QUANTAL RESPONSE EQUILIBRIA FOR POSTED OFFER-MARKETS

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Resumen: Existe una creciente literatura experimental sobre teoría de juegos y organización industrial que revela desviaciones sistemáticas del equilibrio de Nash. El presente trabajo generaliza el análisis de equilibrio de Nash en juegos del tipo de Bertrand para permitir errores de decisión determinados endógenamente. Este trabajo presenta soluciones explícitas de las distribuciones de precios de equilibrio con errores endógenos para ciertos modelos experimentales del tipo de Bertrand. Para otros modelos más complejos se muestra que el Equilibrio con Respuestas Discretas, *ERD*, es sensible a cambios en variables estructurales, mientras que el equilibrio de Nash permanece inalterado. El *ERD* es importante porque explica las desviaciones sistemáticas del equilibrio de Nash observadas en experimentos controlados.

Abstract: There is a growing body of data from game theory and industrial organization experiments that reveals systematic deviations from Nash equilibrium behavior. In this paper, the perfectly rational decision-making embodied in Bertrand-Nash equilibrium is generalized to allow for endogenously determined decision errors. Closed-form solutions for equilibrium price distributions with endogenous errors are derived for several different models. In some of these models, the price distribution in a quantal response equilibrium, QRE, is affected by changes in structural variables although the Nash equilibrium remains unaltered. The quantal response approach is appealing since it thereby accounts for systematic deviations from the Bertrand-Nash equilibrium.

1. Introduction

Although the Nash equilibrium is widely used in economic theory, there is some dissatisfaction with this concept. One criticism is that rationality is less restrictive than the Nash equilibrium implies. In general, there are many more strategies that may be considered rational choices, according to some beliefs, than merely those choices described as Nash equilibrium strategies, Bernheim (1984) and Piearce (1984).[1] Other critics find the Nash equilibrium concept to be too unrestrictive because it allows for behavior that is intuitively unreasonable. The literature on *refinements*, beginning with Selten (1975), has developed a series of proposed rules for eliminating such implausible equilibria. Articles on refinements typically begin with examples having several Nash equilibria, some of which are intuitively implausible because, for instance, they are based on strategies that can be interpreted as noncredible threats.[2] Some economists have therefore proposed that an analysis of learning and adjustment is the most useful way to proceed. The literature on the topic of evolutionary game theory consistently adopts this assumption, going back at least to Alchian's (1950) seminal paper and to Simon's (1957) work on bounded rationality.[3] There is also much recent work on *naive* (non-strategic) learning models, showing behavior that converges to a Nash equilibrium.[4]

Most theorists are uneasy about models of limited rationality, in part because of the looseness and the multiplicity of possible approaches. However, data from laboratory games with human subjects provide empirical regularities that can guide theoretical work on learn-

[1] For Benheim and Pearce, a strategy is rationalizable if it is a best reply to strategies that the other players might actually use. A particular collection of strategy sets, one for each player, is rationalizable if each strategy in the set belonging to player i is a best reply to some strategy profile in the sets belonging to the other players.

[2] For further discussion of related issues, see Kohlberg and Mertens (1986) and Kreps and Wilson (1982).

[3] This literature is characterized by models with individuals who make choices based on rules of thumb or who have some very rigid method of choice.

[4] See Kalai and Lehrer (1991) and the references therein. Brandts and Holt (1992), (1993), show that adaptive behavior in laboratory games can result in equilibrium patterns that are ruled out by almost all standard refinements of the sequential Nash equilibrium.

ing and adjustment. As a first approximation, evidence from game experiments tends to conform to Nash equilibrium predictions (Davis and Holt, 1993, chapter 2). However, some features of the data from market experiments have been difficult to explain in this way. Systematic deviations from rational behavior have been observed in experiments where the Nash equilibrium is located at the boundary of the set of feasible decisions, e.g., in ultimatum and public goods games.[5] For instance, in a 1-dollar ultimatum bargaining game, the sender proposes a split which the receiver must either accept or reject. A rejection results in earnings of 0 for both players. For this game, a subgame perfect Nash outcome is 1 penny for the receiver and 99 cents for the sender. Yet the actual outcomes of experimental ultimatum games are not nearly so asymmetric.

The ultimatum game can be given a simple market interpretation, with a single seller proposing a price that the buyer must either accept or reject. In market games with multiple price-setting sellers, however, the Nash equilibrium may involve randomization if sellers' production capacities are limited. Experimental data seem to track the qualitative features of Nash equilibria in such games, but prices are often much higher than the Nash equilibrium predictions (Davis and Holt, 1994).

In order to sort out the reasons for the observed departures from the Nash prediction, a useful positive theory of behavior in games could begin by qualifying the assumption that individuals are perfect maximizers of their own money payoffs. Several authors have relaxed the perfect rationality assumption in experimental games: Brown and Rosenthal (1990), Camerer and Weigelt (1988), McKelvey and Palfrey (1992, 1993), Banks *et al.* (1994), Brandts and Holt (1992), and Palfrey and Rosenthal (1991, 1992). One way is to introduce decision error, i.e. in choosing their strategies players make mistakes. Players 'tremble' and therefore every strategy (even a dominated strategy) is played with a strictly positive probability. In the case of vanishingly small strategy errors, this approach was originally used to rule out unintuitive criteria, especially in extensive form games (Selten, 1975).

[5] See Davis and Holt (1993), chapters 5 and 6. One way to move the Nash equilibrium away from the boundary in these games is suggested by Palfrey and Prisbrey (1993) and Prisbrey (1994).

As a first step, it is useful to distinguish between two sources of deviations from the Nash equilibria as calculated from expected money payoffs. First, *systematic* deviations may be due to the importance of neglected factors, such as altruism, envy, fairness, etc. These factors are likely to be more important in bargaining and public goods games than in impersonal market situations. Second, *non-systematic* or random "errors" can follow from mistakes in recording decisions, from time constraints as in chess games, or from random errors in evaluating small differences in expected payoffs. Experimental evidence suggests that non-systematic errors can occur in strategic situations (McKelvey and Palfrey, 1993) and also in simpler individual decision-making tasks (Anderson, 1994).

There are many ways to model decision errors. One particularly simple approach is based on the discrete choice theory first proposed by Luce (1959). Let u_1 and u_2 denote the expected utility associated with decisions 1 and 2 respectively. Luce proposed a model in which choice probabilities are determined by ratios of expected utilities:

$$Pr(choose\ decision\ i) = \frac{u_i}{u_1 + u_2} \quad i = 1, 2. \qquad (1)$$

Clearly the probability that the player chooses decision 1 is increasing in the expected utility associated with it, but decreasing in the expected utility associated with decision 2. These choice probabilities reflect boundedly rational behavior because the player does not always choose the decision with the highest utility.

McFadden's (1984) random utility interpretation of discrete choice theory is conceptually very different from the notion of decision errors. In McFadden's approach, the modeler can only imperfectly observe the characteristics influencing an individual's choice. For example, the u_i in equation (1) represent the observed parts of an individual's utility, but the optimal decision may also depend on unobserved utility elements that are random from the point of view of an outside observer. The distribution of the random utility elements determines the form of the probabilistic choice function (e.g., logit, probit), as discussed below. These choice functions are also called "quantal response functions".

The quantal response functions discussed above are used to model individual decisions. Capturing decision error in a way that is clearly spelled out and not ad hoc is a difficult task. The quantal response equilibrium does this based on elements borrowed from the discrete or quantal choice theory developed by Luce (1959), McFadden (1984) and Thurstone (1927). The added complexity of applying the quantal response equilibrium to game theory —in contrast to individual choice— is that the choice probabilities of the players have an important interactive component, since they are simultaneously determined in equilibrium. In a quantal response equilibrium, a player's beliefs about others' actions will determine the player's own expected payoffs, which in turn determine the player's choice probabilities via a quantal response function. The model is closed by requiring the choice probabilities to be consistent with the initial beliefs.

To illustrate the effects of decision errors in a market model, consider the quantal response equilibrium for a simple Bertrand game with zero production cost and two price choices. In this game, each seller simultaneously chooses between a high price P_H, and a low price P_L. The combination of prices determines payoffs as shown in the table below, where seller 1's payoff is listed to the left in each cell. The profits from defection, π_d, exceed those from cooperation, π_c, which in turn exceed the profit π_n from the Nash equilibrium: $\pi_d > \pi_c > \pi_n > 0$. The only Nash equilibrium outcome is (π_n, π_n).

		Seller 2	
		P_H	P_L
Seller 1	P_H	π_c, π_c	$0, \pi_d$
	P_L	$\pi_d, 0$	π_n, π_n

Next, consider the effects of decision errors determined in (1). Let σ denote the probability that seller 2 chooses the cooperative decision P_H. Given this probability, seller 1's expected payoff is $u_H = \sigma\pi_c$ for decision P_H and $u_L = \sigma\pi_d + (1 - \sigma)\pi_n$ for decision P_L. Using the Luce choice function (1), player 1 will choose decision P_i with probability $u_i/(u_H + u_L)$, or equivalently,

$$Pr(chose\ decision\ P_H) = \frac{\sigma\pi_c}{\sigma\pi_c + \sigma\pi_d + (1 - \sigma)\pi_n}. \qquad (2)$$

The equilibrium consistency requirement is that choice probabilities correspond tó beliefs. In particular, the right side of equation (2) must equal σ, which provides an equation that can be solved for o:[6]

$$\sigma = \frac{\pi_c - \pi_n}{\pi_c - \pi_n + \pi_d} \tag{3}$$

Clearly, the probability o that a seller chooses the high "cooperative" price P_H is positively related to the gain from cooperation, $(\pi_c - \pi_n)$, and negatively related to the payoff from defection, π_d. Since P_L is a dominant strategy in the Nash game without errors as long as $\pi_d > \pi_c$, σ can be interpreted as the probability of making an error.

This work investigates the quantal response equilibrium in markets in which sellers post prices simultaneously. The laboratory implementation of this model is commonly called a "posted offer auction." The posted offer auction was chosen because laboratory data on such auctions suggest that prices deviate from Bertrand-Nash predictions in a systematic manner. In this work, the degree of rationality in the probabilistic choice function is parameterized so that, at one extreme, players choose randomly, independent of expected payoffs. At the other extreme, players always choose the decision with the highest expected payoff, as in a conventional Nash equilibrium. Explicit analytical solutions and numerical methods are used to examine the effects of changes in market structure on the endogenous price distributions. Although this research is mainly theoretical, it is driven by experimental data and has clear implications for such data. In particular, there are structural variables that do not affect the Nash equilibrium but do alter the price distribution in a quantal response equilibrium. This observation provides a natural null hypothesis for experimental analysis.

This paper is organized as follows. Section 2 contains a more detailed discussion of probabilistic choice and the quantal response equilibrium. In section 3, a useful parametric class of quantal response functions is derived from a model of multiplicative random errors. This class of functions is used in section 4 to derive the equilibrium distribution of prices in posted offer auction markets. Section 5 draws some conclusions and outlines some future directions of the present research.

[6] Note that the Nash equilibrium condition, o = 0, does not satisfy (3).

2. Equilibrium with Endogenous Errors

Boundedly rational players have been most commonly characterized by either the random choice or the random utility version of discrete or quantal choice theory. In the first interpretation the utility is constant but the decision rule is random (Luce, 1959; Tversky, 1972a).[7] By contrast, the second interpretation assumes that utility is random while the decision rule is constant (Thurstone, 1927; McFadden, 1984). Hence the two approaches can be distinguished according to the interpretation of the random mechanism that governs choice.

To illustrate the random utility interpretation, consider the following example. Let the seller's expected payoffs associated with decisions 1 and 2 be u_1 and u_2, and let the random error term ε_i be i.i.d. log Weibull with parameter λ. Thus,

$$\begin{aligned} Pr(choose\ 1) &= Pr(u_1 + \varepsilon_1 > u_2 + \varepsilon_2) \\ &= Pr(u_1 - u_2 > \varepsilon_2 - \varepsilon_1) \\ &= F(u_1 - u_2) \end{aligned}$$

[7] One criticism of the Luce model (and thus of independence of irrelevant alternatives) is that it may not hold true in situations where the choice is divided in some manner. To illustrate this criticism, Debreu (1960) offered the following example. Assume that the choice set contains three elements: a recording of the Debussy quartet, D; a recording of a Beethoven symphony conducted by f, B_f; and a recording of the same symphony conducted by k, B_k. Let U be the entire recording music menu and J be the subset containing the Beethoven recordings, i.e., B_f and B_k. Suppose that a subject selects B_f with probability 1/2, when presented with $\{B_k, B_f\}$, so that these alternatives have the same scale values, i.e., $u_{Bf} = u_{Bk}$. Further, when the subject is confronted with either $\{D, B_k\}$ or $\{D, B_f\}$, D is selected with probability 3/5. From (1), the probability 3/5 implies $u_D = (3/2)u_{Bf} = (3/2)u_{Bk}$. According to the Luce model with these scale values, when presented with $\{D, B_f, B_k\}$, D must be chosen with probability 3/7. Thus when making a decision between D and B_f, the subject would rather have Debussy. However, when choosing between D, B_f, and B_k, while being indifferent between B_f and B_k, the subject is more likely to choose one of the Beethoven recordings. Debreu concluded that the Luce choice axiom is only appropriate when the choice sets have equally dissimilar alternatives. Another possible explanation of subjects' incorrect choices is that they are due to mistakes in recording decisions, so the addition of "irrelevant" choice alternatives can affect choice probabilities.

where $F(*)$ denotes a cumulative distribution. McFadden (1984) showed that the probability that an individual chooses alternative 1 can be expressed in terms of the logistic error function:

$$Pr(choose\ 1) = \frac{e^{\lambda u_1}}{e^{\lambda u_2} + e^{\lambda u_1}} \tag{4}$$

Notice that λ parameterizes the degree of rationality for the quantal response function in (4). If λ is zero, the individual will choose between options 1 and 2 with equal probability, regardless of the expected payoffs. If λ is ∞, the individual will always choose the option with the higher expected payoff.

Following McKelvey and Palfrey (1993) a quantal response equilibrium is a fixed point in choice probabilities. Define π as the set of all possible combinations of the expected payoffs for all players in a finite normal form game. Let δ be the Cartesian product of the mixed strategies for all players, and let $\hat{p}$ be an element of δ, i.e., $\hat{p}$ specifies a particular mixed strategy for each player. Denote the vector of all expected payoffs as $e(\hat{p})$. Thus, $e(\hat{p})$ maps a particular array of mixed strategies, $\hat{p}$, into a vector of players' expected payoffs, π. A discrete choice function σ maps expected payoffs into a mixed strategy for a single player. The function σ is assumed to be continuous and monotonically increasing in the payoffs. Let T^{σ} represent the resulting mapping from the set of all possible combinations of players' expected payoffs to their choice probabilities, $T^{\sigma} : \pi \to \delta$.

To summarize, $e(\hat{p}) : \delta \to \pi$ maps mixed strategy probabilities to expected payoffs, and $T^{\sigma} : \pi \to \delta$ maps expected payoffs to mixed strategy probabilities.

The equilibrium is a fixed point:

DEFINITION 1. *A Quantal Response Equilibrium is a* $\hat{p}$ *such that* $\hat{p} = T^{\sigma}(e(\hat{p}))$.

The Brouwer fixed point theorem implies the existence of such an equilibrium, since $T^{\sigma}(e(\hat{p}))$ is a continuous function that maps a compact set δ into itself.

3. Power Function Decision Rule

The Luce framework provides a rather rigid relationship between the underlying utilities and the choice probabilities of the individuals. The Luce model choice probabilities can be expressed as ratios of expected payoffs, rather than utilities, as in (1):

$$Pr(choose\ i) = \frac{\pi_i}{\pi_1 + \pi_2} \quad \text{for } i = 1, 2. \tag{5}$$

The above expression can be parameterized in a more general form that permits an arbitrary degree of bounded rationality, with fully rational individuals at one extreme. At the other extreme, there is absolutely no connection between expected payoffs and choice probabilities. The power-function quantal response equilibrium derived in this section generalizes equation (5) by having each expected payoff raised to a power. This functional form turns out to be a useful way to model decision errors in models of price competition since it often leads to tractable solutions and comparative statics results. The power-function quantal response equilibrium is based on random utility maximization with multiplicative error terms as discussed in section 1. In the power-function quantal response equilibrium, each player's quantal response function will have a power parameter which, when equal to 1, yields the Luce model. The parameter, however, can take on any value between 0 and ∞.

For simplicity in exposition assume that a single decision maker must choose between two alternatives, 1 and 2. The corresponding expected payoffs, π_1 and π_2, are assumed to be strictly positive. Under the power function model, the probability of choosing alternative 1 is given by:

$$Pr(choose\ 1) = \frac{\pi_1^{\lambda}}{\pi_1^{\lambda} + \pi_2^{\lambda}} \tag{6}$$

where the ratio of expected payoffs is raised to a power λ. In (6), λ is a nonnegative parameter that measures the degree of rationality of the individuals. As λ goes to 0, the individual chooses each decision with equal probability, regardless of expected payoffs. As λ goes to ∞, the decision with the highest expected payoff is selected with probability 1.

There are many ways to model the stochastic behavior of the error term in the payoff function in (6). Previous research has focused on either normally or log Weibull distributed errors, which yield the probit and logit decision rules respectively.[8] The power function decision rule can be derived from random utility expressed as a product: $U_i = \pi_i \kappa_i$, where κ_i is an identical and independently distributed multiplicative error term known to a player, and π_i is a nonnegative expected payoff. With two alternatives, the probability that a player selects decision 1 is

$$Pr(choose\ 1) = Pr(\pi_1 \kappa_1 > \pi_2 \kappa_2). \tag{7}$$

Making a logarithmic transformation, we have

$$Pr(choose\ 1) = Pr(\ln\pi_1 + \ln\kappa_1 > \ln\pi_2 + \ln\kappa_2). \tag{8}$$

Let G(*) denote the distribution of κ, such that [9]

$$G(\kappa) = e^{-(\kappa)^{-\lambda}} \qquad \kappa \in [0, \infty],\ \lambda > 0. \tag{9}$$

Define a transformation of the error term: $\varepsilon = \ln\kappa$ or $\kappa = e^{\varepsilon}$. Substitute e^{ε} for κ in (9) to obtain the distribution function:

$$H(\varepsilon) = e^{-e^{-\lambda\varepsilon}} \tag{10}$$

which is a log Weibull distribution with parameter λ. When an additive random utility error, ε, is log Weibull distributed, Luce and Suppes (1967) have shown that the standard logit decision rule in (4) is derived from (10). Since the logarithmic transformation of the multiplicative error is additive in the logarithm of π_i, the relevant probabilistic choice function is the logit formulation with the expected payoff, v_i, replaced by log π_i. Hence

[8] There are a number of other papers that use explicit models of the error structure. Logit and probit specifications of the errors in the analysis of experimental data are used by Palfrey and Rosenthal (1991), Palfrey and Prisbrey (1993), Stahl and Wilson (1993), Anderson (1994), and Harless and Camerer (1994). Zauner (1994) uses a Harsanyi (1973) equilibrium model with independent normal errors to explain data from a centipede game reported by McKelvey and Palfrey (1992).

[9] In the analysis that follows, the *i* subscript is dropped from the error terms since the errors are i.i.d.

$$e^{\lambda v_i} = e^{\lambda \log \pi_i} = \pi_i^{\lambda} \tag{11}$$

and the logistic choice rule in equation (4) reduces to the power function rule in equation (5) of this section.

To summarize:

PROPOSITION 1. *If the payoff function is random and multiplicative, $\pi_i \kappa_i$, with the error terms identically and independently distributed as $G(\kappa_i) = e^{-(\kappa_i)^{-\lambda}}$, the probabilistic choice function is the power function: Pr(choose i) $= (\pi_i)^{\lambda} / \Sigma(\pi_j)^{\lambda}$ para toda $j \neq i$*

4. Market Models

The laboratory implementation of the Bertrand model is a posted-offer auction. In this institution, sellers submit prices simultaneously and then randomly designated buyers purchase at the posted prices. The Bertrand-Nash equilibrium will differ from the competitive equilibrium when sellers set prices above the competitive level. However, Nash equilibrium prices are not often observed in such situations (Holt and Davis, 1990; Davis and Holt, 1994 and Brown-Kruse *et al.*, 1994). Certain factors have been associated with systematic price deviations from Bertrand-Nash equilibrium in posted offer markets: cost structure, low excess supply at prices above the competitive price, small numbers of sellers, and market power (Davis and Williams, 1990; Wellford *et al.*, 1990; Davis and Holt, 1994; and Brown-Kruse *et al.*, 1994).

This section uses the quantal response equilibrium to model behavior in posted offer markets. The objective is to derive testable propositions about the effects of changes in market structure such as cost structure, market power and seller concentration on equilibrium price distributions.

This section is structured as follows: First, the quantal response equilibrium is calculated for posted offer markets with severe capacity constraint. In these designs, the competitive equilibrium price is the Nash equilibrium. An interesting feature of some of these models is that the quantal response equilibrium proves to be sensitive to changes in the cost and demand parameters that do not affect the Bertrand-Nash equilibrium. Next, we use the quantal response equilibrium to investigate the effects of market power on equilibrium prices. The Nash equilibrium in these markets involves mixed strategies. It is shown that the Nash equilibrium in mixed-strategies and quantal response equilibrium differ. However, this is not true in general (Lopez, 1995). It is also shown, in a more complex market model, that the quantal response equilibrium stochastically dominates the Nash equilibrium in mixed-strategies. Finally, the effects of seller concentration on the quantal response equilibrium are examined.

4.1. *A Basic Model with no Market Power*

We begin with a duopoly model, of the type used in market experiments, shown in figure 1a. Sellers' units are indicated on the market supply curve by designations, S1 and S2, for sellers 1 and 2 respectively. It is assumed that sellers choose prices simultaneously and share demand in the event of a tie. A well-known result is that the Bertrand-Nash equilibrium is for both sellers to charge the competitive price. In this sense, sellers have no market power in this design.

The quantal response equilibrium for the market design, 1a, is characterized by a price distribution for each seller, $F(p)$. Thus, $F(p)$ is the probability that p is the highest price posted. A seller who chooses price p sells the unit with probability $1 - F(p)$. The expected profit to a seller as a function of p is

$$\pi(p) = p[1 - F(p)] \tag{12}$$

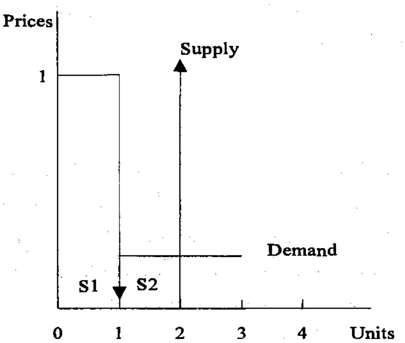
In the present market context, the power function decision rule with $\lambda > 0$ results in the following condition for a quantal response equilibrium, as shown in appendix I:

$$f(p) = \frac{(p[1 - F(p)])^{\lambda}}{\mu}, \tag{13}$$

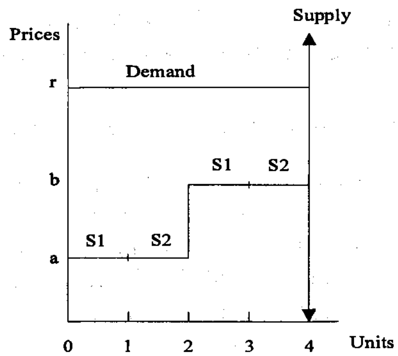
$$\mu = \int_0^1 (x[1 - F(x)])^{\lambda} dx,$$

where μ is a constant, independent of p. The above equation parameterizes the set of possible equilibrium response functions $f(p)$ with the

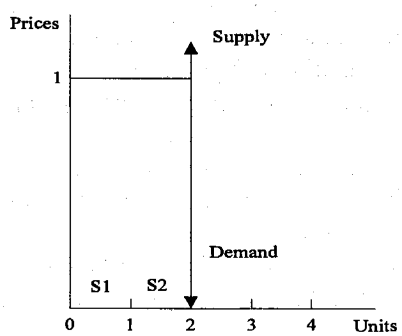
Figure 1



1a. A Basic Model with no Market Power and Cost = 0



1b. A Basic Model with Market Power and Positive Costs



1c. A Basic Model with Market Power and Cost = 0

parameter λ, which is inversely related to the level of error. For $\lambda < 1$, the equilibrium price distribution is

$$F(p) = 1 - [1 - p^{\lambda + 1}]^{\frac{1}{1 - \lambda}}, \tag{14}$$

with the corresponding equilibrium price density:

$$f(p) = \frac{\lambda + 1}{1 - \lambda} p^{\lambda} [1 - p^{\lambda + 1}]^{\frac{\lambda}{1 - \lambda}} \tag{15}$$

It is shown in appendix I that, as λ goes to 1, all the mass of probability is concentrated on the set of prices near 0. This result is appealing, since the model thereby accounts for systematic deviations from the Bertrand-Nash equilibrium.[10]

4.2. *A Basic Model with Market Power*

Consider a more complex market model, figure 1b, where each seller has 1 unit with a low cost denoted by a and 1 unit with a high cost denoted by b, with $a < b < r$. Here r is the reservation price. The market demand is rectangular with 4 units demanded for prices below r. The intersection of the high marginal cost with demand determines the range of competitive prices, $[b, r]$. The Nash equilibrium is the highest competitive price, r, as shown in 1b and 1c of figure 1, where 1c is characterized by zero costs, as in 1a. Clearly, a seller posting a price above r earns 0 profits, while a unilateral price reduction does not increase sales. When demand is divided equally at the Nash price, each seller sells 2 units.

Although the market designs 1b and 1c may share identical Nash equilibria, in empirical experiments different median prices were observed. Since the quantal response equilibrium price distribution is typically sensitive to factors that do not affect the Nash equilibrium, it might be able to explain this experimental result. We next compute the quantal response equilibrium for the market design in 1b. The calculation involves two parts, distinguished by the relation of price to the high cost, b. Note that for any p below the high cost step, b, both sellers sell 1 unit with probability 1, so

$$\pi(p) = (p - a), \quad \text{for } p \in [a, b) \tag{16}$$

Similarly, both sell 2 units for prices above the high cost step b:

$$\pi(p) = (2p - b - a), \quad \text{for } p \in [b, r] \tag{17}$$

For any given $\lambda > 0$, the power-function conditions for a quantal response equilibrium are:

[10] As noted in appendix II, when $\lambda \geq 1$, one quantal response equilibrium is the degenerate distribution $F(p) = 1$ for $p \geq 0$.

$$f(p) = \frac{(p-a)^\lambda}{\mu}, \qquad \text{for } p \in [a, b)$$
$$f(p) = \frac{(2p-b-a)^\lambda}{\mu}, \qquad \text{for } p \in [b, r] \qquad (18)$$

Notice that the densities in (18) must integrate to 1. Hence μ is written as

$$\mu = \int_a^b (x-a)^\lambda dx + \int_b^r (2x-b-a)^\lambda dx \qquad (19)$$

Integrating the densities in (19), we have

$$F(p) = \frac{(p-a)^{\lambda+1}}{\mu(\lambda+1)} + k_1, \qquad \text{for } p \in [a, b)$$
$$F(p) = \frac{(2p-b-a)^{\lambda+1}}{2\mu(\lambda+1)} + k_2, \quad \text{for } p \in [b, r] \qquad (20)$$

where k_1 and k_2 are the constants of integration. Note that $F(a) = 0$ implies $k_1 = 0$. The constant k_2 is chosen so that $F(b)^+ = F(b)^-$. It follows from (20) that $k_2 = (b-a)^{\lambda+1}/2\mu(\lambda+1)$. The constant μ is next determined. First, consider the upper bound $\bar{p}$. Equation (18) implies that the probability density is 0 for $p > r$ since the expected payoff is 0 for prices in this range. Therefore it must be the case that $F(r) = 1$. Substituting this result back into (20) evaluated at $p = r$ and using the formula for k_2, it can be shown that $\mu = [(2r-b-a)^{\lambda+1} + (b-a)^{\lambda+1}]/2(\lambda+1)$. Substituting the formula for μ back into (20), it follows that the equilibrium probability functions are written as:

$$F(p) = \frac{2(p-a)^{\lambda+1}}{(2r-a-b)^{\lambda+1} + (b-a)^{\lambda+1}}, \quad \text{for } p \in [a, b)$$

$$F(p) = \frac{(2p-b-a)^{\lambda+1} + (b-a)^{\lambda+1}}{(2r-a-b)^{\lambda+1} + (b-a)^{\lambda+1}}, \quad \text{for } p \in [b, r] \qquad (21)$$

with the corresponding equilibrium probability densities:

$$f(p) = \frac{2(p-a)^\lambda(\lambda+1)}{(2r-b-a)^{\lambda+1} + (b-a)^{\lambda+1}}, \quad \text{for } p \in [a, b)$$

$$f(p) = \frac{2(2p - b - a)^{\lambda}(\lambda + 1)}{(2r - b - a)^{\lambda + 1} + (b - a)^{\lambda + 1}}, \quad \text{for } p \in [b, r] \tag{22}$$

It follows from equations (21) and from the definition of μ that equations (22) satisfy the quantal response equilibrium conditions in (18). As λ goes to 0, $F(p)$ goes to $(p - a)/(r - a)$, which is a uniform distribution resulting from maximal decision error. Next we apply L'Hopital's rule to evaluate (21) as λ goes to ∞. Differentiating both parts of the fraction in (21) with respect to λ, for $p \in [a, b)$, it can be shown (Lopez, 1995), that as λ goes to ∞, $F(p)$ goes to 0 and all of the probability is on the upper price range, where the Nash equilibrium is located. Applying L'Hopital's rule and hence differentiating both parts of the fraction in (21) with respect to λ for $p \in [b, r]$ it can be shown that $F(p)$ still goes to 0 as λ goes to ∞, and the price distribution converges to the Nash equilibrium price of r.

Another interesting property of this model is that a change in the cost structure does not alter the Nash equilibrium as long as b remains below r. However, a change in the cost parameters may alter the price distribution in a quantal response equilibrium. Next, we examine how changes in the cost parameters affect the quantal response equilibrium. As before, each price range must be considered separately. For $p \in [a, b)$, the first partial derivatives of $F(p)$ in the top part of (21) with respect to a and b are:

$$i) \; \frac{\partial F(p)}{\partial a} = \frac{2(\lambda + 1)(p - a)^{\lambda}[(2r - b - a)^{\lambda}(p - 2r + b) + (p - b)(b - a)^{\lambda}]}{[(2r - a - b)^{\lambda + 1} + (b - a)^{\lambda + 1}]^2} < 0$$

$$\text{for } p \in [a, b)$$

$$ii) \; \frac{\partial F(p)}{\partial b} = \frac{2(\lambda + 1)(p - a)^{\lambda + 1}[(b - a)^{\lambda} - (2r - b - a)^{\lambda}]}{[(2r - a - b)^{\lambda + 1} + (b - a)^{\lambda + 1}]^2} > 0 \tag{23}$$

$$\text{for } p \in [a, b)$$

where the inequality claims are verified below. The sign of the equation 23(i) is negative if $p - 2r + b < 0$ and $p - b < 0$, which is true since $p < b$ and $b < r$. Thus, an increase in a *decreases* $F(p)$. The sign of the equation 23(h) is positive since $(2r - b - a) > b - a$, or equivalently, $r > b$. Hence, an increase in b *increases* $F(p)$ on $[a, b)$.

For $p \in [b, r]$, the first partial derivative of $F(p)$ with respect to a is

$$\frac{\partial F(p)}{\partial a} = \big[2(\lambda + 1)[(b - a)^{\lambda}[(2r - a - b)^{\lambda}(b - r) + (2p - b - a)^{\lambda}(p - b)] + (2r - b - a)^{\lambda}(2p - b - a)^{\lambda}(p - r)\big]/\omega < 0 \quad \text{for } p \in [b, r) \qquad (24)$$

where the denominator of (24) is given by

$$\omega = [(2r - a - b)^{\lambda + 1} + (b - a)^{\lambda + 1}]^2.$$

The sign in equation (24) is negative if the following is true

$$(2r - a - b)^{\lambda}(b - r) > (2p - a - b)^{\lambda}(p - b), \text{ for } p \in [b, r)$$

$$\left(\frac{2r - a - b}{2p - a - b}\right)^{\lambda} > \left(\frac{p - b}{b - r}\right) \qquad (25)$$

Notice that $(2r - a - b)/(2p - a - b) > 1$ in (25) since $r > p$. On the other hand, the term $(p - b)/(b - r)$ is always negative since $p > b$ and $r > b$. Therefore, an increases in a decreases $F(p)$ on $[b, r)$. The first partial derivative of $F(p)$ with respect to the high cost unit b yields

$$\frac{\partial F(p)}{\partial b} = \big[2(\lambda + 1)[(b - a)^{\lambda}[(2r - a - b)^{\lambda}(r - a) + (2p - b - a)^{\lambda}(a - p)] + (2r - b - a)^{\lambda}(2p - b - a)^{\lambda}(p - r)\big]/\omega > 0 \quad \text{for } p \in [b, r) \qquad (26)$$

The sign in equation (26) is positive if

$$(2r - b - a)^{\lambda}/(2p - a - b)^{\lambda} > (a - p)/(r - a)$$

Since $r > p$ and $p > a$, it follows that the left hand side of the inequality is positive, and the right hand side is negative. Therefore, an increase in b increases $F(p)$ in (26).

In summary, the models presented in this section are characterized by the fact that the Nash equilibrium is unaffected by changes in the cost structure as long as b remains below r. For the quantal response equilibria, an increase in the low-cost step stochastically raises prices in the whole range of prices. Thus, sellers in a quantal response equilibrium post stochastically higher prices when they face an increase in the low-cost step. By contrast, an increase in the high-cost step raises the distribution function for the whole range of prices. Hence, sellers post stochastically *lower* prices given an increase in the high-cost step. The intuition behind this last result is that an increase in b reduces profits for the second unit that is only sold at prices above b, which causes sellers to post stochastically lower prices in a quantal response equilibrium.

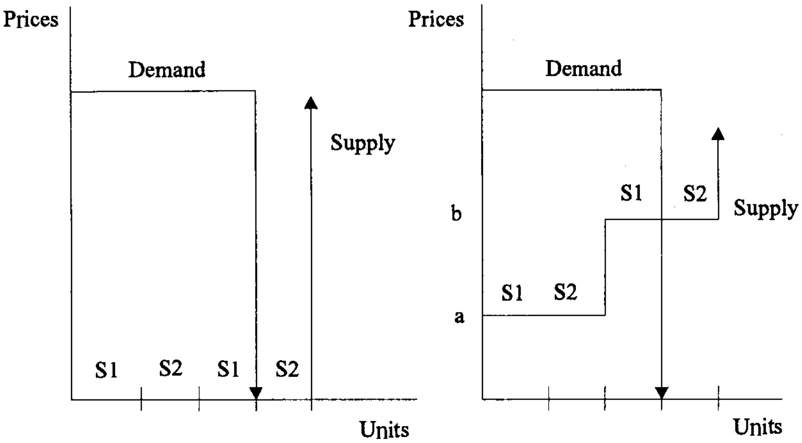
4.3. *Market Power and Constant Costs with Continuous Prices*

The market model in 2a of figure 2 illustrates the situation when there is excess supply and the Nash equilibrium involves randomization. To understand how randomization may arise, consider the duopoly model in 2a. Each of the 2 sellers has the capacity to supply 2 units at 0 cost. The market quantity demanded is 3 units at any price less than or equal to 1, and 0 at any price above 1. Assume that sellers split the market in the case of ties. Further, suppose that only two prices can be posted. For example, if seller 1 offers 2 units at a price of 0 and seller 2 posts a price equal to 1/2, buyers would like to buy from seller 1. This seller will sell two units, netting a profit of 0. Seller 2 will face a residual demand of 1 unit and will net 1/2. Hence each seller has a unilateral incentive to raise price above a common competitive price of 0.

The Nash equilibrium will involve randomization.[11] Given seller 2's capacity constraint, seller 1 can always obtain a safe payoff of 1 by charging the price of 1 and selling to the residual demand. For seller 1 to be indifferent between posting some arbitrary price p and the limit price 1, it must be the case that seller 2 prices according to a distribution $F(p)$

[11] As can be verified, there is no equilibrium in pure strategies. There is randomization over the set of prices (1/2,1). For instance, if seller 1 posts a price of 1, seller 2's best response is to slightly cut this price and sell the 2 units. Then, seller 1's best response is to cut this price. This Edgeworth cycle of best responses continues until the price falls to 1/2. At this price, the expected payoff from selling one unit equals the expected payoff from selling 2 units at the price of 1.

Figure 2



2a. Market Power and Constant Costs with Continuous Prices

2b. Market Power and Constant Costs with Integer-valued Prices

that makes seller 1's expected earnings at p equal to certain earnings in the charges 1. When seller 1 chooses a price of p, he has the highest price with probability $F(p)$ and the lowest price with probability $1 - F(p)$. Therefore, seller 1 sells 1 unit with probability $F(p)$ and 2 units with probability $1 - F(p)$. Hence the expected profit function for a seller as a function of p is

$$\pi(p) = pF(p) + 2p[1 - F(p)]. \tag{27}$$

In a mixed strategy Nash equilibrium, seller 1 must be indifferent among all prices over which randomization occurs. Hence, the distribution $F(p)$ must equate the expected profit at each p in the support $[\underline{p}, \overline{p}]$, to the certain profit of 1. The resulting equation yields

$$F(p) = 2 - \frac{1}{p}, \quad \text{mixed Nash equilibrium} \tag{28}$$

with the corresponding probability density

$$f(p) = \frac{1}{p^2}. \tag{29}$$

Next, we determine the upper and the lower bound of the price distribution. Notice that no price above the reservation price will be charged since the payoff to a seller is zero. From (29), the boundary condition $F(\underline{p}) = 0$ implies that $\underline{p} = 1/2$

In the analysis that follows, the Nash equilibrium is generalized to incorporate decision errors. First note that the expected profit in (27) can be expressed: $\pi(p) = p[2 - F(p)]$. Therefore, the power-function quantal response equilibrium condition is:

$$f(p) = \frac{[p(2 - F(p))]^{\lambda}}{\mu}$$
$$\mu = \int_0^r [x(2 - F(x))]^{\lambda} dx \qquad (30)$$

As before, μ is a constant independent of p. Before deriving the quantal response equilibrium price distribution, it is worth pointing out one interesting property of (30). By substituting (28) into the right side of (30), it follows that a quantal best response to the other seller's Nash equilibrium in mixed-strategies is the uniform distribution, $1/\mu$. This is because the expected profits are equal at all prices in a mixed-strategy Nash equilibrium. Hence, if the rival is using his Nash equilibrium, the seller's best response is to spread price decisions uniformly. This result shows why the quantal response equilibrium and Nash distribution cannot be the same when the Nash mixed distribution is not uniform to begin with.

To derive F(p) in (30), we integrate from 0 to some p^* to obtain

$$\int_0^{p^*} \frac{f(p)}{[2 - F(p)]^{\lambda}} dp = \int_0^{p^*} \frac{p^{\lambda}}{\mu} dp \qquad (31)$$

Let $c = F(p)$ and $dc = f(p)dp$. As p goes from 0 to p^*, c goes from $F(0) = 0$ to $F(p^*)$. Assuming $\lambda \neq 1$,[12]

$$-\frac{[2 - c]^{1 - \lambda}}{1 - \lambda} \Big|_0^{F(p^*)} = \frac{p^{\lambda + 1}}{\mu(\lambda + 1)} \Big|_0^{p^*}. \qquad (32)$$

[12] If $\lambda = 1$, the same method provides an explicit solution. The derivation of the price distribution is provided in appendix 2.

Equation (32) can be expressed as

$$-\frac{[2 - F(p^{*})]^{1-\lambda}}{1-\lambda} + \frac{2^{1-\lambda}}{1-\lambda} = \frac{(p^{*})^{\lambda+1}}{\mu(\lambda+1)} - 0 \tag{33}$$

To simplify the notation let p denote p* and rearrange:

$$[2 - F(p)]^{1-\lambda} = 2^{1-\lambda} - \frac{(1-\lambda)}{\mu(1+\lambda)} p^{\lambda+1}. \tag{34}$$

It follows from (34) that the price distribution is

$$F(p) = 2 - \left[2^{1-\lambda} - \frac{(1-\lambda)}{\mu(1+\lambda)} p^{\lambda+1}\right]^{\frac{1}{1-\lambda}}. \tag{35}$$

The next task is to determine μ. The boundary condition, $F(1) = 1$, implies that the term in square brackets in (35) is equal to 1. Hence, $\mu = (1-\lambda)/(1+\lambda)(2^{1-\lambda} - 1)$ Thus, replacing μ in (35) yields the power-function equilibrium distribution:

$$F(p) = 2 - \left[2^{1-\lambda} - (2^{1-\lambda} - 1)\, p^{\lambda+1}\right]^{\frac{1}{1-\lambda}} \tag{36}$$

which makes it possible to calculate the quantal response equilibrium, QRE. The corresponding equilibrium density is

$$f(p) = \frac{(2^{1-\lambda} - 1)(\lambda+1)p^{\lambda}}{(1-\lambda)}\left[2^{1-\lambda} - 2^{1-\lambda} - 1)\, p^{\lambda+1}\right]^{\frac{\lambda}{1-\lambda}} \tag{37}$$

It can be shown using (36) and the definition of μ that the density in (37) satisfies the equilibrium condition in (31). As λ goes to 0, $F(p)$ goes to p and prices are uniformly distributed.

By comparing equations (28) and (36) it can be verified that the Nash equilibrium distribution at the lower bound of the price support, $F_N(1/2) = 0$, is less than the quantal response equilibrium distribution, $F_Q(1/2)$. Therefore, consider the possibility that the Nash equilibrium distribution, $F_N(p)$, stochastically dominates (in terms of first degree dominance) the quantal response equilibrium, $F_Q(p)$, or equivalently, $F_N(p) < F_Q(p)$. From equations (28) and (36), $F_N(p) < F_Q(p)$ implies

$1/p > [2^{1-\lambda}(1-p^{\lambda+1}) + p^{\lambda+1}]^{1/1-\lambda}$. Raising both sides of the inequality to the power $1-\lambda$ yields $p^{\lambda-1} > [2^{1-\lambda}(1-p^{\lambda+1}) + p^{\lambda+1}]$. Dividing by $p^{\lambda-1}$ and arranging terms we have $1 > p^{1-\lambda}2^{1-\lambda}(1-p^{1+\lambda}) + p^2$. As λ goes to 1, the right hand side of the inequality goes to $2(1-p^2)+p^2 = 2-p^2$, which is greater than 1 for $p \in (0, 1)$. Thus, the Nash equilibrium does not stochastically dominate the QRE.

4.4. *Market Power and Constant Costs with Integer-Valued Prices*

We next examine the quantal response equilibria for the market design in 2b of figure 2. In laboratory experiments the set of allowable price decisions often is finite (e.g., pennies). In what follows, the mixed-strategy Nash equilibrium for integer-valued prices is calculated. The equilibrium expected payoff, S, must satisfy (38) below. This equation is similar to the one for the continuous case. However, this equation also accounts for the payoff function that determines earnings when a seller's price matches the other's price. At this price, demand is divided equally, so each seller earns the average profit: $[(2p-a-b)+(p-a)]/2$. The density, $f(p_i)$, is the equilibrium probability that a price selected is p_i, $f(p_i) \geq 0$, for $p_i = p_1, \ldots, r$:

$$S = \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i)\right](p_k - a) + f(p_k)\frac{[(2p_k - a - b) + (p_k - a)]}{2}$$

$$+\left[1 - \sum_{p_i=p_1}^{p_k} f(p_i)\right](2p_k - a - b) \qquad (38)$$

where $p_k = p_1, \ldots, r$ Equation (38) can also be expressed as

$$S = \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2}\right](p_k - a) + \left[1 - \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2}\right]\right](2p_k - a - b) \quad (39)$$

The $G(p_k)$ in equation (40) is a *modified* "distribution function", that allows for the event of ties:

$$G(p_k) = \sum_{p_i = p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2} = \frac{(2p_k - a - b) - S}{p_k - b}, \tag{40}$$

In order to obtain the support of the equilibrium mixed-strategy Nash equilibrium, consider a set of consecutive integer-valued prices: $[p_1, \ldots, r]$, where r is the largest price. Define p_L and p_H as the lowest and highest prices respectively that are selected with strictly positive probability, where $p_1 \le p_L < p_H \le r$. By evaluating (38) at p_H and using the fact that the sum of the densities up to $f(p_H)$ equals one, one obtains

$$S = [1 - f(p_H)](p_H - a) + f(p_H)\left[\frac{(2p_H - a - b) + (p_H - a)}{2}\right]$$
$$= (p_H - a) + \frac{f(p_H)}{2}[p_H - b] \quad . \tag{41}$$

Since $f(p_H) > 0$, $r > p_H$ and $r > b$, it follows from (41) that $S > p_H - a$. Now, we calculate the mixed equilibrium probabilities for this model. For example, suppose that $a = 0$, $b = 4$ and $r = 9$. Suppose that $f(p_k) = (r - b)/(b - p_k)^2$, with the upper bound $p_H = 9$ and the lower bound $p_L = 7$, is a Nash equilibrium in mixed-strategies. In equilibrium, the seller must be indifferent between the prices 7, 8 and 9. Next, we verify that the seller has no incentive to deviate by choosing an outside price with positive probability. Using equation (39) and the assumption, $f(p_k) = (r - b)/(b - p_k)^2$, one can show that $S_7 = S_8 = S_9 \approx 9.16$.[13] The equilibrium probabilities are: $f(7) = 5/9, f(8) = 5/16$ and $f(9) = 5/25$ The equilibrium distribution function that results is $G(7) = 5/9$, $G(8) = 125/144$ and $G(9) = 1$. The mixed-strategy Nash

[13] These three calculations are:

$$S_7 = \frac{f(7)}{2}[(2 * 7 - 4 - 0) + (7 - 0)] + [1 - f(7)](2 * 7 - 4 - 0),$$

$$S_8 = f(7)[8 - 0] + \frac{f(8)}{2}[(2*8 - 4 - 0) + (8 - 0)] + [1 - f(7) - f(8)](2 * 8 - 4),$$

$$S_9 = [f(7) + f(8) + \frac{f(9)}{2}]((9 - 0) - (2 * 9 - 4)) + (2 * 9 - 4).$$

The Nash and QRE price distributions are illustrated in figure 3. In this figure the distribution functions are indicated in the vertical axis, while prices are represented in the horizontal axis. The quantal response equilibrium is plotted for different error rates. The upper and lower bound of the mixed-strategy Nash equilibrium is 9 and 7 respectively. The figure shows that, in the QRE, as the error rate, $1/\lambda$, decreases the reservation price of 9 becomes more probable. In the Nash equilibrium, however, a price of 7 is a more probable outcome. With respect to the equilibrium strategies, it is interesting to note that $F_Q(p)$ dominates stochastically $F_N(p)$, in terms of first degree dominance, or equivalently, $F_Q(p) < F_N(p)$. This means that sellers are posting stochastically higher prices in a QRE, than in the mixed-strategy Nash equilibrium.

4.5. *Seller Concentration*

Next we examine the QRE in the presence of a change in seller concentration. This structural variable is another factor that has been associated with systematic deviations from the Bertrand-Nash equilibrium in posted-offer markets (Holt and Davis, 1984).

Consider a generalization of the baseline model introduced at the beginning of this section to the case of N sellers. As before, each seller has 1 unit to sell at a zero cost. The quantity demanded is 1 unit for all prices less than or equal 1. A well known result is that for $N \geq 2$, where N is the number of sellers, the Bertrand-Nash equilibrium is to set price equal to marginal cost.

Now consider the calculation of the quantal response equilibrium. When a seller charges p it may be that p is the smallest price being posted. This happens only if the other sellers charge prices higher than p, an event which has probability $[1 - F(p)]^{N-1}$. Therefore, the expected profit of the seller is

$$\pi(p) = [1 - F(p)]^{N-1} p \tag{46}$$

In the present market context, the power-function decision rule implies that the choice probabilities must satisfy:

$$f(p) = \frac{(p[1 - F(p)]^{N-1})^{\lambda}}{\mu},$$

$$\mu = \int_0^1 (x[1 - F(x)]^{N-1})^{\lambda} dx \tag{47}$$

Equation (47) can be expressed as

$$\frac{f(p)}{[1 - F(p)]^{(N-1)\lambda}} = \frac{p^{\lambda}}{\mu} \tag{48}$$

It can be shown (Lopez, 1995) that the probability price distribution is

$$F(p) = 1 - \left[1 - \frac{(1 + \lambda - N\lambda)}{\mu(1 + \lambda)} p^{\lambda + 1}\right]^{\frac{1}{1 + \lambda - N\lambda}}, \tag{49}$$

where μ is a constant to be determined. The boundary condition, $F(1) = 1$, implies that $\mu = (1 + \lambda - \lambda N)/(\lambda + 1)$. The quantal response equilibrium price distribution is [14]

$$F(p) = 1 - [1 - p^{\lambda + 1}]^{\frac{1}{1 + \lambda - N\lambda}} \tag{50}$$

with the corresponding equilibrium price density:

$$f(p) = \left(\frac{\lambda + 1}{1 + \lambda - \lambda N}\right) p^{\lambda}[1 - p^{\lambda + 1}]^{\frac{-\lambda + \lambda N}{1 + \lambda - \lambda N}} \tag{51}$$

Next we examine the effect of the number of sellers, on the endogenous equilibrium price distribution. The partial derivative of (50) with respect to N is

$$\frac{\partial F}{\partial N} = -\frac{\partial}{\partial N}\left(e^{\frac{1}{1 + \lambda - N\lambda}\ln(1 - p^{\lambda + 1})}\right) \tag{52}$$

Equation (52) is expressed as follows

$$\frac{\partial F}{\partial N} = -e^{\frac{1}{1 + \lambda - N\lambda}\ln(1 - p^{\lambda + 1})} \frac{\partial}{\partial N}\left(\frac{1}{1 + \lambda - N\lambda}\ln(1 - p^{\lambda + 1})\right), \tag{53}$$

[14] From (47), a QRE for $(1/N) - 1 \geq \lambda$ is the degenerate distribution $F(p) = 1$ for $p \geq 0$.

Rearranging, the partial derivative of $F(p)$ with respect to N is

$$\frac{\partial F}{\partial N} = - [1 - p^{\frac{1}{1 + \lambda - N\lambda}}] \frac{\ln(1 - p^{\lambda + 1})\lambda}{(1 + \lambda - N\lambda)^2} . \tag{54}$$

The logarithm in (54) is negative since $p \in (0, 1)$. Therefore, as N increases the price distribution, $F(p)$ increases, and price declines stochastically.

To summarize:

As the number of sellers, N, increases, the power-function price equilibrium increases. Thus, given an increase in N, sellers post stochastically lower prices.

5. Conclusions and Outline of Future Work

Observed patterns of behavior for both game theory and industrial organization experiments reveal systematic deviations from the Nash equilibrium. A variety of factors have been associated with systematic price deviations in laboratory markets: the rules of the market, low excess supply at prices above the competitive price, and few sellers (Smith, 1978; Davis and Williams, 1990; Wellford *et al.*, 1990). The approach used in this paper is the quantal response equilibrium, which incorporates decision error in a way that is clearly spelled out and not ad hoc, using elements borrowed from the discrete choice theory of Luce (1959), McFadden (1984) and Thurstone (1927). As discussed in section 2, the added complexity in applying the quantal response equilibrium to game theory —in contrast to individual choice— is that the choice probabilities of the players have an important interactive component, since they are simultaneously determined in equilibrium. In a QRE, a player's beliefs about others' actions will determine the player's own expected payoffs, which in turn determine the player's choice probabilities via a quantal response function. The model is closed by requiring the choice probabilities to be consistent with the initial beliefs. Closed-form or analytical solutions for equilibrium price distributions with endogenous errors were derived in simple models of price competition.

The main finding of this paper is that the quantal response approach is consistent with the higher-than-competitive prices observed in posted-offer markets. Specific conclusions for the models analyzed in this paper include:

a) With severe capacity constraints, the quantal response equilibrium predicts systematic departures from the Bertrand-Nash equilibrium for finite error parameters, and convergence to the Nash equilibrium as the errors vanish. Accordingly, the model is able to account for the systematic price deviations observed in past experiments.

b) With severe capacity constraints and increasing costs, it is shown that in the QRE, sellers post stochastically higher prices when they face an increase in the low cost parameter. By contrast, the Nash equilibrium is unaffected by changes in the cost parameters as long as the high cost parameter is below the reservation price.

c) With market power and constant costs, it is shown that the Nash equilibrium in mixed-strategies and the QRE differ. Specifically, prices tend to be lower in the QRE regardless of whether prices are discrete or continuous.

d) With market power and increasing costs, the quantal response equilibrium stochastically dominates the Nash equilibrium (in terms of first degree dominance).

e) A well known result is that for $N \geq 2$, where N is the number of sellers, the Bertrand-Nash equilibrium is to set price equal to marginal cost. However, in a quantal response equilibria, an increase in the number of sellers generates a stochastic decrease in prices which may not converge to the Nash equilibrium.

The theoretical results obtained in this paper are motivated by stylized patterns in experimental data and will be used to suggest designs for further experiments. Even though one important feature of the approach derived in this paper was its simplicity we would like to outline some extensions of the approach and give directions for further research.

One extension is to apply the quantal response equilibrium to posted-offer experimental data. The posted-offer triopolies conducted by Holt and Davis (1990) are especially interesting since the observed median prices for the first 15 market periods reveal systematic deviations from the Bertrand-Nash prediction. As a first step, these market models were examined using numerical methods. In these simulations prices tend to be 12% higher than the Nash equilibrium. The next step in this research will be a statistical analysis of the data, using standard maximum likelihood techniques in a structural model incorporating QRE, Another extension is to examine ultimatum bargaining games in which there are decision errors in the buyer's purchase choices. There is a large experimental literature that documents systematic deviations from the Nash equilibrium in bargaining games. Systematic deviations in these games have been attributed to perceptions of fairness, focalness, and to random "errors" (Prisbrey, 1994). In principle, it is not difficult to extend the model to allow for buyer's decision errors. However, further work is needed in order to determine whether the model can predict systematic price deviations in ultimatum experiments. Another promising direction is to account for price choice decisions under horizontal product differentiation, in which the "error" rate is a measure of location or differentiation. Experimental evidence shows that in a Hotelling duopoly model, sellers' prices did not converge to the Nash prediction. In fact prices seem to be higher with greater distance between firms, even when different locations have identical Nash equilibrium (Brown-Kruse, 1989). The QRE is well-suited to explain such deviations since structural variables (such as distance) affect the equilibrium price distribution but typically do not affect the Nash equilibrium. Under product differentiation, however, the quantal response equilibrium condition becomes a complex second-order nonlinear differential equation in the price distribution. Therefore performance of the model may be based on simulations.

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Appendix 1

In the present market context, the continuous power function rule implies that the choice probability density must satisfy (13), where the density is proportional to expected profit raised to a power λ:

$$f(p) = \frac{(p[i - F(p)])^{\lambda}}{\int_0^1 (x[1 - F(x)])^{\lambda} dx} \qquad (55)$$

Let μ denote the denominator of the right hand side of (55), which is a constant independent of p. Thus

$$f(p) = \frac{p^{\lambda}[1 - F(p)]^{\lambda}}{\mu} \tag{56}$$

Equation (56) is a nonlinear differential equation that can be used to determine the distribution, $F(p)$. The main result is given in the next proposition:

PROPOSITION 2. *The price density*

$$f(p) = \frac{\lambda + 1}{1 - \lambda} p^{\lambda} [1 - p^{\lambda + 1}]^{\frac{\lambda}{1 - \lambda}} \tag{57}$$

For $\lambda \in [0, 1)$, *equation (57) is a power-function quantal response equilibrium.*

PROOF: Notice that equation (13) can be expressed as

$$\frac{f(p)}{[1 - F(p)]^{\lambda}} = \frac{p^{\lambda}}{\mu} \tag{58}$$

Integrating both sides of equation (58) from a nonnegative p_a to some $p^* > p_a$, we have

$$\int_{p_a}^{p^*} \frac{f(p)}{[1 - F(p)]^{\lambda}} dp = \int_{p_a}^{p^*} \frac{p^{\lambda}}{\mu} dp. \tag{59}$$

Making a change of variables on the left hand side of the above integral and defining $c = F(p)$ and $dc = f(p)dp$, this becomes

$$\int_{F(p_a)}^{F(p^*)} \frac{dc}{[1 - c]^{\lambda}} = \int_{p_a}^{p^*} \frac{p^{\lambda}}{\mu} dp. \tag{60}$$

Note that the lower bound of the power-function price distribution must be zero because negative prices produce no profits, which contradicts the power function QRE in (13). Thus if $p = 0$, $c = F(0) = 0$. As p goes from 0 to p^*, c goes from 0 to $F(p^*)$. Assuming $\lambda \neq 1$, equation (60) can be integrated:

$$-\frac{[1-c]^{1-\lambda}}{1-\lambda}\Bigg|_0^{F(p^*)} = \frac{p^{\lambda+1}}{\mu(\lambda+1)}\Bigg|_0^{p^*} \tag{61}$$

equation (61) also yields

$$-\frac{[1-F(p)]^{1-\lambda}}{1-\lambda}+\frac{1}{1-\lambda}=\frac{p^{\lambda+1}}{\mu(\lambda+1)}-0 \tag{62}$$

Simplifying the notation, we obtain

$$[1-F(p)]^{1-\lambda}=1-\frac{(1-\lambda)}{\mu(1+\lambda)}p^{\lambda+1} \tag{63}$$

or equivalently,

$$1-F(p)=\left[1-\frac{(1-\lambda)}{\mu(1+\lambda)}p^{\lambda+1}\right]^{\frac{1}{1-\lambda}} \tag{64}$$

The price probability function is given by

$$F(p)=1-\left[1-\frac{(1-\lambda)}{\mu(1+\lambda)}p^{\lambda+1}\right]^{\frac{1}{1-\lambda}} \tag{65}$$

where μ is a constant to be determined. Note that the lower boundary condition, $F(0)=0$, is satisfied. The other boundary condition, $F(1)=1$, in turn implies that $\mu=(1-\lambda)/(1+\lambda)>0$. Thus replacing λ in (65), one obtains the power function cumulative probability:

$$F(p)=1-\left[1-p^{\lambda+1}\right]^{\frac{1}{1-\lambda}} \tag{66}$$

The corresponding price choice probability is

$$f(p)=\frac{\lambda+1}{1-\lambda}p^{\lambda}\left[1-p^{\lambda+1}\right]^{\frac{\lambda}{1-\lambda}} \tag{67}$$

It follows from (66) and from the definition of μ that the density in (67) satisfies the equilibrium condition in (13). As $\lambda \to 0, f(p) \to 1$, and the distribution of prices becomes the uniform distribution. Recall that the Nash equilibrium price is zero in this Bertrand model. The QRE in (67) therefore, produces systematic departures from the Nash equilibrium with $p > 0$, even though the expected value of the error term is 0. The proposition below shows that the power function equilibrium price distribution, $f(p)$, captures the extent to which a player's behavior deviates from the Nash equilibrium zero price outcome.

To summarize:

PROPOSITION 3. *As the error rate decreases,* $\lambda \to 1$, *the power function cumulative probability converges to the Nash equilibrium:* $F(p) \to 1$ *for all* $p > 0$.

PROOF. We need to show that, for any p value, $[1 - p^{(1+\lambda)}]^{1/1-\lambda}$ in equation (66) vanishes as $\lambda \to 1$. Consider $\lambda < 1$ and notice that $p^{(1+\lambda)} > p^2$ for all $p \in (0, 1)$, so $1 - p^{(1+\lambda)} < 1 - p^2 < 1$. Hence

$$[1 - p^{(1+\lambda)}]^{1/1-\lambda} < [1 - p^2]^{1/1-\lambda}.$$

Since $1 - p^2 < 1$ and the exponent, $1/1 - \lambda$, goes to ∞ as $\lambda \to 1$, it follows that $[1 - p^{(1+\lambda)}]^{1/1-\lambda}$ converges to 0. Therefore $F(p) \to 1$.

Now consider $\lambda \geq 1$. The power-function quantal response equilibrium condition in equation (13) implies that a QRE is the degenerate distribution $F(p) = 1$ for $p \geq 0$. ∎

Appendix 2

For $\lambda = 1$, the power function quantal response equilibrium condition becomes

$$f(p) = \frac{p(2 - F(p))}{\mu} \tag{68}$$

This equation can be also arranged as follows

$$-\frac{f(p)}{(2 - F(p))} = -\frac{p}{\mu}. \tag{69}$$

The above equation can also be expressed as

$$\partial \ln(2 - F(p)) = -\frac{p}{\mu} \tag{70}$$

Integrating from 0 to some p^*, we have

$$\int_0^{p^*} \partial \ln(2 - F(p)) = \int_0^{p^*} -\frac{p}{\mu t} \tag{71}$$

Equation (71) yields the following result:

$$\ln(2 - F(p^*)) = -\frac{p^{*2}}{2\mu} \tag{72}$$

To simplify the notation let p denote p^*. It follows from the above equation that the price distribution is

$$F(p) = 2[1 - e^{-\frac{p^2}{2\mu}}] \,. \tag{73}$$

The next task is to determine μ. The boundary condition, $F(1) = 1$, implies that $\mu = -1/(2 * \ln(1/2))$.