View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Research Papers in Economics

SEÐLABANKI ÍSLANDS

WORKING PAPER

CENTRAL BANK OF ICELAND

No. 52

Longevity, Growth and Intergenerational Equity - The Deterministic Case

By

Torben M. Andersen and Marias H. Gestsson

October 2010

Central Bank of Iceland Working Papers are published by the Economics Departments of the Central Bank of Iceland. The views expressed in them are those of their authors and not necessarily the views of the Central Bank of Icleand.

Also available on the Central Bank of Iceland World Wide Web site (http://www.sedlabanki.is)

All rights reserved. May be reproduced or translated provided the source is stated.

ISSN 1028-9445

Longevity, Growth and Intergenerational Equity - The Deterministic Case*

Torben M. Andersen
School of Economics and Management, Aarhus University†
CEPR, CESifo and IZA

Marias H. Gestsson
Central Bank of Iceland‡

October 2010

Abstract

Challenges raised by ageing (increasing longevity) have prompted policy debates featuring policy proposals justified by reference to some notion of intergenerational equity. However, very different policies ranging from pre-savings to indexation of retirement ages have been justified in this way. We develop an overlapping generations model in continuous time which encompasses different generations with different mortality rates and thus longevity. Allowing for trend increases in both longevity and productivity, we address the issue of intergenerational equity under a utilitarian criterion when future generations are better off in terms of both material and non-material well being. Increases in productivity and longevity are shown to have very different implications for intergenerational distribution.

JEL classification: J11

Keywords: OLG models, demographics, longevity, taxes, transfers, retirement age, dependency ratio, healthy ageing, decentralization

*The authors wish to thank Ben Heijdra, Jan Rose Skaksen, Bent Jesper Christensen as well as seminar participants at Aarhus University and the Central Bank of Iceland for constructive and helpful comments.

†School of Economics and Management, Aarhus University, Building 1322, DK-8000 Aarhus C, Denmark, E-mail: tandersen@econ.au.dk.

‡Central Bank of Iceland, Kalkofnsvegi 1, 101 Reykjavik, Iceland, E-mail: marias.h.gestsson@sedlabanki.is.

1 Introduction

A trend increase in longevity is a major driver underlying demographic shifts in all OECD countries. According to Wilmoth (2000), longevity (life expectancy at birth) increased from 67 to 72 years for men and 75 to 79 years for women over the period 1970 to 1995. Further, according to recent UN forecasts (see United Nations, 2008) the growth rate of longevity is expected to be 0.2 % per year. Hence, the effects are non-trivial and a main cause behind projected increases in demographic dependency ratios. The policy debate thus centres around the paradox that increases in longevity on the one hand constitute a major welfare improvement[1] and on the other hand threaten the financial viability of various welfare arrangements. To reap the benefits of increased longevity, policy adjustments are needed, and in policy debates it is often stressed that the adjustments made should ensure intergenerational equity. But what is the precise meaning attached to the notion of intergenerational equity when different cohorts have different longevity? An issue which becomes even more complicated when taking into account that future generations may also be richer due to productivity growth.

In policy formulations specific proposals are often justified with reference to intergenerational equity although this seems to lead to very different implications. To illustrate, the UK pension committee interpreted this to imply that retirement ages should be proportional to longevity:

> "Over the long run, fairness between generations suggests that average pension ages should tend to rise proportionately in line with life expectancy, with each generation facing the same proportion of life contributing to and receiving state pensions" (UK Pensions Commission, 2005, p. 4).

In contrast the Swedish fiscal policy framework has taken pre-saving or consolidation of public finances prior to changing demographics to be called for by intergenerational equity:[2]

> "A current high level of public saving is basically motivated by the need to ensure a more equal distribution of consumption possibilities across generations" (Swedish Government, 2008, p 170).

The aim of this paper is to clarify the notion of intergenerational equity when overlapping generations have different mortality and thus longevity. We approach this from a utilitarian perspective (further discussed in section 6) and consider the socially optimal allocation across generations. More precisely, the aim of the paper is to analyse how consumption of produced goods and leisure should be allocated to individuals belonging to different generations. This paper can thus be seen as reverting to the classical paper by Samuelson (1958) considering the social optimal allocation across generations. The novel aspect in this paper is to allow for overlapping generations with

[1] The human development index (HDI) published by the UNDP has longevity to weight by 1/3 (see United Nations Development Programme, 2008).

[2] Balasonne et al. (2009) also argue that intergenerational equity calls for pre-savings. The now common metric of fiscal sustainability S2 (see e.g. European Commission (2006, 2009)) giving the needed permanent change in the primary budget balance implies pre-savings if the underlying demographic changes cause a trend deterioration in the primary budget balance.

different mortality and thus longevity. Rather than appealing to money as a social contrivance, we assume a small open economy facing a given (and time invariant) interest rate at which consumption possibility can be intertemporally substituted (see section 6). This also allows us to avoid the complications arising when endogenizing the capital stock (see Diamond, 1965).

In a seminal paper, Calvo and Obstfeld (1988) consider the role of mortality for the social optimal allocation (utilitarian) in a continuous time setting with an age dependent survival rate.[3] Basic consumption smoothing arguments imply that the social optimum has consumption to be invariant to age. The models presented in Sheshinski (2006, 2008) allow comparison between allocation of two individuals with different survival rates, which gives that consumption smoothing entails redistribution from individuals with high mortality (low longevity) to individuals with low mortality (high longevity). However, the issue of retirement (and thus leisure on par with consumption) is not considered.

This paper makes two important extensions to the abovementioned papers. First, we consider different mortality rates across generations, capturing the empirically observed trend increase in longevity. This implies that demographics are in transition over time/generations, precluding a steady state analysis. The overlap and redistribution across generations with different mortality (longevity) raise particular modelling issues which cannot be handled in standard models or by comparing steady state equilibrium under various assumptions concerning longevity. We present a model in continuous time in which there is overlap of cohorts with different mortality paths and thus longevity.[4] Second, allowing for changes in longevity makes the usual OLG simplification of dividing life length in periods of exogenous length denoted "young" and "old" dubious. We allow for an endogenous determination of these phases of life by including the retirement age as an endogenous variable under rather general specifications of the utility functions, which enables us to capture various age and health effects. The endogeneity of the retirement age implies that the economic environment may display certain stationarity properties, although the underlying demographics do not necessarily do so. To focus on the issue of longevity, we assume fertility to be constant, implying that all demographic shifts are generated by changes in mortality rates.

We link changes in longevity explicitly to changes in mortality rates which are cohort specific. Hence, we allow for a trend increase and an overlap of generations with different survival rates (and thus longevity). The specific modelling of mortality rates is inspired by the approach in Boucekkine et al. (2002) featuring age/cohort specific mortality rates.[5] This approach can be seen as a generalization of the Yaari-Blanchard approach assuming stochastic survival with age independent survival rates (see Blan-

[3] They also allow the social planner to weight the utility of future generations differently than implied by the subjective discount rate of the individuals. In the egalitarian or pure utilitarian case where the two are the same, the flat consumption profile follows.

[4] These issues have also been addressed in standard two-period overlapping generations models where longevity changes are interpreted as extending the length of the second period by Auerbach and Hasset (2007) and Andersen (2008).

[5] Heijdra and Romp (2008) adopt a similar approach. Both modelling approaches give a fairly good approximation to observed mortality rates. The main difference is that here there is a given maximum age, while in the Heijdra and Romp (2008) model survival approaches zero in the limit for a high age.

chard, 1985). The latter is obviously in contradiction to the empirical evidence,[6] and the formulation adopted here captures that mortality rates are (almost) constant (and low) up to a certain age after which they are increasing in age.[7] We take mortality rates to be exogenous to focus on the basic issues on intergenerational equity when different cohorts have different longevity.

This paper is organized as follows: The modelling of demographics including trend changes in mortality rates is laid out in section 2 together with the specification of individual utility functions. The social planner allocation problem under a utilitarian social welfare function is formulated and analysed in section 3, and the optimal allocation is interpreted in section 4. Decentralization of this allocation is discussed in section 5. Finally, a few concluding remarks are given in section 6.

2 An Overlapping Generations Model with Cohort Dependent Longevity

Consider a setting in continuous time where a given (and constant) number of individuals are born at each instant. Individual life time is stochastic, but the fraction of a given cohort surviving is deterministic. The survival rates and thus longevity (life expectancy at birth) are allowed to change over time and thus to differ across cohorts. Hence, overlapping generations alive at a given point in time differ not only in age but also longevity. Life has two phases "young" and "old", where young refers to the phase in life when individuals are working, and old refers to the phased when they are not working (retired). The length of these two phases is endogenous since the retirement age is a choice variable. The social planner (utilitarian) decides on consumption and work (retirement) profiles; that is, allocations across "young" and "old" at a given point in time and across time under the intertemporal resource/budget constraint. The economy is small and open in global capital markets, implying that the interest rate r is exogenous, and for simplicity, assumed time-invariant.[8]

2.1 Demographics

2.1.1 Survival functions

The number of individuals born at each point in time is assumed to be constant and normalized to 1. Following Boucekkine et al. (2002), it is assumed that the unconditional probability for an individual born in time t of reaching age a (the survival

[6] This holds when mortality rates apply to individuals, as is the case in this paper. When these apply to families, the assumption of constant mortality rates is more acceptable as is discussed in Blanchard (1985).

[7] This is in accordance with empirical evidence, see e.g. Wilmoth (2000).

[8] This is a simplifying assumption since increases in longevity are a global phenomenon, and it is likely that increased longevity results in changes in the world interest rates. Whether it results in higher or lower rates is, however, generally uncertain. Some empirical studies do show evidence of a positive relationship between longevity and aggregate savings (see discussion in Sheshinski, 2008). This implies that interest rates decrease as longevity increases.

rate) is given by the following function:[9]

$$\hat{m}\left(a, \beta\left(t\right)\right) = \frac{e^{-a\beta(t)} - \alpha}{1 - \alpha} \tag{2.1}$$

where $\alpha > 1$, $\beta\left(t\right) < 0$ and $a \in \left[0, A\left(t\right)\right]$, where $A\left(t\right)$ is the highest age any member of the cohort born at t can reach., i.e., the age at which $\hat{m}\left(A(t), \beta\left(t\right)\right) = 0$:

$$A\left(t\right) = -\frac{\ln \alpha}{\beta\left(t\right)} \tag{2.2}$$

To incorporate demographic shifts in the model, the parameter $\beta\left(t\right)$ is assumed to be time dependent. Hence, the maximum age $A\left(t\right)$ is also time dependent.

Using (2.2) in (2.1) gives the survival rate as

$$m\left(a, A\left(t\right)\right) = \frac{\alpha^{\frac{a}{A(t)}} - \alpha}{1 - \alpha} \tag{2.3}$$

Note that $m\left(0, A\left(t\right)\right) = 1$, i.e., there is no infant mortality, and $m\left(A\left(t\right), A\left(t\right)\right) = 0$, implying that $A\left(t\right)$ is the highest age any member of the cohort born at t can reach, as is discussed above.

In the following we term this maximum age for a given cohort the longevity for the cohort. This can be justified by noting that life expectancy at birth:

$$v\left(t\right) = \int_{a=0}^{A(t)} a \left[-\frac{\partial m\left(a, A\left(t\right)\right)}{\partial a}\right] da = \left[\frac{\alpha \ln \alpha - \left[\alpha - 1\right]}{\ln \alpha \left[\alpha - 1\right]}\right] A\left(t\right) \tag{2.4}$$

is strictly increasing in the maximum age, i.e., $\frac{\partial v(t)}{\partial A(t)} = \left[\frac{\alpha \ln \alpha - [\alpha - 1]}{\ln \alpha [\alpha - 1]}\right] > 0$[10] [11].

It is easily established that

$$\begin{aligned} \frac{\partial m(a,A(t))}{\partial a} &= -\frac{\frac{\ln \alpha}{A(t)} \alpha^{\frac{a}{A(t)}}}{\alpha - 1} < 0 & \frac{\partial^2 m(a,A(t))}{\partial a^2} &= -\frac{\left(\frac{\ln \alpha}{A(t)}\right)^2 \alpha^{\frac{a}{A(t)}}}{\alpha - 1} < 0 \\ \frac{\partial m(a,A(t))}{\partial A(t)} &= \frac{a \frac{\ln \alpha}{A(t)^2} \alpha^{\frac{a}{A(t)}}}{\alpha - 1} > 0 & \frac{\partial^2 m(a,A(t))}{\partial A(t) \partial a} &= \frac{\frac{\ln \alpha}{A(t)^2} \left(1 + a \frac{\ln \alpha}{A(t)}\right) \alpha^{\frac{a}{A(t)}}}{\alpha - 1} > 0 \end{aligned}$$

The survival rate is strictly decreasing and concave in age. Combining this with the results above implies that $m\left(a, A\left(t\right)\right) \in [0, 1]$, i.e., the survival rates are between zero and one. Moreover, it is strictly increasing in longevity, and the effect of higher longevity on the survival rate is increasing in age. Hence, an increase in longevity of a cohort results in higher survival probabilities at every age, but the increase is greater the older an individual is. These properties are independent of the value of the parameter α (as long as $\alpha > 1$). The survival rate function is illustrated in figure 1. The figure also shows that greater longevity $A\left(\tau\right) > A\left(s\right)$ for a generation born at τ than a generation born at s implies an outward shift in the survival curve from the full line (corresponding to $A\left(s\right)$) to the dotted line (corresponding to $A\left(\tau\right)$), and hence the survival to any age is non-decreasing in longevity.

[9] This function captures the "rectangular" shape of the data based survival curve shown in Wilmoth (2000).

[10] This holds since $\lim_{\alpha \to 1^+} \left[\alpha \ln \alpha - \left[\alpha - 1\right]\right] = 0$, $\frac{\partial[\alpha \ln \alpha - [\alpha - 1]]}{\partial \alpha} = \ln \alpha > 0$ and, hence, $\alpha \ln \alpha - \left[\alpha - 1\right] > 0 \ \forall \ \alpha > 1$.

[11] Note that $-\frac{\partial m(a,A(t))}{\partial a}$ is the unconditional probability of passing away at the age of a for an individual born at time t.

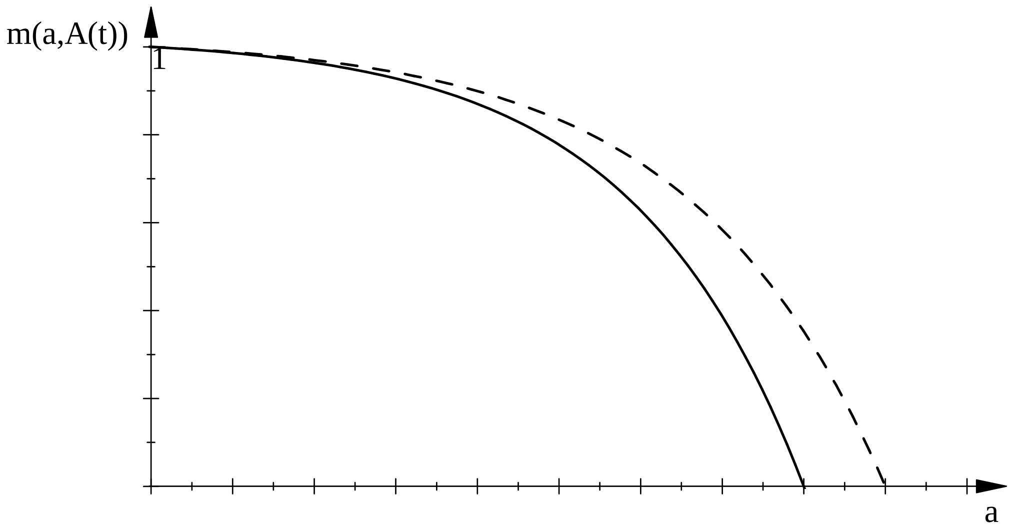

Figure 1: Survival rates for different cohorts with different longevity

In accordance with empirical evidence (see introduction), we assume that there is a trend increase in longevity.[12] Hence, we make the following assumption:

Assumption 1. *Longevity of the generation born at time t, i.e., $A(t)$, follows the process:*

$$dA(t) = \mu_A(t)\,dt \quad ; \quad \mu_A(t) \geq 0 \tag{2.5}$$

Hence, longevity of the generation born at some time t relates to longevity for the generation born at $s < t$ as

$$A(t) = A(s) + \int_{j=s}^{t} \mu_A(j)\,dj \quad for \quad t \geq s$$

Equation (2.5) implies that each new generation has a longevity that is no less than that of the previous generation. Further, it is assumed that $\frac{\partial \mu_A(t)}{\partial t} \leq 0$, i.e., growth in longevity is non-increasing over time.[13] Note that since we can have $\mu_A(t) = 0$ for some t and $\frac{\partial \mu_A(t)}{\partial t} \leq 0$ for all t, it follows that a special case of this setup is one where there is an upper bound to longevity.

As is discussed above, longevity of an individual born at time t is denoted by $A(t)$. Hence, longevity of an individual aged a in time t is denoted by $A(t-a)$ since he is born at time $t-a$. At any point in time, the last person from some generation passes away,

[12] Historically, there has been a trend increase in longevity (see Oeppen and Vaubel (2002)), and demographic evidence does not show signs that human life spans are approaching fixed limits imposed by biology or other factors, see also Wilmoth (2000) and Christensen et al. (2009).

[13] The idea is that the growth in longevity is non-increasing in longevity. Since there is a one-to-one relationship between longevity and time in the model, this is identical to having the growth in longevity non-increasing in time. In appendix E we show how the results of the paper generalize to the case where there is a constant upward trend in longevity to some upper bound:

$$\begin{aligned} dA(t) &= \mu_A dt \quad for \quad A(t) < \overline{A} \\ dA(t) &= 0 \quad for \; A(t) = \overline{A} \end{aligned}$$

All qualitative results in the paper hold under this assumption.

i.e., the generation becomes extinct. At time t this happens for the generation born at time $t-\tilde{A}(t)$ with longevity $\tilde{A}(t)$, i.e., $\tilde{A}(t)$ denotes the longevity of the generation that becomes extinct at time t. Using this and (2.5), longevity of the generation aged a at time t relates to longevity of the generation that becomes extinct at time t as

$$A(t-a)=\tilde{A}(t)+\int\limits_{j=t-\tilde{A}(t)}^{t-a}\mu_A(j)\,dj \quad for \quad \tilde{A}(t)\geq a\geq 0 \tag{2.6}$$

Conveniently, this relation allows us to restate the survival probability function (2.3) in terms of the longevity of the generation that becomes extinct today (at time t), i.e.,

$$\tilde{m}\left(a,\tilde{A}(t)\right)\equiv m\left(a,A(t-a)\right) \tag{2.7}$$

where $A(t-a)$ is given in (2.6). From (2.3) and (2.6), it is obvious that $\tilde{m}\left(0,\tilde{A}(t)\right)=1$ and $\tilde{m}\left(\tilde{A}(t),\tilde{A}(t)\right)=0$. Moreover[14]

$$\frac{\partial\tilde{m}\left(a,\tilde{A}(t)\right)}{\partial a}<0 \quad \frac{\partial^2\tilde{m}\left(a,\tilde{A}(t)\right)}{\partial a^2}<0$$
$$\frac{\partial\tilde{m}\left(a,\tilde{A}(t)\right)}{\partial A(t-A)}>0 \quad \frac{\partial^2\tilde{m}\left(a,\tilde{A}(t)\right)}{\partial A(t-A)\partial a}>0$$

Hence, the $\tilde{m}$-function has similar properties as the m-function, $0\leq\tilde{m}\left(a,\tilde{A}(t)\right)\leq 1$, and it is bounded on $a\in\left[0,\tilde{A}(t)\right]$ and, hence, its integral exists. These properties are important since they ensure that the population size, the number of young and the number of old individuals are well defined in the model.

2.1.2 Population composition

While the birth rate is constant, the population size is not, since survival rates and longevity change. Since, by assumption, 1 individual is born at each point in time, the number of individuals aged a in time t is $\tilde{m}\left(a,\tilde{A}(t)\right)$. Hence, the total population at time t is given as

$$N(t)=\int\limits_{a=0}^{\tilde{A}(t)}\tilde{m}\left(a,\tilde{A}(t)\right)da \tag{2.8}$$

The retirement age of an individual born at time t is denoted by $R(t)$. It follows that we may classify individuals born at time t as *young* when their age is between 0 and $R(t)$, i.e., $a\in[0,R(t)]$, and *old* when their age is between $R(t)$ and the maximum age $A(t)$, i.e., $a\in(R(t),A(t)]$. Further, the retirement age of an individual aged a in time t is denoted by $R(t-a)$ since he is born at time $t-a$. At any point in time, individuals from some generation retire. At time t this happens for the generation born at time $t-\tilde{R}(t)$ with retirement age $\tilde{R}(t)$, i.e., $\tilde{R}(t)$ denotes the retirement age of the generation that retires at time t. It follows that individuals aged between 0 and $\tilde{R}(t)$, i.e., $a\in\left[0,\tilde{R}(t)\right]$, are young at time t, and individuals aged between $\tilde{R}(t)$ and

[14] The derivatives are shown in appendix A.

$\tilde{A}(t)$, i.e., $a \in \left(\tilde{R}(t), \tilde{A}(t)\right]$, are old at time t. Note that the retirement age is allowed to depend on time and thus to be cohort specific.

The number of young (working) and old (retired) individuals, respectively, at time t is therefore

$$N_w(t) = \int_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da = N_w\left(\tilde{R}(t), \tilde{A}(t)\right) > 0 \tag{2.9}$$

$$N_o(t) = \int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da = N_o\left(\tilde{R}(t), \tilde{A}(t)\right) > 0 \tag{2.10}$$

Hence, both the number of young and the number of old individuals at time t can be expressed as functions of longevity of the generation that becomes extinct at time t, i.e., $\tilde{A}(t)$, and the retirement age of the generation that retires at time t, i.e., $\tilde{R}(t)$. Obviously, total population size is given as $N(t) = N_w(t) + N_o(t)$ as can be verified from (2.8), (2.9) and (2.10).

The dependency ratio[15] is defined as the ratio between the number of old and young individuals:

$$K(t) = \frac{N_o(t)}{N_w(t)} = \frac{N_o\left(\tilde{R}(t), \tilde{A}(t)\right)}{N_w\left(\tilde{R}(t), \tilde{A}(t)\right)} \equiv K\left(\tilde{R}(t), \tilde{A}(t)\right) \tag{2.11}$$

Note that even though demographic stock variables need not be stationary due to a trend increase in longevity, it follows that compositional variables like the dependency ratio may be stationary since they depend crucially on how the retirement age responds to demographic changes. This turns out to be crucial below.

We can now work out how the upward trend in longevity (2.5) affects the key demographic variables $N(t)$, $N_w(t)$, $N_o(t)$ and $K(t)$. We have (see appendix B) that both the number of young and the number of old increase (for a given retirement age $\tilde{R}(t)$). The latter is straightforward, and the former arises because an increase in longevity implies a decrease in mortality rates at all ages (see figure 1). However the number of old individuals increases by more than the number of young individuals, and hence we have that the dependency ratio unambiguously increases. In summary, we have:

$$\begin{aligned} dN(t) &> 0 \\ dN_w(t) &> 0 \\ dN_o(t) &> 0 \\ dK(t) &> 0 \end{aligned}$$

Note that $\tilde{R}(t)$ is endogenous in the social planner's maximization problem and, hence, so are $N_w(t)$, $N_o(t)$ and $K(t)$. This prevents the dependency ratio $K(t)$ becoming infinitely large over time.

[15] This may be termed the economic dependency ratio since it depends on the retirement age which is endogenous, and not some exogenously given age.

2.2 Individual utility

Individuals consume as young and old, but work as young only. Utility is in the standard way specified over consumption and leisure. However, allowing for changes in mortality rates and thus implicitly in health status across different generations implies that one has to consider carefully the specification of the utility function to capture essential age and health effects on the disutility from work and the value of leisure time. The standard approach features age independent and constant disutility from work (exogenous working hours) or utility from leisure, and hence retirement involves a trade-off between consumption and leisure along the extensive margin (see e.g. Sheshinski (1978), Crawford and Lilien (1981) and Kalemli-Ozcan and Weil (2002)). While this trade-off is fundamental, it is problematic to assume disutility from work and utility from leisure to be age invariant, in particular when analysing changes in mortality and longevity which are clearly related to health. We propose a more general formulation allowing for both age and health effects on the disutility from work as well as utility from other forms of time uses. This is summarized in the utility generated from time uses taking into account various possible time consuming activities like work, leisure activities, rest etc. We make the assumption that the value of time (disutility from work and value of time spent on leisure activities) for working and retired persons depends negatively on age a and positively on health captured by longevity $A(t-a)$.

The utility function for a young (working) individual aged a at time t is given as a separable function specified over consumption[16] $c_w(t)$ and time, i.e.,

$$W_w(t,a) = U(c_w(t)) + L(a, A(t-a)) \quad \forall \quad a \in \left[0, \tilde{R}(t)\right] \tag{2.12}$$

where U is strictly increasing and concave. The function L captures the value of time[17] taking into account both leisure and working time for the young individual.[18] The value of time is assumed to be decreasing in age and increasing in longevity

[16] Assuming that the marginal utility of consumption is dependent on age would create a question of optimal distribution of taxes (consumption) among young individuals, on one hand, and among old individuals, on the other. Hence, we assume that it is independent of age as is implied by the separable formulation adopted here.

[17] Think of time being spent on three main types of activities: (i) work (h_w), (ii) leisure activities such as travel, sports, hobbies, home production (h_t) and (iii) rest (h_r), where $h_w + h_t + h_r = 1$; i.e. the available time is normalized to one, and the various time uses are exogenous. Using the time constraint, the utility from time use measured relative to time spent on the fall back option "rest" can be written as

$$L(a,A) = \kappa_w(a,A)\left[h_w - h_r\right] + \kappa_t(a,A)\left[h_t - h_r\right]$$

where κ_w and κ_t give the respective marginal values of time spent on work and leisure activities. Assuming that $\frac{\partial \kappa_w}{\partial a} < 0$ and $\frac{\partial \kappa_t}{\partial a} < 0$ corresponds to less utility from work (more disutility from work) and leisure activities with age, while $\frac{\partial \kappa_w}{\partial A} > 0$ and $\frac{\partial \kappa_t}{\partial A} > 0$ implies that the utility from work (disutility decreases) and utility from leisure activities increase at any age a when longevity increases. Note that the formulation allows for $\kappa_w(a,A)$ being positive for some values of (a,A), implying that people up to some limit like to work; e.g. to use skills and qualification acquired in education.

[18] Bloom et al. (2006) use a similar specification to capture the disutility from work: $v(z,t)$, where z is life expectancy, or longevity, and t is age. They propose that the v function is homogeneous of degree zero, which can be interpreted as reflecting healthy ageing (see section 4).

(capturing health). Hence, we have that

$$\frac{\partial L\left(a, A\left(t-a\right)\right)}{\partial a} < 0 \tag{2.13}$$

$$\frac{\partial L\left(a, A\left(t-a\right)\right)}{\partial A\left(t-a\right)} > 0 \tag{2.14}$$

The idea behind this is that health and the ability to enjoy time (working or spent on leisure activities etc.) worsen with age, while increasing with longevity. Hence, the value of time for a young individual aged a is greater when longevity of his generation is $A_1(t-a)$ years than if it is $A_2(t-a) < A_1(t-a)$ years.[19] This can be justified using figure 1, which shows that increased longevity results in increased survival rates at all ages implying better health, while survival rates decrease with age implying worse health.

The utility of an old (retired) individual aged a at time t is given as

$$W_o(t,a) = Q(c_o\left(t\right)) + H\left(a, A\left(t-a\right)\right) \quad \forall \quad a \in \left(\tilde{R}(t), \tilde{A}\left(t\right)\right] \tag{2.15}$$

where Q is strictly increasing and concave in consumption $c_o\left(t\right)$ and the function H captures the value of time to old individuals (similar to the L function for the young). Using the same justification as for young, it is assumed that the value of time to old individuals is decreasing in age but increasing in longevity:

$$\frac{\partial H\left(a, A\left(t-a\right)\right)}{\partial a} < 0 \tag{2.16}$$

$$\frac{\partial H\left(a, A\left(t-a\right)\right)}{\partial A\left(t-a\right)} > 0 \tag{2.17}$$

The specification here allows for different utility functions defined over consumption and time for young and old. While the latter follows straightforwardly given disutility from work, the former is open for discussion.[20] In what follows, it is assumed that

$$\frac{\partial L\left(a, A\left(t-a\right)\right)}{\partial a} < \frac{\partial H\left(a, A\left(t-a\right)\right)}{\partial a} \tag{2.18}$$

$$\frac{\partial L\left(a, A\left(t-a\right)\right)}{\partial A\left(t-a\right)} > \frac{\partial H\left(a, A\left(t-a\right)\right)}{\partial A\left(t-a\right)} \tag{2.19}$$

Note that retirement in general involves a trade-off between consumption (life-time income increases with the retirement age) and utility from time uses (disutility from work). Consider the latter which involves the utility from time uses as young and old (L and H), respectively, where both an age and a longevity effect are involved. The age effect (2.18) says that the value of time as working decreases more than the value of time as retired; that is, the gain in the value of time as retired relative to working

[19] In the case of the function being homogenous of degree zero, as is proposed in Bloom et al. (2006), the value of time for a 40 year old individual with longevity 80 years for his generation is the same as the value of time for a 50 year old with longevity 100 years for his generation.

[20] As a special case, we also present results for the case where the utility defined over consumption is the same for young and old below.

increases with age[21]. The longevity effect (2.19) says that increased longevity tends to increase the value of time more as working than retired; that is, the gain in value of time of retiring at a given age decreases with longevity[22]. The latter captures a health effect in the sense that higher longevity is associated with better health reducing the disutility from work at a given age, ceteris paribus. Note that assuming that the utility from time when young is greater than when old ($L > H$) implies that (2.18) ensures that, at some age level a^*, the value of time as old becomes greater than the value of time as young (working), i.e., $L < H$ for all $a > a^*$ which provides a "leisure" motive to retirement.

In the following it turns out to be more convenient to analyse the model when the value of leisure is expressed in terms of the longevity of the generation becoming extinct at time t, i.e., $\tilde{A}(t)$. By use of (2.5) we have that

$$\begin{aligned}\tilde{L}\left(a, \tilde{A}(t)\right) &\equiv L(a, A(t-a)) \\ \tilde{H}\left(a, \tilde{A}(t)\right) &\equiv H(a, A(t-a))\end{aligned}$$

where $A(t-a)$ is given in (2.6), and hence utilities for young and old are given as

$$\begin{aligned}\tilde{W}_w(t, a) &= U(c_w(t)) + \tilde{L}\left(a, \tilde{A}(t)\right) \\ \tilde{W}_o(t, a) &= Q(c_o(t)) + \tilde{H}\left(a, \tilde{A}(t)\right)\end{aligned}$$

where we, by use of (2.6) and the properties of the L and H functions, have

$$\begin{aligned}\frac{\partial \tilde{L}}{\partial a} &= \frac{\partial L}{\partial a} - \mu_A(t-a)\frac{\partial L}{\partial A(t-a)} < 0 \\ \frac{\partial \tilde{L}}{\partial \tilde{A}(t)} &= \left[1 + \mu_A\left(t - \tilde{A}\right)\right]\frac{\partial L}{\partial A(t-a)} > 0 \\ \frac{\partial \tilde{H}}{\partial a} &= \frac{\partial H}{\partial a} - \mu_A(t-a)\frac{\partial H}{\partial A(t-a)} < 0 \\ \frac{\partial \tilde{H}}{\partial \tilde{A}(t)} &= \left[1 + \mu_A\left(t - \tilde{A}\right)\right]\frac{\partial H}{\partial A(t-a)} > 0\end{aligned}$$

It is easily verified that this changes none of the qualitative insights from above. Note that (2.18) and (2.19) are readily shown to hold for the modified values of time for young and old ($\widetilde{L}$, $\widetilde{H}$), i.e., $\frac{\partial \tilde{L}}{\partial a} < \frac{\partial \tilde{H}}{\partial a}$ and $\frac{\partial \tilde{L}}{\partial \tilde{A}(t)} > \frac{\partial \tilde{H}}{\partial \tilde{A}(t)}$ hold.

2.3 Productivity

The earnings capability is assumed exogenous but to be affected by productivity growth. Hence, we set forth the following assumption:

[21] This could, for example, be due to the fact that working often becomes more physically challenging when individuals become older and their health worsens.

[22] This is consistent with empirical evidence found by Haliday and Podor (2009) showing that improvements in health status have large and positive effects on time allocated to home and market production and large negative effects on time spent on watching TV, sleeping, and consumption of other types of leisure activities.

Assumption 2. *Output of a young individual at time t, i.e., $y(t)$, follows the process*

$$dy(t) = \mu_y y(t)\, dt \quad ; \quad \mu_y > 0 \tag{2.20}$$

Total output in the economy $N_w(t)\, y(t)$ is endogenous since $N_w(t)$ is endogenous. Note that productivity is only included for comparative purposes, allowing a comparison of intergenerational distribution arising due to different longevity or economic possibilities across cohorts.

2.4 Planner allocation

The social planner decides on consumption and work (retirement age). It is easiest to characterize and interpret the social planner allocation if it is cast in terms of taxes and transfers relative to the reference outcome where individuals consume their labour income. Hence, consumption of a young and an old individual, respectively, at time t is:

$$c_w(t) = y(t) - T_w(t) \tag{2.21}$$

$$c_o(t) = T_o(t) \tag{2.22}$$

where $T_w(t)$ is net taxes paid by a young individual, and $T_o(t)$ is net transfers to an old individual at time t. We cast the model in this way both because it is analytically more tractable and because it gives a simple relation to the analysis of social security schemes in standard two-period OLG models. Further, since $y(t)$ is exogenous, choosing $T_w(t)$ and $T_o(t)$ is tantamount to choosing $c_w(t)$ and $c_o(t)$ in the model.

We have made the informational restriction that the social planner can not make taxes and transfers age dependent, but only dependent on labour market status (working or not working). This can be motivated in terms of information- and transactions costs, which apparently are large in reality since actual schemes in general satisfy the condition imposed here. Hence, at a given point in time, the social planner has to collect the same amount of tax from each young (working) individual and give the same amount of transfer to each old (retired)[23]. Since output of each young individual is only time dependent, it follows from (2.21) and (2.22) that, at a given point in time, consumption of each young and each old individual is independent of age. This fits with the specification of the utility functions from above; i.e., that the marginal utility from consumption is independent of age (see $U(\cdot)$ and $Q(\cdot)$ in (2.12) and (2.15), respectively).

The policy package thus includes three elements, namely, the tax levied on the young (working), the transfer to the old (retired) and the retirement age. The policy problem is thus to choose $\left\{T_w(i), T_o(i), \tilde{R}(i)\right\}_{i=t}^{\infty}$ at each point in time t.

The primary budget balance of this scheme in any period t reads

$$B(t) = N_w(t)\, T_w(t) - N_o(t)\, T_o(t) \tag{2.23}$$

[23] Hence, the social planner is not able to choose a consumption profile for each generation at the birth of the generation as in Calvo and Obstfeld (1988).

while the intertemporal budget constraint at time t can be written as[24]

$$D(t) = \int_{i=t}^{\infty} e^{-r(i-t)} B(i)\, di \tag{2.24}$$

where r is the interest rate. From (2.24), debt dynamics can be written

$$\begin{aligned} dD(t) &\equiv D(t+dt) - D(t) \\ &= r dt D(t) - B(t)\, dt \\ &= [rD(t) - B(t)]\, dt \end{aligned} \tag{2.25}$$

where it is used that $e^r \approx 1 + r$.

3 Social Optimum

We consider the social optimum under a utilitarian criterion.[25] While this criterion is neither unproblematic nor uncontroversial, it is useful to illustrate some basic trade-offs arising, and we consider it as a useful benchmark case for studying intergenerational distribution, cf. discussion in section 6.

The objective of the social planner at time t is to maximize the sum of the present values of lifetime utilities of generations born at time t or later and the present value of utilities for the remaining lives of generations alive at time t. Hence, the social welfare function is[26]

$$W(t) = \int_{i=t}^{\infty} e^{-\theta(i-t)} Q(i)\, di + \int_{i=t-\tilde{A}(t)}^{t} e^{-\theta(i-t)} Q(i)\, di \tag{3.1}$$

where

$$\begin{aligned} Q(i) = & \int_{a=0}^{R(i)} e^{-\theta a} m(a, A(i))\, W_w(i+a) da \\ & + \int_{a=R(i)}^{A(i)} e^{-\theta a} m(a, A(i))\, W_o(i+a) da \end{aligned}$$

[24]This constraint makes the economy-wide resource constraint hold.

[25]Note that it is more accurate to call this a constrained social optimum since the social planner is constrained by having to collect the same amount of tax from each young individual and give the same amount of transfer to each old one.

[26]The time separability and exponential discounting following Yaari (1965) imply that problems of time inconsistency do not arise and that agents display risk neutrality with respect to the length of life (see Bommier (2006) and Bommier et al. (2009)).

for all $i \geq t$,

$$Q(i) = \int_{a=t-i}^{R(i)} e^{-\theta a} m(a, A(i)) W_w(i+a) da + \int_{a=R(i)}^{A(i)} e^{-\theta a} m(a, A(i)) W_o(i+a) da$$

for all $t > i \geq t - \tilde{R}(t)$ and

$$Q(i) = \int_{a=t-i}^{A(i)} e^{-\theta a} m(a, A(i)) W_o(i+a) da$$

for all $t - \tilde{R}(t) > i \geq t - \tilde{A}(t)$, where $W_w(i+a)$ and $W_o(i+a)$ are given in (2.12) and (2.15). As is shown in appendix C, (3.1) is well defined.

We assume that the social planner discounts at the same subjective rate as individuals (θ) which may be considered the "pure" utilitarian case where utility achieved at any point in time gets the same weight irrespective of who obtains the utility. Under the utilitarian criterion, (3.1) gives the same welfare measure as is obtained by calculating the present value of instantaneous utility generated to all living individuals (see Calvo and Obstfeld (1988)). The social welfare function can therefore be written as

$$W(t) = \int_{i=t}^{\infty} e^{-\theta(i-t)} Z(i) di \tag{3.2}$$

where

$$Z(i) = \int_{a=0}^{\tilde{R}(i)} \tilde{m}\left(a, \tilde{A}(i)\right) \tilde{W}_w(i,a) da + \int_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}\left(a, \tilde{A}(i)\right) \tilde{W}_o(t,a) da$$

where (2.6) and (2.7) are used.

The problem facing the policy maker is

$$\underset{\left\{T_w(i), T_o(i), \tilde{R}(i)\right\}_{i=t}^{\infty}}{Max} W(t) \tag{3.3}$$

subject to the budget constraint from (2.24) (and (2.23)) and given (2.5) and (2.20).

The problem is solved by setting up the Hamilton-Jacobi-Bellman (HJB) equation which in short hand writing (suppressing time indexes) determines the value function

$V(\cdot)$ as

$$\begin{aligned}
\theta V\left(D,y,\tilde{A}\right) &= \underset{T_w,T_o,\tilde{R}}{Max}\left\{\begin{array}{c}\int\limits_{a=0}^{\tilde{R}} \tilde{m}\left(a,\tilde{A}\right)\left[U\left(y-T_w\right)+\tilde{L}\left(a,\tilde{A}\right)\right]da \\ +\int\limits_{a=\tilde{R}}^{\tilde{A}} \tilde{m}\left(a,\tilde{A}\right)\left[Q\left(T_o\right)+\tilde{H}\left(a,\tilde{A}\right)\right]da \\ +\frac{1}{dt}dV\left(D,y,\tilde{A}\right)\end{array}\right\} \\
& \quad s.t. \\
dD &= \left[rD-\left[N_w\left(\tilde{R},\tilde{A}\right)T_w-N_o\left(\tilde{R},\tilde{A}\right)T_o\right]\right]dt \\
d\tilde{A} &= \hat{\mu}_A dt \\
dy &= \mu_y y dt
\end{aligned} \tag{3.4}$$

with N_w and N_o given from (2.9) and (2.10), respectively, and $\hat{\mu}_A \equiv \frac{\mu_A}{1+\mu_A}$[27]. Assuming an interior solution, this gives the following first-order conditions for the optimal T_w, T_o and $\tilde{R}$, respectively:[28]

$$V_D(\cdot) = -U'\left(y-T_w\right) \tag{3.5}$$

$$V_D(\cdot) = -Q'\left(T_o\right) \tag{3.6}$$

$$\left[T_w+T_o\right]V_D(\cdot) = U\left(y-T_w\right)+\tilde{L}\left(\tilde{R},\tilde{A}\right)-Q\left(T_o\right)-\tilde{H}\left(\tilde{R},\tilde{A}\right) \tag{3.7}$$

Further, this gives the following law of motion for the marginal value function:

$$dV_D(\cdot) = (\theta-r)\,V_D dt \tag{3.8}$$

3.1 Neutral generational weighting

The relation between the subjective (θ) and the objective (r) discount rates determines whether the marginal value function is decreasing or increasing over time, cf. (3.8). Similar to models of intertemporal consumption choices, a subjective discount rate higher (lower) than the objective discount rate implies a profile for consumption that is decreasing (increasing) over time, see e.g. Blanchard and Fischer (1989). This source of reallocation across time and thus generations is standard. In the following we therefore only consider the case where the subjective and objective (world market) discount rates are identical $\theta = r$. This may also be interpreted as a case of neutral generational weighting in the sense that the subjective and objective discount rates are equal. Hence, discounting per se does not imply any profile benefitting current or future

[27] Setting $a = 0$ in (2.6) gives the relationship between longevity of the generation born today and the generation that passes away today. Taking total difference of this, using (2.5) and rearranging gives $d\tilde{A}(i) = \frac{\mu_A\left(i-\tilde{A}\right)}{1+\mu_A\left(i-\tilde{A}\right)}di$.

[28] The first-order conditions and the law of motion for the marginal value function are derived in appendix D. Further, it is shown there that (i) the first-order conditions are necessary and sufficient for solving the maximization problem in (3.4) and (ii) the HJB equation and the budget constraint in (2.24) (and (2.23)) are necessary for solving (3.3).

generations in a particular way. In short we refer to this case as neutral generational weighting. Under this assumption we have (relaxing the short hand writing)

$$dV_D\left(D(i), y(i), \tilde{A}(i)\right) = 0 \tag{3.9}$$

for all $i \geq t$, i.e., the optimal policy package is such that the marginal value function $V_D(\cdot)$ is the same for all $i \geq t$.

Applying this to (3.5)-(3.7) gives

$$V_D(t) = -U'(y(i) - T_w(i)) \tag{3.10}$$
$$V_D(t) = -Q'(T_o(i)) \tag{3.11}$$

and

$$\begin{aligned} & [T_w(i) + T_o(i)]\, V_D(t) \\ = \; & U(y(i) - T_w(i)) + \tilde{L}\left(\tilde{R}(i), \tilde{A}(i)\right) \\ & -Q(T_o(i)) - \tilde{H}\left(\tilde{R}(i), \tilde{A}(i)\right) \end{aligned} \tag{3.12}$$

for all $i \geq t$, where $V_D(t)$ is written as a function of t to indicate that it is the same for all $i \geq t$.

3.2 Optimal policy package

An optimal policy package $\left\{T_w(i), T_o(i), \tilde{R}(i)\right\}_{i=t}^{\infty}$ must satisfy (3.10)-(3.12) as well as the intertemporal budget constraint

$$D(t) = \int_{i=t}^{\infty} e^{-r(i-t)} \begin{bmatrix} N_w\left(\tilde{R}(i), \tilde{A}(i)\right) T_w(i) \\ -N_o\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(i) \end{bmatrix} di \tag{3.13}$$

where $N_w\left(\tilde{R}(i), \tilde{A}(i)\right)$ and $N_o\left(\tilde{R}(i), \tilde{A}(i)\right)$ are given in (2.9) and (2.10). Hence, the optimal policy package satisfies conditions for fiscal sustainability (see e.g. European Commission, 2006).

Proposition 1. *The social optimum implies that the following holds for any growth rates of longevity $\mu_A(i)$ and output μ_y from assumptions 1 and 2, respectively:*

(i) Taxes and transfers:

$$\begin{aligned} T_w(i) &= T_w(t) + y(t)\left[e^{\mu_y[i-t]} - 1\right] \\ T_o(i) &= T_o(t) \end{aligned} \tag{3.14}$$

for all $i \geq t$.

(ii) Consumption:

$$\begin{aligned} c_w(i) &= c_w(t) \\ c_o(i) &= c_o(t) \end{aligned} \tag{3.15}$$

for all $i \geq t$.

(iii) Retirement:

$$d\tilde{R}(i) = \eta_y(i)\,dy(i) + \eta_A(i)\,d\tilde{A}(i)$$
where
$$\eta_y(i)\,,\,\eta_A(i) > 0 \tag{3.16}$$

for all $i \geq t$.

Proof. *See appendix F.*

According to proposition 1, the tax payment of young individuals is given as some time invariant component plus the income growth since time t, i.e., all income growth is fully taxed. Old individuals receive a time invariant transfer. This implies that the consumption level of both the young and the old is constant over time. Income growth is thus smoothed across generations; that is, it affects the overall level of consumption but not the profile (note the assumption $\theta = r$). To put it differently, current and future generations all share the gains from future productivity increases in terms of a higher consumption level.

According to (3.10) and (3.11), the optimal policy implies that marginal utility of consumption is equal for young and old, i.e.,

$$U'(y(i) - T_w(i)) = Q'(T_o(i))$$

or

$$U'(c_w(i)) = Q'(c_o(i))$$

from (2.21) and (2.22), capturing the well-known finding that a utilitarian policy maker redistributes to ensure equal marginal utilities of consumption (income). If utility functions over consumption are the same for young and old $U(\cdot) = Q(\cdot)$, it is an immediate implication that they will have the same consumption level. This generalizes the Calvo and Obstfeld (1988) result to a setting with time varying mortality rates and thus different longevity across generations (and income growth). If the marginal utility of consumption is higher when working (for young individuals), the solution implies that the consumption level for young individuals is greater than that of old ones.

The results above on consumption are neither surprising nor new since they follow straightforwardly from standard consumption smoothing arguments under separable utility. More interesting and novel are the implications for the retirement age. Two issues are important here, namely, the level of the retirement age and its profile over time (due to growth in productivity and longevity). The former is discussed in section 5 where allocation under social optimum is compared to allocation under individual decision making, while results for the latter are stated in proposition 1 and discussed below.

Consider the two drivers, productivity growth and increasing longevity. First, we have that productivity growth unambiguously implies that the retirement age should be increasing over time since $\frac{\partial \tilde{R}(i)}{\partial y(i)} > 0$ from (3.16) and since output is increasing over time from (2.20). The intuition is that the generations with the higher productivity should work more than generations with lower productivity. Hence, while productivity growth implies a front-loading of consumption in the sense that both current and future generations benefit from productivity growth in terms of a higher consumption level, it implies a rising profile for the retirement age and thus a shift of the work load onto future generations.

Second, the effect of increasing longevity on the retirement age implies that the retirement age should increase over time since $\frac{\partial \tilde{R}(i)}{\partial \tilde{A}(i)} > 0$ from (3.16). Using (2.6), this implies that

$$\frac{\partial \tilde{R}\left(i\right)}{\partial A\left(i-\tilde{R}\right)} > 0 \tag{3.17}$$

that is, generations with greater longevity should retire later. This reflects that higher longevity (better health) decreases the direct utility gain from retirement (from (2.19)), and therefore retirement is delayed. The retirement age increases basically due to better health measured in terms of the value of time. Hence, although longevity does not influence the consumption profiles for working and retired persons, it does influence retirement ages.

To conclude, longevity and productivity do not affect the socially optimal consumption profiles (development of over time) while they do affect the retirement age profile.

4 Implications

As discussed in the introduction, policies are often motivated by reference to some notion on intergenerational equity, although this has been interpreted in quite different ways. This section interprets the properties of the social optimum in relation to some of these policy views.

4.1 Retirement and healthy ageing

Consider first the perception that intergenerational equity implies that the retirement age should evolve proportionally to longevity. We consider under which assumptions this is implied by the socially optimal policy derived above and how these are related to the notion of healthy ageing. This argument involves only longevity, and we therefore disregard productivity growth, i.e., $dy\left(i\right) = 0$.

The first result is given in the following lemma:

Lemma 1. *Given homogeneity of the $\tilde{L}$ and $\tilde{H}$ functions, the social optimum implies that*

$$\tilde{R}\left(i\right) = \kappa \tilde{A}\left(i\right) \tag{4.1}$$

for all $i \geq t$ where κ is some constant (> 0), iff the $\tilde{L}$ and *$\tilde{H}$ functions are homogeneous of degree zero.*

Proof. *See appendix G.*

Hence, for the class of homogeneous $\tilde{L}$ and $\tilde{H}$ functions, only those that are homogeneous of degree zero imply a retirement age proportional to longevity in the social optimum.

Lemma 1 gives the conditions that need to be fulfilled for the social optimum to imply that the retirement age should be proportional to longevity of the generation that becomes extinct today. Intuitively the retirement age should be related to the longevity of the generation retiring rather than the longevity of the generation becoming extinct. The following lemma shows an equivalence:

Lemma 2. *Given constant growth in longevity, there exist* κ *and* ψ *(*$< \kappa$*) such that*

$$\tilde{R}(i) = \kappa \tilde{A}(i) \quad iff \quad \tilde{R}(i) = \psi A\left(i - \tilde{R}\right)$$

for all $i \geq t$.

Proof. *See appendix G.*

Hence, given constant growth in longevity, a retirement age proportional to longevity of the generation that becomes extinct today ($\tilde{A}(i)$) is equivalent to the retirement age being proportional to the longevity of the generation that retires today ($A\left(i - \tilde{R}\right)$). Therefore lemma 1 also gives the necessary conditions that need to be fulfilled for the social optimum to imply the retirement age being proportional to longevity of the generation that retires today.

Consider healthy ageing in the sense that the health conditions at a given age improve proportionally with longevity. That is, the direct utility consequences of retirement at age a_0 for a person belonging to a generation with longevity $A(0)$ are the same as the consequences of retiring at an age $a_1 = a_0 \frac{A(1)}{A(0)}$ for a person from a generation with longevity $A(1)$ ($\neq A(0)$).[29] This is implied in the present setup if the L and H functions are homogeneous of degree zero,[30] which is implied by zero homogeneity of the $\tilde{L}$ and $\tilde{H}$ functions when there is constant growth in longevity, as the following lemma shows:

Lemma 3. *Given constant growth in longevity, the* $\tilde{L}$ *and* $\tilde{H}$ *functions are homogeneous of degree* λ *iff the* L *and* H *functions are homogeneous of degree* λ.

Proof. *See appendix G.*

This leads us to the following proposition:

Proposition 2. *Given constant growth in longevity and homogeneity of the* L *and* H *functions, social optimum implies*

$$\begin{aligned} \tilde{R}(i) &= \kappa \tilde{A}(i) \\ \tilde{R}(i) &= \psi A\left(i - \tilde{R}\right) \end{aligned}$$

for all $i \geq t$ *where* κ and ψ *are positive constants (*$\psi < \kappa$*), iff the* L and H *functions are homogeneous of degree zero (healthy ageing).*

Proof. *See appendix G.*

Hence, for the class of homogeneous L and H functions, only those that are homogeneous of degree zero imply a social optimal retirement age that is proportional to longevity. This is driven by the fact that the direct utility consequences (disutility from work, utility from leisure) at retirement are constant under the above notion of healthy ageing when the retirement age is proportional to longevity. To phrase it differently, a future generation with longer longevity can retire proportionally later at the same utility consequences as current generations with shorter longevity retiring earlier. Hence, the view that a retirement age proportional to longevity is in accordance with intergenerational equity holds under this notion of healthy ageing, but not generally.

[29] In e.g. OECD (2006) healthy ageing is interpreted in the sense that the need for health care and thus age dependent health expenditures shift proportionally with longevity to higher ages.

[30] The homogeneity assumption is made in e.g. Bloom et al. (2006) and Andersen (2009).

4.2 Dependency ratio

Policy debates centre around the dependency ratio for the obvious reason that it gives the balance between contributing and receiving persons in a Pay-As-You-Go type scheme. The basic problem is that ageing for unchanged policies implies an increasing dependency ratio and worsening budget balance, cf. section 2 and (2.23). When the retirement age is a policy instrument, the choice of the retirement age may ensure that a balance between the share of population working and non-working is maintained. Hence, even if demographic stock variables may display non-stationarity, it does not necessarily follow that the key economic variable - the dependency ratio - does so. We therefore consider the implications of the socially optimal policy for the dependency ratio.

The change in the dependency ratio is given as (from (2.11))

$$dK(i) = \frac{\partial K(i)}{\partial \tilde{R}(i)} d\tilde{R}(i) + \frac{\partial K(i)}{\partial \tilde{A}(i)} d\tilde{A}(i) \tag{4.2}$$

for all $i \geq t$. We have already shown in section 2 that $\frac{\partial K}{\partial \tilde{A}(i)} > 0$, and obviously a higher retirement age reduces the dependency ratio, i.e.,

$$\begin{aligned} \frac{\partial K}{\partial \tilde{R}(i)} &= \frac{\frac{\partial N_o}{\partial \tilde{R}(i)} N_w(i) - N_o(i) \frac{\partial N_w}{\partial \tilde{R}(i)}}{N_w(i)^2} \\ &= \frac{-\frac{\partial N_w}{\partial \tilde{R}(i)} \left[N_w(i) + N_o(i)\right]}{N_w(i)^2} < 0 \end{aligned} \tag{4.3}$$

since $\frac{\partial N_w}{\partial \tilde{R}(i)} = -\frac{\partial N_o}{\partial \tilde{R}(i)} > 0$.

The case of productivity growth is trivial. In this case $\tilde{A}(i)$ is constant and the retirement age is increasing, cf. from (3.16). Hence, the dependency ratio is declining. The following proceeds under the assumption that productivity growth is zero, i.e., $dy(i) = 0$, to focus on the effects of longevity. It follows from (4.2) that

$$\frac{\partial K(i)}{\partial \tilde{A}(i)} \gtreqqless 0 \quad iff \quad \frac{\partial \tilde{R}(i)}{\partial \tilde{A}(i)} \lesseqqgtr -\frac{\frac{\partial K}{\partial \tilde{A}(i)}}{\frac{\partial K}{\partial \tilde{R}(i)}} > 0 \tag{4.4}$$

for all $i \geq t$. This underlines that the dependency ratio increases unless the optimal policy implies a sufficiently strong increase in the retirement age. The question is thus whether this is implied by the socially optimal policy and which conditions need to be met for it to hold.

To address this issue, we consider the case where growth in longevity is constant. Under the assumption of zero growth in productivity, we have from (3.14) that $T_o(i) = T_o(t)$ and $T_w(i) = T_w(t)$ for all $i \geq t$. Using this in (3.12) implies that the development of retirement age over time under the social optimal policy can be written as a function of longevity only

$$\tilde{R}(i) = R\left(\tilde{A}(i)\right) \tag{4.5}$$

for all $i \geq t$ and where we, by use of (3.16), have $R' > 0$. This implies that there exists a function for the development of relative retirement age over time, i.e., retirement age relative to longevity, under social optimal policy as

$$\phi(i) = \frac{\tilde{R}(i)}{\tilde{A}(i)} \equiv \phi\left(\tilde{A}(i)\right) \tag{4.6}$$

for all $i \geq t$. Note that we have $\phi' = 0$ and $\tilde{R}' = \phi(i) = \kappa$ (a constant from before) in the case of a retirement age proportional to longevity. In general, we have the following relationship between the relative retirement age and the dependency ratio:

Lemma 4. *Given constant growth in longevity,*

$$\frac{\partial K}{\partial \tilde{A}(i)} \gtreqless 0 \quad iff \quad \phi' \lesseqgtr 0$$

for all $i \geq t$.

Proof. *See appendix H.*

Hence, if social optimal policy is such that the relative retirement age increases following an increase in longevity, then the dependency ratio decreases. It follows directly that if the retirement age is proportional to longevity, $\phi' = 0$, we have $\frac{\partial K}{\partial \tilde{A}(i)} = 0$. In this case the social optimal policy implies that the dependency ratio is constant over time. Combining this with the results from last section gives:

Corollary 1. *Given constant growth in longevity and healthy ageing, the social optimum implies that the dependency ratio is constant over time.*

The so-called retirement-consumption puzzle states that consumption when young is greater than when old. This is ensured if $U(c_w(i)) \geq Q(c_o(i))$ holds in the social optimal solution. Assuming this gives the following:

Lemma 5. *Given (i) constant growth in longevity, (ii) that utility from consumption when young is no less than when old, and (iii) that the L and H functions are homogeneous of degree λ, where λ is sufficiently close to zero, it holds that*

$$\phi' \lesseqgtr 0 \quad iff \quad \lambda \gtreqless 0$$

for all $i \geq t$.

Proof. *See appendix H.*

Hence, if homogeneity is less than zero, the socially optimal policy is such that the relative retirement age increases over time. This leads to the following proposition which has corollary 1 as a special case:[31]

Proposition 3. *Given (i) constant growth in longevity, (ii) that utility from consumption when young is no less than when old, and (iii) that the L and H functions are homogeneous of degree λ, where λ is sufficiently close to zero, the social optimum implies that*

$$\frac{\partial K}{\partial \tilde{A}(i)} \gtreqless 0 \quad iff \quad \lambda \gtreqless 0$$

for all $i \geq t$.

Proof. *See appendix H.*

Hence, the dependency ratio is increasing (decreasing, constant) over time under socially optimal policy if the homogeneity of the L and H functions is greater than (less than, equal to) zero. Hence, only in the case of healthy ageing do we have that the socially optimal policy implies a constant dependency ratio.

[31] Note that corollary 1 does not require an assumption concerning the utilities from consumption.

4.3 Pre-saving

The idea of pre-savings is very predominant in debates on how to cope with the financial problems arising from increasing dependency ratios. The argument is that consolidation of the government's budget is needed in advance of expenditure increases driven by the demographic transitions. This is sometimes phrased in the way that "unpaid bills should not be left in the nursing room". While such consolidation may seem common sense, it is less obvious from a normative perspective taking into account that future generations may enjoy both higher productivity and longevity. The present framework makes it possible to address this issue.

Having formulated the problem in terms of taxes and transfers makes it possible to assess the extent of intergenerational redistribution by considering the budget profile. From (2.9), (2.10) and (2.23) we have that the primary budget balance at time i reads

$$B(i) = N_w\left(\tilde{R}(i), \tilde{A}(i)\right) T_w(i) - N_o\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(i) \tag{4.7}$$

for all $i \geq t$. Taking total difference and using (3.14), (2.9) and (2.10) give

$$\begin{aligned} dB(i) &= \left[[T_w(i) + T_o(t)] \tilde{m}\left(\tilde{R}(i), \tilde{A}(i)\right) \right] d\tilde{R}(i) \\ &+ \left[\begin{array}{c} T_w(i) \frac{\partial N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}{\partial \tilde{A}(i)} \\ -T_o(t) \frac{\partial N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}{\partial \tilde{A}(i)} \end{array} \right] d\tilde{A}(i) \\ &+ \left[N_w\left(\tilde{R}(i), \tilde{A}(i)\right) \right] dy(i) \end{aligned} \tag{4.8}$$

for all $i \geq t$, where $T_w(i)$ is given by (3.14) and it has been used that $dT_o(i) = 0$ and $\tilde{m}\left(\tilde{A}(i), \tilde{A}(i)\right) = 0$.

Considering (4.8), we have three channels affecting the evolution of the budget balance, namely (i) a higher retirement age improves the budget balance since more individuals work and pay taxes and fewer individuals are retired and receive transfers, (ii) the direct effect of ageing (increasing longevity) is in general ambiguous since it both implies more young (working) individuals and thus higher tax revenue, and more old (retired) individuals and thus expenditures on transfers, and (iii) productivity growth improves the budget via improved tax revenue.

Consider first the case with productivity growth only, i.e., $d\tilde{A}(i) = 0$. Point (iii) above implies positive effects of productivity growth on the budget balance. In addition to this, productivity growth has positive effects on the retirement age and, hence, on the budget balance under optimal policy from (3.16). It can thus be concluded that productivity growth unambiguously implies an "upward" profile for the primary budget balance; that is, the budget balance improves over time. Hence, productivity growth tends to imply current borrowing to be matched by future savings (i.e. no pre-savings). The intuition for this derives directly from the consumption smoothing implied by the optimal policy since if all generations are to share the fruits of future productivity growth then intertemporal substitution calls for current borrowing.

We now turn to the role of increasing longevity in the absence of productivity growth, i.e., $dy(i) = 0$. To simplify, assume moreover that the initial debt level is zero, i.e., $D(t) = 0$. This enables us to determine whether pre-saving is an optimal policy by looking at the primary budget balance at time t, i.e., $B(t)$.

Consider the expression for the budget balance (4.7) rewritten in terms of the dependency ratio from (2.11)

$$B(i) = \left[T_w(i) - K\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(i)\right] N_w\left(\tilde{R}(i), \tilde{A}(i)\right)$$

for all $i \geq t$. Note that in the absence of productivity growth we have $T_w(i) = T_w(t)$ and $T_o(i) = T_o(t)\ \forall\ \ i \geq t$ from (3.14). Hence,

$$\begin{aligned} dB(i) &= -T_o(t) N_w\left(\tilde{R}(i), \tilde{A}(i)\right) dK(\cdot) \\ &\quad + \left[T_w(t) - K\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(t)\right] dN_w(\cdot) \end{aligned}$$

for all $i \geq t$ and where (from (2.9))

$$dN_w(\cdot) = \frac{\partial N_w}{\partial \tilde{R}(i)} d\tilde{R}(i) + \frac{\partial N_w}{\partial \tilde{A}(i)} d\tilde{A}(i) > 0$$

since $\frac{\partial N_w}{\partial \tilde{A}(i)} > 0$, $\frac{\partial N_w}{\partial \tilde{R}(i)} > 0$ and $d\tilde{R}(i) > 0$ from (3.16).

Observe first that if the optimal policy implies that the dependency ratio is constant, i.e., $\frac{\partial K}{\partial \tilde{A}(i)} = 0$, as is the case when there is constant growth in longevity and healthy ageing (from corollary 1), then it follows that the primary budget must balance in all periods to satisfy the intertemporal budget constraint in (3.13), i.e.,

$$T_w(t) - K\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(t) = 0$$

Hence, the initial primary budget balance is zero $B(t) = 0$ and remains so, and pre-saving is not implied by social optimal policy.

More generally, if the optimal policy implies that $\frac{\partial K}{\partial \tilde{A}(i)} > 0$, we must have that the primary budget balance at t is positive, i.e.,

$$T_w(t) - K\left(\tilde{R}(t), \tilde{A}(t)\right) T_o(t) > 0$$

since $dN_w(\cdot) > 0$ and (3.13) must hold. Hence, pre-saving is an optimal policy if the dependency ratio is increasing over time. However, if the social optimal policy is such that $\frac{\partial K}{\partial \tilde{A}(i)} < 0$, we must have the converse

$$T_w(t) - K\left(\tilde{R}(t), \tilde{A}(t)\right) T_o(t) < 0$$

since $dN_w(\cdot) > 0$ and (3.13) must hold.

Hence, for pre-saving to be implied by social optimal policy, the dependency ratio must increase with longevity (after adjustment of the retirement age) over time. Connecting this to the results obtained above, this indicates that it is necessary that the value of time functions for the young and old is homogeneous of a degree greater than zero for pre-saving to be an optimal policy. These results are summarized in the following:

Summary 1. *Given (i) constant growth in longevity, (ii) that utility from consumption when young is greater than when old and (iii) that the L and H functions are* *homogeneous of degree λ, where λ is sufficiently close to zero, social optimum implies that*

Homogeneity	*Dependency ratio*	*Pre-saving*
$\lambda < 0$	$\frac{\partial K}{\partial \tilde{A}(i)} < 0$	*No*
$\lambda = 0$	$\frac{\partial K}{\partial \tilde{A}(i)} = 0$	*No*
$\lambda > 0$	$\frac{\partial K}{\partial \tilde{A}(i)} > 0$	*Yes*

As this discussion indicates, it is not generally the case that pre-saving is implied by the social optimal policy. In the benchmark case of healthy ageing, the dependency ratio is constant, and provided an initial budget balance, there are no intergenerational transfers.[32] Further, relaxing the assumption of zero growth in productivity, it becomes less likely that pre-saving can be a social optimal policy since productivity growth implies current borrowing financed by future savings to smooth consumption.

5 Decentralized Equilibrium

Finally, consider the question whether the socially optimal allocation can be decentralized. Two aspects arise here. First, the social optimum involves transfers across generations, and hence it is crucial whether these can be made in the decentralized equilibrium. Secondly, in the decentralized case individuals choose not only consumption but also the retirement age given the transfers and taxes determined by the policy maker.

Consider first the issue of intergenerational transfers. In an OLG setting with age heterogeneity wrt. mortality risk and perfect annuities markets to insure individuals against uncertain lifetimes, Calvo and Obstfeld (1988) showed that the social optimal allocation can be decentralized provided there is a sufficiently rich set of age and time dependent instruments. In that setting, longevity and output are constant over time and the population is stationary. Similarly, Sheshinski (2008) shows that a first best solution is obtained in decentralized equilibrium when annuities markets are available to insure individuals against uncertain lifetime. In that setup, there are no intergenerational differences; i.e., output and longevity are constant over time.

An interesting question here concerns the retirement decision. Are individual incentives underlying retirement the same as for the social planner? They are not when consumption levels are different since marginal utilities are different. If consumption levels are the same, there is a difference if taxes and transfer are distortionary. In the model presented here, the socially optimal allocation implies that intergenerational transfers are needed. Therefore, even though perfect annuities markets exist, taxes and transfers are needed when trying to restore the socially optimal allocation.

In the following we briefly analyse whether the socially optimal allocation from section 3 can be achieved as a decentralized equilibrium when perfect annuities markets are available, generations are heterogeneous wrt. longevity, and the government can use taxes levied on young individuals and transfers to old individuals to make intergenerational transfers. To solve for the decentralized equilibrium, we follow Sheshinski (2008) in how the individual's problem is set up. An individual chooses consumption

[32] The case of healthy ageing implies the optimality of a balanced budget and, hence, implies the optimality of a Pay-As-You-Go type scheme.

and retirement conditional on taxes and transfers levied on individuals. The government then chooses taxes and transfers now and in the future. Description of the model and derivation of the decentralized equilibrium are shown in appendixes I, J and K.

Note the difference to previous sections. Before, the social planner was choosing $\left\{T_w(i), T_o(i), \tilde{R}(i)\right\}_{i=t}^{\infty}$, but now the government's policy package is $\{T_w(i), T_o(i)\}_{i=t}^{\infty}$, and individuals from different cohorts choose $\left\{\tilde{R}(i)\right\}_{i=t}^{\infty}$ (as well as consumption). Hence, the government has a smaller set of instruments in the decentralized equilibrium.

5.1 Retirement age

Compare first the socially optimal retirement decisions and retirement in the decentralized equilibrium. Consider an arbitrary generation born at time t_0. According to the decentralized equilibrium, this generation retires at time $i > t_0$ at the age of R_d, while, according to the social planner's allocation, the generation retires at time $j > t_0$ at the age R_s.[33]

If we assume that the same consumption levels are arising in the social planner's and decentralized equilibrium allocation (partial equilibrium, direct effects), we have that the retirement age differs since[34]

$$R_s \gtreqless R_d \text{ for } T_w(i) + T_o(i) \gtreqless 0 \tag{5.1}$$

The intuition is that the tax-transfer scheme introduces a distortion of the retirement decision in the decentralized equilibrium. The composite marginal tax effect of retiring is $T_w(i) + T_o(i)$ in the decentralized equilibrium, and hence retirement decisions differ between the social optimum and the decentralized allocation. The direction of these effects depends on the sum of taxes and transfers at every point in time in the decentralized equilibrium. Both of these are likely to be positive.

This is, however, only a part of the story since consumption in the decentralized equilibrium is dependent on (i) taxes and transfers and (ii) the retirement age (consumption effects), as is shown in appendix I. Hence, one needs to derive the general equilibrium effects to give the full story. Let us therefore relax the assumption of identical consumption levels in the social planner's and decentralized equilibrium allocations. Instead, think of the following question: How does the retirement age change when the economy moves away from the social planner's solution? It can be shown (see appendix L) that the retirement age tends to decline when $T_w(i) + T_o(i)$ is positive. This underscores the role of the distortion of individual retirement decisions arising in decentralized equilibrium.

5.2 Can the social planner allocation be attained in decentralized equilibrium?

The results above imply that even though consumption is equal under the two schemes, the retirement age in the decentralized equilibrium will differ from the social planner's

[33] Hence, $t_0 = i - R_d = j - R_s$.

[34] This is shown in appendix L.

solution as long as the sum of taxes and transfers is non-zero. This is a strong indication that the social planner's allocation can not be attained in the decentralized equilibrium. More formally, we have the following proposition[35]:

Proposition 4. *Even given annuities markets, the socially optimal allocation can not in general be decentralized.*

Proof. *See appendix L.*

Note that this holds despite the presence of annuities markets, which eliminate the distortion caused by uncertain lifetime. Relaxing the assumption of annuities markets would make it even less likely that the social planner's allocation could be attained in the decentralized equilibrium.

6 Concluding Remarks

The notion of intergenerational equity has been considered under a utilitarian criterion in an overlapping generations model allowing for both productivity growth and increasing longevity. The latter is the more interesting aspect both because of its relevance in relation to current debates on demographics and because it requires a modelling approach allowing for overlapping generations with different mortality rates and thus longevity. As in Calvo and Obstfeld (1988), the social optimal allocation is found under generational neutral weighting to imply consumption smoothing across generations and time; i.e., young and old have the same consumption flows at all times. This is a straightforward implication of intertemporal substitution under the utilitarian criterion and with separable utility functions. However, in addition the retirement age differs across generations, and both productivity growth and increasing longevity tend to call for increasing retirement ages over time. The former holds generally and derives from increasing labour input when its marginal productivity is high, while the latter follows under mild conditions amounting to increasing longevity being reflected in better health at given ages.

In policy debates strong assertions are often made on the implications of intergenerational equity such as retirement ages to follow longevity proportionally or pre-savings. We show that the former holds under so-called healthy ageing, while the latter arises if the retirement age does not increase sufficiently to avoid an increasing (economic) dependency ratio. Note that even if the socially optimal policy implies some pre-savings, the needed savings are affected by the fact that retirement ages increase across generations due to increasing longevity.

Intergenerational distribution and equity have been considered from a utilitarian perspective. Utilitarianism is the dominating approach in welfare analyses, and for comparative purposes it is thus a natural starting point for addressing the issue of changing mortality rates (and productivity). Working out the implications of utilitarianism makes it easier to discuss its pros and cons. It should be noted that the analysis assumes the same underlying utility function across generations weighted in a way that does not imply any generational preference or bias. The criticism of utilitarianism (see e.g. Konow (2003) for a survey and references) includes aspects such

[35] Necessary conditions for allocation in the decentralized equilibrium being the same as the social planner's only hold when the number of young individuals is very close to the number of old individuals, which gives the above proposition.

as inter-personal (generational) comparability of utility (cardinal measurement) and consequentialism assessing only outcomes measured in terms of utility disregarding the underlying process or elements affecting well-being. Utilitarianism implies redistribution based on the ability to generate utility at the margin (marginal utilities) rather than the level of utility per se. This is reflected in the present analysis, and it may be questioned whether future generations should work more (retire later) because they have higher longevity and are more productive.

This paper has taken mortality rates to be exogenous. While this is a natural starting point to clarify the basic issues involved, it is also clear that changes in mortality rates and thus longevity are driven by both individual (life style, eating habits, housing etc) and public decisions (health care). A small but growing literature is exploring the consequences of endogenizing longevity via these channels (see Philipson and Becker (1998), Leroux et al. (2008)). It is an obvious agenda for future research to endogenize mortality rates

Finally, a small-open economy assumption has been adopted with respect to financial markets. This is a reasonable assumption to make for most countries given financial globalization. Consequences of changes in the interest rate profile can readily be analysed within the model. Since ageing is a global phenomenon, there is however an issue with respect to how ageing itself and the induced policy changes will affect global interest rates. While countries acting non-cooperatively take interest rates as exogenous, important interdependencies are likely to exist. An interesting topic for future research would be to analyse these issues.

References

[1] Andersen, T.M. (2008), Longevity and Social Security Reform - A Legislative Procedure Approach, Journal of Public Economics.

[2] Auerback, A. J. and K. Hassett (2007), Optimal Long-Run Fiscal Policy: Constraints, Preferences and the Resolution of Uncertainty, Journal of Economic Dynamics and Control, 31, pp. 1451-1472.

[3] Balassone, F., J. Cunha, G. Langenus, B. Manzke, J. Pavot, D. Prammer, and P. Tommasino (2009), Fiscal Sustainability and Implications for the Euro Area. ECB Working Paper, 994.

[4] Benveniste, L. M. and J. A. Scheinkman (1979), On the Differentiability of the Value Function in Dynamic Models of Economics, Econometrica, 47, pp. 727-732.

[5] Björk, T. (1998), Arbitrage Theory in Continuous Time, Oxford University Press.

[6] Blanchard, O. J. (1985), Debts, Deficits, and Finite Horizons, Journal of Political Economy, 93, pp. 223-247.

[7] Blanchard, O.J. and S. Fischer (1989), Lectures on Macroeconomics, MIT Press

[8] Bloom, D.E., D. Canning, R. Mansfield and M. Moore (2006), Demographic Change, Social Security Systems, and Savings. NBER Working Paper 12621.

[9] Bommier, A. (2006), Uncertain Lifetime and Intertemporal Choice: Risk Aversion as a Rationale for Time Discounting, International Economic Review, 47, pp. 1223-1246.

[10] Bommier, A., M. L. Leroux and J.-M. Lozachmeur (2009), On the Public Economics of Annuities with Differential Mortality, CORE Discussion Paper.

[11] Boucekkine, R., D. de la Croix and O. Licandro (2002), Vintage Human Capital, Demographic Trends, and Endogenous Growth, Journal of Economic Theory, 104, pp. 340-375.

[12] Calvo, G. A. and M. Obstfeld (1988), Optimal Time-Consistent Fiscal Policy with Finite Lifetimes, Econometrica, 56, pp. 411-432.

[13] Christensen, K, G. Doblhammer, R. Rau and J.W. Vaupel (2009), Ageing Populations: The Challenges Ahead, Lancet, 375, 1196-1208.

[14] Crawford, V.P., and D.M. Lilien (1981), Social Security and the Retirement Decision, Quarterly Journal of Economics, 95, 505-529.

[15] Diamond, P. A. (1965), National Debt in a Neoclassical Growth Model, American Economic Review, 55, pp. 1126-1150.

[16] European Commission (2006), The Long-Term Sustainability of Public Finances in the European Union. A Report by the Community Services.

[17] European Commission (2009), Sustainability Report 2009, Bruxelles.

[18] Haliday, T.J., and M. Podor (2009), Health Status and the Allocation of Time, IZA Working Paper 5368.

[19] Heijdra, B. J., and W. E. Romp (2008), A Life-Cycle Overlapping Generations Model of the Small Open Economy, Oxford Economic Papers, 60, pp. 88-121.

[20] Kalemi-Ozcan, S., and P.Weil (2002), Mortality Change, the Uncertainty Effect, and Retirement, NBER Working Paper 8742.

[21] Konow, J. (2003), Which is the Fairest One of All?, A Positive Analysis of Justice Theories, Journal of Economic Literature, 41, pp. 1188-1239.

[22] Leroux, M. L., P. Pestieau and G. Ponthiere (2008), Optimal Linear Taxation under Endogenous Longevity, CORE Working Paper.

[23] OECD (2006), Projecting OECD Health and Long-Term Care Expenditures: What are the Main Drivers?, OECD Economics Department Working Papers, 477.

[24] Oeppen, J., and J. W. Vaupel (2002), Broken Limits to Life Expectancy, Science, 296, pp. 1029-1031.

[25] Philipson, T. J. and G. S. Becker (1998), Old-Age Longevity and Mortality-Contingent Claims, Journal of Political Economy, 106, pp. 551-573.

[26] Rudin, W., (1976), Principles of Mathematical Analysis, Third Edition, McGraw-Hill International Editions.

[27] Samuelson, P. A (1958), An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money, Journal of Political Economy, 66, pp. 467-482.

[28] Swedish Government (2008), Vårpropositionen, Stockholm.

[29] Sheshinski, E. (1978), A Model of Social Security and Retirement Decisions, Journal of Public Economics, 10, pp. 337-360.

[30] Sheshinski E. (2006), Note on Longevity and Aggregate Savings, Scandinavian Journal of Economics, 108, pp. 353-356.

[31] Shesinski, E. (2008), The Economic Theory of Annuities, Princeton University Press.

[32] UK Pensions Commission (2005), A New Pension Settlement for the Twenty-First Century, The Second Report of the Pensions Commission, London.

[33] United Nations (2008), World Population Prospects - The 2008 Revision, Population database, http://esa.un.org/unpp/.

[34] United Nations Development Programme (2008), Human Development Reports, http://hdr.undp.org/en/statistics/indices/hdi/.

[35] Wilmoth, J. R. (2000), Demography of Longevity: Past, Present and Future Trends, Experimental Gerontology, 35, pp. 1111-1129.

[36] Yaari, M. E. (1965), Uncertain Lifetime, Life Insurance, and the Theory of the Consumer, Review of Economic Studies, 32, pp. 137-150.

Appendix

A. Derivatives of the m̃ function

The derivatives are the following:

$$\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial a} = -\frac{1}{A(t-a)}\left[1+\frac{a}{A(t-a)}\mu_A\left(t-a\right)\right]\frac{\ln\alpha\alpha^{\frac{a}{A(t-a)}}}{\alpha-1}<0$$

$$\frac{\partial^2 \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial a^2} = -\frac{1}{A(t-a)^2}\left[\begin{array}{c}\left[1+\frac{a}{A(t-a)}\mu_A\left(t-a\right)\right]\\ \times\left[\begin{array}{c}\left[1+\frac{a}{A(t-a)}\mu_A\left(t-a\right)\right]\ln\alpha\\ +2\mu_A\left(t-a\right)\\ -a\mu'_A\mu_A\left(t-a\right)\end{array}\right]\end{array}\right]\frac{\ln\alpha\alpha^{\frac{a}{A(t-a)}}}{\alpha-1}<0$$

$$\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} = \frac{\left[1+\mu_A\left(t-\tilde{A}\right)\right]a}{A(t-a)^2}\frac{\ln\alpha\alpha^{\frac{a}{A(t-a)}}}{\alpha-1}>0$$

$$\frac{\partial^2 \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)\partial a} = \frac{\left[1+\mu_A\left(t-\tilde{A}\right)\right]}{A(t-a)^2}\left[\begin{array}{c}1+2\frac{a\mu_A(t-a)}{A(t-a)}\\ +\frac{a\ln\alpha}{A(t-a)}\left[1+\frac{a}{A(t-a)}\mu_A\left(t-a\right)\right]\end{array}\right]\frac{\ln\alpha\alpha^{\frac{a}{A(t-a)}}}{\alpha-1}>0$$

where $A\left(t-a\right)$ is given in (2.6).

B. Calculations for population composition

$dN\left(t\right)>0$**,** $dN_w\left(t\right)>0$ **and** $dN_w\left(t\right)>0$**.** Applying Taylor approximation to (2.8), (2.9) and (2.10), and erasing terms that contain "dt" raised to a higher power than 1 (since time is continuous $(dt\rightarrow 0)$) give the following:

$$dN\left(t\right)=\frac{\partial N}{\partial\tilde{A}(t)}d\tilde{A}\left(t\right)=\left[\int_{a=0}^{\tilde{A}(t)}\frac{\partial\tilde{m}\left(a,\tilde{A}(t)\right)}{\partial\tilde{A}(t)}da\right]d\tilde{A}\left(t\right)>0$$

$$dN_w\left(t\right)=\frac{\partial N_w}{\partial\tilde{A}(t)}d\tilde{A}\left(t\right)=\left[\int_{a=0}^{\tilde{R}(t)}\frac{\partial\tilde{m}\left(a,\tilde{A}(t)\right)}{\partial\tilde{A}(t)}da\right]d\tilde{A}\left(t\right)>0$$

$$dN_o\left(t\right)=\frac{\partial N_o}{\partial\tilde{A}(t)}d\tilde{A}\left(t\right)=\left[\int_{a=\tilde{R}(t)}^{\tilde{A}(t)}\frac{\partial\tilde{m}\left(a,\tilde{A}(t)\right)}{\partial\tilde{A}(t)}da\right]d\tilde{A}\left(t\right)>0$$

where (2.5) and properties of the $\tilde{m}$ function (from appendix A) are used. Hence, $dN\left(t\right)>0$, $dN_w\left(t\right)>0$ and $dN_o\left(t\right)>0$ all hold.

Proof of $dK\left(t\right)>0$

Applying Taylor approximation to (2.11) and erasing terms that contain "dt" raised to a higher power than 1 (since time is continuous $(dt\rightarrow 0)$) give the following:

$$dK\left(t\right)=\frac{\frac{\partial N_o}{\partial\tilde{A}(t)}N_w(t)-N_o(t)\frac{\partial N_w}{\partial\tilde{A}(t)}}{N_w(t)^2}d\tilde{A}\left(t\right)$$

$$=\frac{N_o(t)}{N_w(t)}\left[\frac{\frac{\partial N_o}{\partial\tilde{A}(t)}}{N_o(t)}-\frac{\frac{\partial N_w}{\partial\tilde{A}(t)}}{N_w(t)}\right]d\tilde{A}\left(t\right)$$

Hence, using (2.5), $dK\left(t\right)>0$ iff

$$\frac{\frac{\partial N_o}{\partial\tilde{A}(t)}}{N_o(t)}>\frac{\frac{\partial N_w}{\partial\tilde{A}(t)}}{N_w(t)}$$

or, by using (2.9) and (2.10)

$$\frac{\int\limits_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int\limits_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a,\tilde{A}(t)\right) da} > \frac{\int\limits_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int\limits_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a,\tilde{A}(t)\right) da} \quad \text{(B.1)}$$

Since $\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial a} < 0$ and $\frac{\partial^2 \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)\partial a} > 0$ we have that

$$\int\limits_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da > \tilde{R}(t)\, \tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right)$$

$$\int\limits_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} da < \tilde{R}(t) \left.\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}$$

$$\int\limits_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da < \left[\tilde{A}(t) - \tilde{R}(t)\right] \tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right)$$

$$\int\limits_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} da > \left[\tilde{A}(t) - \tilde{R}(t)\right] \left.\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}$$

Hence

$$\frac{\int\limits_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int\limits_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a,\tilde{A}(t)\right) da} < \frac{\tilde{R}(t) \left.\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\tilde{R}(t)\tilde{m}\left(\tilde{R}(t),\tilde{A}(t)\right)}$$

$$\frac{\int\limits_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int\limits_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a,\tilde{A}(t)\right) da} > \frac{\left[\tilde{A}(t)-\tilde{R}(t)\right] \left.\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\left[\tilde{A}(t)-\tilde{R}(t)\right]\tilde{m}\left(\tilde{R}(t),\tilde{A}(t)\right)}$$

and

$$\frac{\int\limits_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int\limits_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a,\tilde{A}(t)\right) da}$$

$$> \frac{\left[\tilde{A}(t)-\tilde{R}(t)\right] \left.\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\left[\tilde{A}(t)-\tilde{R}(t)\right]\tilde{m}\left(\tilde{R}(t),\tilde{A}(t)\right)}$$

$$= \frac{\left.\frac{\partial \tilde{m}(a,\tilde{A}(t))}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\tilde{m}\left(\tilde{R}(t),\tilde{A}(t)\right)}$$
$$= \frac{\tilde{R}(t)\left.\frac{\partial \tilde{m}(a,\tilde{A}(t))}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\tilde{R}(t)\tilde{m}\left(\tilde{R}(t),\tilde{A}(t)\right)}$$
$$> \frac{\int_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}(a,\tilde{A}(t))}{\partial \tilde{A}(t)} da}{\int_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a,\tilde{A}(t)\right) da}$$
Hence, (B.1) holds as well as $dK(t) > 0$. This completes the proof.

C. The social welfare function

The second term in (3.1) is obviously finite and, hence, well defined. This is, however, not obvious for the first term.

There are three cases for longevity development discussed in the paper: (i) upper bound to longevity, (ii) decreasing growth in longevity, and (iii) constant growth in longevity. The first term in (3.1) is obviously finite and well defined if there is an upper bound to longevity, as is assumed in appendix E below.

The first term in (3.1) can be written as

$$\int_{i=t}^{\infty} e^{-\theta(i-t)} v(i) \frac{Q(i)}{v(i)} di \quad \text{(C.1)}$$

where $Q(i)$ is expected discounted lifetime utility at birth for an individual belonging to generation i and $v(i)$ life expectancy at birth of the individual from (2.4). The ratio $\frac{Q(i)}{v(i)}$ can be interpreted as the expected discounted utility each year alive at birth for an individual belonging to generation i, which can be assumed to be stationary.

From (2.4), we have that

$$v(i) = \left[\frac{\alpha \ln \alpha - [\alpha - 1]}{\ln \alpha [\alpha - 1]}\right] A(i)$$

where $\left[\frac{\alpha \ln \alpha - [\alpha - 1]}{\ln \alpha [\alpha - 1]}\right] > 0$.

We therefore have that $v(i)$ is linear in longevity.

Hence, $v(i)$ converges to a constant if the growth in longevity is decreasing over time. In that case, $e^{-\theta(i-t)}$ determines whether (C.1) is well defined. Performing a ratio test (see, e.g., Rudin (1976)) on $e^{-\theta(i-t)}$ gives

$$\lim_{i\to\infty} \left| \frac{e^{-\theta(i+di-t)}}{e^{-\theta(i-t)}} \right| = e^{-\theta di} < 1$$

and, hence, $e^{-\theta(i-t)}$ converges to zero and (C.1) is finite and well defined.

In the case of constant growth in longevity, $v(i)$ is linear in time

$$v(i) = \left[\frac{\alpha \ln \alpha - [\alpha - 1]}{\ln \alpha [\alpha - 1]}\right] [A(t) + \mu_A [i - t]]$$

where (2.5) has been used. In this case, $e^{-\theta(i-t)} v(i)$ determines whether (C.1) is well defined. Performing a ratio test on $e^{-\theta(i-t)} v(i)$ gives

$$\lim_{i\to\infty} \left| \frac{e^{-\theta(i+di-t)} v(i+di)}{e^{-\theta(i-t)} v(i)} \right| = e^{-\theta di} < 1$$

since

$$\lim_{i\to\infty} \left| \frac{v(i+di)}{v(i)} \right| = \lim_{i\to\infty} \left| \frac{\left[\frac{\alpha \ln \alpha - [\alpha-1]}{\ln \alpha [\alpha-1]}\right][A(t)+\mu_A[i+di-t]]}{\left[\frac{\alpha \ln \alpha - [\alpha-1]}{\ln \alpha [\alpha-1]}\right][A(t)+\mu_A[i-t]]} \right|$$

$= \lim_{i\to\infty} \left| \frac{A(t)+\mu_A[i+di-t]}{A(t)+\mu_A[i-t]} \right| = 1$

Hence, $e^{-\theta(i-t)}v(i)$ converges to zero, and (C.1) is finite and well defined.

The conclusion is that (C.1) and, hence, (3.1) are finite and well defined.

D. Social optimum derived

Applying Taylor approximation to the value function $V\left(D, y, \tilde{A}\right)$ and considering the limit for $dt \to 0$ give the following:

$dV\left(D, y, \tilde{A}\right) = \left[V_D\left[rD - \left[N_w\left(\tilde{R}, \tilde{A}\right)T_w - N_o\left(\tilde{R}, \tilde{A}\right)T_o\right]\right] + V_y\mu_y y + V_A\hat{\mu}_A\right]dt$

Inserting this into (3.4) gives

$$\theta V\left(D, y, \tilde{A}\right) = \underset{T_w, T_o, \tilde{R}}{Max}\left\{\begin{array}{c} \int_{a=0}^{\tilde{R}} \tilde{m}\left(a, \tilde{A}\right)\left[U\left(y - T_w\right) + \tilde{L}\left(a, \tilde{A}\right)\right]da \\ + \int_{a=\tilde{R}}^{\tilde{A}} \tilde{m}\left(a, \tilde{A}\right)\left[Q\left(T_o\right) + \tilde{H}\left(a, \tilde{A}\right)\right]da \\ +V_D\left[rD - \left[N_w\left(\tilde{R}, \tilde{A}\right)T_w - N_o\left(\tilde{R}, \tilde{A}\right)T_o\right]\right] + V_y\mu_y y + V_{\tilde{A}}\hat{\mu}_A \end{array}\right\}$$

(D.1)

We can now find the first-order conditions determining T_w, T_o and $\tilde{R}$. For the tax payment by the young T_w, we have

$V_D = -\frac{1}{N_w\left(\tilde{R}, \tilde{A}\right)} \int_{a=0}^{\tilde{R}} \tilde{m}\left(a, \tilde{A}\right)U'\left(y - T_w\right)da$

or using $N_w\left(\tilde{R}, \tilde{A}\right)$ from (2.9), we get (3.5) as

$V_D = -U'\left(y - T_w\right)$

For the transfer to the old T_o, we have

$V_D = -\frac{1}{N_o\left(\tilde{R}, \tilde{A}\right)} \int_{a=\tilde{R}}^{\tilde{A}} \tilde{m}\left(a, \tilde{A}\right)Q'\left(T_o\right)da$

or using $N_o\left(\tilde{R}, \tilde{A}\right)$ from (2.10), we get (3.6) as

$V_D = -Q'\left(T_o\right)$

For the retirement age $\tilde{R}$, we have

$V_D T_w \frac{\partial N_w}{\partial \tilde{R}} - V_D T_o \frac{\partial N_o}{\partial \tilde{R}} = \tilde{m}\left(\tilde{R}, \tilde{A}\right)\left[U\left(y - T_w\right) + \tilde{L}\left(\tilde{R}, \tilde{A}\right)\right] - \tilde{m}\left(\tilde{R}, \tilde{A}\right)\left[Q\left(T_o\right) + \tilde{H}\left(\tilde{R}, \tilde{A}\right)\right]$

or using $N_w\left(\tilde{R}, \tilde{A}\right)$ and $N_o\left(\tilde{R}, \tilde{A}\right)$ from (2.9) and (2.10), we get (3.7) as

$\left[T_w + T_o\right]V_D = U\left(y - T_w\right) + \tilde{L}\left(\tilde{R}, \tilde{A}\right) - Q\left(T_o\right) - \tilde{H}\left(\tilde{R}, \tilde{A}\right)$ where it has been used that $\frac{\partial N_w}{\partial \tilde{R}} = -\frac{\partial N_o}{\partial \tilde{R}} = \tilde{m}\left(\tilde{R}, \tilde{A}\right)$.

Replacing T_w, T_o and $\tilde{R}$ in (D.1) with their optimal values from (3.5)-(3.7) gives

$$\theta V\left(D,y,\tilde{A}\right)=\left[\begin{array}{c}\int\limits_{a=0}^{\tilde{R}^*}\tilde{m}\left(a,\tilde{A}\right)\left[U\left(y-T_w^*\right)+\tilde{L}\left(a,\tilde{A}\right)\right]da\\+\int\limits_{a=\tilde{R}^*}^{\tilde{A}}\tilde{m}\left(a,\tilde{A}\right)\left[Q\left(T_o^*\right)+\tilde{H}\left(a,\tilde{A}\right)\right]da\\+V_D\left[rD-\left[N_w\left(\tilde{R}^*,\tilde{A}\right)T_w^*-N_o\left(\tilde{R}^*,\tilde{A}\right)T_o^*\right]\right]+V_y\mu_y y+V_{\tilde{A}}\hat{\mu}_A\end{array}\right]$$

Using the envelope theorem gives

$$(\theta-r)\,V_D=V_{DD}\left[rD-\left[N_w\left(\tilde{R}^*,\tilde{A}\right)T_w^*-N_o\left(\tilde{R}^*,\tilde{A}\right)T_o^*\right]\right]+V_{yD}\mu_y y+V_{\tilde{A}D}\hat{\mu}_A$$

Applying a Taylor approximation to V_D, and using the law of motion for D, y and $\tilde{A}$, and considering the limit for $dt\to 0$ give

$$dV_D=V_{DD}\left[rD-\left[N_w\left(\tilde{R},\tilde{A}\right)T_w-N_o\left(\tilde{R},\tilde{A}\right)T_o\right]\right]dt+V_{Dy}\mu_y ydt+V_{D\tilde{A}}\hat{\mu}_A dt$$

Evaluating this at T_w^*, T_o^* and $\tilde{R}^*$, plugging into it and using that $V_{Dy}=V_{yD}$ and $V_{D\tilde{A}}=V_{\tilde{A}D}$ (Young's theorem) give (3.8) as

$$dV_D=(\theta-r)\,V_D dt$$

Above we have shown that (3.5)-(3.7) are necessary conditions for solving the maximization problem in (3.4). Now we show that these are also sufficient. Denote by F the function inside the right hand side bracket in (D.1). Then the necessary conditions in (3.5)-(3.7) are also sufficient for maximum if the Hessian matrix corresponding to the F function is a negative definite. From the F function we have (evaluated at the optimal solution in (3.5)-(3.7))

$$\frac{\partial^2 F}{\partial T_w^2}=\int\limits_{a=0}^{\tilde{R}}\tilde{m}\left(a,\tilde{A}\right)U''\left(y-T_w\right)da<0$$

$$\frac{\partial^2 F}{\partial T_o^2}=\int\limits_{a=\tilde{R}}^{\tilde{A}}\tilde{m}\left(a,\tilde{A}\right)Q''\left(T_o\right)da<0$$

$$\frac{\partial^2 F}{\partial \tilde{R}^2}=\tilde{m}\left(\tilde{R},\tilde{A}\right)\left[\frac{\partial\tilde{L}\left(\tilde{R},\tilde{A}\right)}{\partial\tilde{R}}-\frac{\partial\tilde{H}\left(\tilde{R},\tilde{A}\right)}{\partial\tilde{R}}\right]<0$$

$$\frac{\partial^2 F}{\partial T_w\partial T_o}=\frac{\partial^2 F}{\partial T_w\partial\tilde{R}}=\frac{\partial^2 F}{\partial T_o\partial\tilde{R}}=0$$

which give a negative definite Hessian matrix.

We now show that the HJB equation in (3.4) along with (2.24) (and (2.23)) are necessary for solving (3.3). First, (2.24) (and (2.23)) is a part of the problem in (3.3) and is therefore a necessary and sufficient condition for its solution. To show that the HJB equation is necessary, we roughly apply the method used in Björk (1998), where necessary conditions for the HJB equation in the stochastic case are proved. Although the proof in Björk (1998) can be applied here, we show it below since it gives insights into how the HJB equation is obtained in our case.

Proof of the HJB equation being a necessary condition

The objective function in (3.3) can be written in the following way:

$$\underset{\left\{T_w(i),T_o(i),\tilde{R}(i)\right\}_{i=t}^{\infty}}{Max}W(t)=\int\limits_{i=t}^{\infty}e^{-\theta(i-t)}Z(i)\,di \quad \text{(D.2)}$$

where

$$Z(i) = \left[\begin{array}{l} \int_{a=0}^{\tilde{R}(i)} \tilde{m}\left(a, \tilde{A}(i)\right) \left[U\left(y(i) - T_w(i)\right) + \tilde{L}\left(a, \tilde{A}(i)\right)\right] da \\ \quad + \int_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}\left(a, \tilde{A}(i)\right) \left[Q\left(T_o(i)\right) + \tilde{H}\left(a, \tilde{A}(i)\right)\right] da \end{array} \right]$$

$$= Z\left(y(i), \tilde{A}(i), T_w(i), T_o(i), \tilde{R}(i)\right)$$

Optimal $T_w(i)$, $T_o(i)$ and $\tilde{R}(i)$ are

$$T_w^*(i) = T_w\left(D(i), y(i), \tilde{A}(i)\right)$$

$$T_o^*(i) = T_o\left(D(i), y(i), \tilde{A}(i)\right)$$

$$\tilde{R}^*(i) = \tilde{R}\left(D(i), y(i), \tilde{A}(i)\right)$$

for all $i \geq t$. Now consider $T_w(i)$, $T_o(i)$ and $\tilde{R}(i)$ that are not necessarily optimal

$$T_w(i) = T_w(i) \quad for \quad t+h \geq i \geq t$$
$$= T_w^*(i) \quad \forall \quad i \geq t+h$$
$$T_o(i) = T_o(i) \quad for \quad t+h \geq i \geq t$$
$$= T_o^*(i) \quad \forall \quad i \geq t+h$$
$$\tilde{R}(i) = \tilde{R}(i) \quad for \quad t+h \geq i \geq t$$
$$= \tilde{R}^*(i) \quad \forall \quad i \geq t+h$$

where $h > 0$.

The value function at time t is defined as

$$V\left(D(t), y(t), \tilde{A}(t)\right) = \int_{i=t}^{\infty} e^{-\theta(i-t)} Z\left(y(i), \tilde{A}(i), T_w^*(i), T_o^*(i), \tilde{R}^*(i)\right) di$$

and is assumed to be differentiable.[36] Hence, the value function at time $t+h$ is

$$V\left(D(t+h), y(t+h), \tilde{A}(t+h)\right) = \int_{i=t+h}^{\infty} e^{-\theta(i-t-h)} Z\left(y(i), \tilde{A}(i), T_w^*(i), T_o^*(i), \tilde{R}^*(i)\right) di$$

If the optimal $T_w(i)$, $T_o(i)$ and $\tilde{R}(i)$ are chosen, then the value of $W(t)$ in (D.2) is $V\left(D(t), y(t), \tilde{A}(t)\right)$, while it is the following if $T_w(i)$, $T_o(i)$ and $\tilde{R}(i)$ that are not necessarily optimal are chosen:

$$\int_{i=t}^{t+h} e^{-\theta(i-t)} Z\left(y(i), \tilde{A}(i), T_w(i), T_o(i), \tilde{R}(i)\right) di + e^{-\theta h} V\left(D(t+h), y(t+h), \tilde{A}(t+h)\right)$$

Comparing these two, we have that the following must hold:

$$V\left(D(t), y(t), \tilde{A}(t)\right) \geq \int_{i=t}^{t+h} e^{-\theta(i-t)} Z\left(y(i), \tilde{A}(i), T_w(i), T_o(i), \tilde{R}(i)\right) di$$
$$+ e^{-\theta h} V\left(D(t+h), y(t+h), \tilde{A}(t+h)\right)$$

Rewriting this inequality and dividing by h give

[36] Benveniste and Scheinkman (1979) set forth sufficient conditions for the derivatives of the value function to exist. These are assumed to hold here.

$$\left(e^{\theta h}-1\right)h^{-1}V\left(D\left(t\right),y\left(t\right),\tilde{A}\left(t\right)\right)\geq e^{\theta h}h^{-1}\int_{i=t}^{t+h}e^{-\theta(i-t)}Z\left(y\left(i\right),\tilde{A}\left(i\right),T_w\left(i\right),T_o\left(i\right),\tilde{R}\left(i\right)\right)di$$
$$+h^{-1}\left[V\left(D\left(t+h\right),y\left(t+h\right),\tilde{A}\left(t+h\right)\right)-V\left(D\left(t\right),y\left(t\right),\tilde{A}\left(t\right)\right)\right]$$

Taking the limit when $h\rightarrow 0$ on both sides of the inequality sign gives

$$\theta V\left(D\left(t\right),y\left(t\right),\tilde{A}\left(t\right)\right)\geq Z\left(y\left(t\right),\tilde{A}\left(t\right),T_w\left(t\right),T_o\left(t\right),\tilde{R}\left(t\right)\right)+\frac{1}{dt}dV\left(D\left(t\right),y\left(t\right),\tilde{A}\left(t\right)\right)$$

where L'Hospital's rule has been used.

This inequality holds for all $T_w\left(t\right)$, $T_o\left(t\right)$ and $\tilde{R}\left(t\right)$. Further, it holds with the equality sign if the optimal ones are chosen. This implies

$$\theta V\left(D\left(t\right),y\left(t\right),\tilde{A}\left(t\right)\right)=\underset{T_w(t),T_o(t),\tilde{R}(t)}{Max}\left\{\begin{array}{c}Z\left(y\left(t\right),\tilde{A}\left(t\right),T_w\left(t\right),T_o\left(t\right),\tilde{R}\left(t\right)\right)\\+\frac{1}{dt}dV\left(D\left(t\right),y\left(t\right),\tilde{A}\left(t\right)\right)\end{array}\right\}$$

which, after using short and notation along with (2.24) (and (2.23)), (2.5) and (2.20), gives the HJB equation in (3.4), which completes the proof.

We have therefore shown that if there exist optimal $T_w\left(i\right)$, $T_o\left(i\right)$ and $\tilde{R}\left(i\right)$ and the value function $V\left(D\left(t\right),y\left(t\right),\tilde{A}\left(t\right)\right)$ is differentiable, then the HJB equation in (3.4) is a necessary condition for solving (3.3). Hence, we can conclude that if the value function $V\left(D\left(t\right),y\left(t\right),\tilde{A}\left(t\right)\right)$ is differentiable, then the HJB equation is necessary for solving (3.3).

E. Upper bound on longevity

We now have that

$$dA\left(t\right)=\mu_A dt \quad for \quad A\left(t\right)<\overline{A}$$
$$=0 \quad for \quad A(t)=\overline{A} \quad \text{(E.1)}$$

or

$$\mu_A>0 \quad for \ t<t^*$$
$$\mu_A=0 \quad for \quad t\geq t^*$$

where $A(t^*)=\overline{A}$ and it is used that there is a one-to-one relationship between A and t. This changes (2.6) and the $\tilde{L}$, $\tilde{H}$ and $\tilde{m}$ functions such that

$$A\left(i-a\right)=\tilde{A}\left(i\right)-\mu_A\left[a-\tilde{A}\left(i\right)\right] \quad \forall \quad i-a\leq t^*$$
$$=A\left(t^*\right) \quad \forall \quad i-a>t^* \quad \text{(E.2)}$$

$$\tilde{L}\left(a,\tilde{A}\left(i\right)\right)=L\left(a,\tilde{A}\left(i\right)-\mu_A\left[a-\tilde{A}\left(i\right)\right]\right) \quad \forall \quad i-a\leq t^*$$
$$=L\left(a,A\left(t^*\right)\right) \quad \forall \quad i-a>t^* \quad \text{(E.3)}$$

$$\tilde{H}\left(a,\tilde{A}\left(i\right)\right)=H\left(a,\tilde{A}\left(i\right)-\mu_A\left[a-\tilde{A}\left(i\right)\right]\right) \quad \forall \quad i-a\leq t^*$$
$$=H\left(a,A\left(t^*\right)\right) \quad \forall \quad i-a>t^* \quad \text{(E.4)}$$

$$\tilde{m}\left(a,\tilde{A}\left(i\right)\right)=m\left(a,\tilde{A}\left(i\right)-\mu_A\left[a-\tilde{A}\left(i\right)\right]\right) \quad \forall \quad i-a\leq t^*$$
$$=m\left(a,A\left(t^*\right)\right) \quad \forall \quad i-a>t^* \quad \text{(E.5)}$$

Hence, (3.3) becomes

$$\underset{\left\{T_w(i),T_o(i),\tilde{R}(i)\right\}_{i=t}^{\infty}}{Max}W\left(t\right)=\int_{i=t}^{t^*}e^{-\theta(i-t)}Z_1\left(i\right)di+\int_{i=t^*}^{t^*+A(t^*)}e^{-\theta(i-t)}Z_2\left(i\right)di+\int_{i=t^*+A(t^*)}^{\infty}e^{-\theta(i-t)}Z_3\left(i\right)di$$

(E.6)

subject to the budget constraint from (2.24) (and (2.23)) and given (E.1) and (2.20) and where

$$Z_j(i) = \int_{a=0}^{\tilde{R}(i)} \tilde{m}_j\left(a, \tilde{A}(i)\right) \tilde{W}_{w,j}(i,a)\, da + \int_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}_j\left(a, \tilde{A}(i)\right) \tilde{W}_{o,j}(i,a)\, da,\ \ j = 1,2,3$$

For $Z_1(i)$ we have $i \leq t^*$ and, hence, $i - a \leq t^*$. Together with (E.3)-(E.5) this gives

$$\tilde{W}_{w,1}(i,a) = U(y(i) - T_w(i)) + L\left(a, \tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right)$$
$$\tilde{W}_{o,1}(i,a) = Q(T_o(i)) + H\left(a, \tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right)$$
$$\tilde{m}_1\left(a, \tilde{A}(i)\right) = m\left(a, \tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right)$$

For $Z_2(i)$ we have $t^* \leq i \leq t^* + A(t^*)$ and, hence, $t^* - a \leq i - a \leq t^* - a + A(t^*)$. Since $A(t^*) \geq a$, this implies that there exists an $\bar{a}(i)$ for which $a \lesseqgtr \bar{a}(i)$, and we have that $t^* \lesseqgtr i - a$. Hence, we have from (E.5)

$$\begin{aligned}\tilde{m}_2\left(a, \tilde{A}(i)\right) &= m(a, A(t^*)) \quad for \ \ a \leq \bar{a}(i) \\ &= m\left(a, \tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right) \quad for \ \ \bar{a}(i) < a\end{aligned}$$

and if $\bar{a}(i) \leq \tilde{R}(i)$, we have from (E.3)-(E.4)

$$\begin{aligned}\tilde{W}_{w,2}(i,a) &= U(y(i) - T_w(i)) + L(a, A(t^*)) \quad for \ \ a \leq \bar{a}(i) \\ &= U(y(i) - T_w(i)) + L\left(a, \tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right) \quad for \ \ \bar{a}(i) < a\end{aligned}$$
$$\tilde{W}_{o,2}(i,a) = Q(T_o(i)) + H\left(a, \tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right)$$

and if $\bar{a}(i) > \tilde{R}(i)$, we have from (E.3)-(E.4)

$$\tilde{W}_{w,2}(i,a) = U(y(i) - T_w(i)) + L(a, A(t^*))$$
$$\begin{aligned}\tilde{W}_{o,2}(i,a) &= Q(T_o(i)) + H(a, A(t^*)) \ \ \forall \ \ a \leq \bar{a}(i) \\ &= Q(T_o(i)) + H\left(a, \tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right) \ \ \forall \ \ \bar{a}(i) < a\end{aligned}$$

For $Z_3(i)$ we have $i \geq t^* + A(t^*)$ and, hence, $i - a \geq t^*$. Together with (E.3)-(E.5) this gives

$$\tilde{W}_{w,3}(i,a) = U(y(i) - T_w(i)) + L(a, A(t^*))$$
$$\tilde{W}_{o,3}(i,a) = Q(T_o(i)) + H(a, A(t^*))$$
$$\tilde{m}_3\left(a, \tilde{A}(i)\right) = m(a, A(t^*))$$

The Lagrangian for (E.6) is

$$\begin{aligned}\Gamma(t) = &\int_{i=t}^{t^*} e^{-\theta(i-t)} Z_1(i)\, di + \int_{i=t^*}^{t^*+A(t^*)} e^{-\theta(i-t)} Z_2(i)\, di + \int_{i=t^*+A(t^*)}^{\infty} e^{-\theta(i-t)} Z_3(i)\, di \\ &+ \int_{i=t}^{\infty} \lambda_D(i)\left[\dot{D}(i) - \left[rD(i) - \left[\begin{array}{c} N_w\left(\tilde{R}(i), \tilde{A}(i)\right) T_w(i) \\ -N_o\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(i)\end{array}\right]\right]\right] di \\ &+ \int_{i=t}^{\infty} \lambda_y(i)\left[\dot{y}(i) - \mu_y y(i)\right] di \\ &+ \int_{i=t}^{t^*+A(t^*)} \lambda_A(i)\left[\dot{\tilde{A}}(i) - \hat{\mu}_A\right] di \ \text{(E.7)}\end{aligned}$$

where $\dot{x}(i) \equiv \lim_{di \to 0} \frac{dx(i)}{di}$ for $x(i) = D(i)$, $y(i)$, $\tilde{A}(i)$ and where N_w and N_o are given in (2.9) and (2.10) using the $\tilde{m}$ function from (E.5). Hence, we have

$$N_w\left(\tilde{R}(i), \tilde{A}(i)\right) = \int_{a=0}^{\tilde{R}(i)} \tilde{m}_j\left(a, \tilde{A}(i)\right) da$$

$$N_o\left(\tilde{R}(i), \tilde{A}(i)\right) = \int_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}_j\left(a, \tilde{A}(i)\right) da$$

where $j = 1$ when $i \leq t^*$, $j = 2$ when $t^* \leq i \leq t^* + A(t^*)$ and $j = 3$ for $i \geq t^* + A(t^*)$. Using integration by parts in (E.7) gives

$$\Gamma(t) = \int_{i=t}^{t^*} e^{-\theta(i-t)} Z_1(i)\, di + \int_{i=t^*}^{t^*+A(t^*)} e^{-\theta(i-t)} Z_2(i)\, di + \int_{i=t^*+A(t^*)}^{\infty} e^{-\theta(i-t)} Z_3(i)\, di$$
$$-\int_{i=t}^{\infty} \lambda_D(i) \left[rD(i) - \left[\begin{array}{c} N_w\left(\tilde{R}(i), \tilde{A}(i)\right) T_w(i) \\ -N_o\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(i) \end{array} \right] \right] di$$
$$-\int_{i=t}^{\infty} \lambda_y(i)\, \mu_y y(i)\, di - \int_{i=t}^{t^*+A(t^*)} \lambda_A(i)\, \hat{\mu}_A di$$
$$+\lim_{\bar{t}\to\infty} \left[\lambda_D(\bar{t})\, D(\bar{t})\right] - \lambda_D(t)\, D(t) - \int_{i=t}^{\infty} \dot{\lambda}_D(i)\, D(i)\, di$$
$$+\lim_{\bar{t}\to\infty} \left[\lambda_y(\bar{t})\, y(\bar{t})\right] - \lambda_y(t)\, y(t) - \int_{i=t}^{\infty} \dot{\lambda}_y(i)\, y(i)\, di$$
$$+\lambda_A(t^* + A(t^*))\, A(t^*) - \lambda_A(t)\, \tilde{A}(t) - \int_{i=t}^{t^*+A(t^*)} \dot{\lambda}(i)\, \tilde{A}(i)\, di \quad \text{(E.8)}$$

The first-order conditions are (for all $i \geq t$)

$-e^{-\theta(i-t)} U'(y(i) - T_w(i)) + \lambda_D(i) = 0$ (E.9)

$e^{-\theta(i-t)} Q'(T_o(i)) - \lambda_D(i) = 0$ (E.10)

$$e^{-\theta(i-t)} \left[\begin{array}{c} \left[U(y(i) - T_w(i)) + \tilde{L}\left(\tilde{R}(i), \tilde{A}(i)\right)\right] \\ -\left[Q(T_o(i)) + \tilde{H}\left(\tilde{R}(i), \tilde{A}(i)\right)\right] \end{array} \right] + \lambda_D(i)\left[T_w(i) + T_o(i)\right] = 0 \quad \text{(E.11)}$$

where it has been used that $\frac{\partial N_w}{\partial \tilde{R}(i)} = -\frac{\partial N_o}{\partial \tilde{R}(i)} = \tilde{m}_j\left(\tilde{R}(i), \tilde{A}(i)\right)$ for $j = 1, 2, 3$. Further, maximizing (E.8) with respect to $D(i)$ gives the following first-order condition (for all $i \geq t$):

$-\dot{\lambda}_D(i) - \lambda_D(i)\, r = 0$ (E.12)

Using (E.10) in (E.12) gives

$\dot{T}_o(i) = \frac{Q'(T_o(i))}{Q''(T_o(i))} [\theta - r]$

Assuming neutral generational weighting ($\theta = r$) gives $\dot{T}_o(i) = 0$ and, hence,

$T_o(i) = T_o(t)$ (E.13)

for all $i \geq t$. Using this, (2.20), (E.9) and (E.10) gives

$T_w(i) = T_w(t) + y(t)\left[e^{\mu_y[i-t]} - 1\right]$ (E.14) for all $i \geq t$. Plugging (E.10) into (E.11) gives

$$Q'(T_o(i))[T_w(i)+T_o(i)] = \left[Q(T_o(i)) + \tilde{H}\left(\tilde{R}(i), \tilde{A}(i)\right)\right] - \left[U(y(i)-T_w(i)) + \tilde{L}\left(\tilde{R}(i), \tilde{A}(i)\right)\right]$$

for all $i \geq t$ and, by using (E.13) and (E.14),

$$d\tilde{R}(i) = -\frac{Q'}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} dy(i) - \frac{\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)}}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} d\tilde{A}(i)$$

for $t^* + A(t^*) > i \geq t$

$$d\tilde{R}(i) = -\frac{Q'}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} dy(i)$$

for $i \geq t^* + A(t^*)$

Comparing these results to the ones with no upper bound on $A(t)$ from (3.14) and (3.16), we see that the dynamics in T_w and T_o are identical, while the dynamics in $\tilde{R}$ differ due to the upper bound. Further, the intertemporal budget constraint from (3.13) differs between the two cases since $N_w\left(\tilde{R}(i), \tilde{A}(i)\right)$ and $N_o\left(\tilde{R}(i), \tilde{A}(i)\right)$ differ due to different development of $\tilde{A}(i)$. Hence, the levels for T_w, T_o and $\tilde{R}$ differ.

F. A proof for optimal policy package

Proof of proposition 1

(3.10) and (3.11) imply

$$dT_w(i) = dy(i)$$

$$dT_o(i) = 0$$

for all $i \geq t$. Using (2.20) in these gives (3.14). Using (2.21) and (2.22) along with (2.20) in (3.14) gives (3.15).

(3.12) and (3.14) imply

$$V_D(t)\, dT_w(i) = \left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)} d\tilde{R}(i) + \left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)} d\tilde{A}(i)$$

for all $i \geq t$, implying that the retirement age evolves according to

$$d\tilde{R}(i) = \frac{V_D(t)}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} dy(i) - \frac{\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)}}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} d\tilde{A}(i)$$

for all $i \geq t$, where $V_D(t) < 0$ from (3.10), $\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)} < 0$ from (2.18) and $\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)} > 0$ from (2.19). Hence, this gives (3.16) where

$$\eta_y(i) = \frac{V_D(t)}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} > 0$$

$$\eta_A(i) = -\frac{\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)}}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} > 0$$

for all $i \geq t$. This completes the proof.

G. Proofs for retirement and healthy ageing

Proof of lemma 1

"only if"

(4.1) holds only if

$$\frac{d\tilde{R}(i)}{\tilde{R}(i)} = \frac{d\tilde{A}(i)}{\tilde{A}(i)} \quad \text{(G.1)}$$

for all $i \geq t$. The expression for the evolution of the optimal retirement age from (3.16) implies that this holds only if

$$-\frac{\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)}}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} = \frac{\tilde{R}(i)}{\tilde{A}(i)} \quad \text{(G.2)}$$

for all $i \geq t$. Assume that the $\tilde{L}\ (a, \tilde{A}\,(i))$ and $\tilde{H}(a, \tilde{A}\,(i))$ functions are homogeneous of degree λ, i.e., $\tilde{L}\left(a, \tilde{A}\,(i)\right) = \tilde{A}\,(i)^{\lambda}\,\tilde{L}\left(\frac{a}{\tilde{A}(i)}, 1\right)$ and $\tilde{H}\left(a, \tilde{A}\,(i)\right) = \tilde{A}\,(i)^{\lambda}\,\tilde{H}\left(\frac{a}{\tilde{A}(i)}, 1\right)$. Using this in (G.2), (G.1) holds only if

$$-\lambda\left[\tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right) - \tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right)\right] = 0$$

for all $i \geq t$. Hence, as long as $\tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right) \neq \tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right)$ and therefore $\tilde{L}\left(\tilde{R}\,(i), \tilde{A}\,(i)\right) \neq \tilde{H}\left(\tilde{R}\,(i), \tilde{A}\,(i)\right)$ due to the homogeneity assumption, which is likely to hold from (3.10)-(3.12), we have that (G.2) holds only if $\lambda = 0$. That is, (G.2) holds only if the $\tilde{L}$ and $\tilde{H}$ functions are both homogeneous of degree 0. This completes the first half of the proof.

"if"

From (3.12), (3.14) and (2.20) assuming constant output per young individual $(dy\,(i) = 0)$ gives

$$[T_w\,(t) + T_o\,(t)]\,V_D\,(t) = U\,(y\,(t) - T_w\,(t)) + \tilde{L}\left(\tilde{R}\,(i), \tilde{A}\,(i)\right) - Q\,(T_o\,(t)) - \tilde{H}\left(\tilde{R}\,(i), \tilde{A}\,(i)\right) \quad \text{(G.3)}$$

for all $i \geq t$. Assuming that the $\tilde{L}\ (a, \tilde{A}\,(i))$ and $\tilde{H}(a, \tilde{A}\,(i))$ are homogeneous of degree 0 implies from (G.3)

$$\tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right) - \tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right) = constant.$$

This can only hold if $\frac{\tilde{R}(i)}{\tilde{A}(i)}$ is constant since $\frac{\partial \tilde{L}}{\partial a} < \frac{\partial \tilde{H}}{\partial a}$ from (2.18) and (2.19). Hence, we have that (4.1) must hold. This completes the proof.

Proof of lemma 2

Assuming constant growth in longevity in (2.6) gives the following relationship between longevity of the generation that retires today and longevity of the generation that becomes extinct today:

$$A\left(i - \tilde{R}\right) = \tilde{A}\,(i) - \mu_A\left[\tilde{R}\,(i) - \tilde{A}\,(i)\right] \quad \text{(G.4)}$$

"only if"

Assuming that

$$\tilde{R}\,(i) = \kappa \tilde{A}\,(i)$$

holds and plugging into (G.4) give

$$\tilde{R}\,(i) = \frac{\kappa}{1+\mu_A[1-\kappa]} A\left(i - \tilde{R}\right)$$

Choosing $\psi = \frac{\kappa}{1+\mu_A[1-\kappa]}$ completes the first half of the proof.

"if"

Assuming that

$$\tilde{R}\,(i) = \psi A\left(i - \tilde{R}\right)$$

holds and plugging into (G.4) give

$$\tilde{R}\,(i) = \frac{\psi[1+\mu_A]}{1+\psi\mu_A}\tilde{A}\,(i)$$

Choosing $\kappa = \frac{\psi[1+\mu_A]}{1+\psi\mu_A}$ completes the second half of the proof. This completes the proof.

Proof of lemma 3

"only if"

Using the definition of the $\tilde{L}$ function, (2.6) under constant growth in longevity and assuming that the $\tilde{L}$ function is homogeneous of degree λ give

$k^{\lambda}\tilde{L}\left(a,\tilde{A}\left(i\right)\right)$
$=\tilde{L}\left(ka,k\tilde{A}\left(i\right)\right)$
$=L\left(ka,k\tilde{A}\left(i\right)-\mu_A\left[ka-k\tilde{A}\left(i\right)\right]\right)$
$=L\left(ka,k\left[\tilde{A}\left(i\right)-\mu_A\left[a-\tilde{A}\left(i\right)\right]\right]\right)$
$=L\left(ka,kA\left(i-a\right)\right)$
$=k^{\lambda}L\left(a,A\left(i-a\right)\right)$

where k is some constant (>0) and the sixth line uses that $\tilde{L}\left(a,\tilde{A}\left(i\right)\right)=L\left(a,A\left(i-a\right)\right)$ by definition. Hence, the $\tilde{L}$ function is homogeneous of degree λ only if the L function is also homogeneous of degree λ. Obviously, a similar proof goes for the H function. This completes the first half of the proof.

"if"

Using the definition of the $\tilde{L}$ function, (2.6) under constant growth in longevity and assuming that the L function is homogeneous of degree λ give

$k^{\lambda}L\left(a,A\left(i-a\right)\right)$
$=L\left(ka,kA\left(i-a\right)\right)$
$=L\left(ka,k\left[\tilde{A}\left(i\right)-\mu_A\left[a-\tilde{A}\left(i\right)\right]\right]\right)$
$=L\left(ka,\left[k\tilde{A}\left(i\right)-\mu_A\left[ka-k\tilde{A}\left(i\right)\right]\right]\right)$
$=\tilde{L}\left(ka,k\tilde{A}\left(i\right)\right)$
$=k^{\lambda}\tilde{L}\left(a,\tilde{A}\left(i\right)\right)$

where k is some constant (>0) and the sixth line uses that $\tilde{L}\left(a,\tilde{A}\left(i\right)\right)=L\left(a,A\left(i-a\right)\right)$ by definition. Hence, the $\tilde{L}$ function is homogeneous of degree λ if the L function is also homogeneous of degree λ. Obviously, a similiar proof goes for the H function. This completes the proof.

Proof of proposition 2

This follows directly from lemmas 1 - 3. This completes the proof.

H. Proofs for dependency ratio

Proof of lemma 4

Plugging (4.6) into (2.9), (2.10) and (2.6) assuming constant growth in longevity, and the resulting equations into (2.11) give

$$K\left(i\right)=\frac{\tilde{A}(i)\int\limits_{\tilde{a}=\phi(i)}^{1}\frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}}{\tilde{A}(i)\int\limits_{\tilde{a}=0}^{\phi(i)}\frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}}=\frac{\int\limits_{\tilde{a}=\phi(i)}^{1}\frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}}{\int\limits_{\tilde{a}=0}^{\phi(i)}\frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}}$$

where $\tilde{a}\equiv\frac{a}{\tilde{A}(i)}$ and

$$\frac{\partial K}{\partial \tilde{A}(i)} = -\phi' \frac{\left[\frac{\alpha^{\frac{\phi(i)}{1+\mu_A[1-\phi(i)]}}-\alpha}{1-\alpha}\right]}{\left[\int\limits_{\tilde{a}=0}^{\phi(i)} \frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha} d\tilde{a}\right]} [1 + K(i)]$$

It is clear from this that $\frac{\partial K}{\partial \tilde{A}(i)} \gtreqless 0 \;\; iff \;\; \phi' \lesseqgtr 0$. This completes the proof.

Proof of lemma 5

Assuming that the L *and* H functions, and, hence, the $\tilde{L}$ and $\tilde{H}$ functions from lemma 3, are homogeneous of degree λ, we have from (3.16) that

$$R' = -\left[\lambda \frac{\left[\tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)},1\right)-\tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)},1\right)\right]}{\left[\frac{\partial \tilde{L}}{\partial a}-\frac{\partial \tilde{H}}{\partial a}\right]_{a=\frac{\tilde{R}(i)}{\tilde{A}(i)},\tilde{A}(i)=1}} - \frac{\tilde{R}(i)}{\tilde{A}(i)}\right]$$

From (4.6) we have that $\phi' \gtreqless 0$ iff $R' \gtreqless \phi(i) = \frac{\tilde{R}(i)}{\tilde{A}(i)}$. This implies that $\phi' \gtreqless 0$ iff

$$\underbrace{-\lambda \frac{\left[\tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)},1\right)-\tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)},1\right)\right]}{\left[\frac{\partial \tilde{L}}{\partial a}-\frac{\partial \tilde{H}}{\partial a}\right]_{a=\frac{\tilde{R}(i)}{\tilde{A}(i)},\tilde{A}(i)=1}}}_{LHS} \gtreqless 0$$

Assume that the social optimal solution is such that consumption and utility when young are no less than when old ($c_w(i) \geq c_o(i)$ and $U(c_w(i)) \geq Q(c_o(i))$), which implies from (3.12) that $\tilde{H}\left(\tilde{R}(i), \tilde{A}(i)\right) > \tilde{L}\left(\tilde{R}(i), \tilde{A}(i)\right)$ and, hence, that $\tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right) > \tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right)$ due to the homogeneity assumption. We therefore have that $\phi' = 0$ iff $\lambda = 0$ since $\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\frac{\tilde{R}(i)}{\tilde{A}(i)},\tilde{A}(i)=1} < 0$ from (2.18) and (2.19). Further, we have that $\frac{\partial LHS}{\partial \lambda}_{\lambda=0} < 0$ and, hence, $\phi' > 0$ (< 0) when (and only when) λ is slightly less (greater) than zero. The general result is $\phi' \lesseqgtr 0 \;\; iff \;\; \lambda \gtreqless 0$. This completes the proof.

Proof of proposition 3

This follows directly from lemmas 4 - 5. This completes the proof.

I. Decentralized equilibrium

Individual problem

Let us consider an individual born at time j, i.e., he belongs to generation j. His welfare criterion is given by his expected discounted sum of lifetime utility

$$W_I(j) = \int\limits_{a=0}^{R(j)} e^{-\theta a} m(a, A(j)) W_w(j+a, a) da + \int\limits_{a=R(j)}^{A(j)} e^{-\theta a} m(a, A(j)) W_o(j+a, a) da$$

(I.1)

where output $y(j)$ is assumed to be constant $(= y)$, $W_w(j+a, a)$ and $W_o(j+a, a)$ are given in (2.12) and (2.15), respectively, and $m(a, A(j))$ is from (2.3). His problem is one of choosing consumption over his life and retirement age, conditional on profiles for taxes collected from young $T_w(j+a)$ and transfers to old $T_o(j+a)$ during the maximum length of life of individuals from his generation $a \in [0, A(j)]$ such that his welfare criterion is maximized

$$\underset{\{c_w(j+a),c_w(j+a),R(j)\}_{a=0}^{A(j)}}{Max} W_I(j) \quad (I.2)$$

Assuming that he has no initial assets, his budget constraint reads

$$\int_{a=0}^{R(j)} e^{-ra} m(a, A(j)) [y - T_w(j+a)] da + \int_{a=R(j)}^{A(j)} e^{-ra} m(a, A(j)) T_o(j+a) da$$
$$= \int_{a=0}^{R(j)} e^{-ra} m(a, A(j)) c_w(j+a) da + \int_{a=R(j)}^{A(j)} e^{-ra} m(a, A(j)) c_o(j+a) da \quad (I.3)$$

Since we want to focus on the market failure caused by missing market for lending from future generations to present, we assume that there exits an annuities market (as in Yaari, 1965) to eliminate the market failure caused by uncertain lifetime (see, for example, Blanchard (1985) and Sheshinski, (2008)). This implies that an individual born at any time j buys or sells annuities at any age a. The amount of annuities held by the individual is $n(j+a, a)$ and the amount purchased (sold) is $\dot{n}(j+a, a) > 0$ ($\dot{n}(j+a, a) < 0$), where $\dot{n}(j+a, a) \equiv \lim_{da \to 0} \frac{dn(j+a,a)}{da}$. The price of a unit of annuities is 1, and its instantaneous return is $r + \varphi(j+a, a)$ to an individual aged a and born in time j, where it is assumed that the interest rate facing individuals r is the same as for the government. Annuities owned by an individual expire when he dies.

Applying the results in Sheshinski (2008) here, a sufficient condition for (I.3) to hold is that the following holds:

$$\dot{n}(j+a, a) = [r + \varphi(j+a, a)] n(j+a, a)$$
$$+y - T_w(j+a) - c_w(j+a) \quad (I.4)$$

for $0 \leq a \leq R(j)$,

$$\dot{n}(j+a, a) = [r + \varphi(j+a, a)] n(j+a, a)$$
$$+T_o(j+a) - c_o(j+a) \quad (I.5)$$

for $R(j) < a \leq A(j)$ and

$$n(j + A(j), A(j)) = 0 \quad (I.6)$$

$$\varphi(j+a, a) = \frac{-\frac{\partial m(a,A(j))}{\partial a}}{m(a, A(j))} > 0 \quad (I.7)$$

The right hand side of the last equation is the hazard rate for an individual aged a born at time j; i.e., the probability that an individual dies at age a conditional on being alive at age a. This implies that $\varphi(j+a, a) > 0$ and that the interest rate on annuities is higher than the interest rate r.[37] The first equality states that if alive at maximum age $A(j)$, the individual will not hold any annuities since he knows that he will die with certainty.

Using (I.4)-(I.7) and (I.1) in (I.2)-(I.3) gives

$$\underset{\{c_w(j+a),c_w(j+a),R(j)\}_{a=0}^{A(j)}}{Max} \left\{ \begin{array}{c} \int_{a=0}^{R(j)} e^{-\Phi(j+a,a)} W_w(j+a, a) da \\ + \int_{a=R(j)}^{A(j)} e^{-\Phi(j+a,a)} W_o(j+a, a) da \end{array} \right\} \quad (I.8)$$

[37] This condition ensures that expected profits of insurance firms are zero. Insurance firms are willing to pay higher interest rate on annuities than r since a fraction φ of annuities holders will die and, hence, the insurance firms will not have to pay back the amount of their holding of annuities. In equilibrium, this has to equal the hazard rate to eliminate all profit in the market for annuities.

subject to (I.4)-(I.6), where $\Phi(j+a,a) \equiv \theta a + \int\limits_{z=0}^{a} \varphi(j+z,z)\,dz$.

Assuming interior solution, this gives the following first-order conditions for the optimal c_w, c_o and R, respectively:[38]

$e^{-\Phi(j+a,a)}U'(c_w(j+a)) + \lambda_n(j+a,a) = 0$ (I.9)

for $0 \le a \le R(j)$,

$e^{-\Phi(j+a,a)}Q'(c_o(j+a)) + \lambda_n(j+a,a) = 0$ (I.10)

for $R(j) < a \le A(j)$ and

$e^{-\Phi(j+R,R)}[U(c_w(j+R)) + L(R(j),A(j))] - e^{-\Phi(j+R,R)}[Q(c_o(j+R)) + H(R(j),A(j))]$
$= \lambda_n(j+R,R)[y - T_w(j+R) - c_w(j+R)] - \lambda_n(j+R,R)[T_o(j+R) - c_o(j+R)]$
(I.11)

where $\lambda_n(j+a,a)$ ($a \in [0, A(j)]$) is a Lagrange multiplier (costate variable). Further, this gives the following law of motion for the multiplier:

$-\dot{\lambda}_n(j+a,a) - \lambda_n(j+a,a)[r + \varphi(j+a,a)] = 0$ (I.12)

for all $a \in [0, A(j)]$.

From (I.9)-(I.10) and (I.12) we have

$\dot{c}_w(j+a) = -\frac{U'(\cdot)}{U''(\cdot)}[r-\theta]$

$\dot{c}_o(j+a) = -\frac{Q'(\cdot)}{Q''(\cdot)}[r-\theta]$

for all $a \in [0, R(j)]$ and $a \in (R(j), A(j)]$, respectively. Using the assumption of neutral generational weighting from above ($\theta = r$) implies that $\dot{c}_w(j+a) = \dot{c}_o(j+a) = 0$ and consumption is constant over age when an individual is young and when he is old. Using these results in (I.9)-(I.11) gives $e^{-\Phi(j+a,a)}U'(c_w(j)) + \lambda_n(j+a,a) = 0$ (I.13)

for $0 \le a \le R(j)$,

$e^{-\Phi(j+a,a)}Q'(c_o(j)) + \lambda_n(j+a,a) = 0$ (I.14)

for $R(j) < a \le A(j)$ and

$e^{-\Phi(j+R,R)}[U(c_w(j)) + L(R(j),A(j))] - e^{-\Phi(j+R,R)}[Q(c_o(j)) + H(R(j),A(j))]$
$= \lambda_n(j+R,R)[y - T_w(j+R) - c_w(j)] - \lambda_n(j+R,R)[T_o(j+R) - c_o(j)]$ (I.15)

The solution to the individual problem is characterized by $c_w(j)$, $c_o(j)$ and $R(j)$ conditional on profiles for taxes collected from young individuals ($T_w(j+a)$) and transfers to old ones ($T_o(j+a)$) during the maximum length of life of individuals from his generation ($a \in [0, A(j)]$) that satisfy (I.13)-(I.15) as well as the intertemporal budget constraint in (I.3). This has to hold for individuals born at any time j, i.e., it has to hold for any generation j.

From (I.13)-(I.15) and (I.3), the optimal consumption when young and old and the retirement age can be written as implicit functions of taxes, transfers and the maximum length of life of individuals from the generation in question

$c_w(j) = C^w\left(\{T_w(j+a), T_o(j+a)\}_{a=0}^{A(j)}, A(j)\right)$ (I.16)

$c_o(j) = C^o\left(\{T_w(j+a), T_o(j+a)\}_{a=0}^{A(j)}, A(j)\right)$ (I.17)

$R(j) = R\left(\{T_w(j+a), T_o(j+a)\}_{a=0}^{A(j)}, A(j)\right)$ (I.18)

As is shown in appendix J below by use of the implicit function theorem, these functions exist. Further, as is shown in appendix J, comparative static analysis reveals

[38] The first-order conditions and the law of motion for the shadow price of annuities are derived in appendix J below.

that the signs of the first derivatives are generally indeterminate and depend on the levels of the variables in equilibrium.

Government

At every point in time t, the government decides on the tax levied on the young/working and the transfer to the old/non-working now and in the future $\{T_w(i), T_o(i)\}_{i=t}^{\infty}$ (the policy package) such that its welfare objective from (3.2) (see discussion in the next paragraph) is maximized. In doing so, it is constrained by its budget constraint from (2.24) (and (2.23)), the path for longevity from (2.5) and individual behaviour from (I.16)-(I.18).

Consumption differs between generations in the decentralized equilibrium since generations are heterogeneous. This is reflected in (I.16)-(I.18) for a given policy package since $A(j)$ differs between generations. Hence, at a given point in time, consumption is not only time but also age dependent since a given age a represents a given generation $t-a$ at a given point in time t. (2.12) and (2.15) therefore become

$$W_w(t,a) = U(c_w(t-a)) + L(a, A(t-a)) \quad \forall \quad a \in \left[0, \tilde{R}(t)\right] \quad \text{(I.19)}$$

$$W_o(t,a) = Q(c_o(t-a)) + H(a, A(t-a)) \quad \forall \quad a \in \left(\tilde{R}(t), \tilde{A}(t)\right] \quad \text{(I.20)}$$

Hence, the welfare objective differs slightly from the one in (3.2).

Optimal policy package

As in the social planner problem, the optimal policy problem here is solved by setting up the Hamilton-Jacobi-Bellman (HJB) equation which in short hand writing (suppressing time indexes) determines the value function $V(\cdot)$ as

$$\theta V\left(D,\tilde{A}\right) = \underset{T_w,T_o}{Max}\left\{\begin{array}{c} \int_{a=0}^{\tilde{R}} \tilde{m}\left(a,\tilde{A}\right)\left[U\left(c_w(-a)\right) + \tilde{L}\left(a,\tilde{A}\right)\right] da \\ + \int_{a=\tilde{R}}^{\tilde{A}} \tilde{m}\left(a,\tilde{A}\right)\left[Q\left(c_o(-a)\right) + \tilde{H}\left(a,\tilde{A}\right)\right] da \\ + \frac{1}{dt} dV\left(D,\tilde{A}\right) \end{array}\right\}$$

s.t.

$$dD = \left[rD - \left[N_w\left(\tilde{R},\tilde{A}\right)T_w - N_o\left(\tilde{R},\tilde{A}\right)T_o\right]\right]dt$$

$$d\tilde{A} = \hat{\mu}_A dt$$

$$c_w(-a) = C^w\left(\{T_w(-l), T_o(-l)\}_{l=a}^{A(-a)}, A(-a)\right)$$

$$c_o(-a) = C^o\left(\{T_w(-l), T_o(-l)\}_{l=a}^{A(-a)}, A(-a)\right)$$

$$R(-a) = R\left(\{T_w(-l), T_o(-l)\}_{l=a}^{A(-a)}, A(-a)\right) \quad \text{(I.21)}$$

with N_w and N_o given from (2.9) and (2.10), respectively, and $\hat{\mu}_A \equiv \frac{\mu_A}{1+\mu_A}$ as before. Assuming interior solution, this gives the following first-order conditions for the optimal T_w and T_o, respectively:[39]

$$V_D = \frac{\tilde{m}\left(\tilde{R},\tilde{A}\right)}{N_w\left(\tilde{R},\tilde{A}\right)}\left[\begin{array}{c} U\left(c_w\left(-\tilde{R}\right)\right) + \tilde{L}\left(\tilde{R},\tilde{A}\right) \\ -Q\left(c_o\left(-\tilde{R}\right)\right) - \tilde{H}\left(\tilde{R},\tilde{A}\right) - V_D\left[T_w + T_o\right] \end{array}\right]\frac{\partial \tilde{R}}{\partial T_w}$$

[39] The first-order conditions and the law of motion for the marginal value function are derived in appendix K below.

$$+\frac{1}{N_w\left(\tilde{R},\tilde{A}\right)}\int_{a=0}^{\tilde{R}} \tilde{m}\left(a,\tilde{A}\right)U'\left(c_w\left(-a\right)\right)\frac{\partial c_w(-a)}{\partial T_w}da$$

$$+\frac{1}{N_w\left(\tilde{R},\tilde{A}\right)}\int_{a=\tilde{R}}^{\tilde{A}} \tilde{m}\left(a,\tilde{A}\right)Q'\left(c_o\left(-a\right)\right)\frac{\partial c_o(-a)}{\partial T_w}da \quad \text{(I.22)}$$

$$V_D=-\frac{\tilde{m}\left(\tilde{R},\tilde{A}\right)}{N_o\left(\tilde{R},\tilde{A}\right)}\left[\begin{array}{c} U\left(c_w\left(-\tilde{R}\right)\right)+\tilde{L}\left(\tilde{R},\tilde{A}\right) \\ -Q\left(c_o\left(-\tilde{R}\right)\right)-\tilde{H}\left(\tilde{R},\tilde{A}\right)-V_D\left[T_w+T_o\right]\end{array}\right]\frac{\partial\tilde{R}}{\partial T_o}$$

$$-\frac{1}{N_o\left(\tilde{R},\tilde{A}\right)}\int_{a=0}^{\tilde{R}} \tilde{m}\left(a,\tilde{A}\right)U'\left(c_w\left(-a\right)\right)\frac{\partial c_w(-a)}{\partial T_o}da$$

$$-\frac{1}{N_o\left(\tilde{R},\tilde{A}\right)}\int_{a=\tilde{R}}^{\tilde{A}} \tilde{m}\left(a,\tilde{A}\right)Q'\left(c_o\left(-a\right)\right)\frac{\partial c_o(-a)}{\partial T_o}da \quad \text{(I.23)}$$

Further, this gives the following law of motion for the marginal value function (relaxing the short hand writing):

$$dV_D(\cdot)=\left(\theta-r\right)V_D dt \quad \text{(I.24)}$$

Using the assumption of neutral generational weighting from above ($\theta=r$) implies that

$$dV_D\left(D\left(i\right),y\left(i\right),\tilde{A}\left(i\right)\right)=0 \quad \text{(I.25)}$$

for all $i\geq t$, i.e., the optimal policy package is such that the marginal value function $V_D(\cdot)$ is the same for all $i\geq t$. Applying this to (I.22) and (I.23) gives

$$V_D\left(t\right)=\frac{\tilde{m}\left(\tilde{R}(i),\tilde{A}(i)\right)}{N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}\times\left[\begin{array}{c} U\left(c_w\left(i-\tilde{R}\right)\right)+\tilde{L}\left(\tilde{R}\left(i\right),\tilde{A}\left(i\right)\right) \\ -Q\left(c_o\left(i-\tilde{R}\right)\right)-\tilde{H}\left(\tilde{R}\left(i\right),\tilde{A}\left(i\right)\right)-V_D\left(t\right)\left[T_w\left(i\right)+T_o\left(i\right)\right]\end{array}\right]\frac{\partial\tilde{R}(i)}{\partial T_w(i)}$$

$$+\frac{1}{N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}\int_{a=0}^{\tilde{R}(i)} \tilde{m}\left(a,\tilde{A}\left(i\right)\right)U'\left(c_w\left(i-a\right)\right)\frac{\partial c_w(i-a)}{\partial T_w(i)}da$$

$$+\frac{1}{N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}\int_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}\left(a,\tilde{A}\left(i\right)\right)Q'\left(c_o\left(i-a\right)\right)\frac{\partial c_o(i-a)}{\partial T_w(i)}da \quad \text{(I.26)}$$

$$V_D\left(t\right)=-\frac{\tilde{m}\left(\tilde{R}(i),\tilde{A}(i)\right)}{N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}\times\left[\begin{array}{c} U\left(c_w\left(i-\tilde{R}\right)\right)+\tilde{L}\left(\tilde{R}\left(i\right),\tilde{A}\left(i\right)\right) \\ -Q\left(c_o\left(i-\tilde{R}\right)\right)-\tilde{H}\left(\tilde{R}\left(i\right),\tilde{A}\left(i\right)\right)-V_D\left(t\right)\left[T_w\left(i\right)+T_o\left(i\right)\right]\end{array}\right]\frac{\partial\tilde{R}(i)}{\partial T_o(i)}$$

$$-\frac{1}{N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}\int_{a=0}^{\tilde{R}(i)} \tilde{m}\left(a,\tilde{A}\left(i\right)\right)U'\left(c_w\left(i-a\right)\right)\frac{\partial c_w(i-a)}{\partial T_o(i)}da$$

$$-\frac{1}{N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}\int_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}\left(a,\tilde{A}\left(i\right)\right)Q'\left(c_o\left(i-a\right)\right)\frac{\partial c_o(i-a)}{\partial T_o(i)}da \quad \text{(I.27)}$$

for all $i\geq t$, where $V_D\left(t\right)$ is written as a function of t to indicate that it is the same for all $i\geq t$.

An optimal policy package $\{T_w(i), T_o(i)\}_{i=t}^{\infty}$ must satisfy (I.26) and (I.27) as well as the intertemporal budget constraint from the social planner problem in (3.13).

J. Decentralized equilibrium derived

The Lagrangian for (I.2) is

$$\Gamma(j) = \int_{a=0}^{R(j)} e^{-\Phi(j+a,a)} W_w(j+a,a)da + \int_{a=R(j)}^{A(j)} e^{-\Phi(j+a,a)} W_o(j+a,a)da$$
$$+ \int_{a=0}^{R(j)} \lambda_n(j+a,a) \left[\begin{array}{c} \dot{n}(j+a,a) \\ - \left[\begin{array}{c} [r+\varphi(j+a,a)]\, n(j+a,a) \\ +y - T_w(j+a) - c_w(j+a) \end{array} \right] \end{array} \right] da$$
$$+ \int_{a=R(j)}^{A(j)} \lambda_n(j+a,a) \left[\begin{array}{c} \dot{n}(j+a,a) \\ - \left[\begin{array}{c} [r+\varphi(j+a,a)]\, n(j+a,a) \\ +T_o(j+a) - c_o(j+a) \end{array} \right] \end{array} \right] da$$

where $W_w(j+a,a)$ and $W_o(j+a,a)$ are given in (2.12) and (2.15), respectively. Using integration by parts gives

$$\Gamma(j) = \int_{a=0}^{R(j)} e^{-\Phi(j+a,a)} W_w(j+a,a)da + \int_{a=R(j)}^{A(j)} e^{-\Phi(j+a,a)} W_o(j+a,a)da$$
$$- \int_{a=0}^{R(j)} \lambda_n(j+a,a) \left[\begin{array}{c} [r+\varphi(j+a,a)]\, n(j+a,a) \\ +y(j+a) - T_w(j+a) - c_w(j+a) \end{array} \right] da$$
$$- \int_{a=R(j)}^{A(j)} \lambda_n(j+a,a) \left[\begin{array}{c} [r+\varphi(j+a,a)]\, n(j+a,a) \\ +T_o(j+a) - c_o(j+a) \end{array} \right] da$$
$$+\lambda_n(j+R,R)\, n(j+R,R) - \lambda_n(j,0)\, n(j,0)$$
$$- \int_{a=0}^{R(j)} \dot{\lambda}_n(j+a,a)\, n(j+a,a)\, da$$
$$+\lambda_n(j+A,A)\, n(j+A,A) - \lambda_n(j+R,R)\, n(j+R,R)$$
$$- \int_{a=R(j)}^{A(j)} \dot{\lambda}_n(j+a,a)\, n(j+a,a)\, da$$

The first-order conditions are

$$e^{-\Phi(i+a,a)} U'(c_w(i+a)) + \lambda_n(i+a,a) = 0$$

for $0 \le a \le R(j)$,

$$e^{-\Phi(j+a,a)} Q'(c_o(j+a)) + \lambda_n(j+a,a) = 0$$

for $R(j) < a \le A(j)$ and

$$e^{-\Phi(j+R,R)} [U(c_w(j+R)) + L(R(j), A(j))] - e^{-\Phi(j+R,R)} [Q(c_o(j+R)) + H(R(j), A(j))]$$
$$= \lambda_n(j+R,R)\, [y - T_w(j+R) - c_w(j+R)] - \lambda_n(j+R,R)\, [T_o(j+R) - c_o(j+R)]$$

which give (I.9), (I.10) and (I.11), respectively. Further, the first-order condition with respect to $n(j+a,a)$ is

$$-\dot{\lambda}_n(j+a,a) - \lambda_n(j+a,a)\, [r+\varphi(j+a,a)] = 0$$

for $0 \le a \le A(j)$, which gives (I.12).

Eliminating the costate variable from (I.13)-(I.15) and performing comparative static analysis using the resulting two equations and (I.3) with $c_w(j)$, $c_o(j)$ and $R(j)$ as endogenous variables give

$$\frac{\partial c_w(j)}{\partial T_w(j+a)} = -\frac{e^{-ra}m(a,A(j))\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(j)}}{den} < 0 \text{ for } 0 \le a < R(j)$$

$$\frac{\partial c_w(j)}{\partial T_w(j+R)} = \frac{e^{-rR(j)}m(R(j),A(j))\left[nom1-\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(j)}\right]}{den} \gtreqless 0$$

$$\frac{\partial c_w(j)}{\partial T_o(j+a)} = \frac{e^{-ra}m(a,A(j))\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(j)}}{den} > 0 \text{ for } R(j) < a \le A(j)$$

$$\frac{\partial c_w(j)}{\partial T_o(j+R)} = \frac{e^{-rR(j)}m(R(j),A(j))\left[nom1+\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(j)}\right]}{den} \gtreqless 0$$

$$\frac{\partial c_o(j)}{\partial T_w(j+a)} = U''(\cdot)\,Q''(\cdot)^{-1}\,\frac{\partial c_w(j)}{\partial T_w(j+a)} < 0 \text{ for } 0 \le a < R(j)$$

$$\frac{\partial c_o(j)}{\partial T_w(j+R)} = U''(\cdot)\,Q''(\cdot)^{-1}\,\frac{\partial c_w(j)}{\partial T_w(j+R)} \gtreqless 0$$

$$\frac{\partial c_o(j)}{\partial T_o(j+a)} = U''(\cdot)\,Q''(\cdot)^{-1}\,\frac{\partial c_w(j)}{\partial T_o(j+a)} > 0 \text{ for } R(j) < a \le A(j)$$

$$\frac{\partial c_o(j)}{\partial T_o(j+R)} = U''(\cdot)\,Q''(\cdot)^{-1}\,\frac{\partial c_w(j)}{\partial T_o(j+R)} \gtreqless 0$$

$$\frac{dR(j)}{dT_w(j+a)} = \frac{e^{-ra}m(a,A(j))U''(\cdot)U'(\cdot)^{-1}nom1}{den} \gtreqless 0 \text{ for } 0 \le a < R(j)$$

$$\frac{\partial R(j)}{\partial T_w(j+R)} = \frac{e^{-rR(j)}m(R(j),A(j))U''(\cdot)U'(\cdot)^{-1}nom1+U'(\cdot)nom2}{den} \gtreqless 0$$

$$\frac{dR(j)}{dT_o(j+a)} = -\frac{e^{-ra}m(a,A(j))U''(\cdot)U'(\cdot)^{-1}nom1}{den} \gtreqless 0 \text{ for } R(j) < a \le A(j)$$

$$\frac{\partial R(j)}{\partial T_o(j+R)} = -\frac{U''(\cdot)U'(\cdot)^{-1}e^{-rR(j)}m(R(j),A(j))nom1-U'(\cdot)nom2}{den} \gtreqless 0$$

where

$$den = U''(\cdot)\,U'(\cdot)^{-2}\,e^{-rR(j)}m(R(j),A(j))\,nom1^2 + \left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(j)} nom2 < 0$$

$$nom1 = Q(\cdot) - U(\cdot) + [H(\cdot,\cdot) - L(\cdot,\cdot)]_{a=R(j)} \gtreqless 0$$

$$nom2 = \int_{a=0}^{R(j)} e^{-ra}m(a,A(j))\,da + U''(\cdot)\,Q''(\cdot)^{-1}\int_{a=R(j)}^{A(j)} e^{-ra}m(a,A(j))\,da > 0$$

and it is used that $\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(j)} < 0$ and $\left[\frac{\partial L}{\partial A}-\frac{\partial H}{\partial A}\right]_{a=R(j)} > 0$ by assumption. Note that $nom1 = Q(\cdot)-U(\cdot)+[H(\cdot,\cdot)-L(\cdot,\cdot)]_{a=R(j)}$ is the increase in instantaneous utility from retiring. Also note that the denominator (den) in the derivatives is always non-zero and, hence, the conditions of the implicit function theorem are fulfilled.[40]

K. Optimal policy package derived

Applying Taylor approximation to the value function $V\left(D,\tilde{A}\right)$ and considering the limit for $dt \to 0$ give the following:

$$dV\left(D,\tilde{A}\right) = \left[V_D\left[rD - \left[N_w\left(\tilde{R},\tilde{A}\right)T_w - N_o\left(\tilde{R},\tilde{A}\right)T_o\right]\right] + V_{\tilde{A}}\hat{\mu}_A\right]dt$$

Inserting this into (I.21) gives

[40] It is, of course, also assumed that all the derivatives of the equilibrium conditions in (I.13)-(I.15) and (I.3) are continuous.

$$\theta V\left(D,\tilde{A}\right)=\underset{T_w,T_o}{Max}\left\{\begin{array}{c}\int\limits_{a=0}^{\tilde{R}}\tilde{m}\left(a,\tilde{A}\right)\left[U\left(c_w\left(-a\right)\right)+\tilde{L}\left(a,\tilde{A}\right)\right]da\\+\int\limits_{a=\tilde{R}}^{\tilde{A}}\tilde{m}\left(a,\tilde{A}\right)\left[Q\left(c_o\left(-a\right)\right)+\tilde{H}\left(a,\tilde{A}\right)\right]da\\+V_D\left[rD-\left[N_w\left(\tilde{R},\tilde{A}\right)T_w-N_o\left(\tilde{R},\tilde{A}\right)T_o\right]\right]+V_{\tilde{A}}\hat{\mu}_A\end{array}\right\}\tag{K.1}$$

We can now find the first-order conditions determining T_w and T_o. For the tax payment by the young T_w, we have

$$\tilde{m}\left(\tilde{R},\tilde{A}\right)\left[U\left(c_w\left(-\tilde{R}\right)\right)+\tilde{L}\left(\tilde{R},\tilde{A}\right)-Q\left(c_o\left(-\tilde{R}\right)\right)-\tilde{H}\left(\tilde{R},\tilde{A}\right)\right]\frac{\partial\tilde{R}}{\partial T_w}$$
$$+\int\limits_{a=0}^{\tilde{R}}\tilde{m}\left(a,\tilde{A}\right)U'\left(c_w\left(-a\right)\right)\frac{\partial c_w(-a)}{\partial T_w}da$$
$$+\int\limits_{a=\tilde{R}}^{\tilde{A}}\tilde{m}\left(a,\tilde{A}\right)Q'\left(c_o\left(-a\right)\right)\frac{\partial c_o(-a)}{\partial T_w}da$$
$$-V_DN_w\left(\tilde{R},\tilde{A}\right)-V_D\left[\frac{\partial N_w}{\partial\tilde{R}}T_w-\frac{\partial N_o}{\partial\tilde{R}}T_o\right]\frac{\partial\tilde{R}}{\partial T_w}=0$$

or using $N_w\left(\tilde{R},\tilde{A}\right)$ from (2.9), $N_o\left(\tilde{R},\tilde{A}\right)$ from (2.10) and that $\frac{\partial N_w}{\partial\tilde{R}}=-\frac{\partial N_o}{\partial\tilde{R}}=\tilde{m}\left(\tilde{R},\tilde{A}\right)$, we get (I.22) as

$$V_D=\frac{\tilde{m}\left(\tilde{R},\tilde{A}\right)}{N_w\left(\tilde{R},\tilde{A}\right)}\left[\begin{array}{c}U\left(c_w\left(-\tilde{R}\right)\right)+\tilde{L}\left(\tilde{R},\tilde{A}\right)\\-Q\left(c_o\left(-\tilde{R}\right)\right)-\tilde{H}\left(\tilde{R},\tilde{A}\right)-V_D\left[T_w+T_o\right]\end{array}\right]\frac{\partial\tilde{R}}{\partial T_w}$$
$$+\frac{1}{N_w\left(\tilde{R},\tilde{A}\right)}\int\limits_{a=0}^{\tilde{R}}\tilde{m}\left(a,\tilde{A}\right)U'\left(c_w\left(-a\right)\right)\frac{\partial c_w(-a)}{\partial T_w}da$$
$$+\frac{1}{N_w\left(\tilde{R},\tilde{A}\right)}\int\limits_{a=\tilde{R}}^{\tilde{A}}\tilde{m}\left(a,\tilde{A}\right)Q'\left(c_o\left(-a\right)\right)\frac{\partial c_o(-a)}{\partial T_w}da$$

For the transfer to the old T_o, we have

$$\tilde{m}\left(\tilde{R},\tilde{A}\right)\left[U\left(c_w\left(-\tilde{R}\right)\right)+\tilde{L}\left(\tilde{R},\tilde{A}\right)-Q\left(c_o\left(-\tilde{R}\right)\right)-\tilde{H}\left(\tilde{R},\tilde{A}\right)\right]\frac{\partial\tilde{R}}{\partial T_o}$$
$$+\int\limits_{a=0}^{\tilde{R}}\tilde{m}\left(a,\tilde{A}\right)U'\left(c_w\left(-a\right)\right)\frac{\partial c_w(-a)}{\partial T_o}da$$
$$+\int\limits_{a=\tilde{R}}^{\tilde{A}}\tilde{m}\left(a,\tilde{A}\right)Q'\left(c_o\left(-a\right)\right)\frac{\partial c_o(-a)}{\partial T_o}da$$
$$+V_DN_o\left(\tilde{R},\tilde{A}\right)-V_D\left[\frac{\partial N_w}{\partial\tilde{R}}T_w-\frac{\partial N_o}{\partial\tilde{R}}T_o\right]\frac{\partial\tilde{R}}{\partial T_o}=0$$

or using $N_w\left(\tilde{R},\tilde{A}\right)$ from (2.9), $N_o\left(\tilde{R},\tilde{A}\right)$ from (2.10) and that $\frac{\partial N_w}{\partial\tilde{R}}=-\frac{\partial N_o}{\partial\tilde{R}}=\tilde{m}\left(\tilde{R},\tilde{A}\right)$, we get (I.23) as

$$V_D = -\frac{\tilde{m}\left(\tilde{R},\tilde{A}\right)}{N_o\left(\tilde{R},\tilde{A}\right)} \left[\begin{array}{c} U\left(c_w\left(-\tilde{R}\right)\right) + \tilde{L}\left(\tilde{R},\tilde{A}\right) \\ -Q\left(c_o\left(-\tilde{R}\right)\right) - \tilde{H}\left(\tilde{R},\tilde{A}\right) - V_D\left[T_w + T_o\right] \end{array} \right] \frac{\partial \tilde{R}}{\partial T_o}$$

$$-\frac{1}{N_o\left(\tilde{R},\tilde{A}\right)} \int_{a=0}^{\tilde{R}} \tilde{m}\left(a,\tilde{A}\right) U'\left(c_w\left(-a\right)\right) \frac{\partial c_w(-a)}{\partial T_o} da$$

$$-\frac{1}{N_o\left(\tilde{R},\tilde{A}\right)} \int_{a=\tilde{R}}^{\tilde{A}} \tilde{m}\left(a,\tilde{A}\right) Q\left(c_o\left(-a\right)\right) \frac{\partial c_o(-a)}{\partial T_o} da$$

Replacing T_w and T_o with their optimal values from (I.22) and (I.23) gives the optimal values for the endogenous variables $\tilde{R}$, $c_w\left(-a\right)$ and $c_w\left(-a\right)$: $\tilde{R}^*$, $c_w^*\left(-a\right)$ and $c_o^*\left(a\right)$. Plugging these into (K.1) gives

$$\theta V\left(D,\tilde{A}\right) = \left\{ \begin{array}{c} \int_{a=0}^{\tilde{R}^*} \tilde{m}\left(a,\tilde{A}\right)\left[U\left(c_w^*\left(-a\right)\right) + \tilde{L}\left(a,\tilde{A}\right)\right] da \\ + \int_{a=\tilde{R}^*}^{\tilde{A}} \tilde{m}\left(a,\tilde{A}\right)\left[Q\left(c_o^*\left(-a\right)\right) + \tilde{H}\left(a,\tilde{A}\right)\right] da \\ +V_D\left[rD - \left[N_w\left(\tilde{R}^*,\tilde{A}\right)T_w^* - N_o\left(\tilde{R}^*,\tilde{A}\right)T_o^*\right]\right] + V_{\tilde{A}}\hat{\mu}_A \end{array} \right\}$$

Using the envelope theorem gives

$$\left(\theta - r\right)V_D = V_{DD}\left[rD - \left[N_w\left(\tilde{R}^*,\tilde{A}\right)T_w^* - N_o\left(\tilde{R}^*,\tilde{A}\right)T_o^*\right]\right] + V_{\tilde{A}D}\hat{\mu}_A$$

Applying a Taylor approximation to V_D and using the law of motion for D and $\tilde{A}$ and considering the limit for $dt \rightarrow 0$ give

$$dV_D = V_{DD}\left[rD - \left[N_w\left(\tilde{R},\tilde{A}\right)T_w - N_o\left(\tilde{R},\tilde{A}\right)T_o\right]\right]dt + V_{D\tilde{A}}\hat{\mu}_A dt$$

Evaluating this at $\tilde{R}^*$, $c_w^*\left(-a\right)$ and $c_o^*\left(-a\right)$, plugging into it and using that $V_{Dy} = V_{yD}$ and $V_{D\tilde{A}} = V_{\tilde{A}D}$ (Young's theorem) give (I.24) as

$$dV_D = \left(\theta - r\right)V_D dt$$

L. Retirement age comparison

Partial equilibrium effects

According to the decentralized equilibrium, a generation born at time $i - R_d$ retires at the age R_d at time i such that the following holds from appendix I:

$$Q\left(c_o\left(i-\tilde{R}_d\right)\right) + H\left(\tilde{R}_d\left(i\right), A\left(i-\tilde{R}_d\right)\right) - U\left(c_w\left(i-\tilde{R}_d\right)\right) - L\left(\tilde{R}_d\left(i\right), A\left(i-\tilde{R}_d\right)\right)$$
$$=$$
$$U'\left(c_w\left(i-\tilde{R}_d\right)\right)\left[y - T_w\left(i\right) - T_o\left(i\right) - c_w\left(i-\tilde{R}_d\right) + c_o\left(i-\tilde{R}_d\right)\right] \text{ (L.1)}$$

Using (2.21), (2.22), (3.10)-(3.12) and the definition of the $\tilde{L}$ and $\tilde{H}$ functions give that, according to the social planner's solution, the generation born at $l - R_s$ retires at age R_s at time l such that following holds:

$$Q\left(c_o\left(t\right)\right) + H\left(\tilde{R}_s\left(l\right), A\left(l-\tilde{R}_s\right)\right) - U\left(c_w\left(t\right)\right) - L\left(\tilde{R}_s\left(l\right), A\left(l-\tilde{R}_s\right)\right)$$
$$=$$
$$U'\left(c_w\left(t\right)\right)\left[y - c_w\left(t\right) + c_o\left(t\right)\right] \text{ (L.2)}$$

In both the decentralized equilibrium and the social planner's solution, consumption when young and consumption when old are constant over time for a given generation, and the difference in consumption and utility depends only on the utility functions, i.e., $U(\cdot)$ and $Q(\cdot)$, respectively. Hence, we can use that

$c_o(t) = \omega_c c_w(t)$

$Q(c_o(t)) = \omega_u U(c_w(t))$

for the social planner's solution and

$c_o\left(i - \tilde{R}_d\right) = \omega_c c_w\left(i - \tilde{R}_d\right)$

$Q\left(c_o\left(i - \tilde{R}_d\right)\right) = \omega_u U\left(c_w\left(i - \tilde{R}_d\right)\right)$

for the decentralized equilibrium, where $\omega_c, \omega_u \leq 1$ since it is assumed as before that consumption and utility of young individuals are no less than of old individuals. Using these in (L.1) and (L.2) gives

$$H\left(\tilde{R}_d(i), A\left(i - \tilde{R}_d\right)\right) - L\left(\tilde{R}_d(i), A\left(i - \tilde{R}_d\right)\right) = U'\left(c_w\left(i - \tilde{R}_d\right)\right)[y - T_w(i) - T_o(i)] - U'\left(c_w\left(i - \tilde{R}_d\right)\right)[1-\omega_c]\, c_w\left(i - \tilde{R}_d\right) + [1-\omega_u]\, U\left(c_w\left(i - \tilde{R}_d\right)\right) \quad \text{(L.3)}$$

$$H\left(\tilde{R}_s(l), A\left(l - \tilde{R}_s\right)\right) - L\left(\tilde{R}_s(l), A\left(l - \tilde{R}_s\right)\right) = U'(c_w(t))\, y - U'(c_w(t))[1-\omega_c]\, c_w(t) + [1-\omega_u]\, U(c_w(t)) \quad \text{(L.4)}$$

Let us consider a given generation such that $l - R_s = i - R_d$ and compare its retirement age under the two schemes. When comparing (L.3) and (L.4), the left hand sides only vary with the retirement ages $\tilde{R}_d(i)$ and $\tilde{R}_s(l)$ since we are looking at a given generation and, hence, $A\left(i - \tilde{R}_d\right) = A\left(l - \tilde{R}_s\right)$. Further, we have from (2.18) that $\frac{\partial[H(\cdot,\cdot) - L(\cdot,\cdot)]}{\partial a} > 0$. Hence, it follows that $\tilde{R}_d \gtreqless \tilde{R}_s$ iff

$$U'\left(c_w\left(i - \tilde{R}_d\right)\right)[y - T_w(i) - T_o(i)] - U'\left(c_w\left(i - \tilde{R}_d\right)\right)[1-\omega_c]\, c_w\left(i - \tilde{R}_d\right) + [1-\omega_u]\, U\left(c_w\left(i - \tilde{R}_d\right)\right) \gtreqless U'(c(t))\, y - U'(c_w(t))[1-\omega_c]\, c_w(t) + [1-\omega_u]\, U(c_w(t))$$

Assuming that the same consumption levels arise in the social planner's and decentralized equilibrium allocation, we have that the retirement age differs since

$R_s \gtreqless R_d$ for $T_w(i) + T_o(i) \gtreqless 0$

which gives (5.1).

General equilibrium effects

How does the retirement age change if the economy starts where (L.4) holds such that $c_w\left(i - \tilde{R}_d\right) = c_w(t)$ and $\tilde{R}_d(i) = \tilde{R}_s(l)$ and moves to where (L.3) holds where $T_w(i) \to 0$ and $T_o(i) \to 0$ (and, hence, $dT_w(i) \to 0$ and $dT_o(i) \to 0$)? Let us assume for simplicity that consumption and utility functions are identical when individuals are young and old $\omega_c = \omega_u = 1$. Using (L.3), the general equilibrium effects are

$$\frac{\partial \tilde{R}_d(i)}{\partial T_w(i)} + \frac{\partial \tilde{R}_d(i)}{\partial T_o(i)} = \underbrace{\frac{yU''(\cdot)}{\frac{\partial[H(\cdot,\cdot)-L(\cdot,\cdot)]}{\partial \tilde{R}_d(i)}}}_{<0} \left[\frac{\partial c_w\left(i-\tilde{R}_d\right)}{\partial T_w(i)} + \frac{\partial c_w\left(i-\tilde{R}_d\right)}{\partial T_o(i)}\right] - \underbrace{\frac{2U'(\cdot)}{\frac{\partial[H(\cdot,\cdot)-L(\cdot,\cdot)]}{\partial \tilde{R}_d(i)}}}_{>0} \quad \text{(L.5)}$$

where

$\frac{\partial c_w\left(i-\tilde{R}_d\right)}{\partial T_w(i)} + \frac{\partial c_w\left(i-\tilde{R}_d\right)}{\partial T_o(i)}$

$$= \frac{2e^{-r\tilde{R}_s(l)}yU'(\cdot)m(\cdot,\cdot)}{y^2U''(\cdot)e^{-r\tilde{R}_s(l)}m(\cdot,\cdot)-\frac{\partial[H(\cdot,\cdot)-L(\cdot,\cdot)]}{\partial\tilde{R}_d(i)}\left[\int\limits_{a=0}^{A\left(l-\tilde{R}_s\right)} e^{-ra}m\left(a,A\left(l-\tilde{R}_s\right)\right)da\right]} < 0 \quad (L.6)$$

as can be verified using the results of the comparative static analysis in appendix J. All the derivatives are evaluated at the social planner's solution.

The second term on the right hand side of (L.5) gives the direct effect discussed above; i.e., the retirement age decreases when the economy moves away from the social planner's solution such that $T_w(i) + T_o(i)$ becomes positive, while the first terms give the consumption effects. Using (L.6), it is clear that these two effects work in opposite directions. Combining (L.5) and (L.6), we have

$$\frac{\partial\tilde{R}_d(i)}{\partial T_w(i)} + \frac{\partial\tilde{R}_d(i)}{\partial T_o(i)} < 0$$

$$iff$$

$$\frac{\partial[H(\cdot,\cdot)-L(\cdot,\cdot)]}{\partial\tilde{R}_d(i)}\left[\int\limits_{a=0}^{A\left(l-\tilde{R}_s\right)} e^{-ra}m\left(a,A\left(l-\tilde{R}_s\right)\right)da\right] > 0$$

which always holds. Hence, the direct effects are always stronger than the consumption effects and the retirement age decreases when the economy moves away from the social planner's solution such that $T_w(i) + T_o(i) > 0$. Note that this analysis is only meant to give an idea about how the retirement age is in the decentralized equilibrium compared to in the social planner's solution since all the derivatives are evaluated at the social planner's solution.

M. Proof for decentralizing the social planner's allocation

Proof of proposition 4.

The proof proceeds by assuming that the optimal policy package gives the socially optimal allocation and shows that a contradiction emerges.

From the social planner's solution, we have that (3.10)-(3.12) and (3.15) have to hold. Plugging (3.15) into (3.10)-(3.12) and the resulting equations and (3.15) into (I.26)-(I.27) gives

$$1 = -\frac{1}{N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}\int\limits_{a=0}^{\tilde{R}(i)} \tilde{m}\left(a,\tilde{A}(i)\right)\frac{\partial c_w(i-a)}{\partial T_w(i)}da$$
$$-\frac{1}{N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}\int\limits_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}\left(a,\tilde{A}(i)\right)\frac{\partial c_o(i-a)}{\partial T_w(i)}da$$

and

$$1 = \frac{1}{N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}\int\limits_{a=0}^{\tilde{R}(i)} \tilde{m}\left(a,\tilde{A}(i)\right)\frac{\partial c_w(i-a)}{\partial T_o(i)}da$$
$$+\frac{1}{N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}\int\limits_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}\left(a,\tilde{A}(i)\right)\frac{\partial c_o(i-a)}{\partial T_o(i)}da$$

or

$$\frac{1}{N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}\left[\begin{array}{c}\int\limits_{a=0}^{\tilde{R}(i)}\tilde{m}\left(a,\tilde{A}(i)\right)\frac{\partial c_w(i-a)}{\partial T_o(i)}da\\+\int\limits_{a=\tilde{R}(i)}^{\tilde{A}(i)}\tilde{m}\left(a,\tilde{A}(i)\right)\frac{\partial c_o(i-a)}{\partial T_o(i)}da\end{array}\right]$$
$$=$$
$$\frac{1}{N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}\left[\begin{array}{c}\int\limits_{a=0}^{\tilde{R}(i)}\tilde{m}\left(a,\tilde{A}(i)\right)\left[-\frac{\partial c_w(i-a)}{\partial T_w(i)}\right]da\\+\int\limits_{a=\tilde{R}(i)}^{\tilde{A}(i)}\tilde{m}\left(a,\tilde{A}(i)\right)\left[-\frac{\partial c_o(i-a)}{\partial T_w(i)}\right]da\end{array}\right] \quad \text{(M.1)}$$

Plugging into (M.1) using the results of the comparative static analysis in appendix J, the definition of the $\tilde{m}$ function from (2.7) and the assumption that the optimal policy package gives the same allocation as the social planner's solution, we have

$$\frac{1}{N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}\left[\begin{array}{c}\int\limits_{a=0}^{\tilde{R}(i)}\frac{e^{-ra}\tilde{m}\left(a,\tilde{A}(i)\right)^2\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(i-a)}}{den(i-a)}da\\+\int\limits_{a=\tilde{R}(i)}^{\tilde{A}(i)}\frac{U''(\cdot)Q''(\cdot)^{-1}e^{-ra}\tilde{m}\left(a,\tilde{A}(i)\right)^2\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(i-a)}}{den(i-a)}da\\-\frac{\left[\begin{array}{c}\left[1+U''(\cdot)Q''(\cdot)^{-1}\right]e^{-r\tilde{R}(i)}\tilde{m}\left(\tilde{R}(i),\tilde{A}(i)\right)^2\\\times\left[U(\cdot)-Q(\cdot)+\left[L(\cdot,\cdot)-H(\cdot,\cdot)\right]_{a=\tilde{R}(i)}\right]\end{array}\right]}{den\left(i-\tilde{R}\right)}da\end{array}\right]$$
$$=$$
$$\frac{1}{N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}\left[\begin{array}{c}\int\limits_{a=0}^{\tilde{R}(i)}\frac{e^{-ra}\tilde{m}\left(a,\tilde{A}(i)\right)^2\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(i-a)}}{den(i-a)}da\\+\int\limits_{a=\tilde{R}(i)}^{\tilde{A}(i)}\frac{U''(\cdot)Q''(\cdot)^{-1}e^{-ra}\tilde{m}\left(a,\tilde{A}(i)\right)^2\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(i-a)}}{den(i-a)}da\\+\frac{\left[\begin{array}{c}\left[1+U''(\cdot)Q''(\cdot)^{-1}\right]e^{-r\tilde{R}(i)}\tilde{m}\left(\tilde{R}(i),\tilde{A}(i)\right)^2\\\times\left[U(\cdot)-Q(\cdot)+\left[L(\cdot,\cdot)-H(\cdot,\cdot)\right]_{a=\tilde{R}(i)}\right]\end{array}\right]}{den\left(i-\tilde{R}\right)}da\end{array}\right] \quad \text{(M.2)}$$

where $den(i-a)<0$ for all $0\leq a\leq\tilde{A}(i)$.

Simplifying (M.2) and using (2.9), (2.10) give

$$\underbrace{\left[N_w(i) - N_o(i)\right]\left[\begin{array}{c} \int\limits_{a=0}^{\tilde{R}(i)} \frac{e^{-ra}\tilde{m}\left(a,\tilde{A}(i)\right)^2\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(i-a)}}{den(i-a)}da \\ +\int\limits_{a=\tilde{R}(i)}^{\tilde{A}(i)} \frac{U''(\cdot)Q''(\cdot)^{-1}e^{-ra}\tilde{m}\left(a,\tilde{A}(i)\right)^2\left[\frac{\partial L}{\partial a}-\frac{\partial H}{\partial a}\right]_{a=R(i-a)}}{den(i-a)}da \end{array}\right]}_{LHS}$$

$$= \underbrace{N(i)\,\frac{\left[\begin{array}{c}\left[1+U''(\cdot)\,Q''(\cdot)^{-1}\right]e^{-r\tilde{R}(i)}\tilde{m}\left(\tilde{R}(i),\tilde{A}(i)\right)^2 \\ \times\left[U(\cdot)-Q(\cdot)+\left[L(\cdot,\cdot)-H(\cdot,\cdot)\right]_{a=\tilde{R}(i)}\right]\end{array}\right]}{den\left(i-\tilde{R}\right)}da}_{RHS} \quad \text{(M.3)}$$

Note that due to the continuous time setup we have that $da \to 0$ and, hence, that $RHS \to 0$. Also note that $LHS = 0$ iff $N_w(i) = N_o(i)$. Hence, (M.3) holds only if the number of young individuals is very close to the number of old and a contradiction emerges. This completes the proof.