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1 Introduction

In this paper, we propose to study the existence of Nash equilibria, strong Nash equilibria, strong Berge equilibria and Nash-Pareto equilibria in non-cooperative games with non-ordered preference relations and where strategy sets may be non convex and/or non compact.

Nash [1951] has proved that a finite game has a Nash equilibrium in mixed strategies. Later on, Rosen [1965] has extended Nash's result to infinite games with concave payoff functions and convex strategy sets. Nash equilibrium has been successfully applied in many areas of economics including oligopoly theory, general equilibrium, and social choice theory. These applications have led researchers from different fields to investigate the possibility of weakening Rosen's Nash equilibrium existence conditions to further enlarge its domain of applicability. Several results have already been obtained in this direction. Nishimura and Friedman [1981] considered the existence of Nash equilibrium in games where the payoff functions are not quasi-concave (but satisfying a *strong condition*). Lignola [1997] has proven existence results for Nash equilibrium points for two-person games in topological vector spaces and in reflexive Banach spaces with semicontinuous payoff functions and compact strategy sets. Williams [1980] has established the existence of Nash equilibrium points in n-person games when strategy sets are closed and convex subsets of reflexive Banach spaces, each player's cost functional is concave in that player's strategy, weakly continuous in the strategies of the other players, weakly lower semicontinuous in all strategies, and satisfies a coercivity condition if any of the strategy sets is unbounded. The uniqueness of pure strategy Nash equilibrium is established in Rosen [1965]. Yao [1992] considered the existence of Nash equilibrium in games where the payoff functions are γ-diagonally quasiconcave.

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Baye *et al.* [1993] have established necessary and sufficient conditions for the existence of Nash equilibrium in noncooperative games which may have discontinuous and/or non-quasiconcave payoffs, but satisfying conditions called diagonal transfer quasiconcavity and diagonal transfer continuity. Reny [1999] has established the existence of Nash equilibrium in compact and quasiconcave games where the game is better-reply secure. Dasgupta and Maskin [1986] have established the existence of Nash equilibrium for games where the strategy sets are non-empty, convex and compact, and players have payoff functions that are quasiconcave, upper semicontinuous and graph continuous. In all these papers except the paper of Vives [1990], authors have established the existence of Nash equilibrium for games where the strategy sets are non-empty, and/or non convex and/or non compact, and players have payoff functions that are quasiconcave or not quasiconcave, and/or upper-lower semicontinuous. It is interesting to mention that, using lattice-theoretic methods and Tarski's fixed point theorem, Vives [1990] has established the existence of Nash equilibrium in games where payoffs are upper semicontinuous and satisfy certain monotonicity properties. See also Vives [2005], Echenique [2002], Jackson [1992], Jackson [1996], Palfrey [1991], Topkis [1978].

Aumann [1959] introduced the strong Nash equilibrium (SNE) which ensures a more restrictive stability than the Nash equilibrium. A SNE is a Nash equilibrium and Pareto efficient. Thus, a SNE is not only immune to unilateral deviations, but also to deviations by coalitions. The SNE has been used to study different noncooperative games as coalition formation (Hart and Kurz [1983] , Le Breton and Weber [2005]), congestion games Hotzman and Law-Yone [1997], voting models Moulin [1982], production externality games (Moulin and Shenker [1992], Moulin [1994]). Ichiishi [1981] has shown the existence of social coalitional equilibrium under five assumptions about a society[1]. A social coalitional equilibrium can be specialized to the strong Nash equilibrium. Then, the sufficient conditions for the existence of social coalitional equilibrium are also sufficient for the existence of strong Nash equilibrium. Note that the assumption 4 is very strong to verify. [2]

Berge [1957] introduced the strong Berge equilibrium. The strong Berge equilibrium is stable against deviation of all the players except one of them. Indeed, if a player chooses his strategy in a strong Berge equilibrium, then he obliges all the other players to do so. Larbani and Nessah [2001] have shown a theorem of existence of this equilibrium based on Ky Fan inequality.

The paper is organized as follows. Section 2 introduces the some definitions and notation. Section 3 establishes necessary and sufficient conditions for the existence of a Nash equilibrium, strong Nash equilibrium, strong Berge equilibrium and Nash-Pareto equilibrium in non-cooperative games with non-ordered preference relations and where strategy sets may be non convex and/or non compact.

[1]Given a finite set of agents N, a society is a list of specified data $(\{X^j\}_{j\in N}, \{S^C\}_{C\in\mathfrak{P}}, \{u^j_C\}_{j\in C\in\mathfrak{P}}, \mathfrak{F})$

[2]Assumption 4. For every $x \in X$ and for every $v \in \mathbb{R}^n$, if there exists a balanced collection $\mathfrak{B}$ such that for each $C \in \mathfrak{B}$ there exists $y_C \in S_C(x)$ for which $v_j \leq u^j_C(x, y_C)$ for every $j \in C$, then there exist $P \in \mathfrak{F}$ and $z_D \in S_D(x)$ for every $D \in P$ such that $v_j \leq u^j_D(x, z_D)$.

2 Definitions and Notations

In this section, we give the definition of a strong Nash equilibrium, its interpretation and some of its properties. Consider the following noncooperative game in normal form:

$$G = (I,\ X,\ \succeq) \tag{1}$$

where $I = \{1, ..., n\}$ is the finite set of players, $X = \prod_{i \in I} X_i$ is the set of strategy profiles of the game, where X_i is the set of strategies of player i; $X_i \subset E_i$ and E_i is a topological space and $\succeq = (\succeq_1, \succeq_2, ..., \succeq_n)$[3] where $\succeq_i$ is the weak preference relation of player i which is defined on the set X.

Let $\Im$ denote the set of all coalitions (*i.e.*, nonempty subsets of I). For each coalition $R \in \Im$, we denote by $-R$ the set $-R = \{i \in I$ such that $i \notin R\}$: the remaining of coalition R. If R is reduced to a singleton $\{i\}$, we denote then by $-i$ the set of $-R$. We also denote by $X_R = \prod_{i \in R} X_i$ the set of strategies of the players in coalition R. If $\{K_j\}_{j \in \{1,...,s\} \subset \mathbb{N}}$ is a partition of the set of players I, then any strategy profile $x = (x_1, ..., x_n) \in X$ can be written as $x = (x_{K_1}, x_{K_2}, .., x_{K_s})$ where $x_{K_i} \in X_{K_i}$.

For any coalition K, denote by $\succeq_K = (\succeq_j,\ j \in K)$, and if $K = I$, then $\succeq_I = \succeq = (\succeq_1, \succeq_2, ..., \succeq_n)$. Let us consider the strict preference relation *induced* by the weakly preference relation defined as follows:

$$x \succ y \text{ if and only if } x \neq y, \text{ and } x \succeq y.$$

Denote weakly upper, weakly lower, strictly upper, and strictly lower contour sets of $\succeq$ and $\succ$ by, for each x, $U_w(x) = \{y \in X$ such that $y \succeq x\}$, $L_w(x) = \{y \in X$ such that $x \succeq y\}$, $U_s(x) = \{y \in X$ such that $y \succ x\}$, and $L_s(x) = \{y \in X$ such that $x \succ y\}$, respectively. Denote also weakly and strictly lower section sets relatively to a coalition K of $\succeq$ and $\succ$ by, for each $y_K \in X_K$, $LS_w(y_K, K) = \{x \in X$ such that $(x_K, x_{-K}) \succeq_K (y_K, x_{-K})\}$ and $LS_s(y_K, K) = \{x \in X$ such that $(x_K, x_{-K}) \succ_K (y_K, x_{-K})\}$, respectively. Let Y be any set, we denote by $\langle Y \rangle$ the set of all finite subsets of Y. If S is a subset of Y, then denote by clS the closure of S in Y, $intS$ the interior of S in Y and by $co(S)$ the convex hull of S.

Let us consider the complement of a strictly upper section defined as follows:

$$CUS_s(y_K, K) = \{x \in X \text{ such that } (y_K, x_{-K}) \not\succ_K (x_K, x_{-K})\}.$$

If $K = I$, without loss of generality denote $CUS_s(y_I, I)$ by $CUS_s(y)$.

DEFINITION 2.1 *A strategy profile $\overline{x} \in X$ is said to be a Nash equilibrium for game* (1) *if*

$$\forall i \in I,\ \forall y_i \in X_i,\ (\overline{x}_{-i}, \overline{x}_i) \succeq_i (\overline{x}_{-i}, y_i).$$

Or equivalently if $\overline{x} \in \bigcap_{y_i \in X_i} LS_w(y_i, i)$, *for each* $i \in I$.

[3] A relation R is said an order if: (1) reflexive, *i.e.*, $x \succeq x$; (2) antisymmetric, *i.e.*, if $x \succeq y$ and $y \succeq x$ then $x = y$, and (3) transitive, *i.e.*, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

DEFINITION 2.2 *A strategy profile $\overline{x} \in X$ is said to be a Pareto efficient for game* (1) *if $\nexists y \in X$ such that $y \succ \overline{x}$ or $\overline{x} \in \bigcap_{y \in X} CUS_s(y)$.*

DEFINITION 2.3 *A strategy profile $\overline{x} \in X$ is said to be a strong Nash equilibrium for game* (1)*, if $\forall C \in \Im$, there does not exist a $y_C \in X_C$ such that $(y_C, \overline{x}_{-C}) \succ_C (\overline{x}_C, \overline{x}_{-C})$, which is equivalent to $(y_C, \overline{x}_{-C}) \not\succ_C (\overline{x}_C, \overline{x}_{-C})$, for each $y_C \in X_C$.*

A strategy profile is a strong Nash equilibrium if no coalition (including the grand coalition, *i.e.*, all players collectively) can profitably deviate from the prescribed profile. This definition immediately implies that any strong equilibrium is both Pareto efficient and a Nash equilibrium. Indeed, if a coalition S deviates from its strategy $\overline{x}_S$ in some strong Nash equilibrium $\overline{x}$, then it cannot improve the earning of all its players at the same time if the rest of the players maintains its strategy $\overline{x}_{-S}$ of $\overline{x}$. This equilibrium is stable with regard to the deviation of any coalition.

REMARK 2.1 *By definition 2.3, we obtain $\overline{x} \in X$ is a strong Nash equilibrium if and only if $\overline{x} \in \bigcap_{y_S \in X_S} CUS_s(y_S, S)$, for each $S \in \Im$.*

The following lemma characterizes the strong Nash equilibrium for game (1).

LEMMA 2.1 *The strategy profile $\overline{x} \in X$ is a strong Nash equilibrium for game* (1) *if and only if for each $S \in \Im$, the strategy $\overline{x}_S \in X_S$ is Pareto efficient for the following sub-game $\langle S, X_S, \succ_{(\overline{x}_{-S})} \rangle$ with $x_S \succ_{(\overline{x}_{-S})} y_S$ if $(x_s, \overline{x}_{-S}) \succ_S (y_S, \overline{x}_{-S})$.*

PROOF. It is a straightforward consequence of Definition 2.2 and Definition 2.3. ■

DEFINITION 2.4 *A strategy profile $\overline{x} \in X$ is said to be a strong Berge equilibrium for game* (1)*, if*

$$\forall i \in I,\ \forall j \neq i,\ (\overline{x}_i, \overline{x}_{-i}) \succeq_j (\overline{x}_i, y_{-i}),\ \forall y_{-i} \in X_{-i}.$$

If a player i chooses his strategy $\overline{x}_i$ of a $\overline{x}$ which is a strong Berge equilibrium, then the coalition $-i$ cannot improve the earnings of all its players, *i.e.* by deviating from $\overline{x}$. In other words, strong Berge equilibrium is stable against deviations of any coalition of type $-i$, $i \in I$.

REMARK 2.2 *If $n = 2$, then the concepts of strong Berge equilibrium and Nash equilibrium are identical*

PROPOSITION 2.1 *A strong Berge equilibrium is also a Nash equilibrium.*

PROOF. Let $\overline{x} \in X$ be a strong Berge equilibrium of game (1), and let $i \in I$, suppose that, player i choose a strategy x_i, then for all $j \neq i$, we have $(\overline{x}_j, \overline{x}_{-j}) \succeq_i (\overline{x}_j, \overline{x}_{-\{i,j\}}, x_i) = (x_i, \overline{x}_{-i})$. Since i is arbitrarily chosen in I, then $\overline{x}$ is a Nash equilibrium. ■

DEFINITION 2.5 *$\overline{x} \in X$ is said to be a strong Berge-Pareto equilibrium for game* (1)*, if $\overline{x}$ is a strong Berge equilibrium which is also Pareto efficient for the same game.*

REMARK 2.3 *If $n = 2$, then the concepts of strong Berge-Pareto equilibrium and strong Nash equilibrium are identical.*

Let us consider the following assumption.

ASSUMPTION 2.1 *The preference relation $\succeq$ is antisymmetric,* i.e., *if $x \succeq y$ and $y \succeq x$ then $x = y$.*

We have the following theorem.

THEOREM 2.1 *If the preference relation $\succeq$ is antisymmetric, then any strong Berge-Pareto equilibrium for game* (1) *is also a strong Nash equilibrium for the same game.*

PROOF. Let $\overline{x} \in X$ be a strong Berge-Pareto equilibrium for game (1), then by definition, we have:

$$\begin{cases} 1)\ \forall i \in I,\ \forall j \neq i,\ (\overline{x}_i, \overline{x}_{-i}) \succeq_j (\overline{x}_i, y_{-i}),\ \forall y_{-i} \in X_{-i} \\ 2)\ \overline{x} \text{ is a Pareto efficient.} \end{cases} \tag{2}$$

Suppose that $\overline{x}$ is not a strong Nash equilibrium, then there exists $S_0 \in \Im$ and $\widetilde{y}_{S_0} \in X_{S_0}$ such that: $(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) \succ_{S_0} (\overline{x}_{S_0}, \overline{x}_{-S_0})$. This inequality is equivalent to:

$$\begin{cases} 1)\ (\widetilde{y}_{S_0}, \overline{x}_{-S_0}) \succeq_{S_0} (\overline{x}_{S_0}, \overline{x}_{-S_0}),\ \text{and} \\ 2)\ (\widetilde{y}_{S_0}, \overline{x}_{-S_0}) \neq (\overline{x}_{S_0}, \overline{x}_{-S_0}). \end{cases} \tag{3}$$

$S_0 \neq I$ because $\overline{x}$ is Pareto efficient for the game (1) (assumption 2) of system (2)), then $-S_0 \neq \emptyset$. Let $i_0 \in -S_0$. Thus, $S_0 \subset -i_0$. For all $j \in S_0$, $j \neq i_0$. Let $y_{-i_0} = (\overline{x}_{-S_0/\{i_0\}}, \widetilde{y}_{S_0})$ in the inequality (2), then we obtain

$$(\overline{x}_{S_0}, \overline{x}_{-S_0}) \succeq_{S_0} (\widetilde{y}_{S_0}, \overline{x}_{-S_0}). \tag{4}$$

The first assertion in the system (3), inequality (4) and taking into account the symmetry of $\succeq$ imply that $(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) = (\overline{x}_{S_0}, \overline{x}_{-S_0})$. This contradicts the second assertion of (3). ■

3 Existence Equilibria

Let us in first recall the definition of transfer continuity and transfer FS-convexity concepts introduced by Baye [1993].[4]

DEFINITION 3.1 *(Transfer Continuity) Let X and Y be two topological spaces. A correspondence $G : X \to 2^Y$ is said to be transfer closed-valued on X if for every $x \in X$, $y \notin G(x)$ implies that there exists $x' \in X$ such that $y \notin cl\ G(x')$.*

[4] Baye [1993] characterizes the existence of greatest and maximal elements of weak and strict preferences. Conditions called transfer FS-convexity and transfer SS-convexity are shown to be necessary and, in addition with transfer closedness and transfer openness, sufficient for the existence of greatest and maximal elements of weak and strict preferences, respectively.

DEFINITION 3.2 *(Transfer FS-convexity) Let X be a topological space and let Y be a nonempty convex subset of a vector space F. A correspondence $G : X \to 2^Y$ is said to be transfer FS-convex on X if for any finite subset $X^m = \{x^1, ..., x^m\} \in \langle X \rangle$, there exists a corresponding finite subset $Y^m = \{y^1, ..., y^m\} \in \langle Y \rangle$ such that for any subset $\{y^{k^1}, y^{k^2}, ..., y^{k^s}\} \subset Y^m$, $1 \leq s \leq m$, we have $co\{y^{k^1}, y^{k^2}, ..., y^{k^s}\} \subset \bigcup_{j=1,...,s} G(x^{k^j})$.*

Baye [1993] has established the following lemma.

LEMMA 3.1 *Let X be a topological space and Y be a nonempty compact convex subset in a Hausdorf topological vector space F.Suppose a correspondence $G : X \to 2^Y$ is transfer closed-valued and transfer FS-convex on X. Then, $\bigcap_{x \in X} G(x)$ is nonempty and compact.*

REMARK 3.1 *In the case where G is transfer continuous on X, then the transfer FS-convex on X is also necessary. Indeed, suppose that $\bigcap_{x \in X} G(x)$ is nonempty. Then, there exists $\overline{y} \in Y$ such that $\overline{y} \in \bigcap_{x \in X} G(x)$. Let us consider a finite subset of X; $A = \{x^1, ..., x^m\} \in \langle X \rangle$, then there exists $\{y^1, ..., y^m\} \in \langle Y \rangle$ with $y^j = \overline{y}$ for each j. Therefore, for each $J \subset \{1, ..., m\}$ we have $co\{y^j, j \in J\} = \{\overline{y}\} \subset \bigcap_{x \in X} G(x) \subset \bigcup_{j \in J} G(x^{k^j})$.*

3.1 Existence of Nash Equilibrium

We can now state our main result which characterizes the existence of Nash equilibria when the strategy set is compact and convex.

THEOREM 3.1 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i \in I}$ be a family of Hausdorf topological vector spaces and let X_i be a nonempty compact and convex subset of E_i such that the weakly lower section $LS_w(y_i, i)$ is transfer closed on X_i, for each $i \in I$. Then, the game* (1) *has a Nash equilibrium if and only if the following set $\bigcap_{i \in I} LS_w(y_i, i)$ is transfer FS-convex on X.*

PROOF. Necessity: If $\overline{x} \in X$ is a Nash equilibrium, then $\forall i \in I$, $\forall y_i \in X_i$, $(\overline{x}_{-i}, \overline{x}_i) \succeq_i (\overline{x}_{-i}, y_i)$. Thus, $\overline{x} \in \bigcap_{y \in X} \bigcap_{i \in I} LS_w(y_i, i)$. Let $\{y^1, ..., y^m\} \subset X$, then there exists a corresponding points $x^j = \overline{x}$, $j = 1, ..., m$ such that for each $J \subset \{1, ..., m\}$, we have $co(x^j,\ j \in J) = \{\overline{x}\} \subset \bigcap_{y \in X} \bigcap_{i \in I} LS_w(y_i, i) \subset \bigcup_{j \in J} [\bigcap_{i \in I} LS_w(y_i^j, i)]$.

Sufficiency: Let us consider the following correspondence:

$$G : X \to 2^X \text{ defined by } x \mapsto G(x) = \bigcap_{i \in I} LS_w(x_i, i).$$

The transfer closures of the weakly lower section $LS_w(y_i, i)$ imply that the correspondence G is transfer closed. Taking into account the correspondence $G(x) = \bigcap_{i \in I} LS_w(y_i, i)$

is transfer FS-convex on X, consequently, the correspondence G has a nonempty intersection (Lemma 3.1), thus there exists $\overline{x} \in X$ such that $\overline{x} \in \bigcap_{y \in X} G(y) = \bigcap_{y \in X} \bigcap_{i \in I} LS_w(y_i, i)$. Then, by definition 2.1, $\overline{x}$ is a Nash equilibrium for the game (1). ■

Theorem 3.1 can be generalized by relaxing the compactness and the convexity of X.

THEOREM 3.2 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i \in I}$ be a family of Hausdorf topological vector spaces and let X_i be a nonempty subset of E_i. Then, the game* (1) *has a Nash equilibrium if and only if there exists a nonempty compact and convex subset X^0 of X such that:*

(1) the correspondence defined on X by $G(y) = \{x \in X^0$ such that $(x_i, x_{-i}) \succeq_i (y_i, x_{-i}),\ \forall i \in I\}$ is a transfer closed-valued;

(2) there exists $y^0 \in X$ such that $G(y^0)$ is compact;

(3) the correspondence G is transfer FS-convex on X.

PROOF. Necessity. Suppose that the game (1) has a Nash equilibrium $\overline{x} \in X$. Let $X^0 = \{\overline{x}\}$. Then, the set X^0 is nonempty compact and convex, and we have $G(y) = \{x \in X^0 \text{ such that } (x_i, x_{-i}) \succeq_i (y_i, x_{-i}),\ \forall i \in I\} = \{\overline{x}\}$, for each $y \in X$ which is transfer closed-valued and transfer FS-convex on X.

Sufficiency. For each $y \in X$,

$$G(y) = \{x \in X^0 \text{ such that } (x_i, x_{-i}) \succeq_i (y_i, x_{-i}),\ \forall i \in I\}.$$

Then, $\bigcap_{y \in X} G(y) = \bigcap_{y \in X} \text{cl}\, G(y)$ by condition (1) of Theorem 3.2. The condition (3) of Theorem 3.2 imply that $\{\text{cl}\, G(y) \cap G(y^0);\ y \in X\}$ has the finite intersection property. Since $\{G(y) \cap G(y^0);\ y \in X\}$ is a compact family in the compact $G(y^0)$. Thus, $\emptyset \neq \bigcap_{y \in X} G(y) \cap G(y^0) = \bigcap_{y \in X} G(y)$. Hence, there exists $\overline{x} \in X^0$ such that each $y \in X$, $(x_i, x_{-i}) \succeq_i (y_i, x_{-i})$, $\forall i \in I$. This completes the proof. ■

3.2 Existence of Strong Nash Equilibrium

This section characterizes the existence of strong Nash equilibria.

THEOREM 3.3 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i \in I}$ be a family of Hausdorf topological vector spaces and let X_i be a nonempty compact and convex subset of E_i such that the complementary of strictly upper section $CUS_s(y_K, K)$ is transfer closed on X_K, for each $K \in \Im$. Then, the game* (1) *has a strong Nash equilibrium if and only if the following set $\bigcap_{K \in \Im} CUS_s(y_K, K)$ is transfer FS-convex on X.*

PROOF. Denote by $\hat{X} = \prod_{K \in \Im} X_K$.

Necessity: Let $\overline{x} \in X$ be a strong Nash equilibrium. Thus by Remark 2.1, we obtain $\overline{x} \in \bigcap_{\hat{y} \in \hat{X}} \bigcap_{K \in \Im} CUS_s(y_K, K)$. Let $\{\hat{y}^1, ..., \hat{y}^m\} \subset \hat{X}$, then there exists a corresponding points $x^j = \overline{x}$, $j = 1, ..., m$ such that for each $J \subset \{1, ..., m\}$, we have $co(x^j,\ j \in J) = \{\overline{x}\} \subset \bigcap_{\hat{y}\in\hat{X}} \bigcap_{K\in\Im} CUS_s(y_K, K) \subset \bigcup_{j\in J}[\bigcap_{K\in\Im} CUS_s(y_K, K)]$.

Sufficiency: Let us consider the following correspondence:

$$G : \hat{X} \to 2^X \text{ defined by } \hat{x} \mapsto G(\hat{x}) = \bigcap_{K\in\Im} CUS_s(y_K, K).$$

The transfer closures of the complement of strictly upper section $CUS_s(y_K, K)$ imply that the correspondence G is transfer closed. Taking into account that the correspondence $G(\hat{y}) = \bigcap_{K\in\Im} CUS_s(y_K, K)$ is transfer FS-convex on X, it follows that, the correspondence G has a nonempty intersection (Lemma 3.1). Thus, there exists $\overline{x} \in X$ such that $\overline{x} \in \bigcap_{\hat{y}\in\hat{X}} G(\hat{y}) = \bigcap_{\hat{y}\in\hat{X}} \bigcap_{K\in\Im} CUS_s(y_K, K)$. Then, according to Remark 2.1, $\overline{x}$ is a strong Nash equilibrium for the game (1). ■

Theorem 3.3 can be generalized by relaxing the compactness and convexity of X.

THEOREM 3.4 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i\in I}$ be a family of Hausdorf topological vector spaces and let X_i be a nonempty subset of E_i. Then, the game* (1) *has a Nash equilibrium if and only if there exists a nonempty compact and convex subset $X^0 = \prod_{i\in I} X_i^0$ of X such that:*

(1) the correspondence defined on $\hat{X}$ by $\hat{G}(\hat{y}) = \{x \in X^0$ such that $(y_K, x_{-K}) \not\succ_K (x_K, x_{-K}),\ \forall K \in \Im\}$ is transfer closed-valued;

(2) there exists $\hat{y}^0 \in \hat{X}$ such that $\hat{G}(\hat{y}^0)$ is compact;

(3) the correspondence $\hat{G}$ is transfer FS-convex on X.

PROOF. Necessity. Suppose that the game (1) has a strong Nash equilibrium $\overline{x} \in X$. Let $X^0 = \{\overline{x}\}$. Then, the set X^0 is nonempty compact and convex, and we have $\hat{G}(\hat{y}) = \{x \in X^0$ such that $(y_K, x_{-K}) \not\succ_K (x_K, x_{-K}),\ \forall K \in \Im\} = \{\overline{x}\}$, for each $\hat{y} \in \hat{X}$ which is transfer closed-valued and transfer FS-convex on X.

Sufficiency. For each $\hat{y} \in \hat{X}$,

$$\hat{G}(\hat{y}) = \{x \in X^0 \text{ such that } (y_K, x_{-K}) \not\succ_K (x_K, x_{-K}),\ \forall K \in \Im\}.$$

The remaining proof of sufficiency is the same as that in the proof of Theorem 3.2. ■

3.3 Existence of Strong Berge Equilibrium

We present in this section two theorems that give necessary and sufficient conditions for the existence of strong Berge equilibria when the strategy set is compact convex and in the case where the strategy set is not compact and/or convex.

Recall the definition of a strong Berge equilibrium. A strategy profile $\overline{x} \in X$ is said to be a strong Berge equilibrium of game (1), if

$$\forall i \in I,\ \forall j \neq i,\ (\overline{x}_i, \overline{x}_{-i}) \succeq_j (\overline{x}_i, y_{-i}),\ \forall y_{-i} \in X_{-i}.$$

Define the weakly lower section LS_w^j relatively to the preference relation j as follows:

$$LS_w^j(y_{-i}, -i) = \{x \in X \text{ such that} (x_i, x_{-i}) \succeq_j (x_i, y_{-i})\}.$$

Then, we obtain by definition $\overline{x} \in X$ is a strong Berge equilibrium if and only if

$$\overline{x} \in \bigcap_{i,j \in I, i \neq j} LS_w^j(y_{-i}, -i), \quad \text{for each } y_{-i} \in X_{-i} \tag{5}$$

Then, we obtain the following theorem.

THEOREM 3.5 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i \in I}$ be a family of Hausdorf topological vector spaces and let X_i be a nonempty compact and convex subset of E_i such that the weakly lower section $LS_w^j(y_{-i}, -i)$ is transfer closed on X_{-i}, for each $i, j \in I, i \neq j$. Then, the game* (1) *has a strong Berge equilibrium if and only if the following set $\bigcap_{i,j \in I, i \neq j} LS_w^j(y_{-i}, -i)$ is transfer FS-convex on X.*

PROOF. Denote by $\tilde{X} = \prod_{i,j \in I, i \neq j} X_{-i}^j$ with $X_{-i}^j = X_{-i}$, for each j.

Necessity: Let $\overline{x} \in X$ be a strong Berge equilibrium. Thus by (5), we obtain $\overline{x} \in \bigcap_{i,j \in I, i \neq j} LS_w^j(y_{-i}, -i)$, for each $y_{-i} \in X_{-i}$. Let $\{\tilde{y}^1, ..., \tilde{y}^m\} \subset \tilde{X}$, then there exists a corresponding points $x^j = \overline{x}$, $j = 1, ..., m$ such that for each $J \subset \{1, ..., m\}$, we have $co(x^h,\ h \in J) = \{\overline{x}\} \subset \bigcap_{\tilde{y} \in \tilde{X}} \bigcap_{i,j \in I, i \neq j} LS_w^j(y_{-i}, -i) \subset \bigcup_{h \in J} [\bigcap_{i,j \in I, i \neq j} LS_w^j(y_{-i}, -i)]$.

Sufficiency: Let us consider the following correspondence:

$$G : \tilde{X} \to 2^X \text{ defined by } \tilde{x} \mapsto G(\tilde{x}) = \bigcap_{i,j \in I, i \neq j} LS_w^j(y_{-i}, -i).$$

The remaining proof of sufficiency is the same as in the proof of Theorem 3.3. ■

Theorem 3.5 can be generalized by relaxing the compactness and the convexity of X.

THEOREM 3.6 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i \in I}$ be a family of Hausdorf topological vector spaces and let X_i be a nonempty subset of E_i. Then, the game* (1) *has a Nash equilibrium if and only if there exists a nonempty compact and convex subset X^0 of X such that:*

(1) the correspondence defined on $\tilde{X}$ by $\tilde{G}(\tilde{y}) = \{x \in X^0 \text{ such that } (\overline{x}_i, \overline{x}_{-i}) \succeq_j (\overline{x}_i, y_{-i}),\ \forall i \in I,\ \forall j \neq i\}$ is transfer closed-valued;

(2) there exists $\tilde{y}^0 \in \tilde{X}$ such that $\tilde{G}(\tilde{y}^0)$ is compact;

(3) the correspondence $\tilde{G}$ is transfer FS-convex on X.

PROOF. Necessity. Suppose that the game (1) has a strong Berge equilibrium $\overline{x} \in X$. Let $X^0 = \{\overline{x}\}$. Then, the set X^0 is nonempty compact and convex, and we have $\tilde{G}(\tilde{y}) = \{x \in X^0 \text{ such that } (\overline{x}_i, \overline{x}_{-i}) \succeq_j (\overline{x}_i, y_{-i}),\ \forall i \in I,\ \forall j \neq i\} = \{\overline{x}\}$, for each $\tilde{y} \in \tilde{X}$ which is transfer closed-valued and transfer FS-convex on X.

Sufficiency. For each $\tilde{y} \in \tilde{X}$,

$$\tilde{G}(\tilde{y}) = \{x \in X^0 \text{ such that } (\overline{x}_i, \overline{x}_{-i}) \succeq_j (\overline{x}_i, y_{-i}),\ \forall i \in I,\ \forall j \neq i\}.$$

The remaining proof of sufficiency is the same as in the proof of Theorem 3.4. ■

3.4 Existence of Nash-Pareto Equilibrium

In this section, we examine the existence of a Nash equilibrium which is also a Pareto efficient.

The strategy profile $\overline{x} \in X$ is said to be a Nash-Pareto equilibrium for game (1), if $\overline{x}$ is a Nash equilibrium which is also Pareto efficient for the same game.

REMARK 3.2 *If $n = 2$, then the concepts of Nash-Pareto equilibrium, strong Berge-Pareto equilibrium, and strong Nash equilibrium are identical.*

Denote the set of *Weakly Dominant Nash* strategies by $DN(y)$, defined by

$$DN_w(y) = \{x \in X \text{ such that } (x_{-i}, x_i) \succeq (x_{-i}, y_i),\ \forall i \in I\} = \bigcap_{i \in I} LS_w(y_i, i).$$

Recall that $\overline{x} \in X$ is a Nash equilibrium if and only if $\overline{x} \in \bigcap_{y_i \in X_i} \bigcap_{i \in I} LS_w(y_i, i) = \bigcap_{y \in X} DN_w(y)$ and $\overline{x} \in X$ is Pareto efficient if and only if there does not exists $y \in X$ such that $y \succ \overline{x}$, *i.e.*, $\overline{x} \in \bigcap_{y \in X} CUS_s(y)$.

We obtain the following lemma.

LEMMA 3.2 *The game* (1) *has at least one Nash-Pareto equilibrium if and only if the set $\bigcap_{y \in X} H(y)$ is nonempty where $H(y) = DN_w(y) \cap CUS_s(y)$.*

The two following theorems characterize the existence of Nash-Pareto equilibria.

THEOREM 3.7 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i \in I}$ be a family of Hausdorf topological vector spaces and let X_i be a nonempty compact and convex subset of E_i such that H is transfer closed on X. Then, the game* (1) *has a Nash-Pareto equilibrium if and only if the following set H is transfer FS-convex on X.*

Theorem 3.7 can be generalized by relaxing the compactness and the convexity of X.

THEOREM 3.8 *Let $I = \{1, ..., n\}$ be an indexed finite set. Let $\{E_i\}_{i \in I}$ be a family of Hausdorf topological vector spaces and let X_i be a nonempty subset of E_i. Then, the game* (1) *has a Nash-Pareto equilibrium if and only if there exists a nonempty compact and convex subset X^0 of X such that:*

(1) the correspondence defined on X by $\bar{H}(y) = X^0 \cap H(y)$ is transfer closed-valued;

(2) there exists $y^0 \in X$ such that $\bar{H}(y^0)$ is compact;

(3) the correspondence $\bar{H}$ is transfer FS-convex on X.

4 Conclusion

This paper characterizes the existence equilibria which may have non convexity and/or non compactness assumptions. Necessary and sufficient condition have been obtained for the existence of strategy Nash equilibrium, strong Nash equilibrium, strong Berge equilibrium and Nash-Pareto equilibrium in non-cooperative games with non-ordered preference relations which may have non convex and/or non compactly assumptions. Note that the non-ordered preference relation generalizes the payoffs function of the player. Then, this shows that some of the key assumptions still widely used in the literature on the existence of Nash equilibria, strong Nash equilibrium, strong Berge equilibrium and Nash-Pareto equilibrium can be substantially weakened.

References

Aumann, J.P.: Acceptable Points in General Cooperative n-Person Games. In: Tucker, A.W., Luce, R.D. (eds.) Contributions to the Theory of Games IV. Princeton: Princeton University Press 1959

Baye, M.R., Tian, G., Zhou, J. (1993), Characterizations of the Existence of Equilibria in Games with Discontinuous and Non-Quasiconcave Payoffs, *Review of Economic Studies*, **60**, 935-948

Tian, G. (1993), Necessary and Sufficent Conditions for Maximization of a Clasee of Preference Relations, *Review of Economic Studies*, **60**, 949-958

Berge, C. (1957), *Théorie Génèrale des Jeux à n-Personnes*, (Gauthier Villars, Paris 1957)

Dasgupta, P., Maskin, E. (1986), The Existence of Equilibrium in Discontinuous Economic Games, I: Theory, *Review of Economic Studies*, **53**, 1-26

Echenique, F. (2002), Comparative Statics by Adaptive Dynamics and the Correspondence Principle, *Econometrica*, **70**, 833-844

Friedman, J.W. (1977), *Oligopoly and the Theory of Games*, (North-Holland, Amsterdam 1977)

Hart, S., Kurz, M. (1983), Endogenous Formation of Coalitions, *Econometrica*, **51**, 1047-1067

Hotzman, R., Law-Yone, N. (1997), Strong Equilibrium in Congestion Games, *Games and Economic Bahavior*, **21**, 85-101

Ichiishi, T. (1981), A Social Coalitional Equilibrium Existence Lemma, *Econometrica*, **49**, 369-377

Jackson, M-O. and Srivastava, S. (1996), A Characterization of Game-Theoretic Solutions which Lead to Impossibility Theorems, *Review of Economic Studies*, **63**, 23-38

Jackson, M-O. (1992), Implementation in Undominated Strategies: A Look at Bounded Mechanisms, *Review of Economic Studies*, **59**, 757-775

Larbani, M., Nessah, R. (2001), Sur l'Equilibre Fort selon Berge "Strong Berge Equilibrium", *RAIRO Operations Research* **35**, 439-451

Le Breton, M., Weber, S. (2005), Stable Partitions in a Model with Group-Dependent Feasible Sets, *Economic Theory*, **25**, 187-201

Lignola, M.B. (1997), Ky Fan Inequalities and Nash Equilibrium Points without Semicontinuity and Compactness, *Journal of Optimization Theory and Applications*, **94**, 137-145

Moulin, H. (1982), Voting with Proportional Veto Power, *Econometrica*, **50**, 145-162

Moulin, H., Shenker, S. (1992), Serial Cost Sharing, *Econometrica*, **60**, 1009-1037

Moulin, H. (1994), Serial Cost Sharing of Excludable Public Goods, *Review of Economic Studies*, **61**, 305-325

Nash, J.F. (1951), Noncooperative Games, *Annals of Maths*, **54**, 286-295

Nishimura, K. and Friedman, J. (1981), Existence of Nash Equilibriumin n-Person Games without Quasi-Concavity, *International Economic Review*, **22**, 637-648

Palfrey, T-R. (1991), Nash Implementation Using Undominated Strategies, *Econometrica*, **59**, 479-501

Reny, J.P. (1999), On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games, *Econometrica* **67**, 1029-1056

Rosen, J.B. (1965), Existence and Uniqueness of Equilibrium Point for Concave n-Person Games, *Econometrica* **33**, 520-534

Topkis, D-M. (1978), Minimizing a Submodular Function on a Lattice, *Operations Research*, **26**, 305-321

Vives, X. (1990),Nash Equilibrium with Strategic Complementarities, *Journal of Mathematical Economics*, **19**, 305-321

Vives, X. (2005), Complementarities and Games: New Developments, *Journal of Economic Literature*, **43**, 437-479

Williams, R.J. (1980), Sufficient Conditions for Nash Equilibria in n-Person Games over Reflexive Banach Spaces, *Journal of Optimization Theory and Applications*, **30**, 383-394

Yao, J-C. (1992), Nash Equilibria in n-Person Games without Convexity, *Applied Mathematics Letters*, **5**, 67-69