A NOTE ON THE ARC ELASTICITY OF DEMAND

Andrés Vázquez
Consejo Superior de Investigaciones Científicas

Resumen: Se presenta una fórmula alternativa de la elasticidad arco de la demanda que surge naturalmente y retiene las características sobresalientes de la elasticidad puntual. Una propiedad importante es que ambas elasticidades coinciden en algún punto del intervalo, lo que implica que la medida propuesta conduce a la exacta estimación de la elasticidad punto cuando ésta es constante. Las relaciones entre la elasticidad arco de la demanda y las elasticidades arco de la renta total son también formalmente idénticas a las obtenidas en el caso de la elasticidad puntual.

Abstract: This note presents a simple alternative measure for the arc elasticity of demand which comes naturally and does retain the salient characteristics of the point demand elasticity. A remarkable feature is that it equals the point demand elasticity at some point inside the interval, thus leading to the exact estimation of the point demand elasticity when this is constant. It also ensures that the relationships between the arc demand elasticity and the arc revenue elasticities keep exact formal analogy with those well-known for the point elasticity case.

1. Introduction

The concept of elasticity of demand was introduced in the literature for the purpose of explaining the seemingly paradoxical result, observed at least since Gregory King's times, that a bumper crop yielded lower revenues for farmers than a small one. It was the demand elasticity artifice which made it clear that this is so whenever price changes proportionally more than the amount demanded, or, in other words, whenever the demand is inelastic, to use present terminology. Now, whereas the point demand elasticity is a well defined concept, arc elasticity is essentially an ambiguous concept to which no unique measure can be attached. In his early criticisms to Dalton's (1920) seminal definition, Schultz (1928) already pointed out that percentages were not good measures of proportionate changes when the bases on which they are computed are not kept constant. A much better measure, he said, is to be had by taking the difference between the logarithms of the numbers, since then it makes no difference whatsoever between the point elasticity and the arc elasticity of any demand curve having a constant point demand elasticity. Schultz's proposal, however, did not appear to have made any lasting impression on his contemporaries, particularly Lerner (1933) and Allen (1934), whose now familiar measures, like Dalton's, all are defined under the underlying assumption of a linear demand curve, which in fact implies quite narrow limits within which the percentage changes in price and quantity are allowed to vary.

In this note we show that Schultz's logarithmic approach to the concept leads to a measure for the arc elasticity of demand with some attractive properties. In particular, it equals the point demand elasticity at some point inside the interval, which in fact embraces Schultz's verbal definition as a special case. The note proceeds as follows. Next section first introduces the concept, and then relates it to Allen's dominant measure. In section 3 we show that the relationships between the arc elasticity of demand and the arc revenue elasticities keep exact formal analogy with those derived for the point elasticity case. Finally, section 4 contains some brief concluding remarks.

2. An Alternative Definition of the Arc Price Elasticity of Demand

To tackle the problem, let the traditional demand function be

$$x = f(p) \tag{1}$$

where x stands for quantity, and p for price. The point price elasticity of demand, denoted $\eta(p)$, is defined by

$$\eta(p) = -\frac{pdx}{xdp} = -\frac{d \ln x}{d \ln p} \tag{2}$$

Now, letting $u = \ln x$, and $v = \ln p$, the demand function can be written $u = F(v)$, and by definition we then have

$$\eta(p) = -\frac{du}{dv} = -F'(v) \tag{3}$$

Let next $P_0(x_0, p_0)$ and $P_1(x_1, p_1)$ be any two points on the demand curve (1), and suppose that $x_0 > x_1$, so that, on the usual assumption of a downward sloping demand curve, we have $p_0 < p_1$. Then, we propose to define the arc elasticity of demand as the slope of the chord connecting the two points on the graph of the demand logarithmic transform. More formally, if we call this elasticity $\eta(p_0, p_1)$, we have

$$\eta(p_0, p_1) = -\frac{u_0 - u_1}{v_0 - v_1} = \frac{\ln x_0 - \ln x_1}{\ln p_1 - \ln p_0}, \tag{4}$$

which coincides with Schultz's verbal definition.[1]

Note that (4) comes from (2) or (3) by simply replacing the differentials with finite changes. Note also that $\eta(p_0, p_1)$ is independent of the units of measurement of price and quantity, symmetric with respect to the arc end points, and unity whenever $x_0 p_0 = x_1 p_1$, i.e., whenever total revenue at the two price-quantity combinations are equal. Now, however, a very remarkable feature of formula (4) is that it coincides with the point demand elasticity at some point inside the interval. This attractive property, which covers Schultz's verbal definition as a special case, is also easily seen once we realize that (4) can be rewritten as

$$\eta(p_0, p_1) = \frac{F(v_0) - F(v_1)}{v_1 - v_0} = \frac{\int_{v_1}^{v_0} F'(v)dv}{\int_{v_0}^{v_1} dv} = -\frac{\int_{p_0}^{p_1} \eta d \ln p}{\int_{p_0}^{p_1} d \ln p} \tag{5}$$

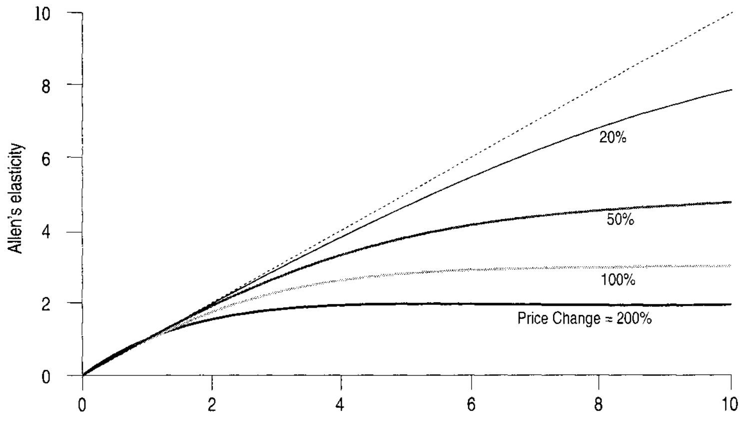
meaning literally that arc elasticity may alternatively be defined as the average value of point demand elasticities over the price logarithmic arc. Then, it is immediate from (4) or (5) and the mean value theorem that even if the underlying demand curve passing through the arc end points

[1] See also Roy (1942), Gallego-Dlaz (1944/1945), and Holt and Samuelson (1946).

is not one with constant point demand elasticity, there exists a point in the interval such that $\eta(p_0, p_1) = \eta(\xi)$, where $\eta(\xi)$ is the point demand elasticity at some point $\xi(p_0 < \xi < p_1)$. Obviously, in the special case of an isoelastic demand curve, $\eta(p_0, p_1) \equiv \eta(p) \equiv \eta(\xi)$, a constant. Expresed in words, this means that if the actual demand curve has a constant point demand elasticity, then the arc elasticity also yields the same constant value irrespective of the span of the arc. It is important to note that in this case two points are enough to determine the equation of the demand curve, and that the elasticity coefficient is precisely given by formula (4).

On the other hand, it may be of some interest to relate our arc demand elasticity definition to Allen's, which is by far the most widely-used measure in the literature. This is illustrated in figure 1 for some price percentage changes.[2]

Figure 1



[2] As it is known, Allen's elasticity, denoted η^A, is defined by

$$\eta^A = -\frac{\frac{1}{2}(p_0 + p_1)(x_0 - x_1)}{\frac{1}{2}(x_0 + x_1)(p_0 - p_1)} = \frac{(1 + p_1 / p_0)(1 - x_1 / x_0)}{(p_1 / p_0 - 1)(1 + x_1 / x_0)},$$

so that, rather than an arc elasticity, this popular measure is nothing but the point demand elasticity calculated at the chord midpoint. Now, from (4) we get

$$\frac{x_1}{x_0} = (p_0 / p_1)^{\eta(p_0, p_1)}$$

Inspection of the figure shows that only for values less than or near to unity can the divergences between these two measures be called negligible. For greater values, however, Allen's dominant formula yields increasingly distorted results which underestimate the proposed logarithmic measure, even for price percentage changes and elasticity values often found in the real world. Obviously, the more the elasticity differs from unity and the greater the price percentage changes, the greater the magnitude of the bias. Of course, it is not hard to show that both Dalton's and Lerner's formulae yield even larger biases. We may also note that distortion is present even for relatively small values of the elasticity. This is so because the linear demand assumption which underlies Allen's definition in fact implies the upper bound of 200 percent in the price and quantity relative changes, beyond which it asymptotically converges to the unitary limit. By contrast, no finite bounds exist for logarithmic changes.

3. Relationships between the Arc Price Elasticity of Demand and the Arc Revenue Elasticities

In this section we derive the relationships between the arc elasticity of demand and the arc elasticity of total revenue, the purpose for which the concept was originally defined. To this end, let E_{rp} be the arc revenue elasticity with respect to price, or the arc price revenue elasticity for short. In a likewise manner to the arc demand elasticity, the arc price revenue elasticity is defined by

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which upon substitution in η^A, and rearranging, yields:

$$\eta^A = \{1 + \frac{2}{\lambda}\}\{\frac{(1+\lambda)^{\eta(p_0,p_1)} - 1}{(1+\lambda)^{\eta(p_0,p_1)} + 1}\}$$

where λ stands for the relative change in price, i.e., $\lambda = (p_1 - p_0)/p_0$, or, alternatively, $\lambda = (p_0 - p_1)/p_1$. Notice that $\eta^A = 0$, as $\eta(p_0, p_1) = 0$ and $\eta^A = 1$, as $\eta(p_0, p_1) = 1$. Further, it is not hard to show that η^A is strictly increasing and strictly concave. In the limit, we have

$$\lim_{\eta(p_0,p_1)\to\infty} \eta^A = 1 + \frac{2}{\lambda} > 1.$$

$$\lim_{\lambda\to\infty} \eta^A = 1.$$

$$E_{rp} = \frac{\ln r_1 - \ln r_0}{\ln p_1 - \ln p_0} \, , \tag{6}$$

where $r_0 = x_0 p_0$, and $r_1 = x_1 p_1$. Similarly, the quantity arc revenue elasticity, denoted E_{rx}, is defined by

$$E_{rx} = \frac{\ln r_1 - \ln r_0}{\ln x_1 - \ln x_0} \, . \tag{7}$$

Now, combined with (4), (6) and (7) yield the following relationships[3]:

$$E_{rp} = 1 - \eta(p_0, p_1) \tag{8}$$

$$E_{rx} = 1 - \frac{1}{\eta(p_0, p_1)} \, , \tag{9}$$

which are formally identical to those that can be derived for the point demand elasticity case.[4] Thus, the parallel propositions can here be stated as follows: the response of total revenue of a certain commodity to a discrete change in its price (quantity) is positive (negative), zero, or negative (positive) depending on whether the arc elasticity of demand is respectively greater than, equal to, or smaller than unity, and vice versa. In other words, for any two points on a given demand curve, the total revenue at the higher (smaller) price (quantity) point will be greater than, equal to, or less than the total revenue at the other point, depending on whether the arc elasticity of demand is less than, equal to, or greater than unity, and vice versa.

Finally, it should be mentioned that even though both Allen's and Lerner's formulae, but not Dalton's, also allow us to establish analytical relationships similar to (8) and (9), provided the arc revenue elasticities be conveniently defined[5], only these latter keep exact formal analogy with the point elasticity case. This remarkable feature is an immediate consequence of our arc elasticity definitions and the so-called Cauchy's (1821) functional equation of the logarithm, according to which only for

[3] It may also be noted that the arc revenue elasticities are in turn related by the identity $E_{rp}^{-1} = 1 - E_{rx}^{-1}$.

[4] See Vázquez (1985a) for a graphical derivation.

[5] See Vázquez (1985b, 1989). It may be noted that in the case of Allen's formula, the arc revenue elasticities equal too the point revenue elasticities evaluated at the chord

logarithmic changes the relative change in total revenue exactly equals the sum of the relative changes in price and quantity.

4. Conclusions

To conclude, in this note we have developed a simple alternative measure for the arc elasticity of demand which preserves the salient characteristics of the point demand elasticity, and yields narrower values than Allen's widely used measure. Another important implication concerns the analytical relationships between the arc elasticity of demand and the arc revenue elasticities, which keep exact formal analogy with those well-known for the point demand elasticity case. Mention, however, should be made that these important relationships, whether they concern point or arc elasticity, are merely identities and therefore empty of empirical substance. Of course, they are actually useful in the sense of providing the same information about the sign and magnitude of the revenue elasticities from the knowledge of the demand elasticity, or vice versa. As a final remark, our arc demand elasticity measure is independent of the path connecting the arc end points, a feature which may be of some interest where no clear guidelines for choosing the appropriate functional form for estimation exist.

References

Allen, R. G. D. (1934). "The Concept of Arc Elasticity of Demand", *Review of Economic Studies,* vol. 1, pp. 226-230.

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midpoint. In fact, letting E^*_{rp} and E^*_{rx} be respectively the price and quantity arc revenue elasticity, we get

$$E^*_{rp} = \frac{p^*(r_0 - r_1)}{r^*(p_0 - p_1)} = 1 - \eta^A$$

$$E^*_{rx} = \frac{x^*(r_0 - r_1)}{r^*(x_0 - x_1)} = 1 - \frac{1}{\eta^A},$$

where $p^* = \frac{1}{2}(p_0 + p_1)$, $x^* = \frac{1}{2}(x_0 + x_1)$, and $r^* = x^* p^*$.

Cauchy, A. L. (1821). *Cours d'analyse de l'École Royale Polytechnique*, 1ère partie, *Analyse algébrique* (Oeuvres complètes, II^{e}, série III). Paris, Gauthier-Villars et Fils, Imprimeurs-Libraires.

Dalton, H. (1920). *Some Aspects of the Inequality of Incomes in Modern Communities*, London, George Routledge & Sons.

Gallego-Díaz, J. (1944/1945). "A Note on the Arc Elasticity of Demand", *Review of Economic Studies*, vol. 12, pp. 114-115.

Holt, Ch. C., and P. A. Samuelson (1946). "The Graphic Depiction of Elasticity of Demand", *Journal of Political Economy*, vol. 54, pp. 354-357.

Lerner, A. P. (1933). "The Diagrammatical Representation of Elasticity of Demand", *Review of Economic Studies*, vol. 1, pp. 39-44.

Roy, R. (1942). "Henry Schultz, Pionnier de l'Économétrie", in O. Lange, F. McIntyre and T. Yntema (eds.), *Studies in Mathematics and Econometrics*, Chicago, The University of Chicago Press, pp. 18-31.

Schultz, H. (1928). *Statistical Laws of Demand and Supply with Special Application to Sugar*, Chicago, The University of Chicago Press.

Vázquez, A. (1985a). "Geometrical Measurement of the Elasticity of Demand and the Elasticities of Total Revenue", *Giornale degli Economisti e Annali di Economia*, voi. 44, pp. 657-665.

——— (1985b). "Geometrical Analysis of the Arc Elasticity of Demand and the Changes in Total Revenue", *Canadian Journal of Agricultural Economics*, vol. 33, pp. 387-395.

——— (1989). "A Note on the Application of the Derivation of Price Elasticity of Demand from Total Revenue Response to Price Changes: Some Clarifications", *Economic Notes*, vol. 19, pp. 423-431.