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Abstract

We address the problem of ranking distributions of opportunity sets in terms of poverty. In order to accomplish this task, we identify a suitable notion of 'multidimensional poverty line' and we characterize axiomatically several poverty rankings of opportunity profiles. Among them, the Head-Count and the Opportunity-Gap poverty rankings, which are the natural counterparts of the most widely used income poverty indices.

Key words: Poverty, opportunity sets, head-count, poverty-gap.
JEL classification: D31; D63; I31.

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1. Introduction

The present work is devoted to the problem of ranking distributions of individual opportunity sets in terms of poverty.

Poverty reduction plays a prominent role in political debates in many countries, and methods and techniques to make poverty comparisons are necessary tools in order to design and to evaluate policies aimed at decreasing poverty.

Since the publication of Sen's (1976) pioneering paper on poverty measurement, in the last quarter century a great deal has been written on this subject and several measures of poverty are now available in the literature. However, most of the existing literature on poverty measurement regards income or consumption expenditures as the only relevant explanatory dimensions of poverty. This approach now appears as inadequate because poverty is essentially a multidimensional phenomenon and the exclusive reliance on just one indicator can hide crucial aspects of economic deprivation. Indeed, if we consider for example two societies with the same distribution of monetary earnings, we can hardly think of them as equivalent in terms of poverty if in one of them a fraction of the population is denied a number of basic rights and liberties such as the right to vote, freedom of speech, freedom of movement and so on. In that connection, many scholars like Rawls (1971), Sen (1980, 1997), Roemer (1996) have defended in their influential works the necessity to move from an income-based evaluation of social inequities towards a more comprehensive domain.

Although the inadequacy of a unidimensional approach to evaluating social inequities is well recognized, it is nevertheless a common practice of economists to do so. One of the basic reasons for this is linked to the difficulties met in data collection and data analysis. Moreover, in addition to data limitation and empirical constraints, a multidimensional evaluation of poverty is by no means straightforward from a theoretical point of view.

The focus of the present work is, in fact, this specific measurement problem. In particular, we consider the problem of ranking distributions of opportunities on the basis of poverty. An individual's opportunities are described by a set rather than by a scalar, as it is the case with income or consumption. As a consequence, the problem becomes that of ranking different distributions of opportunity sets.

To keep the approach as general as possible, the notion of "opportunity" is treated in an abstract way: we define an opportunity set as any finite set in some arbitrary space. Opportunities may be thought of as non welfare characteristics of agents such as basic liberties, political rights, and individual freedoms; or as access to certain welfare enhancing traits. A further interpretation is in terms of functionings à la Sen (such as being educated, being well-nourished, avoiding premature mortality, and so on): in this case the opportunity set corresponds to the capability set of an individual.

The present paper is linked to two different branches of literature.

On the one hand, it is related to the literature on the measurement of multidimensional poverty (see, among others, Alkire and Foster (2008), Chakravarty, Mukherjee and Ranade (1998), Bourguignon and Chakravarty (1999, 2002), Tsui (2002),). However, the approach we propose is different and possibly more general with respect to such a literature. Our abstract setting for modeling the different dimensions of individual deprivation relies on a finite domain as opposed to the domain considered in the literature on multidimensional poverty indices that is the Cartesian product of multivariate Euclidean spaces. Moreover, it is a well established result that any multivariate distribution, real-valued or otherwise, typically admits only partial rankings (e.g. dominance orderings) of the latter as natural and non-controversial. On the contrary, the literature on multidimensional poverty measurement is concerned with synthetic measures of the degree of poverty among individuals and, in so doing, it reduces all variables we want to compare to scalars. Such an information loosing exercise is in fact disputable in a multivariate context and is the opposite in spirit to what we are going to develop here. Indeed, we propose to characterize poverty rankings that are preorders, rather than controversial total ordering of multidimensional distributions, which rely on some suitable minimalist requirements.

On the other hand our paper is linked to the literature which focuses on the ranking problems for different distributions of opportunity sets. This problem has been first addressed by Kranich (1996) and Ok (1997), who however focused only on inequality rankings. There is now an extensive literature concerned with the measurement of inequality of opportunity: see, for example, Arlegi and Nieto (1999), Bossert, Fleurbaey, and Van de gaer (1999), Herrero (1997), Herrero, Iturbe-Ormaetxe, and Nieto (1998), Kranich (1996, 1997), Ok (1997), Ok and Kranich (1998), and Savaglio and Vannucci (2007). A survey of this literature may be found in Barberà, Bossert and Pattanaik. (2004).

The issue of ranking different distributions of opportunities in terms of the poverty they exhibit has never been addressed before. The present paper fills this gap. We address the problem of ranking profiles of opportunity sets on the basis of poverty.

A natural approach towards devising a poverty ranking for opportunity distributions is try and extend the basic income poverty measures into our richer setting. In this vein, we study alternative ways of extending the familiar notion of "poverty line" and the most well known poverty measures in the context of opportunity distributions. In order to identify the different value systems involved in the use of different poverty criteria we use the axiomatic metodhology, we propose a number of properties that a poverty-ranking relation on the possible distributions (profiles) of finite opportunity sets should satisfy, and study their logical implications. We introduce a threshold or minimum standard for opportunity sets, which mimics the "poverty line" of the unidimensional case, and characterize two fundamental orderings: the Head-Count and the Opportunity-Gap poverty rankings. Such rankings are the natural counterparts of the most widely used income poverty measures, namely the head count ratio and the income poverty gap. Indeed, the head-count ranking is produced by counting the number of population units whose endowments fail to meet the minimum standard. On the other hand, the opportunity gap ranking is produced by counting the number of extra-opportunities (or functionings) each population unit should be endowed with in order to achieve the minimum standard, and by summing them.

In addition, we axiomatically characterize two lexicographic orderings based on the HC and OG rankings and a third one based on a linear combination of the head-count and gap criteria.

The paper is organized as follow. The next section introduces the analytical setting and defines formally the basic problem studied in this paper. Section 3 introduces and discusses a first set of axioms and contains the main results of the paper: the characterization of the Head-Count and the

Opportunity-Gap poverty rankings. Section 4 provides and discusses an additional set of axioms aimed at characterizing composite rankings based on the HC and OG. Section 5 concludes with a brief discussion of the results and of directions for future research, while an appendix collects all proofs.

2. The framework

We start by identifying a universal non-empty set of opportunities, denoted by X. We assume that each element in X is desirable in some universal sense. Moreover, following the existing literature, we assume that opportunities are nonrival, so that a given opportunity is potentially available to everyone simultaneously, and that opportunities are excludable, so that providing an opportunity to some individuals does not necessarily imply that everyone has this opportunity.

Let $N = \{1, ..., n\}$ denote the finite set of relevant population units and $\mathcal{P}[X]$ the set of all finite subsets of X. Elements of $\mathcal{P}[X]$ are referred to as opportunity sets, and mappings $\mathsf{Y} = (Y_1, ..., Y_n) \in \mathcal{P}[X]^N$ as profiles of opportunity sets, or simply opportunity profiles.

Hence, each individual in a society is endowed with an opportunity set and a society is represented by an opportunity profiles. We are interested in ranking such opportunity profiles in terms of poverty.[1]

Following Sen's approach, the evaluation of poverty can be divided into two steps: (i) the identification step, in which the poor are identified in a given society; (ii) the aggregation step, in which the characteristics of the poor are aggregated in order to obtain an assessment of the poverty in a society.

As for the identification step, in the unidimensional context an income poverty line is chosen that divides the population into two sets: the poor and the non-poor. The identification of that level of income below which people are described as poor can follow an absolute or a relative approach. While with an absolute approach the poverty line is defined in an exogenous way and is the same across distributions, with a relative approach the poverty line in a distribution is a function of the distribution itself (e.g. the poverty line can be fixed at half the median income level in that society).

[1]Notice that our general model can be related to behaviorally oriented notions of opportunity set by the following interpretation. Let X be a possibly multidimensional space of relevant, observable functionings, N^*a population, $\mathsf{x} \in X^{N^*}$ the profile of achieved functionings within the population under consideration, and $\boldsymbol{\pi} = \{\pi_1, ..., \pi_n\} \in \Pi(N)$ a partition of the population into a finite set $N = \{1, ..., n\}$ of types according to a fixed set of verifiable criteria. Then, the opportunity set of type $i \in N$ at $(\mathsf{x}, \boldsymbol{\pi})$ is $X_i = \{x \in X : \text{there exists } j \in \pi_i \text{ such that } x_j = x\}$

In a multidimensional setting the identification step is not as simple. There are two different choices to be made. The first is the choice of a threshold for each relevant dimension. The second is the aggregation along the different dimensions in order to evaluate the poverty of each individual.

As for the first problem, our choice is implicit by the domain we are working with: each dimension is modeled as a binary variable. One individual doeas have access to a specific opportunity or he does not. There are not levels, either cardinal or ordinal, of access at a given dimension of well being. As for the aggregation of the different dimensions, there are two main approaches in the existing literature on multidimensional poverty[2]: one is the union approach, which declares one person as poor if he is below the threshold in a single dimension; the intersection approach instead regards one person as poor if he is below the threshold in all the relevant dimensions.

In our framework, a poverty threshold (or poverty line) is a set $T \in \mathcal{P}[X]$, which identifies a set of essential alternatives: an individual is declared as poor or, equivalently, he is declared to be below the poverty threshold if her opportunity set does not contain all the essential alternatives, i.e., all the alternatives contained in T. For instance, let $X = \{x_1, x_2, ..., x_k\}$ be the set of all opportunities, $N = \{1, 2, 3\}$ the relevant pupulation and $T = \{x_1, x_3\}$. Then, at opportunity profile $\mathsf{Y} = (Y_{1=} \{x_1, x_2, x_4, x_5, x_6\}, Y_2 = \{x_3, x_4\}, Y_3 = \{x_1, x_2, x_3\})$, population units 1 and 2 are poor, while 3 is rich because $Y_1 \not\supseteq T$, $Y_2 \not\supseteq T$ and $T \subset Y_3$. Hence we follow the union approach to identification but we restrict it to the essential alternatives. As a matter of convenience, in our presentation the set T is not dependent on the specific profile: i.e., in the identification step we adopt an absolute approach. However, our threshold can also be taken to be contingent on suitable profiles of opportunity sets.[3]

It is worth emphasizing here that the distinction between essential and non essential alternatives plays a crucial role in our axiomatic construction. One possible interpretation of such essential alternatives is linked to the basic needs approach: having access to all essential alternatives in this interpretation means being able to satisfy all basic needs. An alternative interpretation suggests that the essential alternatives could represent certain basic functionings (see Sen (1985)) such as, for example, life expectancy, literacy, and so on, or a set of primary goods (see Rawls (1971)). We

[2]For an "intermediate" solution, based on a variable minimal number of dimensions of deprivation see Alkire and Foster (2008).

[3]In particular, the threshold T may be defined ad the median of individual opportunity sets of a given distribution (or perhaps more to the point as the median of the interval of opportunity sets ranging from the smallest to the median opportunity set of the original distribution). Moreover, the threshold T may be regarded as the median of a set of proposals advanced by members of a panel of experts. That is so, because the set of possible thresholds (i.e. the set of subsets of X) is a in fact a distributive lattice with respect to the set-inclusion and therefore the median of any subset of possible thresholds is well-defined.

are aware or the crucial role played by the selection of the relevant dimensions in any empirical analysis of poverty. However, we believe that the issue of selecting the relevant essential alternatives lies substantially beyond the scope of the present paper: we assume that appropriate judgments on this have been made, and we concern ourselves with the remaining theoretical challenges.

As for aggregation, the problem is that of amalgamating information on the deprivation suffered by the poor in order to produce a suitable assessment of aggregate poverty. In the present setting, our first step involves the definition of a metric in the space of opportunity sets: in other words, we need first to define a criterion to compare individuals endowed with different opportunity sets. Thus, when is one person poorer (richer) than another person in terms of opportunities? There is an extensive literature devoted to the problem of ranking opportunity sets (see on this the excellent survey by Barberà, Bossert and Pattanaik (2004)). In order to answer such a question, we propose a criterion such that all the sets above the poverty threshold are each other indifferent; as for the sets below the poverty thresholds, they are ranked by set inclusion. Therefore, we consider a unique indifference class within the universe of the non–poor and we propose the very mild condition of set inclusion as the reference ranking rule within the poor.

Formally, our starting point is a preorder $\succcurlyeq_T^*$ on $\mathcal{P}[X]$ induced by the poverty threshold T and defined as follows: for any $Y, Z \in \mathcal{P}[X]$,

$$Y \succcurlyeq_T^* Z \text{ if and only if } [Y \supseteq Z \text{ or } Y \supseteq T].$$

The notation $\mathsf{Y}_{|T}$ will be employed in the rest of this paper to denote opportunity profile $(Y_i \cap T)_{i \in N}$. A poverty ranking of opportunity profiles under threshold T is a preorder $\succcurlyeq_T$ on $\mathcal{P}[X]^N$ such that for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$ whenever $Z_i \succcurlyeq_T^* Y_i$ for each $i \in N$.

In the present work, we generalize two of the most widely used income poverty measures, namely the head count ratio and the income poverty gap in a context of opportunity profiles:

Definition 1. The head-count (HC) poverty ranking under threshold T is the preorder $\succcurlyeq_T^h$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$,

$$\begin{aligned} \mathsf{Y} &\succcurlyeq_T^h \mathsf{Z} \text{ if and only if } h_T(\mathsf{Y}) \geq h_T(\mathsf{Z}), \text{ where for each } \mathsf{W} \in \mathcal{P}[X]^N, \\ h_T(\mathsf{W}) &= \# H_T(\mathsf{W}) \text{ and } H_T(\mathsf{W}) = \{i \in N : W_i \nsupseteq T\}. \end{aligned}$$

The head-count poverty ordering ranks two distributions on the basis of the number of individuals that are below the poverty threshold T, hence, it captures the incidence of poverty. Although such a measure gives useful information on the poverty in a distribution, the head-count does not take into account the depth or the severity of the deprivation suffered by the poor. In order to capture this aspect of the aggregate poverty, we also propose the opportunity-gap (OG) poverty ranking which measures the aggregate intensity of poverty.

Definition 2. The opportunity-gap (OG) poverty ranking under threshold T is the preorder $\succcurlyeq_T^g$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$,

$$\begin{aligned} \mathsf{Y} &\succcurlyeq_T^g \mathsf{Z} \text{ if and only if } g_T(\mathsf{Y}) \geq g_T(\mathsf{Z}), \text{ where} \\ \text{for each } \mathsf{W} &\in \mathcal{P}[X]^N,\ g_T(\mathsf{W}) = \sum_{i \in H_T(\mathsf{W})} \# \{x : x \in T \setminus W_i\}. \end{aligned}$$

Thus, for each poor individual, the intensity of poverty, the "individual poverty gap", is measured by the number of essential alternatives she does not have access to. That is, for each poor individual i, with opportunity set W_i, the "individual poverty gap" $g_T(W_i)$ is given by the following refined cardinality difference[4] with respect to the threshold set T : $g_T(W_i) = |\#(T) - \#(W_i \cap T)|$.

The opportunity-gap poverty ranking[5] aggregates this information by summing the individual gaps; hence, it tells us how poor are the poor. In the following, we propose some desirable properties that a poverty ranking should satisfy.

3. The basic characterizations

The axiomatic structure we propose will lead us to the characterization of the foregoing two poverty criteria defined above.

3.1. The axioms.

We introduce now the following basic properties for a poverty ranking $\succcurlyeq_T$ of $\mathcal{P}[X]^N$:

Axiom 1 (Anonymity (AN)). For any permutation π of N, and any $\mathsf{Y} \in \mathcal{P}[X]^N$: $\mathsf{Y} \sim_T \pi\mathsf{Y}$ (where $\pi\mathsf{Y} = (Y_{\pi(1)}, ..., Y_{\pi(n)})$).

Axiom 2 (Irrelevance of Inessential Opportunities (IIO)). For any $\mathsf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \setminus T$: $\mathsf{Y} \sim_T (\mathsf{Y}_{-i}, Y_i \setminus \{x\})$.

Axiom 3 (Irrelevance of Poor's Opportunity Deletions (IPOD)). For any $\mathsf{Y} \in \mathcal{P}[X]^N$, $i \in H_T(\mathsf{Y})$, and $x \in Y_i$: $\mathsf{Y} \sim_T (\mathsf{Y}_{-i}, Y_i \setminus \{x\})$.

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[4] The cardinality difference relation was introduced and axiomatically characterized by Kranich (1996).

[5] This is admittedly a quite crude 'metric' of opportunity gap that relies on the cardinality total preorder, which has been widely studied in the literature on rankings of opportunity sets. To be sure, the latter criterion has been also the target of sustained criticism. However, we submit, our version of the OG-poverty ranking may make much sense as a first approximation to a sound assessment of the aggregate 'intensity of poverty', whenever combined with suitable definitions of the opportunity space and the poverty threshold.

Axiom 4 (Dominance at Essential Profiles (DEP)). For any Y, Z $\in \mathcal{P}[X]^N$ such that both $\{Y_1, ..., Y_n\} \subseteq \{T, \emptyset\}$ and $\{Z_1, ..., Z_n\} \subseteq \{T, \emptyset\}$,

$$\mathsf{Y} \succ_T \mathsf{Z} \text{ if and only if } \#\{i \in N : Y_i = \emptyset\} > \#\{i \in N : Z_i = \emptyset\}.$$

The first three axioms are invariance properties, in the sense that they require our poverty rankings to ignore certain aspects of the opportunity distributions and to focus on others. The first, Anonymity, is an axiom that requires a symmetric treatment of individuals, thereby preventing from paying attention to the identities of individuals. Irrelevance of Inessential Opportunities says that if the opportunity set of an individual i is reduced by the subtraction of an alternative which is not essential, then the new profile of opportunity sets exhibits the same degree of poverty as the original profile. This axiom is reminiscent of the Focus axiom, used in the income poverty paradigm, which requires invariance with respect to reduction in the incomes of the non-poor; however, instead of distinguishing between the poor and the non-poor, in the current scenario the basic distinction is between essential and non essential alternatives. Irrelevance of Poor's Opportunity Deletions says that if the opportunity set of a poor individual i is reduced by the subtraction of an alternative, then the new profile of opportunity sets exhibits the same degree of poverty as the original profile.

While the previous invariance properties are useful in identifying the information that our poverty rankings should use, the last axiom is a dominance property, which identifies classes of transformation that have a certain effect on the poverty rankings, thereby restricting the set of poverty criteria. Dominance at Essential Profiles indeed considers a particular case in which two 'degenerate' profiles are composed of either empty sets or sets coinciding with the poverty threshold T. In this special case, one profiles exhibit more poverty than the other if the number of people endowed with the empty set in the former is higher than the number of individuals endowed with an empty set in the latter.

Our first proposition shows that these axioms are necessary and sufficient conditions for the characterization of the HD-poverty ranking $\succcurlyeq_T^h$:

Proposition 1. Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$. Then $\succcurlyeq_T$ is the HC ranking $\succcurlyeq_T^h$ if and only if $\succcurlyeq_T$ satisfies AN, IIO, IPOD and DEP. Moreover, such a characterization is tight.

We now introduce two further axioms:

Axiom 5 (Strict Monotonicity with respect to Essential Deletions (SMED)). For any Y $\in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \cap T$: $(\mathsf{Y}_{-i}, Y_i \setminus \{x\}) \succ_T \mathsf{Y}$.

Axiom 6 (Independence of Balanced Essential Deletions (IBED)). For any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $i \in N$, $y \in Y_i \cap T$ and $z \in Z_i \cap T$: $\mathsf{Y} <_T \mathsf{Z}$ if and only if $(\mathsf{Y}_{-i}, Y_i \setminus \{y\}) <_T (\mathsf{Z}_{-i}, Z_i \setminus \{z\})$.

Strict Monotonicity with respect to Essential Deletions is another dominance property which says that if the opportunity set of an individual i is reduced by the subtraction of an essential alternative, then the new profile of opportunity sets exhibits a higher degree of poverty than the original profile. This axiom is a direct translation in our context of the Monotonicity axiom used in the income inequality paradigm (see Foster (2006)); again the difference relies on the fact that in the current scenario the crucial distinction is between essential and non essential alternatives rather than between poor and non-poor individuals.

Finally, we propose a standard independence axiom, Independence of Balanced Essential Deletions, which pertains to the deletion of an essential alternative from the set of an individual i in two opportunity profiles Y, Z. Such balanced deletions preserves the ranking of the two opportunity profiles.

The first two and the last two axioms of this section are necessary and sufficient to characterize our poverty-gap criterion:

Proposition 2. Let $<_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$. Then $<_T$ is the OG ranking $<_T^g$ if and only if $<_T$ satisfies AN, IIO, SMED and IBED. Moreover, such a characterization is tight.

Thus, we provide two simple characterizations of the most basic poverty rankings of opportunity profiles. We would like to stress that, to the best of our knowledge, those results have no counterpart in the standard literature on poverty indices of income distributions, though the head-count and poverty-gap are the most widely used criteria in the theoretical and empirical literature on poverty.

4. Composite rankings

In this section, we propose and axiomatically characterize two lexicographic orderings based on the HC and OG rankings and third one based on a linear combination of the head-count and gap criteria.

The first composite criterion, the (HG)- lexicographic poverty ranking, combines in a lexicographic order the HG and the OG rankings, with priority given to the HC criterion.

Definition 3. A (HG)- lexicographic poverty ranking of opportunity profiles under threshold T is a binary relational system $\left(\mathcal{P}[X]^N, <_T^{hg}\right)$ where $<_T^{hg}$ is a preorder defined as follow: for any

$\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$,

$$\mathsf{Y} \succcurlyeq_T^{hg} \mathsf{Z} \text{ if and only if either } \mathsf{Y} \succ_T^h \mathsf{Z} \text{ or } (\mathsf{Y} \sim_T^h \mathsf{Z} \text{ and } g_T(\mathsf{Y}) \geq g_T(\mathsf{Z})).$$

The (GH)- lexicographic poverty ranking also combines in a lexicographic order the HG and the OG rankings, but with priority given to the OG criterion.

Definition 4. A (GH)- lexicographic poverty ranking of opportunity profiles under threshold T is a binary relational system $\left(\mathcal{P}[X]^N, \succcurlyeq_T^{gh}\right)$ where $\succcurlyeq_T^{gh}$ is a preorder defined as follow: for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$,

$$\mathsf{Y} \succcurlyeq_T^{gh} \mathsf{Z} \text{ if and only if either } \mathsf{Y} \succ_T^g \mathsf{Z} \text{ or } (\mathsf{Y} \sim_T^g \mathsf{Z} \text{ and } h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})).$$

Finally, the (HG)- weighted poverty ranking linearly combines the HG and the OG criteria.

Definition 5. A (HG)-weighted poverty ranking of opportunity profiles under threshold T is a binary relational system $\left(\mathcal{P}[X]^N, \succcurlyeq_T^{w}\right)$ where $\succcurlyeq_T^{w}$ is a preorder defined as follow: there exist w_1, $w_2 \in \mathbb{R}_{++}$ such that, for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$,

$$\mathsf{Y} \succcurlyeq_T^{w} \mathsf{Z} \text{ if and only if } w_1 h_T(\mathsf{Y}) + w_2 g_T(\mathsf{Y}) \geq w_1 h_T(\mathsf{Z}) + w_2 g_T(\mathsf{Z}).$$

4.1. **More axioms.** In order to characterize such composite rankings, we now propose the following axioms:

Axiom 7 (Qualified Independence of Balanced Essential Deletions (Q-$IBED$)). For any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, for any $y, z \in X$ and for any $i \in N$, such that $Y_i \subset T$, $Z_i \subset T$, $y \in Y_i \cap T$ and $z \in Z_i \cap T$:

$$\mathsf{Y} \succcurlyeq_T \mathsf{Z} \text{ if and only if } (\mathsf{Y}_{-i}, Y_i \setminus \{y\}) \succcurlyeq_T (\mathsf{Z}_{-i}, Z_i \setminus \{z\}).$$

The Q-IBED axiom pertains to the deletion of an essential alternative from the set of a poor individual i in two opportunity profiles Y, Z. Such balanced deletions preserves the ranking of the two opportunity profiles. This axiom is implied by the axiom introduced before, Independence of Balanced Essential Deletions.

Axiom 8 (Conditional Dominance (CD)). Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbb{R}^{\mathcal{P}[X]^N}$, such that for all $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $f_i(\mathsf{Y}) = f_i(\mathsf{Z})$, $i = 1, ..., k$ entails $\mathsf{Y} \sim_T \mathsf{Z}$. Then, for all $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $(f_1(\mathsf{Y}), ..., f_k(\mathsf{Y})) \neq (f_1(\mathsf{Z}), ..., f_k(\mathsf{Z}))$ and $f_i(\mathsf{Y}) \geq f_i(\mathsf{Z})$, $i = 1, ..., k$ entails $\mathsf{Y} \succ_T \mathsf{Z}$.

In order to compare any two opportunity profiles in terms of poverty, we set a finite number of real-valued evaluation function on $\mathcal{P}[X]^N$. Then, according to the Conditional Dominance axiom, if any function evaluates the first distribution as equivalent to the second one, we conclude that both opportunity profiles have the same degree of poverty no matter how we decide to measure it. On the other hand, if all evaluation functions consider the value associated to the first distribution as great as the value associated to the other one, then the CD axiom says that we are forced to conclude that the first profile shows at least as much poverty as the second one in terms of opportunity.

Axiom 9 (Non-Compensation (NC)). Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbb{R}^{\mathcal{P}[X]^N}$, such that:

(i): for all $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$: if $f_i(\mathsf{Y}) = f_i(\mathsf{Z})$, $i = 1, ..., k$, then $\mathsf{Y} \sim_T \mathsf{Z}$,

(ii): there exist $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ and $i^* \in \{1, ..., k\}$, such that $f_{i^*}(\mathsf{Y}) > f_{i^*}(\mathsf{Z})$ and $f_j(\mathsf{Z}) > f_j(\mathsf{y})$for any $j \in \{1, ..., k\}$, $j \neq i^*$, and $\mathsf{Y} \succ_T \mathsf{Z}$.

Then for all $\mathsf{U}, \mathsf{V} \in \mathcal{P}[X]^N$: $\mathsf{U} \succ_T \mathsf{V}$ whenever $f_{i^*}(\mathsf{U}) > f_{i^*}(\mathsf{V})$.

Condition NC prevents trade-offs among factors and is therefore needed to set the basic feature of lexicographic rankings (see e.g. Fishburn (1975) where a similar condition is introduced in order to characterize lexicographic orderings over products of ordered sets).

The next two axioms propose two different and alternative dominance conditions, based on two basic transformation. Consider a profile Y and two individuals, i and j, which are just below the poverty thresholds: that is, they miss just one essential opportunity, say x and y respectively. Now consider two different transformations of profile Y: (i) the transfer of opportunity y from j to i; (ii) the deletion of one opportunity from the opportunity set available to j. By the joint effect of this double transformation, the number of individuals below the poverty thresholds has decreased, as now i is not poor anymore while j is still poor (poorer than before); however the aggregate number of opportunities that individuals i and j do not have has increased. What is the net effect on our poverty ranking? The answer will depend on the specific weight we give to the number of poor in our society vis a vis to the aggregate severity of poverty. The two axioms we propose give different and opposite answers: according to the Local Head-Count-Priority, poverty decreases; according to Local Gap-Priority, poverty increases. Formally,

Axiom 10 (Local Head-Count Priority (HP)). Let $\succcurlyeq_T$ be a poverty ranking with threshold T, such that $\#T \geq 3$. For any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, if [there exist $i, j \in N$ and $x, y, z \in T$, with $x \neq y \neq z \neq x$, such that for any $l \neq i, j$, $Y_l = Z_l$, $Y_i = T \setminus \{x\}$, $Y_j = T \setminus \{y\}$, $Z_i = T$, and $Z_j = \emptyset$],then $\mathsf{Y} \succ_T \mathsf{Z}$.

Axiom 11 (Local Gap-Priority (GP)). Let $\succcurlyeq_T$ be a poverty ranking with threshold T, such that $\#T \geq 3$. For any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, if [there exist $i, j \in N$ and $x, y, z \in T$, with $x \neq y \neq z \neq x$, such that for any $l \neq i, j$, $Y_l = Z_l$, $Y_i = T \setminus \{x\}$, $Y_j = T \setminus \{y\}$, $Z_i = T$, and $Z_j = \emptyset$],then $\mathsf{Z} \succ_T \mathsf{Y}$.

The foregoing two final axioms set the basis for a lexicographic combination of the Head Count and Opportunity Gap criteria.

The last axiom is a quite standard and technical axiom in social choice theory, generally used for the characterization of utilitarian social welfare functions.

Axiom 12 (Cardinal Unit-Comparability (CUC)). Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbb{R}^{\mathcal{P}[X]^N}$, such that for all $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$: if $f_i(\mathsf{Y}) = f_i(\mathsf{Z})$, $i = 1, ..., k$ entails $\mathsf{Y} \sim_T \mathsf{Z}$. Posit

$$\Phi = \left\{ \begin{array}{c} \varphi = (\varphi_1, ..., \varphi_k) : \varphi_i \in \mathbb{R}^{\mathbb{R}}, i = 1, ..., k \text{ such that there exist} \\ \alpha > 0, \beta_i \in \mathbb{R} \text{ with } \varphi_i(x) = \alpha x + \beta_i \text{ for any } x \in \mathbb{R} \end{array} \right\}.$$

Then, for all $\mathsf{Y}, \mathsf{Z}, \mathsf{V}, \mathsf{U} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$, $(f_1(\mathsf{U}), ..., f_k(\mathsf{U})) = ((\varphi_1 \circ f_1)(\mathsf{Y}), ..., (\varphi_k \circ f_k)(\mathsf{Y}))$ and $(f_1(\mathsf{V}), ..., f_k(\mathsf{V})) = ((\varphi_1 \circ f_1)(\mathsf{Z}), ..., (\varphi_k \circ f_k)(\mathsf{Z}))$ with $\varphi = (\varphi_1, ..., \varphi_k) \in \Phi$ entail $\mathsf{U} \succcurlyeq_T \mathsf{V}$.

CUC induces an information environment where the admissible transformations are increasing affine functions and, in addition, the scaling unit must be the same for all individuals. This assumption allows for interpersonal comparisons of differences in the selected parameters (namely, in our case, h_T and g_T). However, parameter levels cannot be compared interpersonally because the intercepts of the affine transformations may differ arbitrarily across individuals.

4.2. **More results.** The next characterizations rely on a simple Lemma showing that if a ranking of opportunity profiles satisfies AN, IIO, DEP and $Q-IBED$ then two opportunity profiles must be indifferent whenever they exhibit the same number of poor and the same aggregate poverty gap. In other words, this Lemma shows that any such a ranking is specified by just two parameters (namely h_T and g_T).

Lemma 1. Let $\succcurlyeq_T$ be a poverty ranking on $\mathcal{P}[X]^N$ and a total preorder which satisfies AN, IIO, DEP and $Q - IBED$. Then, for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) = (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$ entails $\mathsf{Y} \sim_T \mathsf{Z}$.

We are now able to characterize our composite rankings. The first proposition characterizes the (HG)- lexicographic poverty ranking $\succcurlyeq_T^{hg}$ which uses the opportunity-gap criterion as a refinement of the head-count one.

Proposition 3. Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$, such that $\#T \geq 3$, and a total preorder. Then, $\succcurlyeq_T = \succcurlyeq_T^{hg}$ if and only if $\succcurlyeq_T$ satisfies AN, IIO, DEP, Q-IBED, CD, NC, and HP. Moreover, such a characterization is tight.

The next proposition characterizes the (GH)- lexicographic poverty ranking $\succcurlyeq_T^{gh}$ which employs the head-count criterion in order to refine the opportunity-gap one.

Proposition 4. Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$ such that $\#T \geq 3$, and a total preorder. Then, $\succcurlyeq_T = \succcurlyeq_T^{gh}$ if and only if $\succcurlyeq_T$ satisfies AN, IIO, DEP, Q-IBED, CD, NC, and GP. Moreover, such a characterization is tight.

Our final proposition characterizes the class of (HG)- weighted poverty rankings.

Proposition 5. Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$ and a total preorder, and suppose $n > \#T \geq 2$. Then, $\succcurlyeq_T = \succcurlyeq_T^w$ for some $w = (w_1, w_2) \in (\mathbb{R} \setminus \{0\})^2$ if and only if $\succcurlyeq_T$ satisfies AN, IIO, DEP, Q-IBED, CD and CUC. Moreover, such a characterization is tight.

Thus, our characterizations of $\succcurlyeq_T^{hg}$, $\succcurlyeq_T^{gh}$, and $\succcurlyeq_T^w$ rely on a common core properties. Within the class of rankings that satisfies those properties, the Non-Compensation axiom identifies the lexicographic combinations of the head-count and opportunity-gap criteria, while Cardinal Unit Comparability axiom is strong enough to characterize the class of rankings induced by a their linear combination.[6]

5. Final remarks

The need for complementing the traditional evaluation of income poverty by an analysis of the deprivation suffered in many dimensions of individual and social life has been forcefully defended by many economists in the last decades. Such a measurement extension may substantially improve our understanding of the poverty in a society and may well have far-reaching policy implications. To keep the analysis as general as possible, in this paper the different dimensions have been treated in an abstract way: we have defined an opportunity set as any finite set in some arbitrary space and we have attempted to outline an axiomatic theory for the measurement of poverty of opportunity. To the best of our knowledge, there have been no previous attempts to compare profiles of opportunity sets on the basis of poverty.

We have characterized two fundamental rankings, the Head-Count and the Opportunity-Gap poverty rankings, which generalize the most known poverty measures used in the income poverty framework, namely the head count ratio and the income poverty gap. In addition, we have characterized axiomatically two lexicographic rankings based on the HC and OG rankings and a third one based on a linear combination of the head-count and gap criteria.

[6]It is worth noticing that the structure of our proof of Proposition 5 replicates to a large extent the style of proof of Theorem 4.4 in Dutta and Sen (1996) as subsequently amended by Alcalde-Unzu and Ballester (2005).

We are aware of the critique of the head-count and poverty-gap measures, formulated by Sen within the income poverty framework, and based on their inability to take into account the inequality among the poor. This critique has led to the characterization of richer families of income poverty indices (see Clark et al. (1981) and Foster et al. (1984)). It would be interesting to study such an extension in our setting.

Moreover, we have only considered comparisons of opportunity profiles for a fixed population. A possible extension of our analysis would be to compare the opportunities available to societies with different numbers of individuals. This would make it possible to rank opportunity profiles for different countries, different demographic groups, and for different time periods.

Finally, the recent availability of individual data on different dimensions of poverty makes it possible an empirical application based on the rankings characterized in this paper. All these topics will be the object of future research.

6. Appendix: Proofs

Proof of Proposition 1. (A) It is straightforward to check that $\succcurlyeq_T^h$ is a poverty ranking and does indeed satisfy AN, IIO, DEP and IPOD.

Conversely, suppose $\succcurlyeq_T$ is a poverty ranking that satisfies AN, NT, IIO, and IPOD. Now, consider $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$. Then, by repeated application of IIO and transitivity, $\mathsf{Y}_{|T} \succcurlyeq_T \mathsf{Z}_{|T}$.

Next, observe that $(T^{N \backslash H_T(\mathsf{Y})}, \emptyset^{H_T(\mathsf{Y})}) \sim_T \mathsf{Y}_{|T} \succcurlyeq_T \mathsf{Z}_{|T} \sim_T (T^{N \backslash H_T(\mathsf{Z})}, \emptyset^{H_T(\mathsf{Z})})$, by repeated application of IPOD. Let us now suppose that $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$: then, by AN and DEP, $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. Hence, $h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})$, i.e. $\mathsf{Y} \succcurlyeq_T^h \mathsf{Z}$.

To prove the reverse inclusion, suppose that $\mathsf{Y} \succcurlyeq_T^h \mathsf{Z}$, i.e. $h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})$. Then, consider $(T^{N \backslash H_T(\mathsf{Y})}, \emptyset^{H_T(\mathsf{Y})})$, $(T^{N \backslash H_T(\mathsf{Z})}, \emptyset^{H_T(\mathsf{Z})})$ and a permutation π of N such that $\pi(H_T(\mathsf{Z})) \subseteq \pi(H_T(\mathsf{Y}))$.

By IIO, $\mathsf{Y} \sim_T (T^{N \backslash H_T(\mathsf{Y})}, \emptyset^{H_T(\mathsf{Y})})$ and $\mathsf{Z} \sim_T (T^{N \backslash H_T(\mathsf{Z})}, \emptyset^{H_T(\mathsf{Z})})$; by AN, $(T^{N \backslash H_T(\mathsf{Y})}, \emptyset^{H_T(\mathsf{Y})}) \sim_T (T^{\pi(N \backslash H_T(\mathsf{Y}))}, \emptyset^{\pi(H_T(\mathsf{Y}))})$ and $(T^{N \backslash H_T(\mathsf{Z})}, \emptyset^{H_T(\mathsf{Z})}) \sim_T (T^{\pi(N \backslash H_T(\mathsf{Z}))}, \emptyset^{\pi(H_T(\mathsf{Z}))})$.

Clearly, if $\pi(H_T(\mathsf{Z})) = \pi(H_T(\mathsf{Y}))$, then $(T^{\pi(N \backslash H_T(\mathsf{Y}))}, \emptyset^{\pi(H_T(\mathsf{Y}))}) = (T^{\pi(N \backslash H_T(\mathsf{Z}))}, \emptyset^{\pi(H_T(\mathsf{Z}))})$, hence, by transitivity of $\succcurlyeq_T$, $\mathsf{Y} \sim_T \mathsf{Z}$.

Let us then suppose that $\pi(H_T(\mathsf{Z})) \subset \pi(H_T(\mathsf{Y}))$. By DEP, it follows that:

$$(T^{\pi(N \backslash H_T(\mathsf{Y}))}, \emptyset^{\pi(H_T(\mathsf{Y}))}) \succ_T (T^{\pi(N \backslash H_T(\mathsf{Z}))}, \emptyset^{\pi(H_T(\mathsf{Z}))}),$$

hence, in particular, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$.

(B) The characterization provided above is tight. To check the validity of this claim, consider the following examples.

i) To begin with, consider the non-anonymous refinement of HG defined by the following rule: $\mathsf{Y} \preccurlyeq_T^{h_1} \mathsf{Z}$ if and only if:

a) $\mathsf{Y} \preccurlyeq_T^{h} \mathsf{Z}$ and $\{Y_i, Z_i\} \subseteq \{T, \emptyset\}$ for each $i \in N$ or

b) $\mathsf{Y} \succ_T^{h} \mathsf{Z}$ or

c) $\mathsf{Y} \sim_T^{h} \mathsf{Z}$, there exist $i, j \in N$ such $\{Y_i, Z_j\} \cap \{T, \emptyset\} = \emptyset$, and $Y_1 \not\supseteq T$.

Clearly, $\preccurlyeq_T^{h_1}$ is a poverty ranking that satisfies IIO, IPOD and DEP, but violates AN.

ii) Consider the refinement of HC defined by the following rule: $\mathsf{Y} \preccurlyeq_T^{h^*} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{h} \mathsf{Z}$ or $\mathsf{Y} \sim_T^{h} \mathsf{Z}$ and $\#\{i \in N : Y_i \supset T\} \leq \#\{i \in N : Z_i \supset T\}$. Such a preorder is a poverty ranking that satisfies AN, DEP and IPOD but violates IIO.

iii) Consider the universal indifference poverty ranking: i.e. $\mathsf{Y} \preccurlyeq^I \mathsf{Z}$ for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$. That ranking does satisfy AN, IIO and IPOD but violates DEP.

iv) Consider the OG-refinement of HC as defined by the following rule: $\mathsf{Y} \preccurlyeq_T^{h_g} \mathsf{Z}$ if and only if either $\mathsf{Y} \succ_T^{h} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{h} \mathsf{Z}$ and $g_T(\mathsf{Y}) \geq g_T(\mathsf{Z})$). Such a preorder is a poverty ranking that satisfies AN, IIO and DEP, but fails to satisfy IPOD. ¤

Proof of Proposition 2. (A) It is easily checked that $\preccurlyeq_T^{g}$ is a poverty ranking and does satisfy AN, IIO, SMED and IBED.

Conversely, suppose $\preccurlyeq_T$ is a poverty ranking that satisfies AN, IIO, SMED and IBED.

Then, consider $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Y} \preccurlyeq_T \mathsf{Z}$. Again, by repeated application of IIO and transitivity, $\mathsf{Y}_{|T} \preccurlyeq_T \mathsf{Z}_{|T}$. Now, suppose that $g_T(\mathsf{Z}) > g_T(\mathsf{Y})$. Then, by repeated application of IBED, $\mathsf{Z}'_{|T} \sim_T \mathsf{Y}_{|T}$ for some Z' such that $Z'_i \subseteq Z_i$ for each $i \in N$, and $g_T(\mathsf{Z}') = g_T(\mathsf{Y})$. It follows that, by repeated application of SMED, $\mathsf{Z}_{|T} \succ_T \mathsf{Z}'_{|T}$, hence by transitivity, $\mathsf{Z}_{|T} \succ_T \mathsf{Y}_{|T}$. Thus, by repeated application of IIO and transitivity again, $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction.

On the other hand, suppose that $\mathsf{Y} \preccurlyeq_T^{g} \mathsf{Z}$, i.e. $g_T(\mathsf{Y}) \geq g_T(\mathsf{Z})$, and consider $\mathsf{T} = (T, ..., T) \in \mathcal{P}[X]^N$. Of course, $\mathsf{T} \sim_T \mathsf{T}$, by reflexivity. Then, by AN and repeated application of IBED to $\mathsf{T} \sim_T \mathsf{T}$, it follows that $\mathsf{Y}' \preccurlyeq_T \mathsf{Z}$ for some Y' such that $Y'_i \setminus T = Y_i \setminus T$ and $Y_i \subseteq Y'_i$ for each $i \in N$, and $g_T(\mathsf{Y}') = g_T(\mathsf{Z})$. If, in particular, $g_T(\mathsf{Y}') = g_T(\mathsf{Y})$ then $\mathsf{Y}' = \mathsf{Y}$, hence $\mathsf{Y} \preccurlyeq_T \mathsf{Z}$, and we are done. Otherwise, there exist $i \in N$ and $x \in T \cap (Y'_i \setminus Y_i)$, hence $\mathsf{Y} \succ_T \mathsf{Z}$ by transitivity and repeated application of SMED. In any case, $\mathsf{Y} \preccurlyeq_T \mathsf{Z}$ as required.

(B) The foregoing characterization is also tight. To verify that claim consider the following examples.

$i)$ Take the following non-anonymous refinement of the OG poverty ranking: $\mathsf{Y} \succcurlyeq_T^{g_1} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{g} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{g} \mathsf{Z}$, $Y_1 \not\supseteq T$ and $Z_1 \cap T \supseteq Y_1 \cap T$). That ranking satisfies IIO, SMED and IBED but fails to satisfy AN.

$ii)$ Consider the following refinement of the OG poverty ranking: $\mathsf{Y} \succcurlyeq_T^{g^*} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{g} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{g} \mathsf{Z}$ and $\sum_{i\in N} \#(Y_i \smallsetminus T) \leq \sum_{i\in N} \#(Z_i \smallsetminus T)$). That ranking satisfies AN, SMED and IBED but fails to satisfy IIO.

$iii)$ Consider again the universal indifference ranking: i.e. $\mathsf{Y} \succcurlyeq^I \mathsf{Z}$ for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$. That preorder is a poverty ranking which does satisfy AN, IIO and IBED but violates SMED.

$iv)$ Consider the HC-refinement of the OG poverty ranking: $\mathsf{Y} \succcurlyeq_T^{g_h} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{g} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{g} \mathsf{Z}$ and $h_T(\mathsf{Y}) \geq h_T(\mathsf{Z})$). That poverty ranking satisfies AN, IIO, SMED but violates IBED. ¤

Proof of Lemma 2. Let us suppose $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$, $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$. Also, notice that for any $\mathsf{U} \in \mathcal{P}[X]^N$, $h_T(\mathsf{U}) = h_T\left(\mathsf{U}_{|T}\right)$ and $g_T(\mathsf{U}) = g_T\left(\mathsf{U}_{|T}\right)$ by definition of h_T and g_T respectively. Therefore, $h_T\left(\mathsf{Y}_{|T}\right) = h_T\left(\mathsf{Z}_{|T}\right) = m$ and $g_T\left(\mathsf{Y}_{|T}\right) = g_T\left(\mathsf{Z}_{|T}\right) = k$ for some m, k non-negative (observe that $m = 0$ if and only if $k = 0$). Next, posit $\widetilde{\mathsf{V}} = \left(\widetilde{V}\right)_{i=1,\ldots,n}$ with $\widetilde{V}_i = T$ if $V_i \supseteq T$, and $\widetilde{V}_i = \emptyset$ if $V_i \not\supseteq T$ and note that $h_T\left(\mathsf{V}_{|T}\right) = h_T\left(\widetilde{\mathsf{V}}\right)$ since $\widetilde{\mathsf{V}}$ does not alter the set of poor population units in $\mathsf{V}_{|T}$. Next, $\mathsf{Y}_{|T} \succcurlyeq_T \mathsf{Z}_{|T}$ if and only if $\widetilde{\mathsf{Y}} \succcurlyeq_T \widetilde{\mathsf{Z}}$ by AN and a repeated application of $Q-IBED$ ($(m\,|T| - k)$ times). Moreover, since $h_T\left(\widetilde{\mathsf{Y}}\right) = h_T\left(\widetilde{\mathsf{Z}}\right)$, it follows by DEP that neither $\widetilde{\mathsf{Y}} \succ_T \widetilde{\mathsf{Z}}$ nor $\widetilde{\mathsf{Z}} \succ_T \widetilde{\mathsf{Y}}$. Therefore, $\widetilde{\mathsf{Y}} \sim_T \widetilde{\mathsf{Z}}$ because $\succcurlyeq_T$ is a total preorder. Finally, $\mathsf{Y} \sim_{\mathbf{T}} \mathsf{Y}_{|T}$ and $\mathsf{Z} \sim_{\mathbf{T}} \mathsf{Z}_{|T}$ by repeated applications of IIO. It follows, by transitivity, that $\mathsf{Y} \sim_{\mathbf{T}} \mathsf{Z}$. ¤

Proof of Proposition 3. (A) Again, it is easily checked that $\succcurlyeq_T^{hg}$ does indeed satisfy AN, IIO, DEP, Q-IBED, CD, NC and HP.

On the other hand, let $\succcurlyeq_T$ be a poverty ranking and a total preorder that satisfies AN, IIO, DEP, Q-IBED, CD, NC and HP.

Let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} \succcurlyeq_T^{hg} \mathsf{Z}$, then one of the following cases obtains:

a) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$
b) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$
c) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ and $g_T(\mathsf{Z}) > g_T(\mathsf{Y})$
d) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$
e) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$

Under case a) b), d) $\mathsf{Y} \succ_T \mathsf{Z}$ by CD. Under case c), $\mathsf{Y} \succ_T \mathsf{Z}$ by Lemma 1 and NC and HP. In e) by Lemma 1 $\mathsf{Y} \sim_T \mathsf{Z}$. Hence, in any case, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$.

Conversely, let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$, then the following cases should be distinguished:

1) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$
2) $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$
3) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$
4) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Z}) > g_T(\mathsf{Y})$
5) $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$ and $g_T(\mathsf{Z}) = g_T(\mathsf{Y})$.

Under case 1), 3), $\mathsf{Y} \succ_T^{hg} \mathsf{Z}$ by definition. Under case 2), two subcases should be distinguished, namely either $g_T(\mathsf{Z}) \geq g_T(\mathsf{Y})$ or $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$. If $g_T(\mathsf{Z}) \geq g_T(\mathsf{Y})$ then by CD $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. If, on the contrary, $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$ then, by Lemma 1 and NC and HP, $\mathsf{Z} \succ_T \mathsf{Y}$ a contradiction again. Moreover, under case 4) by CD $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. Finally, under case 5), we have that $\mathsf{Y} \sim_T^{hg} \mathsf{Z}$ by definition. Hence, the desired result follows.

(B) The characterization of $\succcurlyeq_T^{hg}$ provided above is tight. Indeed, consider the following examples:

$i)$ Take the following non-anonymous refinement of the (HG)- lexicographic poverty ranking: $\mathsf{Y} \succcurlyeq_T^{hg_1} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{hg} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{hg} \mathsf{Z}$, $\{Y_1, ..., Y_n, Z_1, ..., Z_n\} \nsubseteq \{T, \varnothing\}$ and $\#(Y_1 \cap T) \leq \#(Z_1 \cap T)$). That ranking is a total preorder that satisfies IIO, DEP, Q-IBED, CD, NC and HP but fails to satisfy AN.

$ii)$ Consider the following refinement of the (HG)- lexicographic poverty ranking: $\mathsf{Y} \succcurlyeq_T^{hg_2} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{hg} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{hg} \mathsf{Z}$ and $\sum_{i \in N} \#(Y_i \smallsetminus T) \leq \sum_{i \in N} \#(Z_i \smallsetminus T)$). That ranking is a total preorder that satisfies AN, DEP, Q-IBED, CD, NC and HP but fails to satisfy IIO.

$iii)$ Consider the following poverty ranking: $\mathsf{Y} \succcurlyeq_T^{hg_3} \mathsf{Z}$ if and only if $\sum_{i \in N, Y_i \nsupseteq T} \#(Y_i \cap T) \geq \sum_{i \in N, Z_i \nsupseteq T} \#(Z_i \cap T)$. That ranking is a total preorder that satisfies AN, IIO, Q-IBED, CD, NC and HP but fails to satisfy DEP.

$iv)$ Choose $x^* \in T$, and consider the following refinement of the (HG)- lexicographic poverty ranking: $\mathsf{Y} \succcurlyeq_T^{hg_4} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{hg} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{hg} \mathsf{Z}$ and either $[x^* \in (Y_{i^*} \cap Z_{j^*})]$ or $[x^* \notin Y_{i^*}]$, where $i^* = \min\{i \in N : Y_i \nsupseteq T\}$ and $j^* = \min\{i \in N : Z_i \nsupseteq T\}$). That ranking is a total preorder that satisfies AN, IIO, DEP, CD, NC and HP but fails to satisfy Q-IBED.

$v)$ Consider the following poverty ranking: $\mathsf{Y} \succcurlyeq_T^{hg_5} \mathsf{Z}$ if and only if either $[\{Y_i, Z_i\} \subseteq \{T, \emptyset\}$ for each $i \in N]$ or [there exists $i \in N$ such that $Y_i \notin \{T, \emptyset\}$]. It can be shown that $\succcurlyeq_T^{hg_5}$ is indeed a total preorder: moreover, it satisfies AN, IIO, DEP, Q-IBED, NC and HP but not CD.

$vi)$ Consider the following poverty ranking: $\mathsf{Y} \succcurlyeq_T^{hg_6} \mathsf{Z}$ if and only if $w_1 \cdot h_T(\mathsf{Y}) + w_2 \cdot g_T(\mathsf{Y}) \geq w_1 \cdot h_T(\mathsf{Z}) + w_2 \cdot g_T(\mathsf{Z})$ with $w_1 = t - 2 + \epsilon$, $\epsilon \in \mathbb{R}_+ \setminus \{0\}$, $\epsilon \approx 0$, and $w_2 = 1$. It can be shown that $\succcurlyeq_T^{hg_6}$ is a total preorder that satisfies AN, IIO, DEP, Q-IBED, CD, and HP but fails to satisfy NC.

vii) Consider the poverty ranking $\succcurlyeq_T^{gh}$: clearly enough, it is a total preorder that satisfies AN, IIO, DEP, Q-IBED, CD, and NC, but not HP. ¤

Proof of Proposition 4. (A) The proof replicates almost verbatim the previous one. We reproduce it here for the sake of completeness.

It is easily checked that $\succcurlyeq_T^{gh}$ does indeed satisfy AN, IIO, DEP, Q-IBED, CD, NC and GP.

On the other hand, let $\succcurlyeq_T$ be a poverty ranking and a total preorder that satisfies AN, IIO, DEP, Q-IBED, CD, NC and GP.

Let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} \succcurlyeq_T^{gh} \mathsf{Z}$, then one of the following cases obtains:

a) $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$

b) $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$

c) $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$ and $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$

d) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$

e) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) = h_T(\mathsf{Z})$

Under case a) b), d) $\mathsf{Y} \succ_T \mathsf{Z}$ by **CD**. Under case c), $\mathsf{Y} \succ_T \mathsf{Z}$ by Lemma 1 and NC and HP. In e) by Lemma 1 $\mathsf{Y} \sim_T \mathsf{Z}$. Hence, in any case, $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$.

Conversely, let $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} \succcurlyeq_T \mathsf{Z}$, then the following cases should be distinguished:

1) $g_T(\mathsf{Y}) > g_T(\mathsf{Z})$

2) $g_T(\mathsf{Z}) > g_T(\mathsf{Y})$

3) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$

4) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$

5) $g_T(\mathsf{Y}) = g_T(\mathsf{Z})$ and $h_T(\mathsf{Z}) = h_T(\mathsf{Y})$.

Under case 1), 3), $\mathsf{Y} \succ_T^{gh} \mathsf{Z}$ by definition. Under case 2), two subcases should be distinguished, namely either $h_T(\mathsf{Z}) \geq h_T(\mathsf{Y})$ or $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$. If $h_T(\mathsf{Z}) \geq h_T(\mathsf{Y})$ then, by CD, $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. If, on the contrary, $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ then, by Lemma 1 and NC and GP, $\mathsf{Z} \succ_T \mathsf{Y}$ a contradiction again. Moreover, under case 4), by CD, $\mathsf{Z} \succ_T \mathsf{Y}$ a contradiction. Finally, under case 5), we have that $\mathsf{Y} \sim_T^{gh} \mathsf{Z}$ by definition. Hence, the desired result.

(*B*) The characterization of $\succcurlyeq_T^{gh}$ provided above is also tight. Indeed, consider the following examples.

i) Take the following non-anonymous refinement of the (*GH*)- lexicographic poverty ranking: $\mathsf{Y} \succcurlyeq_T^{gh_1} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{gh} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{gh} \mathsf{Z}$, $\{Y_1, ..., Y_n, Z_1, ..., Z_n\} \nsubseteq \{T, \varnothing\}$ and $\#(Y_1 \cap T) \leq \#(Z_1 \cap T)$). That ranking is a total preorder that satisfies IIO, DEP, Q-IBED, CD, NC and GP but fails to satisfy AN.

ii) Consider the following refinement of the (GH)- lexicographic poverty ranking: $\mathsf{Y} \preccurlyeq_T^{gh_2} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{gh} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{gh} \mathsf{Z}$ and $\sum_{i \in N} \#(Y_i \smallsetminus T) \leq \sum_{i \in N} \#(Z_i \smallsetminus T)$). That ranking is a total preorder that satisfies AN, DEP, Q-IBED, CD, NC and GP but fails to satisfy IIO.

iii) Consider the following poverty ranking: $\mathsf{Y} \preccurlyeq_T^{gh_3} \mathsf{Z}$ if and only if $\sum_{i \in N, Y_i \not\supseteq T} \#(Y_i \cap T) \leq \sum_{i \in N, Z_i \not\supseteq T} \#(Z_i \cap T)$. That ranking is a total preorder that satisfies AN, IIO, Q-IBED, CD, NC and GP but fails to satisfy DEP.

iv) Choose $x^* \in T$, and consider the following refinement of the (GH)- lexicographic poverty ranking: $\mathsf{Y} \preccurlyeq_T^{gh_4} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^{gh} \mathsf{Z}$ or ($\mathsf{Y} \sim_T^{gh} \mathsf{Z}$ and either $[x^* \in (Y_{i^*} \cap Z_{j^*})]$ or $[x^* \notin Y_{i^*}]$, where $i^* = \min\{i \in N : Y_i \not\supseteq T\}$ and $j^* = \min\{i \in N : Z_i \not\supseteq T\}$). That ranking is a total preorder that satisfies AN, IIO, DEP, CD, NC and GP but fails to satisfy Q-IBED.

v) Consider the following poverty ranking: $\mathsf{Y} \preccurlyeq_T^{gh_5} \mathsf{Z}$ if and only if either $[\{Y_i, Z_i\} \subseteq \{T, \emptyset\}$ for each $i \in N$] or [there exists $i \in N$ such that $Z_i \notin \{T, \emptyset\}$]. It can be shown that $\preccurlyeq_T^{gh_5}$ is indeed a total preorder: moreover, it satisfies AN, IIO, DEP, Q-IBED, NC and HP but not CD.

vi) Consider the following poverty ranking: $\mathsf{Y} \preccurlyeq_T^{gh_6} \mathsf{Z}$ if and only if $w_1 \cdot h_T(\mathsf{Y}) + w_2 \cdot g_T(\mathsf{Y}) \geq w_1 \cdot h_T(\mathsf{Z}) + w_2 \cdot g_T(\mathsf{Z})$ with $w_1 = t - 2 - \epsilon$, $\epsilon \in \mathbb{R}_+ \setminus \{0\}$, $\epsilon \approx 0$, and $w_2 = 1$. It can be shown that $\preccurlyeq_T^{gh_6}$ is a total preorder that satisfies AN, IIO, DEP, Q-IBED, CD, and GP but fails to satisfy NC.

vii) Consider the poverty ranking $\preccurlyeq_T^{hg}$: clearly, it is a total preorder that satisfies AN, IIO, DEP, Q-IBED, CD, and NC, but not GP. ¤

Proof of Proposition 5. (A) Checking that $\preccurlyeq_T^{w}$ is a poverty ranking which satisfies AN, IIO, DEP, Q-IBED, CD, and CUC is straightforward. Then, we only need to prove the 'if' part.

First, notice that for any $\mathsf{Y} \in \mathcal{P}[X]^N$, $h_T(\mathsf{Y}), g_T(\mathsf{Y}) \in \mathbb{Z}_+$, $h_T(\mathsf{Y}) \leq n$, and $h_T(\mathsf{Y}) \leqslant g_T(\mathsf{Y}) \leqslant n \cdot t$, where $t = \#T$. Now, take any poverty ranking $\preccurlyeq_T$ that is a total preorder and satisfies AN, IIO, DEP, Q-IBED, CD, and CUC.

We distinguish two basic cases, namely:

Case I: There exist $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $A = (h_T(\mathsf{Y}), g_T(\mathsf{Y})) \neq (h_T(\mathsf{Z}), g_T(\mathsf{Z})) = B$ and $\mathsf{Y} \sim_T \mathsf{Z}$.

Then, observe that all points lying on line joining A and B are $\sim_T$ indifferent. Indeed, $A \sim_T B$ by hypothesis. Then, $A - A \sim_T B - A$, i.e. $O \sim_T B - A$ by CUC. Hence, for any $\lambda > 0$, $O \sim_T \lambda(B - A)$ by CUC , which, in turn, entails $A \sim_T \lambda(B - A) + A$. Similarly, $O \sim_T B - A$ implies that $-(B - A) \sim_T O$. Then, for any $\lambda > 0$, $\lambda(-(B - A)) \sim_T O$, entails $A + \lambda(-(B - A)) \sim_T A$. Let us denote $w_1 x + w_2 y = k$, with $w_1, w_2 \in \mathbb{R}_+ \setminus \{0\}$ and $k \in \mathbb{R}$ the real line joining Y and Z. Moreover, observe that by CUC, $E = (h_T(\mathsf{Y}) + \delta_1, g_T(\mathsf{Y}) + \delta_2) \sim_T (h_T(\mathsf{Z}) + \delta_1, g_T(\mathsf{Z}) + \delta_2) = D$ for any $\delta_1, \delta_2 \in \mathbb{R}$. Therefore, all proper indifference curves are parallel to each other. Of course, there might exist a finite number of isolated points. But, then for each one of them, one can draw a line through it which is parallel to the other indifference curves. Finally, notice that by CD $\mathsf{U} \succ_T \mathsf{V}$ whenever $w_1 h_T(\mathsf{U}) + w_2 g_T(\mathsf{U}) = k_1$, $w_1 h_T(\mathsf{V}) + w_2 g_T(\mathsf{V}) = k_2$ and $k_1 > k_2$. Therefore, $\succcurlyeq_T = \succcurlyeq_T^w$ by definition of $\succcurlyeq_T^w$.

Case II: There is no pair $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) \neq (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$ and $\mathsf{Y} \sim_T \mathsf{Z}$, hence for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) \neq (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$ either $\mathsf{Y} \succ_T \mathsf{Z}$ or $\mathsf{Z} \succ_T \mathsf{Y}$.

Of course, if $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) = (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$ then $w_1 h_T(\mathsf{Y}) + w_2 g_T(\mathsf{Y}) = w_1 h_T(\mathsf{Z}) + w_2 g_T(\mathsf{Z})$ for any $w_1, w_2 \in \mathbb{R}_+$, and, by Lemma 1, $\mathsf{Y} \sim_T \mathsf{Z}$. Moreover, if $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) \geqslant (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$ and $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) \neq (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$, then $w_1 h_T(\mathsf{Y}) + w_2 g_T(\mathsf{Y}) > w_1 h_T(\mathsf{Z}) + w_2 g_T(\mathsf{Z})$ for any $w_1, w_2 \in \mathbb{R}_+ \setminus \{0\}$, and, by CD, $\mathsf{Y} \succ_T \mathsf{Z}$.

Therefore, it suffices to check pairs $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that either:

i) $h_T(\mathsf{Y}) > h_T(\mathsf{Z})$ and $g_T(\mathsf{Z})) > g_T(\mathsf{Y})$ (also denoted, relying on an obvious choice of a Cartesian coordinate system in the real plane, as $\mathsf{Z} \in NW(\mathsf{Y})$ i.e. 'Z is North-West of Y') or

ii) $h_T(\mathsf{Z}) > h_T(\mathsf{Y})$ and $g_T(\mathsf{Y})) > g_T(\mathsf{Z})$ (also denoted as $\mathsf{Y} \in NW(\mathsf{Z})$ i.e. 'Y is North-West of Z').

First, consider any $\succcurlyeq_T$ such that for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succ_T \mathsf{Z}$ whenever $\mathsf{Y} \in NW(\mathsf{Z})$.

It is clearly the case that $w_1 h_T(\mathsf{Y}) + w_2 g_T(\mathsf{Y}) > w_1 h_T(\mathsf{Z}) + w_2 g_T(\mathsf{Z})$ for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Y} \in NW(\mathsf{Z})$, provided

$w_2(g_T(\mathsf{Y}) - g_T(\mathsf{Z})) > w_1(h_T(\mathsf{Z}) - h_T(\mathsf{Y}))$ or equivalently whenever $\frac{w_2}{w_1} > n-1$, i.e. $\frac{w_1}{w_2} < \frac{1}{n-1}$, since $\frac{h_T(\mathsf{Z}) - h_T(\mathsf{Y})}{g_T(\mathsf{Y}) - g_T(\mathsf{Z})} \leq n-1$ for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Y} \in NW(\mathsf{Z})$.

Now, consider any $\succcurlyeq_T$ such that for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, $\mathsf{Y} \succ_T \mathsf{Z}$ whenever $\mathsf{Z} \in NW(\mathsf{Y})$.

Clearly, $w_1 h_T(\mathsf{Y}) + w_2 g_T(\mathsf{Y}) > w_1 h_T(\mathsf{Z}) + w_2 g_T(\mathsf{Z})$ for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Z} \in NW(\mathsf{Y})$, provided $w_1(h_T(\mathsf{Y}) - h_T(\mathsf{Z})) > w_2(g_T(\mathsf{Z}) - g_T(\mathsf{Y}))$, i.e. whenever $\frac{w_1}{w2} > n \cdot (t-1) - t$ since $\frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} \leq (n-1) \cdot t - n = n \cdot (t-1) - t$ for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Z} \in NW(\mathsf{Y})$.

Therefore, it only remains to be considered the case of a total preorder $\succcurlyeq_T$ with the required properties such that there exist $\mathsf{Y}, \mathsf{Z}, \mathsf{Y}', \mathsf{Z}' \in \mathcal{P}[X]^N$, with $\mathsf{Y} \succ_T \mathsf{Z}$ and $\mathsf{Y}' \succ_T \mathsf{Z}'$, while $\mathsf{Y} \in NW(\mathsf{Z})$ and $\mathsf{Z}' \in NW(\mathsf{Y}')$. In this case, posit $m^- = \max\left\{\frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} : \mathsf{Z} \in NW(\mathsf{Y}) \text{ and } \mathsf{Y} \succ_T \mathsf{Z}\right\}$ and $m^+ = \min\left\{\frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} : \mathsf{Z} \in NW(\mathsf{Y}) \text{ and } \mathsf{Z} \succ_T \mathsf{Y}\right\}$ (notice that $m^+$and m^- are both well defined under our special hypothesis on $\succcurlyeq_T$).

Next, we shall prove that $m^- < m^+$ (the proof is along the same lines of that provided in Alcalde-Unzu and Ballester (2005): we reproduce it here for the sake of completeness).

Indeed, let us first suppose that $m^- = m^+$. Then, there exist $\mathsf{Y}, \mathsf{Z}, \mathsf{Y}', \mathsf{Z}' \in \mathcal{P}[X]^N$ such that $\mathsf{Y} \succ_T \mathsf{Z}$ and $\mathsf{Z}' \succ_T \mathsf{Y}'$, $\mathsf{Z} \in NW(\mathsf{Y})$ and $\mathsf{Z}' \in NW(\mathsf{Y}')$, while $\frac{g_T(\mathsf{Z})-g_T(\mathsf{Y})}{h_T(\mathsf{Y})-h_T(\mathsf{Z})} = \frac{g_T(\mathsf{Z}')-g_T(\mathsf{Y}')}{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')} = \rho > 0$.

Then, by some straightforward algebraic manipulations it follows that:
$h_T(\mathsf{Y}') = \left[\frac{g_T(\mathsf{Z}')-g_T(\mathsf{Y}')}{g_T(\mathsf{Z})-g_T(\mathsf{Y})}\right] \cdot h_T(\mathsf{Y}) + \left[h_T(\mathsf{Z}') - (h_T(\mathsf{Z}) \cdot \frac{g_T(\mathsf{Z}')-g_T(\mathsf{Y}')}{g_T(\mathsf{Z})-g_T(\mathsf{Y})})\right]$
$h_T(\mathsf{Z}') = \left[\frac{g_T(\mathsf{Z}')-g_T(\mathsf{Y}')}{g_T(\mathsf{Z})-g_T(\mathsf{Y})}\right] \cdot h_T(\mathsf{Z}) + \left[h_T(\mathsf{Y}') - (h_T(\mathsf{Y}) \cdot \frac{g_T(\mathsf{Z}')-g_T(\mathsf{Y}')}{g_T(\mathsf{Z})-g_T(\mathsf{Y})})\right]$
$g_T(\mathsf{Y}') = \left[\frac{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}{h_T(\mathsf{Y})-h_T(\mathsf{Z})}\right] \cdot g_T(\mathsf{Y}) + \left[g_T(\mathsf{Z}') - (g_T(\mathsf{Z}) \cdot \frac{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}{h_T(\mathsf{Y})-h_T(\mathsf{Z})})\right]$
$g_T(\mathsf{Z}') = \left[\frac{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}{h_T(\mathsf{Y})-h_T(\mathsf{Z})}\right] \cdot g_T(\mathsf{Z}) + \left[g_T(\mathsf{Y}') - (g_T(\mathsf{Y}) \cdot \frac{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}{h_T(\mathsf{Y})-h_T(\mathsf{Z})})\right]$.

Notice that $\frac{g_T(\mathsf{Z})-g_T(\mathsf{Y})}{h_T(\mathsf{Y})-h_T(\mathsf{Z})} = \frac{g_T(\mathsf{Z}')-g_T(\mathsf{Y}')}{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}$ clearly implies $\frac{g_T(\mathsf{Z}')-g_T(\mathsf{Y}')}{g_T(\mathsf{Z})-g_T(\mathsf{Y})} = \frac{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}{h_T(\mathsf{Y})-h_T(\mathsf{Z})}$ whence by some further simple algebra

$$\left[h_T(\mathsf{Z}') - (h_T(\mathsf{Z}) \cdot \frac{g_T(\mathsf{Z}') - g_T(\mathsf{Y}')}{g_T(\mathsf{Z}) - g_T(\mathsf{Y})})\right] = \left[h_T(\mathsf{Y}') - (h_T(\mathsf{Y}) \cdot \frac{g_T(\mathsf{Z}') - g_T(\mathsf{Y}')}{g_T(\mathsf{Z}) - g_T(\mathsf{Y})})\right] =$$

$$= \frac{h_T(\mathsf{Y}) \cdot h_T(\mathsf{Z}') - h_T(\mathsf{Y}') \cdot h_T(\mathsf{Z})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})},$$

and $\left[g_T(\mathsf{Z}') - (g_T(\mathsf{Z}) \cdot \frac{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}{h_T(\mathsf{Y})-h_T(\mathsf{Z})})\right] = \left[g_T(\mathsf{Y}') - (g_T(\mathsf{Y}) \cdot \frac{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}{h_T(\mathsf{Y})-h_T(\mathsf{Z})})\right] =$

$$= \frac{g_T(\mathsf{Y}') \cdot g_T(\mathsf{Z}) - g_T(\mathsf{Y}) \cdot g_T(\mathsf{Z}')}{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}.$$

Therefore,
$(h_T(\mathsf{Y}'), g_T(\mathsf{Y}')) = (\alpha \cdot h_T(\mathsf{Y}) + \beta_1, \alpha \cdot g_T(\mathsf{Y}) + \beta_2)$ and
$(h_T(\mathsf{Z}'), g_T(\mathsf{Z}')) = (\alpha \cdot h_T(\mathsf{Z}) + \beta_1, \alpha \cdot g_T(\mathsf{Z}) + \beta_2)$
with $\alpha = \frac{h_T(\mathsf{Y}')-h_T(\mathsf{Z}')}{h_T(\mathsf{Y})-h_T(\mathsf{Z})} > 0$, $\beta_1 = \frac{h_T(\mathsf{Y})\cdot h_T(\mathsf{Z}')-h_T(\mathsf{Y}')\cdot h_T(\mathsf{Z})}{h_T(\mathsf{Y})-h_T(\mathsf{Z})}$, $\beta_2 = \frac{g_T(\mathsf{Y}')\cdot g_T(\mathsf{Z})-g_T(\mathsf{Y})\cdot g_T(\mathsf{Z}')}{g_T(\mathsf{Z})-g_T(\mathsf{Y})}$.

Thus, CUC applies and $\mathsf{Y} \succ_T \mathsf{Z}$ implies $\mathsf{Y}' <_T \mathsf{Z}'$, a contradiction.

Suppose then that $m^- = \frac{p}{q} > \frac{r}{s} = m^+$ with both $\frac{p}{q}$ and $\frac{r}{s}$ irreducible fractions. Let us consider a few exhaustive subcases, namely

i) $p \geq r$, $q \leq s$. Clearly, it must be the case that $p > r$ or $q < s$ (or both). Then, one may select $\mathsf{U}, \mathsf{V} \in \mathcal{P}[X]^N$ such that $\mathsf{U} \in NW(\mathsf{V})$ and $\frac{g_T(\mathsf{U})-g_T(\mathsf{V})}{h_T(\mathsf{V})-h_T(\mathsf{U})} = \frac{r}{q}$.

If $p = r$ then $q < s$, hence $m^- = \frac{r}{q} > m^+$: thus, consider $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ such that $\mathsf{Y} \in NW(\mathsf{Z})$ and $\frac{g_T(\mathsf{Y})-g_T(\mathsf{Z})}{h_T(\mathsf{Z})-h_T(\mathsf{Y})} = m^-$. By definition of m^-, $\mathsf{Z} \succ_T \mathsf{Y}$. But then, by applying to the pairs (Y, Z),(U, V) the very same argument used above for excluding $m^- = m^+$, and repeating the same calculations, it follows that CUC applies to the effect of implying $\mathsf{V} <_T \mathsf{U}$, hence indeed $\mathsf{V} \succ_T \mathsf{U}$ in view of the general hypothesis of **Case II** we are considering now. On the other hand, consider

$\mathsf{Y}', \mathsf{Z}', \mathsf{V}' \in \mathcal{P}[X]^N$, such that $\mathsf{Y}' \in NW(\mathsf{Z}')$, $\mathsf{Y}' \succ_T \mathsf{Z}'$,

$$\frac{g_T(\mathsf{Y}') - g_T(\mathsf{Z}')}{h_T(\mathsf{Z}') - h_T(\mathsf{Y}')} = \frac{r}{s} = \frac{g_T(\mathsf{U}) - g_T(\mathsf{V}')}{h_T(\mathsf{V}') - h_T(\mathsf{U})} = m^+ < \frac{r}{q} = \frac{g_T(\mathsf{U}) - g_T(\mathsf{V})}{h_T(\mathsf{V}) - h_T(\mathsf{U})},$$

$g_T(\mathsf{U}) - g_T(\mathsf{V}') = r$, and $g_T(\mathsf{V}') = g_T(\mathsf{V})$. Hence, by the same argument as above, CUC applies and implies $\mathsf{U} <_T \mathsf{V}'$(in fact, $\mathsf{U} \succ_T \mathsf{V}'$). Moreover,

$$h_T(\mathsf{V}') = h_T(\mathsf{U}) + s = h_T(\mathsf{U}) + (s - q) + h_T(\mathsf{V}) - h_T(\mathsf{U}) = h_T(\mathsf{V}) + (s - q) > h_T(\mathsf{V}).$$

Thus, by CD, $\mathsf{V}' \succ_T \mathsf{V}$ and by transitivity $\mathsf{U} \succ_T \mathsf{V}$, a contradiction.

If $q = s$ then $p > r$, hence $m^- > \frac{r}{s} = m^+$: consider $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Y} \in NW(\mathsf{Z})$ and $\frac{g_T(\mathsf{Y}) - g_T(\mathsf{Z})}{h_T(\mathsf{Z}) - h_T(\mathsf{Y})} = m^+$. By definition of m^+, $\mathsf{Y} \succ_T \mathsf{Z}$. But then, by applying again to the pairs (Y, Z),(U, V) the same argument used above for excluding $m^- = m^+$, and repeating the same calculations, it follows that CUC applies, implying $\mathsf{U} <_T \mathsf{V}$, hence indeed $\mathsf{U} \succ_T \mathsf{V}$ in view of the general hypothesis of **Case II** we are considering now. On the other hand, consider $\mathsf{Y}', \mathsf{Z}', \mathsf{U}' \in \mathcal{P}[X]^N$, such that $\mathsf{Y}' \in NW(\mathsf{Z}')$, $\mathsf{Y}' \succ_T \mathsf{Z}'$,

$$\frac{g_T(\mathsf{Y}') - g_T(\mathsf{Z}')}{h_T(\mathsf{Z}') - h_T(\mathsf{Y}')} = \frac{r}{q} = \frac{g_T(\mathsf{V}) - g_T(\mathsf{U})}{h_T(\mathsf{V}) - h_T(\mathsf{U})} = m^+ < \frac{p}{q} = \frac{g_T(\mathsf{U}') - g_T(\mathsf{V})}{h_T(\mathsf{V}) - h_T(\mathsf{U}')},$$

$h_T(\mathsf{V}) - h_T(\mathsf{U}') = q$, and $h_T(\mathsf{U}') = h_T(\mathsf{U})$. Hence, by the same argument as above, CUC applies and implies $\mathsf{V} <_T \mathsf{U}'$(in fact, $\mathsf{V} \succ_T \mathsf{U}'$). Moreover,

$g_T(\mathsf{U}') = g_T(\mathsf{V}) + p > g_T(\mathsf{V}) - r = g_T(\mathsf{U})$.

Thus, by CD, $\mathsf{U}' \succ_T \mathsf{U}$ whence by transitivity $\mathsf{V} \succ_T \mathsf{U}$, a contradiction.

If $p > r$ and $q < s$ then $m^- > \frac{r}{q} > m^+$.

In this case, take $\mathsf{Y}, \mathsf{Z}, \mathsf{Y}', \mathsf{Z}', \mathsf{U}', \mathsf{V}' \in \mathcal{P}[X]^N$ such that $\mathsf{Y} \in NW(\mathsf{Z})$, $\mathsf{Y}' \in NW(\mathsf{Z}')$, $\mathsf{Z} \succ_T \mathsf{Y}$, $\mathsf{Y}' \succ_T \mathsf{Z}'$

$$m^+ = \frac{r}{s} = \frac{g_T(\mathsf{Y}') - g_T(\mathsf{Z}')}{h_T(\mathsf{Z}') - h_T(\mathsf{Y}')} = \frac{g_T(\mathsf{U}) - g_T(\mathsf{V}')}{h_T(\mathsf{V}') - h_T(\mathsf{U})} < \frac{g_T(\mathsf{U}) - g_T(\mathsf{V})}{h_T(\mathsf{V}) - h_T(\mathsf{U})} =$$
$$= \frac{r}{q} < \frac{g_T(\mathsf{Y}) - g_T(\mathsf{Z})}{h_T(\mathsf{Z}) - h_T(\mathsf{Y})} = \frac{g_T(\mathsf{U}') - g_T(\mathsf{V})}{h_T(\mathsf{V}) - h_T(\mathsf{U}')} = \frac{p}{q} = m^-,$$

$h_T(\mathsf{V}) - h_T(\mathsf{U}') = q$, $h_T(\mathsf{U}') = h_T(\mathsf{U})$, $g_T(\mathsf{U}) - g_T(\mathsf{V}') = r$, and $g_T(\mathsf{V}') = g_T(\mathsf{V})$. Then, by repeating the previous arguments, we have $\mathsf{V} \succ_T \mathsf{U}'$ and $\mathsf{U} \succ_T \mathsf{V}'$ by CUC; moreover, $g_T(\mathsf{U}') - g_T(\mathsf{V}) = p$, whence $g_T(\mathsf{U}') = g_T(\mathsf{V}) + p = (g_T(\mathsf{U}) - r) + p > g_T(\mathsf{U})$, hence, by CD, $\mathsf{U}' \succ_T \mathsf{U}$, and thus $\mathsf{V} \succ_T \mathsf{U}$ by transitivity. On the other hand, $h_T(\mathsf{V}') - h_T(\mathsf{U}) = s$, hence $h_T(\mathsf{V}') = h_T(\mathsf{U}) + s = (h_T(\mathsf{V}) - q) + s > h_T(\mathsf{V})$: thus, by CD, $\mathsf{V}' \succ_T \mathsf{V}$, hence, by transitivity, $\mathsf{U} \succ_T \mathsf{V}$, a contradiction again.

ii) $p > r, q > s$. Take $\mathsf{Y}, \mathsf{Z}, \mathsf{Z}' \in \mathcal{P}[X]^N$, such that $\mathsf{Y} \in NW(\mathsf{Z})$, $\mathsf{Y} \in NW(\mathsf{Z}')$,

$$\frac{g_T(\mathsf{Y}) - g_T(\mathsf{Z})}{h_T(\mathsf{Z}) - h_T(\mathsf{Y})} = \frac{p}{q} = m^-, \frac{g_T(\mathsf{Y}) - g_T(\mathsf{Z}')}{h_T(\mathsf{Z}') - h_T(\mathsf{Y})} = \frac{r}{s} = m^+.$$

By definition of m^- and m^+, and CUC, $\mathsf{Z} \succ_T \mathsf{Y}$ and $\mathsf{Y} \succ_T \mathsf{Z}'$, hence by transitivity $\mathsf{Z} \succ_T \mathsf{Z}'$. Notice that

$$h_T(\mathsf{Z}) - h_T(\mathsf{Z}') = h_T(\mathsf{Z}) - h_T(\mathsf{Y}) + h_T(\mathsf{Y}) - h_T(\mathsf{Z}') = q - s > 0, \text{ and}$$

$$g_T(\mathsf{Z}') - g_T(\mathsf{Z}) = g_T(\mathsf{Z}') - g_T(\mathsf{Y}) + g_T(\mathsf{Y}) - g_T(\mathsf{Z}) = -r + p > 0,$$

hence $\mathsf{Z}' \in NW(\mathsf{Z})$. Now, observe that

$$\frac{p-r}{q-s} = \frac{p}{q} \cdot (\frac{q}{p} \cdot \frac{p-r}{q-s}) = \frac{p}{q} \cdot (\frac{q \cdot p - q \cdot r}{q \cdot p - p \cdot s}).$$

Since, by definition, $p \cdot s > q \cdot r$, it follows that $(\frac{q \cdot p - q \cdot r}{q \cdot p - p \cdot s}) > 1$, hence $\frac{p-r}{q-s} > \frac{p}{q}$. But then, $\frac{g_T(\mathsf{Z}') - g_T(\mathsf{Z})}{h_T(\mathsf{Z}) - h_T(\mathsf{Z}')} = \frac{p-r}{q-s} > \frac{p}{q} = m^-$. Thus, since $\mathsf{Z}' \in NW(\mathsf{Z})$, it follows, by definition of m^-, that *not* $\mathsf{Z} \succ_T \mathsf{Z}'$, a contradiction.

iii) $p < r$, $q < s$. Take $\mathsf{Y}, \mathsf{Z}, \mathsf{Z}' \in \mathcal{P}[X]^N$, such that $\mathsf{Y} \in NW(\mathsf{Z}), \mathsf{Z}' \in NW(\mathsf{Z})$,

$$\frac{g_T(\mathsf{Y}) - g_T(\mathsf{Z})}{h_T(\mathsf{Z}) - h_T(\mathsf{Y})} = \frac{r}{s} = m^+, \frac{g_T(\mathsf{Z}') - g_T(\mathsf{Z})}{h_T(\mathsf{Z}) - h_T(\mathsf{Z}')} = \frac{p}{q} = m^-.$$

Then, by CUC and the definition of m^+ and m^-, $\mathsf{Y} \succ_T \mathsf{Z}$ and $\mathsf{Z} \succ_T \mathsf{Z}'$, hence by transitivity $\mathsf{Y} \succ_T \mathsf{Z}'$.

On the other hand, $\frac{g_T(\mathsf{Y}) - g_T(\mathsf{Z}')}{h_T(\mathsf{Z}') - h_T(\mathsf{Y})} = \frac{r-p}{s-q} > 0$, i.e. $\mathsf{Y} \in NW(\mathsf{Z}')$, and $\frac{r-p}{s-q} = \frac{r}{s} \cdot (\frac{s}{r} \cdot \frac{r-p}{s-q}) = \frac{r}{s} \cdot (\frac{r \cdot s - p \cdot s}{r \cdot s - r \cdot q})$. Now, since $\frac{p}{q} > \frac{r}{s}$, $p \cdot s > r \cdot q$, whence $(\frac{r \cdot s - p \cdot s}{r \cdot s - r \cdot q}) < 1$. Thus, $\frac{r-p}{s-q} < \frac{r}{s} = m^+$ hence by definition of m^+, *not*$\mathsf{Y} \succ_T \mathsf{Z}'$, a contradiction again.

Summing up the results obtained under cases *i)* - *iii)* above we may conclude that $m^- > m^+$ is also impossible.

Therefore, since we have already shown that $m^- \neq m^+$, it follows that $m^- < m^+$.

Now, take any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Z} \in NW(\mathsf{Y})$, and $\frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} \leq m^-$. Then, $\frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} < m^+$, whence, by definition, *not* $\mathsf{Z} \succ_T \mathsf{Y}$. On the other hand, $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) \neq (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$, hence, by assumption, $\mathsf{Y} \not\sim_T \mathsf{Z}$. Since $\succcurlyeq_T$ is a total preorder, it follows that $\mathsf{Y} \succ_T \mathsf{Z}$. Similarly, take any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, such that $\mathsf{Z} \in NW(\mathsf{Y})$, and $m^+ \leq \frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})}$. Then, $\frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} > m^-$, whence, by definition, *not* $\mathsf{Y} \succ_T \mathsf{Z}$. On the other hand, $(h_T(\mathsf{Y}), g_T(\mathsf{Y})) \neq (h_T(\mathsf{Z}), g_T(\mathsf{Z}))$, hence, by assumption, $\mathsf{Y} \not\sim_T \mathsf{Z}$. Since $\succcurlyeq_T$ is a total preorder, it follows that $\mathsf{Z} \succ_T \mathsf{Y}$, a contradiction. Therefore, for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$, it cannot be the case that $m^- < \frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} < m^+$.

But then, take any $w_1, w_2 \in \mathbb{R}_+ \setminus \{0\}$ such that $m^- < \frac{w_1}{w_2} < m^+$. By our previous observations, for any $\mathsf{Y}, \mathsf{Z} \in \mathcal{P}[X]^N$ (so in particular for any such Y, Z with $\mathsf{Y} \in NW(\mathsf{Z})$) either $\frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} \leq m^-$ or $m^+ \leq \frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})}$. If $\frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} \leq m^-$, then, as shown above, $\mathsf{Y} \succ_T \mathsf{Z}$. Moreover, $\frac{w_2}{w_1} \cdot \frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} < (m^-)^{-1} \cdot m^- = 1$, i.e. $w_2 \cdot (g_T(\mathsf{Z}) - g_T(\mathsf{Y})) < w_1 \cdot (h_T(\mathsf{Y}) - h_T(\mathsf{Z}))$, whence, by definition, $\mathsf{Y} \succ_T^w \mathsf{Z}$ (with $w = (w_1, w_2)$). Conversely, let $\mathsf{Y} \succ_T \mathsf{Z}$. Similarly, if $m^+ \leq \frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})}$, then, as shown above, $\mathsf{Z} \succ_T \mathsf{Y}$ and $\frac{w_2}{w_1} \cdot \frac{g_T(\mathsf{Z}) - g_T(\mathsf{Y})}{h_T(\mathsf{Y}) - h_T(\mathsf{Z})} > (m^+)^{-1} \cdot m^+ = 1$, i.e. $w_2 \cdot (g_T(\mathsf{Z}) - g_T(\mathsf{Y})) > w_1 \cdot (h_T(\mathsf{Y}) - h_T(\mathsf{Z}))$, whence, by definition, $\mathsf{Z} \succ_T^w \mathsf{Y}$ (with $w = (w_1, w_2)$), and the thesis follows.

(B) As the previous ones, the foregoing characterization of $\succcurlyeq_T^w$ is tight, as shown by the following examples:

$i)$ Choose a pair $w = (w_1, w_2)$ of real positive weights, and take the following poverty ranking: $\mathsf{Y} \succcurlyeq_T^{w(1)} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^w \mathsf{Z}$ or ($\mathsf{Y} \sim_T^w \mathsf{Z}$, $\{Y_1, ..., Y_n, Z_1, ..., Z_n\} \nsubseteq \{T, \varnothing\}$ and $\#(Y_1 \cap T) \leq \#(Z_1 \cap T)$). That ranking is a total preorder that satisfies IIO, Q-IBED, CD, NC and CUC but fails to satisfy AN.

$ii)$ Choose a pair $w = (w_1, w_2)$ of real positive weights, and consider the following refinement of the corresponding w-weighted poverty ranking: $\mathsf{Y} \succcurlyeq_T^{w(2)} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^w \mathsf{Z}$ or ($\mathsf{Y} \sim_T^w \mathsf{Z}$ and $\sum_{i \in N} \#(Y_i \smallsetminus T) \geq \sum_{i \in N} \#(Z_i \smallsetminus T)$). That ranking is a total preorder that satisfies AN, DEP, Q-IBED, CD and CUC but fails to satisfy IIO.

$iii)$ Consider again the following poverty ranking: $\mathsf{Y} \succcurlyeq_T^{hg_3} \mathsf{Z}$ if and only if $\sum_{i \in N, Y_i \nsupseteq T} \#(Y_i \cap T) \leq \sum_{i \in N, Z_i \nsupseteq T} \#(Z_i \cap T)$. That ranking is a total preorder that satisfies AN, IIO, Q-IBED, CD and CUC but fails to satisfy DEP.

$iv)$ Choose $x^* \in T$ and a pair $w = (w_1, w_2)$ of real positive weights, and consider the following refinement of the corresponding w- weighted poverty ranking: $\mathsf{Y} \succcurlyeq_T^{w_4} \mathsf{Z}$ if and only if $\mathsf{Y} \succ_T^w \mathsf{Z}$ or ($\mathsf{Y} \sim_T^w \mathsf{Z}$ and either $[x^* \in (Y_{i^*} \cap Z_{j^*})]$ or $[x^* \notin Y_{i^*}]$, where $i^* = \min\{i \in N : Y_i \nsupseteq T\}$ and $j^* = \min\{i \in N : Z_i \nsupseteq T\}$). That ranking is a total preorder that satisfies AN, IIO, DEP, CD and CUC but fails to satisfy Q-IBED.

$v)$ Choose a pair $w = (w_1, w_2)$ of real positive weights, and consider again the following poverty ranking: $\mathsf{Y} \succcurlyeq_T^{gh_5} \mathsf{Z}$ if and only if either $[\{Y_i, Z_i\} \subseteq \{T, \emptyset\}$ for each $i \in N$] or [there exists $i \in N$ such that $Z_i \notin \{T, \emptyset\}$]. It can be shown that $\succcurlyeq_T^{gh_5}$ is indeed a total preorder: moreover, it satisfies AN, IIO, DEP, Q-IBED and CUC, but not CD.

$vi)$ Consider $\succcurlyeq_T^{hg}$ (or $\succcurlyeq_T^{gh}$): clearly both of them are poverty rankings, total preorders, and do satisfy AN, IIO, DEP, Q-IBED and CD,while violating CUC. ¤

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