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WP 2010-11
December 2010

Departamento de Teoría e Historia Económica
Facultad de Ciencias Económicas y Empresariales
Universidad de Málaga
ISSN 1989-6908

A coalitional procedure leading to a family of bankruptcy rules

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September 29, 2010

Abstract

We provide a general coalitional procedure that characterizes a family of rules for bankruptcy problems inspired by the Talmud.

Keywords: *bankruptcy, coalitions, claims, Talmud.*

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1 Introduction

A bankruptcy problem refers to a situation in which one has to distribute, among a group of agents, a perfectly divisible commodity whose available amount is not enough to cover all agents' demands on it. This is a classic allocation problem, which encompasses many different situations like the bankruptcy of a firm, the division of an insufficient estate, or the collection of a given amount of taxes. Although instances of bankruptcy problems (and solutions for them) are already documented in ancient sources, such as the Talmud, Aristotle's books or Maimonides' essays, their formalization was not presented till the early eighties [6]. The reader is referred to [7] for a review of the fast-expanding literature concerning this model. An early (and influential) contribution within this field is due to Aumann and Maschler [1] who, among other things, initiated the so-called game-theoretical approach to bankruptcy problems. Aumann and Maschler also provided a specific formula to rationalize the (apparently unrelated) solutions to several bankruptcy situations that appear in the Talmud. Their formula, which came to be known as the *Talmud rule*, implements a basic principle by which individual rationing is of the same type as collective rationing. More precisely, if the amount to divide is below half of the aggregate claim then no agent gets more than half of her claim, whereas if the amount exceeds half of the aggregate claim then no agent gets less than half of her claim. The same principle can be implemented while considering all possible shares of the amount to divide in the aggregate claim. That is, for any given value $\theta \in [0, 1]$, we may construct the rule that distributes the amount accordingly so that nobody gets more than a fraction θ of her claim if the amount to divide is smaller than θ times the aggregate claim and nobody gets less than a fraction θ of her claim if the amount to divide exceeds θ times the aggregate claim [5]. The family, so constructed, would have the Talmud rule as a focal element, but would also encompass, as extreme cases, two classical rules that can be traced back to Maimonides; namely, the so-called constrained equal awards and constrained equal losses rules.

We provide in this note a general coalitional procedure characterizing each of the rules within the family described above. Our procedure is also inspired by a Talmudic principle, regarding bankruptcy problems, according to which the creditors empower each other. For instance, in a three-agent problem in which agents are increasingly ranked according to their claims, the third empowers the second to deal with the first [1].

The note is organized as follows. Section 2 introduces the model and basic concepts. Section 3 presents the coalitional procedure. Finally, Section 4 concludes.

2 Model and basic concepts

We study bankruptcy problems in a variable population model. The set of potential creditors, or *agents*, is identified with the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. Let n denote the cardinality of N. For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's *claim* and $c \equiv (c_i)_{i\in N}$ the claims profile. Without loss of generality, we assume that $c_1 \le c_2 \le \cdots \le c_n$. A *bankruptcy problem* is a triple consisting of a population $N \in \mathcal{N}$, a claims profile $c \in \mathbb{R}^n_+$, and an amount to be divided $E \in \mathbb{R}_+$ such that $\sum_{i\in N} c_i \ge E$. Let $C \equiv \sum_{i\in N} c_i$. To avoid unnecessary complication, we assume $C > 0$. Let $\mathcal{D}^N$ be the set of rationing problems with population N and $\mathcal{D} \equiv \bigcup_{N\in\mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, c, E) \in \mathcal{D}^N$, an *allocation* is a vector $x \in \mathbb{R}^n$ satisfying the following three conditions:

- *Boundedness*: for each $i \in N$, $0 \le x_i \le c_i$,
- *Balance:* $\sum_{i\in N} x_i = E$, and
- *Order Preservation:* for each $i, j \in N$ such that $c_i \le c_j$, then $x_i \le x_j$, and $c_i - x_i \le c_j - x_j$.

A *bankruptcy rule* on $\mathcal{D}$, $R\colon \mathcal{D} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^n$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$ for the problem. Some classical rules are the *constrained equal awards* rule, which distributes the amount equally among all agents, subject to no agent receiving more than what she claims; the *constrained equal losses* rule, which imposes that losses are as equal as possible, subject to no one receiving a negative amount; and the *proportional* rule, which yields awards proportionally to claims. The following family of rules encompasses two of those rules, while generalizing another one. The family is defined by means of a parameter $\theta \in [0, 1]$. The rule R^θ in the family resolves bankruptcy problems according to the following principle: Nobody gets more than a fraction θ of her claim if the amount to divide is less than θ times the aggregate claim and nobody gets less than a fraction θ of her claim if the amount to divide exceeds θ times the aggregate claim. Formally:

*The **TAL-family** consists of all rules with the following form: For some $\theta \in [0, 1]$, for all $(N, c, E) \in \mathcal{D}$, and all $i \in N$,*

$$R_i^\theta (N, c, E) = \begin{cases} \min\{\theta c_i, \lambda\} & \text{if } E \le \theta C \\ \max\{\theta c_i, c_i - \mu\} & \text{if } E \ge \theta C \end{cases},$$

where $\lambda > 0, \mu > 0$ are chosen so that $\sum_{i\in N} R_i^\theta (N, c, E) = E$.

The constrained equal losses rule corresponds to the case $\theta = 0$, whereas the constrained equal awards rule corresponds to the case $\theta = 1$. The so-called Talmud rule [1] is obtained for $\theta = \frac{1}{2}$.

One can visualize the rule R^{θ} as follows. First, it applies equal division until the creditor with the smallest claim has obtained a fraction θ of her claim. Then, that agent stops receiving additional units and the remaining amount is divided equally among the other agents until the creditor with the second smallest claim gets the fraction θ of her claim. The process continues until every agent has received a fraction θ of her claim, or the available amount is distributed. If there is still something left after this process, agents are invited back to receive additional shares. Now agents receive additional amounts sequentially starting with those with larger claims and applying equal division of their losses.

It is interesting to provide the following alternative definition of the rules within the family for the two-agent case. Formally, given $\theta \in [0,1]$, each rule in the TAL-family has the following expression, in the two-agent case:

$$R^{\theta}(N,c,E) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq 2\theta c_1 \\ (\theta c_1, E - \theta c_1) & \text{if } 2\theta c_1 \leq E \leq c_2 - c_1 + 2\theta c_1 \\ \left(c_1 - \frac{C-E}{2}, c_2 - \frac{C-E}{2}\right) & \text{if } c_2 - c_1 + 2\theta c_1 \leq E \end{cases} \tag{1}$$

It is straightforward to show from (1) that any possible allocation x, for a given two-agent problem (N,c,E), can be obtained as a realization of a given rule within the family. Formally, there exists $\theta \in [0,1]$ such that $R^{\theta}(N,c,E) = x$.

It is also worth mentioning that (1) can also be seen as a two-stage allocation process. In the first stage, agents' claims are weighted to reflect exogenous factors that do not appear in our benchmark model of bankruptcy problems. For instance, the liquidation of a firm might have a cost *per se*, which should be borne by the creditors. Alternatively, if two heirs agree on a procedure to divide a bequest, without resorting to an outside authority, they might be saving part of their awards. The former case could be reflected by reducing (equally) the claims of both agents. The second one could be reflected by increasing (equally) the claims of both agents. In the second stage, the "standard solution" (conceding each agent what the other does not claim, and dividing the remainder equally) is applied to the resulting problem.

3 A characterization

In this section, we design coalition formation mechanisms leading to the outcomes of the rules in the TAL-family. More precisely, fix some $\theta \in [0, 1]$, and consider the following procedure. First, in the case of a two-agent problem, we apply the solution (1). Suppose now that we have a problem with three creditors. Then, we proceed in the following way. First, creditors 2 and 3 pool their claims an act as a single agent vis-a-vis 1. The solution (1) of the resulting problem yields awards to agent 1, and to the coalition of agents 2 and 3; to divide its award among its members, the coalition again applies solution (1). The result is order preserving if and only if $3\theta c_1 \leq E \leq C - 3(1-\theta)c_1$. To see this, note that if $3\theta c_1 > E$, then the award of creditor 1, θc_1, would be strictly greater than the one of creditor 2, which is $\frac{E-\theta c_1}{2}$, as a result of the awards sharing in the coalition of creditors 2 and 3. Analogously, if $E > C - 3(1-\theta)c_1$, then the loss of creditor 1, $(1-\theta)c_1$, would be greater than $\frac{c_2+c_3-E+\theta c_1}{2}$, the resulting loss associated to creditor 2, after dividing the awards in the coalition. If one divides the awards equally when $E \leq 3\theta c_1$, and the losses equally when $E \geq C - 3(1-\theta)c_1$, it is obtained, precisely, the solution provided by the rule R^θ, over the entire range $0 \leq E \leq C$.

By using induction, one may generalize this in a natural way to an arbitrary n. Suppose we already know the solution for $(n-1)$-agent problems. Depending on the values of the amount to divide and the vector of claims, we treat a given $n-$person problem in one of the following three ways:

(i) Divide E between $\{1\}$ and $M = \{2, ..., n\}$, in accordance with the solution (1) to the two-agent problem $(\{1, M\}, E, (c_1, c_2 + ... + c_n))$, and then use the $(n-1)$-agent rule, which we know by induction, to divide the amount assigned to the coalition M between its members.

(ii) Assign equal awards to all creditors.

(iii) Assign equal losses to all creditors.

Specifically, (i) is applied whenever it yields an order-preserving result, which is precisely, when $n\theta c_1 \leq E \leq C - n(1-\theta)c_1$. We apply (ii) when $E \leq n\theta c_1$. Finally, we apply (iii) when $E \geq C - n(1-\theta)c_1$. We call this generalization, the $\theta-$*coalitional procedure*. In the particular case of $\theta = \frac{1}{2}$, the θ-*coalitional procedure* corresponds to the coalitional procedure stated by Aumann and Maschler [1]. To summarize the previous discussion, we can state the following result:

Theorem 1 *For each $\theta \in [0, 1]$, and for each bankruptcy problem, the $\theta-$coalitional procedure and the rule R^θ in the TAL-family yield the same solution to the problem.*

Theorem 1 describes an orderly step-by-step process, which by its very definition must lead to a unique result, therefore characterizing the TAL-family of rules. We now illustrate the process by means of some examples.

Let $n = 3$, $c_i = 100 \cdot i$, $E = 200$, and $\theta = \frac{1}{3}$. At the first stage, the coalition $\{2, 3\}$ forms, and its joint claim is 500. Applying rule (1), when $\theta = \frac{1}{3}$, it yields $\frac{100}{3}$ to creditor 1, and $\frac{500}{3}$ to the coalition. Now, $\frac{500}{3}$ is shared among creditors 2 and 3 as rule (1), providing $\frac{200}{3}$ to creditor 2 and 100 to creditor 3. Therefore, the final result is order preserving, and it coincides with the outcome that the corresponding member of the TAL-family $R^{\frac{1}{3}}$, yields for the problem at stake.

Let $n = 5$, $c_i = 100 \cdot i$ and $E = 510$. If $\theta = \frac{1}{2}$, they show that the $\theta-$coalitional procedure yields the solution $(50, 100, 120, 120, 120)$. Suppose now, that $\theta = \frac{2}{3}$. At the first stage, the coalition $\{2, 3, 4, 5\}$ forms, and its joint claim is 1400. If we apply rule (1), then creditor 1 obtains $100 \cdot \theta = \frac{200}{3}$, and the coalition, $510 - 100 \cdot \theta = \frac{1330}{3}$. If we would split again the coalition, among creditor 2 and the remaining ones, an apply the same rule, then the allocation would not be order preserving, due to the fact that $4\theta \cdot 200 = \frac{1600}{3} > 510 - 100 \cdot \theta = \frac{1330}{3}$. Thus, the $\theta-$coalitional procedure would yield for equal awards. As a result, the proposed allocation would be $\left(\frac{200}{3}, \frac{665}{6}, \frac{665}{6}, \frac{665}{6}, \frac{665}{6}\right)$, which is order preserving. Now, if $\theta = \frac{1}{3}$, it yields $\frac{100}{3}$ to creditor 1, and $510 - \frac{100}{3}$ to the coalition. It is straightforward to see that we can apply (1) until the last step, obtaining the order preserving allocation, $\left(\frac{100}{3}, \frac{200}{3}, \frac{300}{3}, \frac{400}{3}, \frac{530}{3}\right)$. Finally, if $\theta = \frac{1}{4}$, then the procedure gives subsequently θc_i to each of the first three creditors. Now, it remains 360, to be divided among the coalition $\{4, 5\}$. In order to make this division order preserving, then we have to assign equal losses to both creditors. Therefore, the allocation would be $(25, 50, 75, 130, 230)$. In each case, the $\theta-$coalitional procedure yields a division of the amount to divide which is order preserving, and it coincides with the allocation proposed for this problem by the corresponding member of the TAL-family R^θ.

4 Final remarks

We have presented in this note a coalitional procedure that characterizes a one-parameter family of bankruptcy rules encompassing three of the most well known rules. The normative appeal of such family seems unquestionable as it has been shown that it satisfies a wide variety of properties reflecting ethical and operational principles [5]. There is, however, no characterization result for the family as a whole, although independent characterizations results for the three focal members of the family abound in the literature [4], [7], [8]. This note helps to fill that gap.

To conclude, it is worth mentioning the connections between this work and another interesting topic within the literature on bankruptcy problems that refers to coalitional manipulations of bankruptcy rules [2], [3].

If we let agents the possibility of consolidating their claims and be treated as a single creditor or, conversely, we let a particular creditor to divide her claim and be considered as several different creditors then the resulting awards may be altered. It is then worth identifying the precise cases for which creditors will not be able to manipulate the outcomes in their interest via merging or splitting their claims. To do so, let $\tau(N, c, E) = \frac{E}{C}$ stand for the share of the amount to divide in the aggregate claim of a given problem and define $\mathcal{D}^{\delta} = \{(N, c, E) \in \mathcal{D} : \tau(N, c, E) = \delta\}$, for each $\delta \in (0, 1)$. In other words, $\mathcal{D}^{\delta}$ is the set of problems whose ratio between the amount to divide and the aggregate claim is δ. Obviously, $\{\mathcal{D}^{\delta}\}_{\delta\in(0,1)}$ is a partition of $\mathcal{D}$. It is natural to consider this collection of sets in the context of coalitional manipulation, as if a problem belongs to a particular set $\mathcal{D}^{\delta}$ then any resulting problem after merging or splitting agents' claims belongs to the same set.

The following result is obtained:

Proposition 1 *Let $\theta \in [0, 1]$, and $\delta \in (0, 1)$ be given. The following statements hold:*

(a) If $\theta < \delta$, the $\theta-$coalitional procedure is a non manipulable by splitting mechanism, when restricted to bankruptcy problems on $\mathcal{D}^{\delta}$.

(b) If $\theta > \delta$, the $\theta-$coalitional procedure is a non manipulable by merging mechanism, when restricted to bankruptcy problems on $\mathcal{D}^{\delta}$.

(c) If $\theta = \delta$, the $\theta-$coalitional procedure is a non manipulable (by merging or splitting) mechanism, when restricted to bankruptcy problems on $\mathcal{D}^{\delta}$.

It follows from Proposition 1 that if $\theta = 0$ then the corresponding coalitional mechanism is non manipulable by merging for the unrestricted domain of bankruptcy problems. Similarly, if $\theta = 1$ then the corresponding coalitional mechanism is non manipulable by splitting for the unrestricted domain of bankruptcy problems.

Acknowledgements

Thanks are due to Dunia López-Pintado, William Thomson and Antonio Villar for helpful comments and suggestions. Financial support from the Spanish Ministry of Science and Innovation (ECO2008-03883) and from Junta de Andalucía (SEJ-04154) is gratefully acknowledged.

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