > 0$. Moreover, (constrained) consumption-portfolio demand satisfies the various (corresponding weak) forms of Walras' law*

$$p^0 x_h^0 + \sum_i b_h^i \leqq p^0 \omega_h^0,\ \ p^{1s} x_h^{1s} \leqq p^{1s}\omega_h^{1s} + \sum_i r^{is} b_h^i \ \textit{for}\ s \in S,\ \textit{and}\ \boldsymbol{p}x_h \leqq \boldsymbol{p}\omega_h, \tag{20}$$

with equalities if $x_h << \bar{x}$, for $(x_h, b_h) \in (x_h(\boldsymbol{p}, r), b_h(\boldsymbol{p}, r))$.

Finally, if $(x_h, b_h) \in (x_h(\boldsymbol{p}, r), b_h(\boldsymbol{p}, r))$ and $x_h << \bar{x}$, then (x_h, b_h) is also an optimal solution to (7).

(ii) Given $r \in R^{\iota}$, if $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b})$ is a financial equilibrium in this new model (where (7) has been replaced by (16) for $h = 1$ and by (17) for $h > 1$), then it is also a financial equilibrium in the original model.

Materials balance (12) implies that $x_h \leqq \sum_h \omega_h << \bar{x}$ for $h \in H$. So all that needs to be shown is that if x_1 is an optimal solution to (19) and $b_1 = -\sum_{h>1} b_h$, then (x_1, b_1) is an optimal solution to (7) for $h = 1$.

Since $r \in R^{\iota} \subset R$, adding up the budget constraints in (7) for $h = 1$ yields the budget constraint in (19).

$$\begin{aligned} p^0 x_1^{0\prime} + \sum_s p^{1s} x_1^{1s\prime} + \sum_i b_1^{i\prime} &\leqq p^0\omega_1^0 + \sum_s p^{1s}\omega_1^{1s} + \sum_s\sum_i r^{is} b_1^{i}{}' \\ &\leqq p^0\omega_1^0 + \sum_s p^{1s}\omega_1^{1s} + \sum_i (\sum_s r^{is}) b_1^{i}{}' \\ &\leqq p^0\omega_1^0 + \sum_s p^{1s}\omega_1^{1s} + \sum_i b_1^i \end{aligned}$$

or

$$\boldsymbol{p} x_1^{'} \leqq \boldsymbol{p}\omega_1.$$

Hence, feasible solutions to (7) for $h = 1$ are also feasible solutions to (19). Furthermore, Walras' laws (18) and (20) together with materials balance (12) yield the equalities

$$p^0(x_1^0 - \omega_1^0) = -\sum_{h>1} p^0(x_h^0 - \omega_h^0) = \sum_{h>1}\sum_i b_h^i = \sum_i(\sum_{h>1} b_h^i) = -\sum_i b_1^i$$

and

$$p^{1s}(x_1^{1s} - \omega_1^{1s}) = -\sum_{h>1} p^{1s}(x_h^{1s} - \omega_h^{1s}) = -\sum_{h>1}\sum_i r^{is} b_h^i = -\sum_i r^{is}(\sum_{h>1} b_h^i) = \sum_i r^{is} b_h^i$$

for $s \in S$. Hence, (x_1, b_1) is a feasible solution, and thus necessarily also an optimal solution to (7) for $h = 1$.

(iii) Given $r \in R^\iota$, the correspondence $\phi^\varepsilon : P^\varepsilon \times X \rightrightarrows P^\varepsilon \times X$ defined by

$$\boldsymbol{p} \to \{\boldsymbol{x}' : x_1' \in x_1(\boldsymbol{p}) \text{ and } x_h' \in x_h(\boldsymbol{p}, r) \text{ for } h > 1\} \tag{21}$$

and

$$\boldsymbol{x} \to \{\boldsymbol{p}' : \boldsymbol{p}' \text{ is an optimal solution to the problem } \max_{\boldsymbol{p}'' \in P^\varepsilon} \boldsymbol{p}'' \sum_h (x_h - \omega_h)\} \tag{22}$$

has a fixed point $(\boldsymbol{p}^*, \boldsymbol{x}^*) \in \phi^\varepsilon(\boldsymbol{p}^*, \boldsymbol{x}^*)$.

This result follows from a standard argument based on the regularity properties asserted in (i) above, and employing Kakutani's Theorem.

(iv) Given $r \in R^\iota$, if ε is sufficiently small, then a fixed point $(\boldsymbol{p}^*, \boldsymbol{x}^*) \in \phi^\varepsilon(\boldsymbol{p}^*, \boldsymbol{x}^*)$ yields a financial equilibrium $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b})$, where $\boldsymbol{p} = \boldsymbol{p}^*$, $\boldsymbol{x} = \boldsymbol{x}^*$ and (x_h, b_h) is an optimal solution to (7) with $\boldsymbol{p} = \boldsymbol{p}^*$ for $h \in H$.

This result too follows from a common sort of argument, but now based, in particular, on Walras' laws (18) and (20) and the weak monotonicity assumption on u_1. I therefore forego a detailed demonstration (in a small, but well intentioned attempt to conserve length).■

As I have already mentioned in the introduction, the principal virtue of this theorem is that it replaces ad hoc bounds on trades in (fixed combinations of) contingent commodities (as in Hart's argument, pursuing Radner's approach) by economically meaningful restrictions on (permissible configurations of) prices of and returns to contingent claims (culminating in the requirement described by (15)). One might legitimately raise the objection that, while my formulation may encompass some forms of financial instruments (e.g., money or fixed-term bonds), others have returns which are better conceived as being determined endogenously (e.g., stocks or commodity futures), and are therefore not covered by the model. I will return to this question in Section V, where I will outline, inter alia, how Theorem 1 can be utilized to analyze existence of financial equilibrium even under such a richer specification.

2. A "Stability" Result

The trick employed in establishing Theorem 1 easily enables me to derive an interesting result concerning the relation of financial equilibrium to rates of return. (I plan to illustrate its usefulness in forthcoming joint work with Martin Hellwig, in which we investigate the structure of the set of financial equilibria, especially as it pertains to the effects of extrinsic uncertainty in the present model). Let

$$E(r) = \{(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b}) : \text{ given } r, (\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b}) \text{ is a financial equilibrium}\}$$

and

$$\tilde{E}(r) = \{(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b}) : (\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b}) \in E(r) \text{ and } x_1 \text{ is an optimal solution to (19)}\}$$

for $r \in R^{\iota}$.

Theorem 2. *$\tilde{E}(r)$ is nonempty and upper semi-continuous for $r \in R^{\iota}$.*

Proof of Theorem 2. In the course of proving Theorem 1 I have already shown that $\tilde{E}(r) \neq \emptyset$ for $r \in R^{\iota}$. So consider a sequence $\{r^{\upsilon}\}$ such that $r^{\upsilon} \in R^{\iota}$ for $\upsilon \geqq 1$ and $\lim_{\upsilon \to \infty} r^{\upsilon} = r \in R^{\iota}$ with corresponding sequence $\{(\boldsymbol{p}^{\upsilon}, \boldsymbol{x}^{\upsilon}, \boldsymbol{b}^{\upsilon})\}$ such that $(\boldsymbol{p}^{\upsilon}, \boldsymbol{x}^{\upsilon}, \boldsymbol{b}^{\upsilon}) \in \tilde{E}(r^{\upsilon})$ for $\upsilon \geqq 1$ and $\lim_{\upsilon \to \infty} (\boldsymbol{p}^{\upsilon}, \boldsymbol{x}^{\upsilon}, \boldsymbol{b}^{\upsilon}) = (\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{b})$. I only need to demonstrate that $\boldsymbol{p} \in P$, that x is nonnegative and satisfies (12) (so that, a fortiori, $x_h << \bar{x}$ for $h \in H$), and that x_1 is an optimal solution to (9), while (x_h, b_h) is an optimal solution to (7) for $h > 1$ (so that, a fortiori, $(x_1, -\sum_{h>1} b_h)$ is also an optimal solution to (7) for $h = 1$).

Since $\boldsymbol{p}^\upsilon \in P$ for $\upsilon \geqq 1$, obviously $\boldsymbol{p} \in P$. Also, since $\boldsymbol{x}^\upsilon$ is nonnegative and satisfies (12) for $\upsilon \geqq 1$, obviously $\boldsymbol{x}$ is nonnegative and satisfies (12). Hence, since $x_1(\boldsymbol{p})$ is upper semi-continuous on $\boldsymbol{p} \in P$, x_1 is an optimal solution to (19). Hence, by virtue of the weak monotonicity assumption on u_1, $\boldsymbol{p} \in P^\varepsilon$ for some $0 < \varepsilon < 1/(1+\sigma)$. Hence, since $(x_h(\boldsymbol{p}, r), b_h(\boldsymbol{p}, r))$ is upper semi-continuous on, for example, $P^{\varepsilon/2} \times R^\iota, (x_h, b_h)$ is an optimal solution to (7) for $h > 1$, and the proof is complete.■

IV. **Optimality of Financial Equilibrium**

It is now widely understood that, when the market structure is incomplete, competitive allocation is, at best, Pareto optimal among a relatively small subset of feasible allocations. (See, especially, the basic papers I cited earlier, Hart [7], Grossman [6] and Gale [5].) In my formulation, this limitation can be seen very clearly by considering the following sorts of **price** transaction-constrained perturbations (in contrast to the **quantity** transaction-constrained perturbations which were natural in previous analyses of Radner-like models).

Given a particular feasible allocation, say, $\bar{\boldsymbol{x}}$, let

$$\bar{X}^{1s} = \{x^{1s} : \text{ there are } x_h^{1s} \geqq 0 \text{ for } h \in H \text{ such that } u_h(\bar{x}_{h,}^0, \ldots, x_{h,}^{1s}, \ldots, \bar{x}_h^{1\sigma}) \geqq u_h(\bar{x}_h) \text{ for } h \in H \text{ and } x^{1s} \geqq \sum_h x_h^{1s}\} \text{ for } s \in S,$$

$$\bar{P}^{1s} = \{p^{1s} : p^{1s} > 0 \text{ and } p^{1s}x^{1s} \geqq p^{1s}\omega^{1s} \text{ for } x^{1s} \in \bar{X}^{1s}\} \text{ for } s \in S, \text{ and}$$

$$\bar{P}^1 = \times_s \bar{P}^{1s},$$

where it will be convenient to let $\omega = (\omega^0, \omega^{11}, \ldots, \omega^{1\sigma}) \gg 0$ represent the total **resources** available to the economy (so that, with private ownership, $\omega = \sum_h \omega_h$). Also, given $r \in R^\iota$ (and still given $\bar{\boldsymbol{x}}$), define a **type 0 perturbation** to be a goods allocation $\boldsymbol{x}$ with the property that there are bond holdings $\boldsymbol{b}$ such that

$$x_h \geqq 0 \text{ and } p^{1s}x_h^{1s} \leqq p^{1s}\bar{x}_h^{1s} + \sum_i r^{is} b_h^i \text{ for } s \in S, \text{ for } h \in H,$$
$$\text{while } \sum_h x_h^0 \leqq \omega^0 \text{ and } \sum_i \sum_h b_h^i \leqq 0 \tag{23}$$

for some $p^1 \in \bar{P}^1$. Finally, define, for each state s, a **type 1-s perturbation** to be simply a goods allocation $\boldsymbol{x}$ with the property that

$$x_h^{1s} \geqq 0, x_h^0 = \bar{x}_h^0 \text{ and } x_h^{1s'} = \bar{x}_h^{1s'} \text{ for } s' \neq s, \text{ for } h \in H, \\ \text{while } \sum_h x_h^{1s} \leqq \omega^{1s}. \tag{24}$$

Thus, for example, a price vector $p^{1s} \in \bar{P}^{1s}$ represents a support to the set $\bar{X}^{1s}$ (the potential resource bundles required in state s in order to improve upon just the consumption bundles received if state s occurs) at the point $\omega^{1s} \geqq \sum_h \bar{x}_h^{1s}$ (the actual resource bundle available in state s), while a type 1-s perturbation $\boldsymbol{x}$ represents a (feasible) allocation which differs from $\bar{\boldsymbol{x}}$ only in the consumption bundles which will be received if state s occurs.

The point of these definitions is that they allow a straightforward characterization of the optimality properties exhibited by financial equilibria. In particular, they yield an immediate analogue of the first basic theorem of welfare.

Theorem 3. *Given $r \in R^\iota$, suppose that $(\bar{\boldsymbol{p}}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{b}})$ is a financial equilibrium. Then $\bar{p}^1 \in \bar{P}^1$, and $\bar{x}$ is Pareto optimal among both type 0 and type 1-s perturbations (for $p^1 = \bar{p}^1$ and $s \in S$, respectively).*

Furthermore, and perhaps even more striking, they also yield a partial converse, the corresponding analogue of the second basic theorem of welfare.

Theorem 4. *Suppose $\bar{\boldsymbol{x}} \gg \boldsymbol{0}$, $\bar{\boldsymbol{x}}$ is feasible allocation, and $\bar{\boldsymbol{x}}$ is Pareto optimal among both type 0 and type 1-s perturbations (for some $r \in R^\iota$ and $s \in S$, respectively). Then there is a $\bar{\boldsymbol{p}} \in P$ such that $(\bar{\boldsymbol{p}}, \bar{\boldsymbol{x}}, \boldsymbol{0})$ is a financial equilibrium given initial endowments $\omega_h = \bar{x}_h$ for $h \in H$.*

Remark. For this optimality characterization, it makes no fundamental difference whether rates of return are linearly independent, normalized, or even nonnegative. I have specified $r \in R^\iota$ here merely because this again involves no real loss of generality (and also maintains consistency with my existence theorem).

Proof of Theorem 3. The argument is almost identical to Grossman's. Consider each of the asserted conclusions in turn.

(i) $p^1 \in \bar{P}^1$.

Suppose otherwise, i.e., $\bar{p}^{1s} \notin \bar{P}^{1s}$ for some s. Then there are $x_h^{1s} \geqq 0$ for $h \in H$ such that $u_h(\bar{x}_h^0, \ldots, x_h^{1s}, \ldots, \bar{x}_h^{1\sigma}) \geqq u_h(\bar{x}_h)$ for $h \in H$ and (using goods market clearing in (14) and Walras' law (20))

$$\sum_h \bar{p}^{1s} x_h^{1s} = \bar{p}^{1s} \sum_h x_h^{1s} < \bar{p}^{1s} \omega^{1s} = \bar{p}^{1s} \sum_h \bar{x}_h^{1s} = \sum_h (\bar{p}^{1s} \omega_h^{1s} + \sum_i r^{is} \bar{b}_h^i),$$

and thus $x_h^{1s} \geqq 0$ for some h such that $u_h(\bar{x}_h^0, \ldots, x_h^{1s}, \ldots, \bar{x}_h^{1\sigma}) \geqq u_h(\bar{x}_h)$ and

$$\bar{p}^{1s} x_h^{1s} < \bar{p}^{1s} \omega_h^{1s} + \sum_i r^{is} \bar{b}_h^i,$$

and thus (now using the weak monotonicity assumption on u_h) $x_h^{1s\prime} \gg x_h^{1s}$ for some h such that $u_h(\bar{x}_{h,}^0 \ldots, x_h^{1s\prime}, \ldots, \bar{x}_h^{1\sigma}) > u_h(\bar{x}_h)$ and

$$\bar{p}^{1s} x_h^{1s\prime} \leqq \bar{p}^{1s} \omega_h^{1s} + \sum_i r^{is} \bar{b}_h^i.$$

But the last implication contradicts the hypothesis that $(\bar{x}_h, \bar{b}_h)$ is an optimal solution to (7).

(ii) $\bar{\boldsymbol{x}}$ is Pareto optimal among type 0 perturbations for $p^1 = \bar{p}^1$.

Suppose otherwise, i.e., there is a type 0 perturbation $\boldsymbol{x}'$ satisfying (13). Then (using Walras' law (20))

$$\bar{p}^{1s} x_h^{1s\prime} \leqq \bar{p}^{1s} \bar{x}_h^{1s} + \sum_i r^{is} b_h^{i\prime} = \bar{p}^{1s} \omega_h^{1s} + \sum_i r^{is} (b_h^{i\prime} + \bar{b}_h^i) \text{ for } s \in S,$$

for $h \in H$, so that (using household maximization (7)) it must be the case that

$$\bar{p}^0 x_h^{0\prime} + \sum_i (b_h^{i\prime} + \bar{b}_h^i) \geqq \bar{p}^0 \omega_h^0 \text{ for } h \in H, \text{ with strict inequality for some } h. \tag{25}$$

But (25) (using bond market clearing in(14)) leads to the following contradiction of my initial supposition:

$$\begin{aligned} \sum_h (\bar{p}^0 x_h^{0\prime} + \sum_i (b_h^{i\prime} + \bar{b}_h^i)) &= \sum_h \bar{p}^0 x_h^{0\prime} + \sum_i (\sum_h b_h^{i\prime} + \sum_h \bar{b}_h^i) \\ &= \sum_h \bar{p}^0 x_h^{0\prime} + \sum_i \sum_h b_h^{i\prime} > \sum_h p^0 \omega_h^0, \end{aligned}$$

that is,

$$\sum_h x_h^{0c\prime} > \omega^{0c} \text{ for some } c \text{ or } \sum_i \sum_h b_h^{i\prime} > 0.$$

(iii) $\bar{\boldsymbol{x}}$ is Pareto optimal among type 1-s perturbations for $s \in S$.

In this case an argument similar to that in (ii), but now based on the revealed preference inequality

$$\bar{p}^{1s} x_h^{1s\prime} \geqq \bar{p}^{1s} \omega_h^{1s} + \sum_i r^{is} \bar{b}^i \text{ for } h \in H, \text{ with strict inequality for some } h,$$

leads to the following contradiction of the opposite supposition:

$$\sum_h x_h^{1sc\prime} > \omega^{1sc} \text{ for some } c \text{ and } s. \blacksquare$$

Proof of Theorem 4. The concluding step in this argument requires the following version of the familiar relationship between expenditure minimization and utility maximization; the essential idea of its proof is also briefly remarked in the Appendix.

Lemma 2b. *If* (x_h, b_h) *is an optimal solution to the problem*

$$\begin{aligned} &\textit{minimize} && p^0 x_h^{0\prime} + \sum_i b_h^{i\prime} \\ &\textit{subject to} && u_h(x_h^{\prime}) \geqq u_h(x_h), \\ & && p^{1s} x_h^{1s\prime} \leqq p^{1s} \omega_h^{1s} + \sum_i r^{is} b_h^{i\prime} \text{ for } s \in S, \\ &\textit{and} && x_h^{\prime} \geqq 0 \end{aligned} \tag{26}$$

and $p^0 x_h^0 + \sum_i b_h^i = p^0 \omega_h^0 > 0$, *then* (x_h, b_h) *is also an optimal solution to (7).*

The argument itself proceeds as follows:

Consider first type 1-s perturbations (24). The hypothesis that $\bar{\boldsymbol{x}}$ is Pareto optimal in this class of perturbations together with the weak monotonicity assumption on u_h (in terms of x_h^{1s}) implies that ω^{1s} is a boundary point of the set $\bar{X}^{1s}$ for $s \in S$. Moreover, it is easily seen that $\bar{X}^{1s}$ is also "disposable" (i.e., if $x^{1s} \in \bar{X}^{1s}$ and $x^{1s\prime} \geqq x^{1s}$, then $x^{1s\prime} \in \bar{X}^{1s}$) and convex (using the quasi-concavity assumption on u_h) for $s \in S$. Hence, by the usual application of the supporting hyperplane theorem, the set of supports $\bar{P}^{1s}$ is nonempty for $s \in S$, and hence the product of these sets $\bar{P}^1$ is nonempty as well.

Consider next type 0 perturbations (23), given some $p^1 \in \bar{P}^1$. Let $z = (x^0, z^b)$ and $q = (q^0, q^b)$ represent $(\kappa + 1)$ – dimensional vectors, and focus on the sets of such vectors defined by

$$\bar{Z}(p^1) = \{z: \text{ there is } (\boldsymbol{x}, \boldsymbol{b}) \text{ such that } x_h \geqq 0, p^{1s}x_h^{1s} \leqq p^{1s}\bar{x}_h^{1s} + \sum_i r^{is}b_h^i$$

$$\text{for } s \in S, \text{ and } u_h(x_h) \geqq u_h(\bar{x}_h) \text{ for } h \in H \text{ and } z \geqq (\sum_h x_h^0, \sum_i \sum_h b_h^i)\}$$

and

$$\bar{Q}(p^1) = \{q : q > 0 \text{ and } qz \geqq p^0\omega^0 \text{ for } z \in \bar{Z}(p^1)\}.$$

Then the hypothesis that $\bar{\boldsymbol{x}}$ is Pareto optimal among type 0 perturbations together with the weak monotonicity assumption on u_h (now in terms of x_h^0) implies that $(\omega^0, 0)$ is a boundary point of $\bar{Z}(p^1)$. Moreover, once again it is easily seen that this set is "disposable" and convex. Hence, as previously, the set of supports $\bar{Q}(p^1)$ must be nonempty.

Since $\bar{\boldsymbol{x}}$ is hypothesized to be a feasible allocation, materials balance (12) is satisfied. So all that remains to be proven is that, given the hypothesis that $\bar{\boldsymbol{x}} \gg \mathbf{0}$, and given some $(p^0, q^b) \in \bar{Q}(p^1)$ with $p^1 \in \bar{P}^1$, one can construct $\bar{\boldsymbol{p}} \in P$ such that $(\bar{x}_h, 0)$ is an optimal solution to (7) with $\omega_h = \bar{x}_h$ and $\boldsymbol{p} = \bar{\boldsymbol{p}}$ for $h \in H$. Arguing somewhat informally (in order to avoid too many "ε's" and "δ's"), this is established by the following series of results.

(i) $q^b > 0$ (so that, without loss of generality, $q^b = 1$).

Suppose otherwise, i.e., $p^0 > 0$ and $q^b = 0$. Then (using $r \in R^\iota$ and the continuity and weak monotonicity assumptions on u_h) we know that, for any h, increasing b_h^i for some i permits increasing x_h^{1s} for some s, and thus decreasing x_h^0 while (at least) maintaining u_h. In other words, there is $z \in \bar{Z}(p^1)$ such that

$$x^0 = \sum_h x_h^0 << \omega^0,$$

so that if $q^b = 0$, then

$$p^0x^0 + q^bz^b = p^0x^0 < p^0\omega^0 = p^0\omega^0 + q^b0,$$

which contradicts $(p^0, q^b) \in \bar{Q}(p^1)$.

(ii) $p^0 > 0$.

Suppose otherwise, i.e., $p^0 = 0$ and $q^b = 1$. Then since by derivation we have $p^{1s} > 0$ for $s \in S$, we know that, for any h, increasing x_h^0 permits decreasing x_h^{1s} for $s \in S$, and thus decreasing b_h^i for some i while again (at least) maintaining u_h. In other words, there is $z \in \bar{Z}(p^1)$ such that

$$z^b = \sum_i \sum_h b_h^i < 0,$$

so that if $p^0 = 0$, then

$$p^0 x^0 + z^b = z^b < 0 = p^0 \omega^0 + 0,$$

which contradicts $(p^0, 1) \in \bar{Q}(p^1)$.

(iii) $(\bar{x}_h, 0)$ is an optimal solution to (26) with $x_h = \bar{x}_h$, and therefore, by Lemma 2b, also an optimal solution to (7) for $h \in H$.

This result follows immediately from the facts established above ($p^0 > 0$ and $q^b = 1$) upon considering the particular $z \in \bar{Z}(p)$ generated by perturbating just (x_h, b_h), for $h \in H$.

The argument is then completed simply by choosing

$$\bar{\boldsymbol{p}} = (p^0, p^1) / \sum_c (p^{0c} + \sum_s p^{1sc}). \blacksquare$$

I emphasize once again that the central thrust of this welfare characterization is – a point equally well stressed by Hart, Grossman and Gale – that incomplete markets inflict the invisible hand with almost terminal paralysis – contrary to plausible conjecturing based on Diamond's seminal example [4]. In fact, given my formulation in terms of financial markets, one can even construct examples in which there is only a single good and yet financial equilibria can be Pareto ranked; I leave this as an exercise for the interested reader.

V. **Extensions**

My basic approach is avowedly Walrasian: Households have full information (perfect foresight) and behave competitively (take prices as given no matter how large their contemplated transactions). Even within these confines, however, my results can be almost trivially extended in several important directions.

1. **Multiple Periods**

Introducing a (finite) number of future periods, say, τ, presents no fundamental analytical difficulty, though (as usual) it does entail horrendous notational complexity. The only point I would emphasize about this particular extension is that, in general, the model can still be cast in terms of one-period bonds. This follows from the observation that, if there are active resale markets, then (for example) one τ-period bond with prices ψ^{t-1s} for $s \in S^{t-1}$ and returns P^{ts} for $s \in S^t$, for $1 \leqq t \leqq \tau$, must be equivalent to a sequence of τ one-period bonds with the same prices, and returns $P^{ts} + \psi^{ts}$ for $s \in S^t$, $1 \leqq t < \tau$ and P^{ts} for $s \in S^t, t = \tau$. (Here, of course, S^t is an element of a partition $S(t)$ of all possible states of the world S, and so on, *à la mode de* Debreu [3, Chapter 7].)

2. **Structural Interdependence**

To my mind, one of the most fascinating outcomes of the Arrow-Debreu-McKenzie development of the solution to the equilibrium existence problem was the explicit recognition of the singular importance of some minimal commonality between households. This specific aspect of my model deserves further serious study (requiring subtle analysis of the interplay between endowments, preferences and financial opportunities). Without undertaking this intricate task myself, I can still say a bit more about weakening my strong assumption on endowments.

Let

$$\begin{aligned} S^+(r) &= \{s : s \in S \text{ and } \sum_i r^{is} > 0\} \\ &= \text{ the subset of states in which some return is positive} \end{aligned} \tag{27}$$

for $r \in R^\iota$. Then, careful scrutiny of the proofs of Theorems 1 and 4 (taking into account Lemma A2 in the Appendix) reveals that, in these arguments, strict positivity (3) could be replaced by the requirement that, say,

$$\omega_1 \gg 0$$

and

$$\omega_h^0 \gg 0 \text{ and } \omega_h^{1s} \gg 0 \text{ for } s \notin S^+(r) \tag{3'}$$

for $h > 1$. Presumably, (3′) and (4) could be further weakened to some "irreducibility" condition (guaranteeing appropriately positive incomes, in the manner of McKenzie [8]), but this remains to be seen.

3. **Restricted Participation**

Another extension of my analysis, and one which is very significant from an interpretive viewpoint – but also not too difficult from a mathematical viewpoint – is the imposition of institutional restrictions on trading activity in the bond markets. The broadest formulation of such restricted participation (which is, at the same time, relatively easy to handle; see, in particular, the second subsection in the Appendix) is the following: Assume that, in addition to budget constraints, households face the financial constraints $b_h \in B_h \subset \mathbb{R}^\iota$ for $h \in H$, where, say, (i) B_h is a closed, convex set containing zero and (ii) $-\sum_{h>1} B_h \subset B_1$. (Obviously, my present heavy reliance on overall arbitrage arguments only makes sense when, in fact, $B_1 = \mathbb{R}^\iota$.) Thus, for instance, for $h > 1$, Mr. h might not be able to buy bond 1 at all ($b_h \in B_h$ implies $b_h^1 = 0$), or might be able to buy some subset of bonds $I^m \subset I$ only as a mutual fund ($b_h \in B_h$ implies $b^i = m^i b^m$ with $m^i > 0$ for $i, m \in I^m$), or might have an (absolute) upper bound on short-selling ($b_h \in B_h$ implies $\sum_i \min[0, b_h^i] \geqq d$ with $-\infty < d \leqq 0$).

Such restrictions (and many more) have been extensively investigated in the finance literature, of course. But uncovering their implications within this particular model of a financial equilibrium seems to me a problem well worth deeper analysis in its own right.

4. **Endogenous Returns**

In "reality", the returns to many financial instruments (stocks, commodity futures, option contracts,...) depend on tomorrow's market conditions. Within my framework, this "fact" can be formalized by assuming that the rates of return to some subset of bonds, say, $I' = \{1, 2, ..., \iota'\}$ with $\iota' \leqq \iota$, are functions of tomorrow's spot prices or, even more generally, all spot prices, say, $r^i : P \to \{\rho : \rho \in \mathbb{R}^\sigma, \rho \geqq 0 \text{ and } \sum_s \rho^s = 1\}$ for $i \in I'$. It is relatively straightforward to verify that almost nothing in the foregoing analysis (of existence and optimality) is substantially affected by this generalization – **provided that rates of return $r = (r^1(\boldsymbol{p}), ..., r^{\iota'}(\boldsymbol{p}), ..., r^\iota)$ are continuous and satisfy the dimensionality condition**

$$r \in R^\iota \tag{28}$$

for $\boldsymbol{p} \in P$. The only real difficulty is that many "natural" specifications inherently lead to violations of this last requirement. (On this point I strongly recommend looking at Hart's careful analysis of a specific example in [7].) However, such an unfortunate circumstance needn't raise an insurmountable obstacle to obtaining sensible results, at least not when r can be closely approximated by a continuous function, say, r^δ with $\|r^\delta - r\| < \delta$ for $\boldsymbol{p} \in P$, for $\delta > 0$, for which (28) is satisfied.[5]

[5] My belated appreciation of the crucial importance of the distinction between R and R^ι in this context is due largely to Martin Hellwig. An alternative approach for avoiding the difficulty is to guarantee that financial equilibrium occurs (or "usually" occurs) at spot prices $\boldsymbol{p} \in P$ such that $r(\boldsymbol{p}) \in R^\iota$. See, in particular, the very interesting initial analysis developed, following this path, by McManus [9] (for the polar case $\iota = \sigma$).

I can clearly illustrate these ideas in terms of the leading example where there are just two bonds, the first essentially a futures contract on good 1, the second simply (inside) money.

Suppose that $\iota = 2$, and that returns are described by

$$P^{1s} = 1 + p^{1s1} - f^1$$

and

$$P^{2s} = 1$$

for $s \in S$, where the nonnegative number $f^1 \geqq 0$ is to be determined (as part of the financial equilibrium). Then, these returns can be converted to rates of return (conforming with my maintained normalization $r \in R$) by choosing $\psi^s > 0$ for $s \in S$ such that $\sum_s \psi^s = 1$ and considering

$$f^1 = \sum_{s'} \psi^{s'} p^{1s'}$$

and

$$r^{is} = \psi^s P^{is}$$

for $s \in S$, $i = 1, 2$. (To see that, given this specification, the first bond is indeed "essentially a futures contract on good 1," notice that it is equivalent to the mutual fund consisting of one unit of money and one unit of a financial instrument which costs 0 dollars today and returns, given $\mathbf{p} \in P, p^{1s} - f^1 = p^{1s} - \sum_{s'} \psi^{s'} p^{1s'}$ in state s tomorrow. Thus, in effect, f^1 is the contract price for delivery of one unit of good 1 tomorrow, in any state. For an interesting elaboration of a general model built around such futures contracts, see Gale [5]; also, see my related analysis of financial instruments which bear zero price in the first subsection of the Appendix.)

Given such a return structure, obviously r is continuous, while $r \in R^{\iota}$ if and only if $p^{1s'1} \neq p^{1s''1}$ for some s', $s'' \in S$ (since otherwise $r^1 = r^2$). In order to repair this latter defect, it is enough, for instance, simply to perturb rates of return on the first bond slightly near the offending prices. Thus, r^1 might be replaced by

$$r^{1\delta}(\boldsymbol{p}) = \begin{cases} r^1(\boldsymbol{p}), \text{ for } \boldsymbol{p} \in P, \|r^1(\boldsymbol{p}) - r^2\| \geqq \delta/2 \\ \frac{\|r^1(\boldsymbol{p}) - r^2\|}{\delta/2} r^1(\boldsymbol{p}) + (1 - \frac{\|r^1(\boldsymbol{p}) - r^2\|}{\delta/2}) r^{1\prime}, \text{ otherwise,} \end{cases}$$

where $r^{1\prime} \geqq 0, \sum_s r^{1s\prime} = 1$ and $0 < \|r^{1\prime} - r^2\| < \delta/2$, and $\delta > 0$.

I believe that a similar procedure could be employed more generally, but once again, this remains to be seen.

Appendix

My main purpose here is to justify Lemmas 1 and 2. Since in the final section of the paper I appeal to somewhat stronger versions of both these results, I will state and briefly sketch proofs for generalizations from which each of the two lemmas is an immediate corollary.

A1. General Characterization of "No Fast Bucks"

In fact, positivity of bond returns (and therefore of bond prices) plays no essential part in characterizing the absence of arbitrage possibilities on financial markets. So instead of (6), now consider the general case in which, for some $\iota' \leqq \iota$,

$$\psi^i \neq 0 \text{ for } i \in I' = \{1, 2, \ldots, \iota'\}, = 0 \text{ otherwise.}$$

Then the argument which before led to (9) now leads to the general condition

$$\begin{aligned}\{(b', \beta') : \quad & b' \in \mathbb{R}^{\iota'}, \beta' \in \mathbb{R}^{\iota-\iota'} \text{ and} \\ & (-\sum_{i \in I'} b^{i\prime}, \sum_{i \in I'} r^i b^{i\prime} + \sum_{i \in I \setminus I'} P^i \beta^{i\prime}) > 0\} = \emptyset,\end{aligned} \tag{A1}$$

where, as in the text,

$$b^i \equiv \psi^i \beta^i \text{ and } r^i \equiv P^i / \psi^i \text{ for } i \in I'.$$

Furthermore, based on (A1), we also have a general version of Lemma 1 (which reduces to the same thing for the special case in which (2) is satisfied, and thus necessarily $\iota' = \iota$).

Lemma A1. *(A1) obtains if and only if there are $\psi^s > 0$ for $s \in S$ such that*

$$\sum_s \psi^s r^{is} = 1 \textit{ for } i \in I' \tag{A2}$$

and

$$\sum_s \psi^s P^{is} = 0 \textit{ for } i \in I \setminus I'. \tag{A3}$$

Proof of Lemma A1. (sufficiency) Suppose that the conclusion is false, i.e., that

$$(-\sum_{i \in I'} b^{i\prime}, \sum_{i \in I'} r^i b^{i\prime} + \sum_{i \in I \setminus I'} P^i \beta^{i\prime}) > 0$$

for some (b', β'). Then, in view of (A2) and (A3), we immediately get the contradiction that

$$0 \left\{ \begin{array}{c} < \\ = \end{array} \right\} \quad \sum_s \psi^s (\sum_{i \in I'} r^{is} b^{i\prime} + \sum_{i \in I \setminus I'} P^{is} \beta^{i\prime}) =$$

$$\sum_{i \in I'} (\sum_s \psi^s r^{is}) b^{i\prime} + \sum_{i \in I - I'} (\sum_s \psi^s P^{is}) \beta^{i\prime} = \sum_{i \in I'} b^{i\prime} \left\{ \begin{array}{c} \leqq \\ < \end{array} \right\} 0.$$

(necessity) Consider the closed, convex cone

$$W = \{w : \text{ there are } b' \in \mathbb{R}^{\iota'} \text{ and } \beta' \in \mathbb{R}^{\iota - \iota'} \text{ such that } w = (-\sum_{i \in I'} b^{i\prime}, \sum_{i \in I'} r^i b^{i\prime} + \sum_{i \in I - I'} P^i \beta^{i\prime})\},$$

together with its dual

$$\Psi = \{\psi : \psi \in \mathbb{R}^{\sigma+1} \text{ and } w\psi \leqq 0 \text{ for } w \in W\}.$$

Then the hypothesis (A1) is equivalent to the property that

$$W \cap \mathbb{R}^{\sigma+1}_{+} = \{0\}, \tag{A4}$$

while the relationship between a cone and its dual entails that

$$W = \{w : w \in \mathbb{R}^{\sigma+1} \text{ and } w\psi \leqq 0 \text{ for } \psi \in \Psi\}. \tag{A5}$$

It can therefore be easily seen that

(i) There is $\psi \in \Psi$ such that $\psi \gg 0$, i.e., there are $\psi^0 > 0$ and $\psi^s > 0$ for $s \in S$ such that

$$-\psi^0 \sum_{i \in I'} b^{i\prime} + \sum_s \psi^s (\sum_{i \in I'} r^{is} b^{i\prime} + \sum_{i \in I \setminus I'} P^{is} \beta^{i\prime}) \leqq 0 \text{ for } (b', \beta') \in \mathbb{R}^{\iota}. \tag{A6}$$

Suppose otherwise, i.e., $\Psi \cap \mathbb{R}^{\sigma+1}_{++} = \emptyset$. Then, since the dual Ψ is also convex, by the separating hyperplane theorem there is $w \neq 0$ such that

$$w\psi \leqq w\psi' \text{ for } \psi \in \Psi, \psi' \in \mathbb{R}^{\sigma+1}_{+},$$

or

$$w > 0 \text{ such that } w\psi \leqq 0 \text{ for } \psi \in \Psi,$$

or, from (A5), $w \in W$ such that $w > 0$, which contradicts (A4).

Moreover, it can therefore also be easily seen that

(ii) By normalizing ψ so that $\psi^0 = 1$, (A2) and (A3) are satisfied.

On the one hand, taking

$$b^{i\prime} = \begin{cases} \pm 1 \text{ for } i = i' \\ 0 \text{ otherwise} \end{cases}$$

for arbitrary $i' \in I'$ and $\beta' = 0$ in (A6), we have

$$\pm \sum_s \psi^s r^{i's} \leqq \pm 1 \text{ for } i' \in I',$$

which yields (A2).

On the other hand, taking $b' = 0$ and

$$\beta^{i\prime} = \begin{cases} \pm 1 \text{ for } i = i' \\ 0 \text{ otherwise} \end{cases}$$

for arbitrary $i' \in I \diagdown I'$ in (A6), we have

$$\pm \sum_s \psi^s P^{i's} \leqq 0 \text{ for } i' \in I \diagdown I',$$

which yields (A3), and the proof is complete.■

I should note that the essence of the foregoing argument (an application of the Farkas-Minkowski Lemma) is well-known, and can be found in many standard references on convex analysis. I chose to elaborate rather than reference it for a simple reason; spelling it out requires about the same space as translating notation and then tailoring results from a primary source.

It is also worth remarking explicitly that Lemma A1 permits extending my entire analysis of existence and optimality to encompass the general situation where

$$b^i = \begin{cases} \psi^i \beta^i \text{ for } i \in I' \\ \beta^i \text{ otherwise} \end{cases}$$

and

$$R = \{r : \sum_s r^{is} = 1 \text{ for } i \in I', = 0 \text{ otherwise}\}.$$

Thus, for instance (taking into account the brief discussion of endogenous returns at the end of Section V), bond $\iota' + 1$ might represent an ordinary futures contract on good 1 (with "price" $\psi^{\iota'+1} = 0$ and returns $P^{\iota'+1s} = p^{1s1} - f^1$ for $s \in S$, where again f^1 is the delivery price in the contract), while bond 1 might represent a futures contract on good 1 requiring partial payment in advance (with "price" $\psi^1 = f^{01}$ and returns $P^{1s} = p^{1s1} - f^{11}$ for $s \in S$, where now f^{01} is the down payment in the contract).

A2. Regularity Properties of Consumption-Portfolio Demand

Rather than focusing specifically on spot prices which are nontrivial market-by-market, switch attention to the general $\kappa(\sigma+1)$ – dimensional unit simplex P itself. Also, instead of considering unrestricted portfolio choice, introduce the financial constraints $b_h \in B_h \subset \mathbb{R}^\iota$, where, as in the text, B_h is a closed, convex set containing zero, for $h \in H$. Finally, parallel with the latter generalization, now let

$$S_h^+(r) = \{s : s \in S \text{ and there is } b \in B_h$$
$$\text{such that } \sum_i r^{is'} b^i > 0 \text{ for } s' = s, \geqq 0 \text{ otherwise}\}$$

for $r \in R^\iota$, for $h \in H$. Then we have the following general version of Lemmas 2a and 2b (where, to simplify notation, I write p for $\boldsymbol{p}$ and omit the household subscript).

Lemma A2. *The optimal solutions to the problem*

$$\begin{aligned} &\textit{maximize} && u(x') \\ &\textit{subject to} && p^0 x^{0\prime} + \sum_i b^{i\prime} \leqq p^0 \omega^0, \\ & && p^{1s} x^{1s\prime} \leqq p^{1s}\omega^{1s} + \sum_i r^{is} b^{i\prime} \textit{ for } s \in S, \\ & && 0 \leqq x' \leqq \bar{x}, \\ &\textit{and} && b' \in B \end{aligned} \tag{A7}$$

are well-defined and convex-valued for $(p, r) \in P \times R^\iota$. Furthermore, (i) they are upper semi-continuous at (p, r) if $p^0\omega^0 > 0$ and $p^{1s}\omega^{1s} > 0$ for $s \notin S^+(r)$; (ii) they satisfy the various (weak) forms of Walras' law

$$p^0 x^0 + \sum_i b^i \leqq p^0\omega^0, p^{1s}x^{1s} \leqq p^{1s}\omega^{1s} + \sum_i r^{is} b^i \textit{ for } s \in S, \textit{ and}$$
$$px \leqq p\omega, \textit{ with equalities if } x << \bar{x};$$

and (iii) they yield optimal solutions to the same problem without the constraint $x' \leqq \bar{x}$ *at* (p, r) *if* $x << \bar{x}$, $p^0\omega^0 > 0$ *and* $p^{1s}\omega^{1s} > 0$ for $s \notin S^+(r)$. *Finally, optimal solutions to (A7) without the constraint* $x' \leqq \bar{x}$ *yield optimal solutions to problem*

$$\begin{aligned} &\textit{minimize} && p^0x^{0\prime} + \sum_i b^{i\prime} \\ &\textit{subject to} && u(x') \geqq u(x), \\ & && p^{1s}x^{1s\prime} \leqq p^{1s}\omega^{1s} + \sum_i r^{is}b^{i\prime} \textit{ for } s \in S, \\ & && x' \geqq 0, \\ &\textit{and} && b' \in B \end{aligned} \tag{A8}$$

with $p^0x^0 + \sum_i b^i = p^0\omega^0$, *and conversely if* $p^0\omega^0 > 0$.

Proof of Lemma A2. Since verifying these properties is (after the "Walrasian" revolution) such a routine procedure, I will only indicate how the usual arguments need to be modified here. Such modifications are basically of two kinds. Let

$$F(p, r) = \{(x, b) : (x, b) \text{ is a feasible solution to (A7)}\}$$

for $(p, r) \in P \times R^\iota$.

(i) Compactness of $F(p, r)$.

Obviously, $F(p, r)$ is closed. That it is also bounded follows directly from the constraint $0 \leqq x' \leqq \bar{x}$ and the hypothesis $(p, r) \in P \times R^\iota$. To see this, suppose otherwise, i.e., there is a sequence $\{(x^\upsilon, b^\upsilon)\}$ such that $(x^\upsilon, b^\upsilon) \in F(p, r)$ for $\upsilon \geqq 1$ but $\lim_{\upsilon \to \infty} \|b^\upsilon\| = \infty$. Then consider the normalized sequence of portfolio holdings $b^{\upsilon\prime} = b^\upsilon / \|b^\upsilon\|$ for $\upsilon \geqq 1$. Without loss of generality we can assume that $\lim_{\upsilon \to \infty} (x^\upsilon, b^{\upsilon\prime}) = (x, b')$ with $\|b'\| = 1$. Moreover, when the budget constraints in (A7) are also normalized by $1/\|b^\upsilon\|$, we have that

$$(p^0/\|b^\upsilon\|)x^{0\upsilon} + \sum_i b^{i\upsilon\prime} \leqq (p^0/\|b^\upsilon\|)\omega^0$$

and

$$(p^{1s}/\|b^\upsilon\|)x^{1s\upsilon} \leqq (p^{1s}/\|b^\upsilon\|)\omega^{1s} + \sum_i r^{is}b^{i\upsilon\prime} \text{ for } s \in S$$

for $v \geqq 1$. Hence, in the limit, it must be the case that both $b' \neq 0$ and (using $p \in P$)

$$(-\sum_i b^{i\prime}, \sum_i r^{is} b^{i\prime}) \geqq 0,$$

which (using $r \in R^\iota$) contradicts the definition (15).

(ii) Continuity of $F(p, r)$.

Since B is closed, while the other constraints defining F are linear inequalities, this correspondence is obviously upper semi-continuous on $P \times R^\iota$. That it is also lower semi-continuous when $p^0 \omega^0 > 0$ and $p^{1s} \omega^{1s} > 0$ for $s \notin S^+(r)$ follows directly from the observation that, since B is convex, there is $b^+(r) \in B$ such that

$$\sum_i r^{is} b^{i+}(r) > 0 \text{ for } s \in S^+(r), \geqq 0 \text{ otherwise.}$$

Since B also contains 0, this entails that if $(x, b) \in F(p, r)$, then

$$p^0 x^{0\prime} + \sum_i b^{i\prime} < p^0 \omega^0,$$

$$p^{1s} x^{1s\prime} < p^{1s} \omega^{1s} + \sum_i r^{is} b^{i\prime} \text{ for } s \in S,$$

$$0 \leqq x' \leqq \bar{x}$$

and

$$b' \in B$$

for $(x', b') = (1 - \varepsilon)(x, b) + \varepsilon\delta(0, b^+(r))$ and $\varepsilon, \delta > 0$ sufficiently close to 0. Hence, $(x', b') \in F(p', r')$ for $(p', r') \in P \times R^\iota$ sufficiently close to (p, r).■

Perhaps I should also note in passing that the arguments establishing that, under suitable conditions, optimal solutions to (A7) or (A8) yield optimal solution to (A7) without the constraint $x' \leqq \bar{x}$ are also essentially identical to those for the textbook model of household behavior. Again, I leave the details to the interested reader.

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