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General equilibrium without utility functions: How far to go?*

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Abstract

How far can we go in weakening the assumptions of the general equilibrium model? Existence of equilibrium, structural stability and finiteness of equilibria of regular economies, genericity of regular economies and an index formula for the equilibria of regular economies have been known not to require transitivity and completeness of consumers' preferences. We show in this paper that if consumers' non-ordered preferences satisfy a mild version of convexity already considered in the literature, then the following properties are also satisfied: 1) the smooth manifold structure and the diffeomorphism of the equilibrium manifold with a Euclidean space; 2) the diffeomorphism of the set of no-trade equilibria with a Euclidean space; 3) the openness and genericity of the set of regular equilibria as a subset of the equilibrium manifold; 4) for small trade vectors, the uniqueness, regularity and stability of equilibrium for two version of tatonnement; 5) the pathconnectedness of the sets of stable equilibria.

Keywords: General equilibrium; non-ordered preferences; equilibrium manifold; natural projection; demand functions.
JEL-classification: C62, D11, D51.

1. Introduction

The assumptions that underlie the general equilibrium model have become more and more general through time. Walras assumed that consumers' preferences are represented by separable cardinal utility functions [33]. Pareto showed that ordinal utility

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functions suffice for the formulation of the general equilibrium model with finite numbers of goods and consumers [25]. Ordinal utility functions are equivalent to having ordered preferences or, in other words, preferences that are complete and transitive. Transitivity and completeness have traditionally been associated with the very idea of consumer's rationality. Nevertheless, situations have been identified where these properties fail to be satisfied: see for example [22] and [23]. For a philosophically oriented discussion that attempts at disentangling the notion of consumer's rationality from the transitivity of preferences, see [26].

The relevance of the general equilibrium model can only be improved by dropping transitivity and completeness from the basic assumptions of the model. But what properties remain true in the more general setup?

Existence, for example, has been shown not to require completeness (for economies with a continuum of consumers) by Schmeidler [29], a result extended to non-transitive preferences for economies with finite numbers of goods and consumers by Borglin and Keiding [14], Gale and Mas-Colell [19] and Shafer and Sonnenschein [31] and to infinite numbers of goods and consumers by Yannelis and Prabhakar [34]. But existence is peculiar in the sense that it needs only continuity, not differentiability, whether preferences are ordered or not. Most other properties of the general equilibrium model require suitable forms of differentiability. The standard smoothness assumptions for utility functions consist in conditions on the first and second order derivatives of the utility functions [17]. In the case of non-ordered preferences that cannot be represented by utility functions, it is still possible to define a concept of differentiability. The solution consists in using a vector field that plays for non-ordered preferences the role played by the gradient vector field in the case of smooth utility functions [1, 13, 20].

Al-Najjar identifies in [1] a class of non-ordered smooth preferences such that preference maximization subject to a budget constraint yields an individual demand function that is smooth and satisfies Walras law as well as the weak revealed preference property. This class of non-ordered preferences contains the smooth ordered preferences considered by Debreu [17]. Though the main focus of that paper is the study of a specific class of smooth non-ordered preferences, that paper ends with the statement that regular economies (equilibria being finite and structurally stable for regular economies) are generic. The first rigorous study of that question in the setup of smooth non-ordered preferences is due to Bonnisseau who considers by the same token preferences that are not even convex [13]. Using the equilibrium manifold and natural projection approach of Balasko [6], Bonnisseau proves the genericity of regular economies and an index formula à la Dierker [18], a formula that implies the existence of equilibrium. Bonnisseau's paper shows us that the properties proved by Smale [32] for economies with smooth nonconvex ordered preferences extend to the more general setup of non-ordered preferences.

In the case of ordered preferences representable by smooth utility functions, convexity adds the following economically meaningful properties to the general equilibrium model: 1) the set of no-trade equilibria is diffeomorphic to a Euclidean space [5]; 2) the equilibrium manifold has a smooth manifold structure and is diffeomorphic to a Euclidean space [5, 28]; 3) the set of regular equilibria is an open and dense subset of the equilibrium manifold [10]; 4) for small trade vectors, the uniqueness, regularity and stability of equilibrium for two versions of tatonnement (with exogenous and endogenous adjustment speeds respectively [6, 7, 8, 12]; 5) the pathconnectedness of the set of stable equilibria (for the two versions of tatonnement) provided some endowment parameters can take non-positive values [7, 12]. The question is then how these five properties do fare when transitivity and completeness of preferences are dropped.

We show in this paper that general equilibrium models with non-ordered preferences belonging to the class of preferences identified by Al-Najjar satisfy all these five properties. The key element in our approach is that the equilibrium manifold and its projection map (the natural projection) have simple expressions in terms of individual demand functions. We therefore start this paper with consumers who are just equipped with demand functions. We show that Walras law, differentiability, negative quasidefiniteness of the matrix of substitution effects (the Slutsky matrix) and two reasonable properties about the behavior of individual demands when prices tend to the boundary of the price set suffice for the five properties to be satisfied by the associated general equilibrium model. We continue by showing that these minimal properties of the individual demand functions are satisfied by the demand functions generated by the smooth non-ordered preferences considered by Al-Najjar. In order to facilitate comparisons with the case of ordered preferences (representable by utility functions), this paper ends with the special case of complete preferences represented by Shafer's antisymmetric "utility functions" [30].

In extending the properties of the general equilibrium model from utility maximization to the case of non-ordered preferences, we often refer to already known proofs when the latter make no use of the assumption of utility maximization in order to save precious space.

This paper is organized as follows. Section two lists the properties of the individual demand functions that we use in our study of the general equilibrium model. Section three is devoted to the proof of the main properties of the general equilibrium model with demand functions satisfying the assumptions of section two. Section four deals with the derivation of demand functions from Al-Najjar's class of smooth non-ordered preferences. Section five expresses these properties in terms of Shafer's representation of nontransitive complete preferences by antisymmetric functions. Section six concludes this paper with further remarks and open problems.

2. General equilibrium with demand functions

2.1. Goods, prices, endowments and wealth

Goods

There is a finite number $\ell \geq 2$ of divisible goods. A commodity bundle is an element of the commodity space $\mathbb{R}^\ell$. We assume that consumption can only occur in strictly positive quantities. All consumers have as consumption set the strictly positive orthant $X = \mathbb{R}^\ell_{++}$.

Prices

The price vector $p = (p_1, \dots, p_\ell) \in \mathbb{R}^\ell_{++}$ is normalized by the convention $p_1 + \dots + p_\ell = 1$. The relative interior of the unit simplex S of $\mathbb{R}^\ell$ is the set of price vectors. We will explicitly drop this normalization assumption on a few occasions when it will be neither useful nor appropriate.

Consumer's endowment vectors

There is a finite number $m \geq 2$ of consumers. Consumer i, with $1 \leq i \leq m$, is endowed with the commodity bundle $\omega_i \in \mathbb{R}^\ell$ before the opening of the market. We denote by $\omega = (\omega_i)$ the m-tuple representing the endowments of all the consumers in the economy. In some questions, we may consider negative quantities, in which case the endowment space is the set $(\mathbb{R}^\ell)^m$. In many questions, however, endowments are strictly positive. We denote by $\Omega = X^m$ the set of all strictly positive endowments.

Wealth

The wealth of consumer i given the endowment vector $\omega_i \in \mathbb{R}^\ell$ and the price vector $p \in S$ is equal to $w_i = p \cdot \omega_i$. In what follows, only strictly positive wealths will be considered.

The price-income space

The price-income space $B = S \times \mathbb{R}^m_{++}$ consists of the vector $p \in S$ and the wealth distribution $(w_1, \dots, w_m) \in \mathbb{R}^m_{++}$ throughout the economy.

We denote by $\varphi : S \times \Omega \to B$ the map

$$(p, \omega_1, \dots, \omega_m) \to (p, p \cdot \omega_1, \dots, p \cdot \omega_m)$$

that associates with the price-endowment vector $(p, \omega_1, \dots, \omega_m) \in S \times \Omega$ the price-income vector $(p, p \cdot \omega_1, \dots, p \cdot \omega_m) \in B$.

2.2. Demand functions

Consumer i's demand function is a map $f_i : S \times \mathbb{R}_{++} \to X$ where $f_i(p.w_i)$ represents consumer i's demand given the price vector $p \in S$ and consumer i's wealth $w_i > 0$.

We now define the following properties for the demand functions.

Smoothness (S)

Definition 1. *The demand function $f_i : S \times \mathbb{R}_{++} \to X$ is smooth.*

Differentiability can easily be weakened to first and second order differentiability for example.

Walras law (W)

Definition 2. *Walras law (W) for the demand function $f_i : S \times \mathbb{R}_{++} \to X$ is the identity*

$$p \cdot f_i(p, w_i) = w_i$$

for every $(p, w_i) \in S \times \mathbb{R}_{++}$.

Walras law (W) means that the value of consumer i's demand is equal to the consumer's wealth. It is satisfied whenever the budget constraint $p \cdot x_i \leq w_i$ (where $x_i \in X$ is the consumer's demand) is binding.

Boundedness from below (B)

For $r \in X$, we define the set $J(r) = \{z_i \in X \mid z_i \leq r\}$ coordinatewise. Elements of $J(r)$ are commodity bundles that are bounded from above by $r \in X$.

Definition 3. *The demand function $f_i : S \times \mathbb{R}_{++} \to X$ satisfies boundedness from below (B) if the intersection $\{f_i(p, p \cdot \omega_i) \mid \omega_i \in K_i$ and $p \in S\} \cap J(r)$ is bounded away from 0 for any compact subset K_i of X and $r \in X$.*

Property (B) excludes the possibility that if prices of some goods tend to zero, then demand of some goods tend to zero while demand for the other goods remains bounded from above for a fixed endowment vector.

Properness (P)

Definition 4. *The demand function $f_i : S \times \mathbb{R}_{++} \to X$ is proper (P) if the set $f_i^{-1}(K_i)$ is compact for every compact subset K_i of X.*

Weak revealed preference (WRP)

Definition 5. *The demand function $f_i : S \times \mathbb{R}_{++} \to X$ satisfies the weak revealed preference property (WRP) if:*

$$(p, w_i) \neq (p', w_i') \in S \times \mathbb{R}_{++} \text{ and } p \cdot f_i(p', w_i') \leq w_i \quad \Longrightarrow \quad p' \cdot f_i(p, w_i) > w_i'.$$

Negative quasidefiniteness of the Slutsky matrix (NQD)

The Slutsky matrix $Sf_i(b)$ of the smooth demand function f_i at $(p, w_i) \in S \times \mathbb{R}_{++}$ is the $\ell \times \ell$ matrix with (j, k) coefficient for $1 \leq j, k \leq \ell$ equal to

$$s_{jk}(p, w_i) = \frac{\partial f_i^j(p, w_i)}{\partial p_k} + \frac{\partial f_i^j(p, w_i)}{\partial w_i} \; f_i^k(p, w_i) \cdot$$

Note that $p^T Sf_i(p, w_i) = Sf_i(p, w_i)p = 0$.

Definition 6. *The smooth demand function $f_i : S \times \mathbb{R}_{++} \to X$ satisfies property (NQD) if the restriction of the quadratic form $z \to z^T Sf_i(p, w_i)z$ to the hyperplane $H(p) = \{z \in \mathbb{R}^\ell \mid p^T z = 0\}$ perpendicular to p is negative definite for every $(p, w_i) \in S \times \mathbb{R}_{++}$.*

This is equivalent to the $z^T Sf_i(p, w_i)z < 0$ when $z \neq 0$ is not collinear with the price vector p. Note that (NQD) does not imply nor require the symmetry of the Slutsky matrix.

Negative semiquasidefiniteness of the Slutsky matrix (NSQD)

Definition 7. *The smooth demand function $f_i : S \times \mathbb{R}_{++} \to X$ satisfies property (NSQD) if the Slutsky matrix $Sf_i(p, w_i)$ is negative semidefinite for every $(p, w_i) \in S \times \mathbb{R}_{++}$.*

Some relations between these properties

Proposition 1. *(NQD) implies (NSQD).*

Proof. Obvious. □

Proposition 2. *(NSQD) implies (WRP).*

Proof. See [21]. □

Corollary 1. *(NQD) implies (WRP).*

2.3. The general equilibrium setup: definitions

Consumer i is characterized by the pair (f_i, ω_i) where f_i is a demand function $S \times \mathbb{R}_{++} \to X$ and $\omega_i \in X$ represents consumer i's endowments. We assume that only endowment vectors can be varied. An economy is therefore identified with an m-tuple $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$ representing the endowments of all the consumers in the economy.

Equilibrium and the equilibrium manifold

Definition 8 (Equilibrium)**.** *The pair $(p, \omega) \in S \times \Omega$ is an equilibrium if*

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i. \tag{1}$$

The equilibrium manifold E is the subset of $S \times \Omega$ defined by equation (1)*.*

At this point, the equilibrium manifold E is nothing more than a subset of $S \times \Omega$. That E is indeed a smooth submanifold of $S \times \Omega$ of dimension ℓm is Theorem 3.

For the characterization of the set of tatonnement stable equilibria it is convenient to allow for some coordinates of the endowment vector ω to be negative. This leads us to extend the concept of equilibrium to such economies:

Definition 9. *The pair $(p, \omega) \in S \times (\mathbb{R}^\ell)^m$ is an extended equilibrium if equation* (1) *is defined and satisfied. We denote by $\tilde{E}$ the extended equilibrium manifold.*

The equilibrium equation (1) makes sense whenever $p \cdot \omega_i > 0$ for every i and $\sum_i \omega_i \in X$. We obviously have $E \subset \tilde{E}$.

The set of no-trade equilibria

Definition 10. *The pair $(p, \omega) \in S \times \Omega$ is a no-trade equilibrium if the equality $f_i(p, p \cdot \omega_i) = \omega_i$ is satisfied for $i = 1, 2, \ldots, m$.*

Let T denote the set of no-trade equilibria. Obviously, a no-trade equilibrium is an equilibrium, which implies the inclusion $T \subset E$.

The fibers of the equilibrium manifold

Definition 11. *The fiber $F(b)$ associated with the price-income vector $b = (p, w_1, \ldots, w_m) \in B$ is the set*

$$F(b) = \left\{ (p, \omega) \in S \times \Omega \;\middle|\; \begin{array}{c} p \cdot \omega_i = w_i, \quad i = 1, \ldots, m \\ \sum_i \omega_i = \sum_i f_i(p, w_i) \end{array} \right\}.$$

The fiber $F(b)$ is a dimension $(\ell-1)(m-1)$ relatively open convex polytope in $S \times \Omega$.

The extended fiber $\widetilde{F(b)}$ is defined by the same equations as the fiber $F(b)$ except that the endowment parameter $\omega = (\omega_1, \dots, \omega_m) \in (\mathbb{R}^\ell)^m$ is not restricted anymore to have strictly positive coordinates. The extended fiber $\widetilde{F(b)}$ is parameterized by $\bar{\omega}_i = (\omega_i^1, \dots, \omega_i^{\ell-1})$ for $i = 1, \dots, m-1$ and is therefore linearly isomorphic to the Euclidean space $V = \mathbb{R}^{(\ell-1)(m-1)}$.

The natural projection

Definition 12. *The natural projection $\pi : E \to \Omega$ is the restriction to the equilibrium manifold of the projection map $S \times \Omega \to \Omega$.*

The property that the equilibrium manifold E is a smooth submanifold of $S \times \Omega$ (Theorem 3) implies that the natural projection $\pi : E \to \Omega$ is smooth. It is then possible to consider the critical and regular points of that map since it is differentiable.

Critical and regular equilibria

Definition 13. *The equilibrium $(p, \omega) \in E$ is critical if it is a critical point of the natural projection $\pi : E \to \Omega$. The equilibrium $(p, \omega) \in E$ is regular if it is not critical.*

By definition of a critical point (see e.g., [24]), the equilibrium $(p, \omega) \in E$ is critical if the derivative of π at (p, ω) is not onto. Since E and Ω have the same dimension ℓm, this is equivalent to the derivative of the map $\pi : E \to \Omega$ not being a bijection. Using local coordinates, this derivative can be identified with a matrix. At a critical point, the determinant of that matrix is equal to zero. It will follow from Theorem 1 that this condition is equivalent to $\operatorname{rank} J(p, \omega) \leq \ell - 2$ where $J(p, \omega)$ is the Jacobian matrix of the aggregate excess demand map for the given endowment vector ω and non-normalized price vector p:

$$p \in \mathbb{R}^\ell_{++} \to z(p, \omega) = \sum_i f_i(p, p \cdot \omega_i)) - \sum_i \omega_i \in \mathbb{R}^\ell.$$

The matrix $J(p, \omega)$ satisfies $p^T J(p, \omega) = J(p, \omega) p = 0$.

Singular and regular economies

Definition 14. *The economy ω in Ω is a singular economy if it is a singular value of the natural projection. The economy $\omega \in \Omega$ is regular if it is not singular.*

The economy ω in Ω is singular if there exists an equilibrium price vector $p \in S$ such that the equilibrium (p, ω) is critical.

Remark 1. What makes regular economies interesting and important is that they can have only regular equilibria. In the early phases of the theory of smooth economies, however, the concept of regular economy overshadowed the one of regular equilibrium. The reason comes from Sard's theorem. This very general theorem states that the set of singular values of a smooth map has Lebesgue measure zero [24]. This implies that the set of singular economies has measure zero [16] and therefore that the set of regular economies has full Lebesgue measure. There is no theorem at a comparable level of generality for the set of regular points of a smooth map. The proof that the set of regular equilibria is an open subset with full Lebesgue measure of the equilibrium manifold [10] followed by two decades Debreu's result on the genericity of regular economies. The genericity of regular equilibria is a stronger property than the genericity of regular economies since the former implies the latter. But it is also the more interesting one from an economic perspective, equilibrium being a smooth function of the fundamentals of an economy in neighborhoods of regular equilibria.

3. Properties of the general equilibrium model

3.1. Smoothness (S) and Walras law (W) for all consumers

From now on, we assume that all individual demand functions are smooth (S) and satisfy Walras law (W). In this section, these are the only properties satisfied by demand functions.

Equilibrium allocations as the projection of the no-trade equilibria

Let $(p, \omega) \in E$ be an equilibrium. Let $b = (p, w_1. \dots, w_m) = \varphi(p, \omega) \in B$. The equilibrium allocation associated with the equilibrium $(p, \omega) \in E$ is equal to

$$x = \big(f_1(p, w_1), \dots, f_m(p, w_m)\big) \in \Omega.$$

We then have:

Proposition 3. *The set of equilibrium allocations is the set* $\pi(T)$*.*

Proof. Let x be the equilibrium allocation associated with some $(p, \omega) \in E$. Then, the pair (p, x) is a no-trade equilibrium and $x = \pi(p, x)$ belongs to $\pi(T)$.

Conversely, let $x \in \pi(T)$. There exists a price vector $p \in S$ such that (p, x) is a no-trade equilibrium. Let $b = (p, w_1, \dots, w_m) = \varphi(p, x)$. It follows from the definition of a no-trade equilibrium that $x_i = f_i(p, w_i)$ for $i = 1, \dots, m$. □

On critical and regular equilibria

Let $H(p) = \{z \in \mathbb{R}^\ell \mid p^T z = 0\}$ be the hyperplane of $\mathbb{R}^\ell$ perpendicular to the price vector $p \in \mathbb{R}^\ell_{++}$. Let $\mathbb{R}p = \{\lambda p \mid \lambda \in \mathbb{R}\}$ be the one-dimensional vector subspace of $\mathbb{R}^\ell$ generated by p. We have $\mathbb{R}^\ell = H(p) \oplus \mathbb{R}p$. In addition, the subspace $\mathbb{R}p$ is contained in the kernel of the linear map defined by matrix $J(p, \omega)$ and is therefore invariant by that map. The subspace $H(p)$ is also invariant by the same map. This decomposition of $\mathbb{R}^\ell$ into these two invariant orthogonal subspaces plays a crucial role in understanding the properties of the matrix $J(p, \omega)$ in relationship to its submatrices of order $\ell - 1$.

Theorem 1. *The equilibrium $(p, \omega) \in E$ is critical if and only if* $\operatorname{rank} J(p, \omega) \leq \ell - 2$.

Proof. Let $J_{\ell\ell}$ be the principal submatrix obtained by deleting the last row and column of matrix $J(p, \omega)$. One checks readily that $\det J_{\ell\ell} = 0$ implies that all submatrices of $J(p, \omega)$ of order $\ell - 1$ have their determinant equal to 0.

It follows from [9], Theorem (4.3.1) that the equilibrium (p, ω) is critical if and only if $\det J_{\ell\ell} = 0$.

□

Corollary 2. *The equilibrium $(p, \omega) \in E$ is regular if* $\operatorname{rank} J(p, \omega) = \ell - 1$.

Theorem 2. *The equilibrium $(p, \omega) \in E$ is regular if and only if the restriction of the linear map defined by matrix $J(p, \omega)$ to its invariant subspace $H(p)$ is a bijection.*

Proof. Assume that the restriction to $H(p)$ is not a bijection. Then there exists a vector $z \in H(p)$ such that $J(p, \omega)z = 0$. The kernel of the linear map defined by $J(p, \omega)$ contains the linearly independent vectors p and z and has a dimension at least equal to two. The sum of the dimension of the kernel and the dimension of the range (i.e., the rank of matrix $J(p, \omega)$) being equal to ℓ, the dimension of $\mathbb{R}^\ell$, the rank of $J(p, \omega)$ is therefore less than or equal to $\ell - 2$, which implies that the equilibrium $(p, \omega) \in E$ is critical.

Conversely, if the restriction to $H(p)$ is a bijection, then the image of that map coincides with $H(p)$ and is therefore $\ell - 1$ dimensional. The rank of the map defined by $J(p, \omega)$ being less than or equal to $\ell - 1$ has to be equal to $\ell - 1$, in which case the equilibrium $(p, \omega) \in E$ is regular.

□

Remark 2. It follows from the implicit function theorem that the equilibrium price vector $p \in S$ can be expressed locally as a smooth function of the endowment vector $\omega = (\omega_1, \ldots, \omega_m)$ at a regular equilibrium. See [9], Theorem (4.3.1). This property captures the economic importance of the concept of regular equilibrium.

Local structure of the equilibrium manifold

Theorem 3. *The equilibrium manifold E is a smooth submanifold of $S \times \Omega$ of dimension ℓm.*

Proof. See [9], Theorem (3.1.2). ☐

The local structure of the equilibrium manifold E is the one of a smooth manifold of dimension ℓm. In practice, this implies that every sufficiently small open subset U of the equilibrium manifold E can be parameterized by ℓm real coordinates. The open set U is known in mathematics as a chart and is diffeomorphic to $(\mathbb{R}^{\ell})^{m}$.

The set of no-trade equilibria and its structure

Theorem 4. *The set of no-trade equilibria T is a smooth submanifold of the equilibrium manifold E diffeomorphic to $S \times \mathbb{R}^{m}_{++}$.*

Proof. See [9], Proposition (3.3.2). ☐

Global structure of the equilibrium manifold

In general, a smooth manifold is not diffeomorphic to a Euclidean space. For example, the sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$ where $\|x\|_2^2 = \sum_j (x^j)^2$ denotes the Euclidean norm is a dimension n smooth manifold that is easily seen not to be diffeomorphic to $\mathbb{R}^n$. Therefore, the parameterization of all the points of a smooth manifold that is not diffeomorphic to a Euclidean space requires more than one chart. (Incidentally, a collection of charts that cover the manifold is also known in mathematics as an atlas.) Properties of elements of a smooth manifold that is diffeomorphic to a Euclidean space are therefore much easier to study because their study can be done using one global coordinate system. Examples of such properties are the stability and regularity of equilibria. Incidentally, global properties of the equilibrium manifold have direct economic implications of their own. For example, diffeomorphism with a Euclidean space implies pathconnectedness, which implies that it is always possible to move continuously in the equilibrium manifold from one equilibrium to another. For more details on economic interpretations of global properties of the equilibrium manifold, see [9].

The following property is important:

Theorem 5. *The equilibrium manifold E is diffeomorphic to $\mathbb{R}^{\ell m}$.*

Proof. See [6] and [28]. ☐

The proof of the diffeomorphism property is somewhat intricate in the case of the parameter space $\Omega = X^m$ with strictly positive endowments. However we prove the weaker homeomorphism property below as the proof involves a map that plays an important role in a different context, namely, the proof of Theorem 7 in Section 3.2.

Let $x = (p, \omega) \in E$. Recall that we denote by $b = \varphi(x) = (p, p \cdot \omega_1, \ldots, p \cdot \omega_m)$ the associated price-income vector. We parameterize the equilibrium $x = (p, \omega)$ by the ℓm coordinates $\big(b = \varphi(p, \omega), (\bar{\omega}_1, \ldots, \bar{\omega}_{m-1})\big)$.

Let $\partial F(b)$ be the boundary of the fiber $F(b)$. In the $(\ell - 1)(m - 1)$ dimensional affine space generated by the fiber $F(b)$, we denote by $B\big(f(b); 1\big)$ the open ball centered at the no-trade equilibrium $f(b) = \big(p, f_1(b), \ldots, f_m(b)\big)$ with radius 1.

The map $\theta : E \to B\big(f(b); 1\big)$ is defined as follows. For $x \in E$ let $b = \varphi(x)$. Then, we have $x \in F(b)$. If $x \neq f(b)$, the half-line with origin $f(b)$ containing the point $x \in F(b)$ intersects the boundary $\partial F(b)$ at a unique point that we denote by $j(x)$. We define

$$\theta(x) = \begin{cases} f(b) & \text{for } x = f(b) \\ f(b) + \dfrac{\|x - f(b)\|}{\|j(x) - f(b)\|}\big(j(x) - f(b)\big) & \text{for } x \neq f(b) \end{cases}$$

The map $\theta(b, .)$ is easily seen to be a homeomorphism between the convex set $F(b)$ and the open ball $B\big(f(b); 1\big)$. In fact, this map is a homeomorphism for any open convex set that contains $f(b)$. It is even a diffeomorphism when the boundary of the convex set (here the set $\partial F(b)$) is a smooth manifold. But this is not the case here because $\partial F(b)$ has "corners," i.e., points where $\partial F(b)$ fails to be a smooth manifold. Therefore, the map $\theta(b, .)$ is not a diffeomorphism.

Let now $B(1)$ be the open ball in $V = (\mathbb{R}^{\ell-1})^{m-1}$ centered at 0 and with radius 1. The two balls $B\big(f(b); 1\big)$ and $B(1)$ are diffeomorphic through the map $h(b, .) : B\big(f(b); 1\big) \to B(1)$ where

$$h(b, \bar{\omega}_1, \ldots, \bar{\omega}_{m-1}) = (\bar{\omega}_1 - \bar{f}_1(b), \ldots, \bar{\omega}_{m-1} - \bar{f}_{m-1}(b))$$

We define the map $\psi : E \to B(1)$ by

$$\psi(x) = h\big(\varphi(x), \theta(\varphi(x), \bar{\omega}_1, \ldots, \bar{\omega}_{m-1})\big).$$

We then have:

Proposition 4. *The map $\varphi \times \psi : E \to B \times B(1)$ is a homeomorphism*

Proof. Follows readily from the fact that the restriction of the map θ to the fiber $F(b)$ is a homeomorphism with the open ball $B\big(f(b); 1\big)$ for every $b \in B$. For details, see [6]. □

Remark 3. If there are no sign restrictions on endowments, the (extended) fiber $\widetilde{F(b)}$ is identified to $V = \mathbb{R}^{(\ell-1)(m-1)}$. The diffeomorphism between the extended equilibrium manifold $\tilde{E}$ and $B \times V$ is then obvious [9].

3.2. Boundedness from below (B) and properness (P) for one consumer

We now assume that in addition to smoothness (S) and Walras law (W) for all consumers as in the previous section, at least one individual demand function satisfies boundedness from below (B) and properness (P). The following properties are then satisfied.

Properness of the natural projection

Proposition 5. *The natural projection $\pi : E \to \Omega$ is proper.*

Proof. There is no loss of generality in assuming that consumer 1's demand function satisfies (B) and (P).

Let K be a compact subset of Ω. Let us show that the preimage $\pi^{-1}(K)$ is a compact subset of the equilibrium manifold. The set $\pi^{-1}(K)$ is closed by the continuity of the natural projection.

The map $\omega = (\omega_1, \dots, \omega_m) \to \omega_1 + \cdots + \omega_m$ is continuous. The image of the compact set K by this map is compact in $X = \mathbb{R}^{\ell}_{++}$ and therefore bounded from above by some $r^* \in X$. Any $(p, \omega) \in \pi^{-1}(K)$ satisfies the inequality

$$f_1(p, p \cdot \omega_1) + \cdots + f_m(p, p \cdot \omega_m) \leq r^*.$$

This implies in particular the inequality $f_1(p, p \cdot \omega_1) \leq r^*$ for $(p, \omega) \in \pi^{-1}(K)$. It then follows from (B) that there exists $x_1^* \in X$ such that

$$x_1^* \leq f_1(p, p \cdot \omega_1) \leq r^*.$$

Property (P) then implies that $(p, p \cdot \omega_1)$ is contained in a compact subset of $S \times \mathbb{R}_{++}$ for $(p, \omega) \in \pi^{-1}(K)$. The projection of this compact set on the price set S is a compact set H. The set $\pi^{-1}(K)$ is therefore a closed subset of the set $H \times K$, a set that is compact as the Cartesian product of the two compact sets H and K. This proves the compactness of $\pi^{-1}(K)$.

□

The natural projection as a ramified covering

The following theorem is at the very root of the genericity of regular economies, structural stability and finiteness of their equilibria. These properties require only the properness and smoothness of the natural projection.

Theorem 6. *The natural projection $\pi : E \to \Omega$ is a finite ramified covering of Ω.*

By finite ramified covering, it is meant the following: 1) for any regular economy $\omega \in \mathcal{R}$, where $\mathcal{R} = \Omega \setminus \Sigma$ is the set of regular economies, there exists an open neighborhood $U \subset \mathcal{R}$ of ω such that the preimage $\pi^{-1}(U)$ is a finite union of disjoint open sets diffeomorphic to U (finite covering property); 2) the set of singular values $\Sigma = \Omega \setminus \mathcal{R}$ is closed with measure zero (ramification of E over the set of singular values Σ).

Proof. The set of singular values $\Sigma = \Omega \setminus \mathcal{R}$ has measure zero by the very general Sard's theorem.

Closedness of the set Σ requires the properness of the map $\pi : E \to \Omega$. The set of critical equilibria (the critical points of the natural projection $\pi : E \to \Omega$) is closed as the set of zeros of a continuous map, the determinant of the Jacobian matrix of aggregate excess demand. The image of a closed set by a proper map being closed, the set of singular economies Σ is then closed as the image of the closed set of critical equilibria.

For the existence of an open set $U \subset \mathcal{R}$ such that $\pi^{-1}(U)$ is a finite union of disjoint open sets diffeomorphic to U for any $\omega \in \mathcal{R}$, see for example [24]. See also [9], Section 4.2, where this property is proved within the setup of the natural projection. □

The set of regular economies $\mathcal{R}$ is partitioned into connected components. It follows from the structure of the map $\pi : E \to \Omega$ over the set of regular economies $\mathcal{R}$ that the number of equilibria is locally constant at regular economies and, therefore, constant in each connected component of $\mathcal{R}$. In addition, it follows again from Theorem 6 that equilibria are locally smooth functions of $\omega \in \mathcal{R}$. This implies that the property known as structural stability is satisfied by the equilibria of regular economies.

Degrees of the natural projection

It follows from Theorem 6 that the number of equilibria, i.e., the number of elements of the set $\pi^{-1}(\omega)$, is finite for the regular economy $\omega \in \mathcal{R}$. This number may depend on the choice of $\omega \in \mathcal{R}$ but whether this number is even or odd does not. This is by definition the degree modulo 2 of the map $\pi : E \to \Omega$.

In order to define the topological degree, we need to orient the two manifolds E and Ω. The equilibrium manifold E and the parameter space $\Omega = (\mathbb{R}^{\ell}_{++})^m$ are both diffeomorphic to $\mathbb{R}^{\ell m}$. Let us pick two such diffeomorphisms. These diffeomorphisms can be viewed as defining two (global) coordinate systems for E and Ω respectively. We associate with every regular equilibrium $(p, \omega) \in E$ the numbers $+1$ or -1 depending on whether the sign of the Jacobian determinant of the natural projection π computed with these coordinates is positive or negative. The sum of these $+1$'s and -1's over all the elements of $\pi^{-1}(\omega)$ does not depend on the choice of the regular economy $\omega \in \mathcal{R}$. The value of this sum is by definition the topological degree of the map

$\pi : E \to \Omega$ for the orientations of E and Ω defined by these diffeomorphisms (and the positive orientation of $\mathbb{R}^{\ell m}$). See for example [24].

In addition to being independent of the choice of the regular value $\omega \in \mathcal{R}$, the modulo 2 and the topological degrees depend only on the proper homotopy class of $\pi : E \to \Omega$, not on the map itself.

Theorem 7. *The modulo 2 degree of the natural projection $\pi : E \to \Omega$ is equal to 1. There also exist orientations of the equilibrium manifold E and parameter space Ω such that the topological degree of $\pi : E \to \Omega$ is equal to 1.*

Proof. The theorem is true for demand functions associated with ordered preferences satisfying the standard assumptions of smooth consumer theory: [6] and [9], Theorems (4.6.1) and (4.6.2). The idea of the proof is therefore to exploit the invariance of both degrees of a proper map by proper homotopy. But the problem is that the domain of the natural projection, the equilibrium manifold E, varies with the m-tuple of demand functions $(f_1, \dots, f_m)$.

We bypass this difficulty by exploiting the homeomorphism between the equilibrium manifold E and the Cartesian product $B \times B(1)$ of Proposition 4.

For given demand functions $(f_1, \dots, f_m)$, we define the following map from $B \times B(1)$ into Ω by

$$\psi_{(f_1,\dots,f_m)} = \pi \circ (\varphi \times \psi)^{-1}.$$

That map can be viewed as being the natural projection expressed with the global coordinate system defined by the homeomorphism between $B \times B(1)$ and the equilibrium manifold E. This implies that $\psi_{(f_1,\dots,f_m)}$ is proper for any m-tuple $(f_1, \dots, f_m)$ where at least one demand function satisfies (B) and (P) (in addition to smoothness (S) and Walras law (W) satisfied by all demand functions). We assume without loss of generality that $f_1 : S \times \mathbb{R}_{++} \to X$ satisfies (B) and (P).

Let $f'_i : S \times \mathbb{R}_{++} \to X$ be the demand function of consumer i associated with some ordered preference relation satisfying the standard assumptions for smooth economies. For simplicity's sake, we assume that consumers' preferences are identical and defined by a log-linear utility function. Consumer i's demand $f'_i(b)$ is then equal to

$$f'_i(p, w_i) = w_i \ \left(\frac{a^1}{p_1}, \dots, \frac{a^\ell}{p_\ell}\right)$$

with $p = (p_1, p_2, \dots, p_\ell)$, $a^j > 0$ for $1 \leq j \leq \ell$ and $\sum_j a^j = 1$.

Obviously, these demand functions $f'_i : S \times \mathbb{R}_{++} \to X$ satisfy (S), (W), (B) and (P).

Proper homotopy between $\psi_{(f_1,\dots,f_m)}$ and $\psi_{(f'_1,\dots,f'_m)}$

The issue is to define a continuous map $\Psi : \big(B \times B(1)\big) \times [0, 1]$ that is proper and such that $\Psi(., 0) = \psi_{(f_1,\dots,f_m)}$ and $\Psi(., 1) = \psi_{(f'_1,\dots,f'_m)}$.

Let $F_i(p, w_i, t) = (1-t)f_i(p, w_i) + tf_i'(p, w_i)$ for $(p, w_i) \in S \times \mathbb{R}_{++}$ and $t \in [0, 1]$. We then define the map $\Psi : \big(B \times B(1)\big) \times [0, 1]$ by

$$\Psi(., t) = \psi_{\big((F_1(.,t),\dots,F_m(.,t))\big)}.$$

The only property that is not obvious is properness.

Let K be a compact subset of Ω and let us show that the preimage $\Psi^{-1}(K)$ is compact. It suffices that we show that any sequence (b^n, z^n, t^n) in $\Psi^{-1}(K)$ has a convergent subsequence. Let $(p^n, \omega^n) = (\varphi \times \psi)^{-1}(b^n, z^n)$ where $\varphi \times \psi : E \to B \times B(1)$ is the homeomorphism between $B \times B(1)$ and the equilibrium manifold associated with the m-tuple of demand functions $\big(F_1(., t), \dots, F_m(., t)\big)$ of Proposition 4. In addition, we have $\omega^n \in K$. Therefore, by considering a suitable subsequence, we can assume that t^n converges to t^* and ω^n to ω^*.

It also follows from the compactness of K that there exists $r^* \in X$ such that $\sum_i \omega_i \le r^*$ for $\omega \in K$. Combined with

$$\sum_i F_i(p^n, w_i^n, t^n) = \sum_i \omega_i^n,$$

it follows $\sum_i F_i(p^n, w_i^n, t^n) \le r^*$ and, in particular, $F_1(p^n, w_1^n, t^n) \le r^*$.

This implies the inequality

$$(1-t^n)f_1(p^n, w_1^n) + t^n f_1'(p^n, w_1^n) \le r^*.$$

Case $t^* \neq 0$. We have $t^n f_1'(p^n, w_1^n) \le r^*$ and, therefore, for n large enough, we have

$$f_1'(p^n, w_1^n) \le \frac{1}{2t^*}\ r^*.$$

It then follows from property (B) that $f_1'(p^n, w_1^n)$ is contained in a compact subset of X as being bounded from above and from below. It then follows from property (P) that the sequence (p^n, w_1^n) belongs to a compact subset of $S \times R_{++}$ and, therefore, has a convergent subsequence.

Case $t^* = 0$. It suffices to reproduce the same line of reasoning but with the demand function f_1 instead of f_1'.

$\square$

Remark 4. An index formula à la Dierker [18] now follows readily from the topological degree of the natural projection $\pi : E \to \Omega$ being equal to one.

3.3. Negative quasi and semiquasidefiniteness: (NSQD) for all consumers and (NQD) for one consumer

We now add properties involving the first order derivatives of individual demand functions to the previous properties satisfied by individual demand functions, namely (S) and (W) for all consumers and (B) and (P) for one consumer. From now on, we assume that all consumers satisfy (NSQD) and that at least one consumer satisfies (NQD). Note that the consumer who satisfies (NQD) does not have to be the same one who satisfies (B) and (P).

Regularity of the no-trade equilibria

Theorem 8. *Every no-trade equilibrium $(p, \omega) \in T$ is regular.*

Proof. Let $b = (p, w_1, \ldots, w_m) = (p, p \cdot \omega_1, \ldots, p \cdot \omega_m)$ be the price-income vector associated with the no-trade equilibrium (p, ω). We then have

$$(p, \omega) = \big(p, f_1(p, w_1), \ldots, f_m(p, w_m)\big)$$

and the Jacobian matrix $J(p, \omega)$ of the aggregate excess demand map $p \to z(p, \omega)$ is the sum of the m Slutsky matrices:

$$J(p, \omega) = Sf_1(p, w_1) + \cdots + Sf_m(p, w_m).$$

Matrix $J(p, \omega)$ defines a quadratic form that is the sum of the quadratic forms defined by the individual Slutsky matrices. The restriction of the quadratic form defined by $J(p, \omega)$ to the hyperplane $H(p)$ of $\mathbb{R}^\ell$ that is perpendicular to the price vector p is therefore negative definite. $\square$

Let $\mathfrak{R}$ denote the subset of the equilibrium manifold E consisting of the regular equilibria. Then, Theorem 8 then implies inclusion $T \subset \mathfrak{R}$. It then follows from the pathconnectedness of the set of no-trade equilibria T that T is contained in one connected component of the set of regular equilibria $\mathfrak{R}$.

Dynamic stability of the no-trade equilibria

Many questions of dynamic stability are more easily handled without price normalization. Therefore, we drop in this section the simplex normalization $\sum_j p_j = 1$.

Dynamic stability refers to the adjustment of the price vector when the latter is not an equilibrium price vector. By dynamic, it is meant that the price adjustment is governed by some differential equation that relates the derivative of the price vector to aggregate excess demand.

Exogenously fixed adjustment speeds: Walras tatonnement

Walras tatonnement assumes that adjustment speeds are exogenously given and that one commodity is the numeraire, for example $p_\ell = 1$. By a suitable choice of commodity units, the adjustment speeds of all the non-numeraire goods can be normalized to become equal to 1

This yields for Walras tatonnement the differential equation

$$\dot{\bar{p}}(t) = \bar{z}(p(t), \omega)$$

where $\bar{p}(t) = (p_1(t), \dots, p_{\ell-1}(t))$ and $\bar{z} = (z^1, \dots, z^{\ell-1})$. (Recall that $p_\ell(t) = 1$.) Stability can be investigated by looking at the eigenvalues of matrix $J_{\ell\ell}(p, \omega)$. For equilibria such that matrix $J_{\ell\ell}(p, \omega)$ has no eigenvalue with zero real parts (tatonnement hyperbolic equilibria), stability is equivalent to all eigenvalues having strictly negative real parts. This is easily seen to be equivalent to all non-zero eigenvalues of $J(p, \omega)$ having strictly negative real parts and one eigenvalue only being equal to zero.

Endogenous variable adjustment speeds

Some more notation is needed. Let $r = \omega_1 + \cdots + \omega_m$ denote the total resources in the economy defined by $\omega = (\omega_1, \dots, \omega_m)$. The $\square$ product in $\mathbb{R}^\ell$ means coordinatewise multiplication, i.e, $p \square x = (p_1 x^1, \dots, p_\ell x^\ell)$.

In some cases [12], the structure of the exchange process itself is sufficient to determine endogenously the adjustment speeds. This is certainly more satisfactory than when adjustment speeds are arbitrarily taken.

The adjustment process in [12] is governed by the differential equation

$$r \square \dot{p}(t) = z(p(t), \omega) \square p(t). \tag{2}$$

Let (p, ω) be an equilibrium. We define the diagonal $\ell \times \ell$ matrix Π by its jth diagonal coefficient that is equal to p_j / r^j.

Linearization of the differential equation (2) yields the matrix $J(p, \omega)\Pi$. The equilibrium (p, ω) is stable if all the non zero eigenvalues of $J(p, \omega)$ have negative real parts. For details, see [12].

Theorem 9. *Every no-trade equilibrium is stable for the two tatonnement processes.*

Proof. The proofs given in [3] and [6] for Walras tatonnement and [12] for tatonnement with endogenous adjustment speeds have to be adapted because matrix $J(p, \omega)$ is not symmetric anymore.

With non symmetric matrices, the eigenvalues are not necessarily real but this has little impact here. What counts is the sign of the real parts of the eigenvalues. The restriction of the quadratic form $z \to z^T J(p, \omega) z$ to the hyperplane $H(p)$ (perpendicular to the price vector p) being negative quasi-definite, stability for Walras tatonnement then follows from [27], p. 438. (See also [4], Lemma 2, p. 448.) Stability for tatonnement with endogenous adjustment speeds results from what Arrow and McManus call the D-stability of quasi-definite negative matrices: [4], Theorem 1, p. 449. $\square$

The set of regular equilibria

Theorem 10. *The set of regular equilibria $\mathfrak{R}$ is an open subset of the equilibrium manifold E with full Lebesgue measure.*

Proof. Let C denote the set of critical equilibria, the complement in E of the set of regular equilibria $\mathfrak{R}$. The theorem is then equivalent to showing that C is closed with measure zero. The equilibrium $(p, \omega) \in E$ is critical if the determinant of $J_{\ell\ell}(p, \omega)$ is equal to 0. Closedness then follows from the continuity of the map $(p, \omega) \to J_{\ell\ell}(p, \omega)$ and the continuity of the determinant function.

The key argument is to observe that the map $(p, \omega) \to \det J_{\ell\ell}(p, \omega)$ restricted to the fiber $F(b)$ is polynomial. Therefore, the set of zeros of that map is a semialgebraic set and is therefore stratified into a collection of smooth manifolds. None of these manifolds can have full dimension, otherwise the polynomial function would be equal to zero on a nonempty open subset of the fiber and would therefore be equal to zero all over the fiber, which contradicts the fact that no-trade equilibria are regular. For details, see [10].

□

Sets of stable equilibria (for the two tatonnement processes)

The next theorem is, as the previous one, a consequence of the fibered structure of the equilibrium manifold. Let us denote by E_{fs} and E_{vs} the sets of extended equilibria (i.e., some components of the endowment vector can be negative) that are stable for the fixed adjustment speed and variable adjustment speed tatonnements respectively.

Theorem 11. *The sets E_{fs} and E_{vs} of extended stable equilibria are pathconnected.*

Proof. We already know that we have the inclusions $T \subset E_{fs}$ and $T \subset E_{vs}$ and that T is pathconnected. It therefore suffices that the intersection of each extended fiber $\widetilde{F(b)}$ with E_{fs} and E_{vs} are pathconnected. In both cases, this becomes a problem in linear algebra and amounts to proving that some sets of matrices with eigenvalues with real negative parts are pathconnected. For details, see [7] for Walras tatonnement and [12] for tatonnement with variable adjustment speeds.

□

Uniqueness of equilibrium at equilibrium allocations

Theorem 12. *Equilibrium is unique for any $\omega \in \pi(T)$.*

Proof. Let $\omega \in \pi(T)$. There exists some $p \in S$ such that $(p, \omega) \in T$. It follows from the definition of no-trade equilibria that we have $\omega_i = f_i(p, p \cdot \omega_i)$ for $i = 1, \ldots, m$.

The proof now proceeds by contradiction. Assume that there exists $(p', \omega) \in E$ with $p' \neq p \in S$. Let $x'_i = f_i(p', p' \cdot \omega_i)$ for $i = 1, \ldots, m$. It follows from (W) that we have

$$p' \cdot (\omega_i - x'_i) = 0.$$

By Proposition 2, we can apply (WRP) which yields the strict inequality

$$p \cdot (x'_i - \omega_i) > 0.$$

It suffices to add up these strict inequalities for i varying from 1 to m to get a contradiction with the equilibrium condition $\sum_i x'_i = \sum_i \omega_i$.

□

Corollary 3. *The set of equilibrium allocations $\pi(T)$ is diffeomorphic to $B = S \times \mathbb{R}^m_{++}$.*

Proof. Obvious. □

Regularity of equilibrium allocations

Theorem 13. *Every equilibrium allocation is regular, i.e., $\pi(T) \subset \mathcal{R}$.*

Proof. Let $x \in \pi(T)$ be an equilibrium allocation. The set $\pi^{-1}(x)$ contains a unique element $(p, x) \in E$ which is a no-trade equilibrium. Therefore (p, x) is regular, which implies that x itself is regular. □

Theorem 14. *The set of equilibrium allocations is contained in a single pathconnected component of the set of regular economies $\mathcal{R}$. There is uniqueness of equilibrium for all economies in that component.*

Proof. The pathconnectedness of $\pi(T)$ which is a subset of $\mathcal{R}$ implies that $\pi(T)$ is contained in just one pathconnected component of $\mathcal{R}$. The uniqueness of equilibrium for $\omega \in \pi(T)$ implies the uniqueness of equilibrium for all economies ω in that component.

□

Remark 5. Theorems 13 and 14 give us an easier way to compute the degrees of the natural projection $\pi : E \to \Omega$ than Theorem 7. It suffices to pick $\omega \in \pi(T)$ and to observe that $\pi^{-1}(\omega)$ just contains one element. Of course, this simpler proof requires stronger assumptions.

4. From non-ordered preferences to individual demand functions

4.1. The vector field representation of preferences

Following an idea that goes as far back as Antonelli [2] and Pareto [25], preference relations can be represented by vector fields defined on the consumer's consumption space. For smooth ordered preferences represented by a smooth utility function, this vector field is typically the utility gradient vector field or some vector field collinear with the latter. Katzner and Al-Najjar have shown that similar vector fields can be defined for a large class of non-ordered preferences [1, 20] and that such vector fields convey all the information associated with these preferences. Bonnisseau goes one step further by taking these vector fields as the primitive concept for consumers' preferences [13]. This is the approach that we follow in this Section.

We therefore assume in this section that consumer i is equipped with a preference relation $\succeq_i$ that is not necessarily transitive nor complete. The preference relation is represented by a map $q_i : X \to \mathbb{R}^\ell$. We assume that the map $q_i : X \to \mathbb{R}^\ell$ satisfies the following properties:

(A.1) The map $q_i : X \to \mathbb{R}^\ell$ is smooth;

(A.2) $q_i(x) \in X$ for every $x \in X$;

(A.3) $z^T Dq_i(x)z < 0$ for all $x \in X$ and $z \in H(q_i(x)) \setminus \{0\}$, the hyperplane of $\mathbb{R}^\ell$ perpendicular to $q_i(x)$;

(A.4) For any sequence $x^{(n)} \in X$ converging to some $x \in \partial X \setminus \{0\}$ and any limit point q of the sequence $q_i(x^n)$, we have $q \cdot x = 0$.

Property (A.1) expresses smoothness and Property (A.2) monotonicity. Property (A.3) can be viewed as a local convexity property of consumer i's preference relation. Property (A.3) implies that consumer's demand has at most one solution. Property (A.4) implies that consumer's demand is well defined for any given price-income pair $(p, w_i) \in S \times \mathbb{R}_{++}$. Property (A.4) is a standard boundary condition and is obviously related with Properties (B) and (P) of individual demand functions considered in Section 2.

Conditions (A.1) to (A.4) are taken from Al-Najjar [1]. The preference relation $\succeq_i$ on X belongs to Al-Najjar's class $\mathcal{P}^1$ of smooth non-ordered preferences.

4.2. Consumer's demand function

Theorem 15. *Given the price-income pair $(p, w_i) \in S \times \mathbb{R}_{++}$, there exists a unique element $f_i(p, w_i) \in X$ that maximizes consumer i preferences given the budget con-*

straint $p \cdot x_i \leq w_i$. The demand function $f_i : S \times \mathbb{R}_{++} \to X$ is smooth (S), satisfies Walras law (W), negative quasi-definiteness (NQD), boundedness from below (B) and properness (P).

Proof. The existence of a demand function $f_i : S \times \mathbb{R}_{++} \to X$ follows from Proposition 3.3 in [1]. Smoothness with respect to (p, w_i) is proved in [1] by a standard argument, namely the implicit function theorem applied to the vector field equivalent of the first order conditions satisfied by $f_i(p, w_i)$. Negative quasi-definiteness (NQD) follows directly from Proposition 4.1 in [1].

In order to prove boundedness from below (B), let K_i be some compact subset of X and $r \in X$. We have to show that the set

$$\{f_i(p, p \cdot \omega_i) \mid p \in S \text{ and } \omega_i \in K_i\} \cap J(r)$$

is bounded from below by an element of X. The proof proceeds by contradiction. If (B) is not satisfied, then there exists a sequence $x^{(k)} = f_i(p^{(k)}, p^{(k)} \cdot \omega_i^{(k)})$ that converges to some $x \in \partial X$. Since the sequence $\omega_i^{(k)}$ belongs to the compact set K_i, there is no loss of generality in assuming that this sequence is converging to $\omega_i \in K_i$. The closed price simplex $\bar{S}$ being compact, we can also assume that the sequence $p^{(k)}$ converges to some vector $p \in \bar{S} = S \cup \partial S$. There is nothing to prove for $p \in S$. Assume $p \in \partial S$.

We have $p^{(k)} \cdot x^{(k)} = p^{(k)} \cdot \omega_i^{(k)}$ by Walras law (W). At the limit, it comes $p \cdot x = p \cdot \omega_i > 0$ since $\omega_i \in K_i \subset X$. This strict inequality implies that x is different from 0. Therefore, the inequality $p \cdot x > 0$ also contradicts (A4).

Properness (P) follows from a more general property that is interesting for its own sake, namely that $f_i : S \times \mathbb{R}_{++} \to X$ is a diffeomorphism: [1], Lemma B.2.

□

Non symmetric Slutsky matrices

At variance with complete and transitive preferences, the Slutsky matrix associated with non-transitive preferences is not necessarily symmetric.

Suppose that $\ell = 3$ and let the vector fields $q_i : X \to \mathbb{R}^3$ be defined by

$$q_i(x_i) \; = \; \begin{pmatrix} \sqrt{\dfrac{x_i^2}{x_i^1}} \left(\dfrac{1}{x_i^1} + \dfrac{1}{x_i^2} \right) \\ \sqrt{\dfrac{x_i^1}{x_i^2}} \left(\dfrac{1}{x_i^1} + \dfrac{1}{x_i^2} \right) \\ \dfrac{2}{x_i^3} \end{pmatrix}.$$

It is easy to see that this vector field satisfies Properties (A.1) to (A.4).

The associated demand function is

$$f_i(p, w_i) = \begin{pmatrix} \frac{1}{2} \cdot \frac{\sqrt{p_1/p_2} + \sqrt{p_2/p_1}}{1 + \sqrt{p_1/p_2} + \sqrt{p_2/p_1}} \cdot \frac{w_i}{p_1} \\ \frac{1}{2} \cdot \frac{\sqrt{p_1/p_2} + \sqrt{p_2/p_1}}{1 + \sqrt{p_1/p_2} + \sqrt{p_2/p_1}} \cdot \frac{w_i}{p_2} \\ \frac{1}{1 + \sqrt{p_1/p_2} + \sqrt{p_2/p_1}} \cdot \frac{w_i}{p_3} \end{pmatrix}.$$

For $p = (p_1, p_2, p_3)$, where $p_2 \neq p_1$ and $w_i > 0$, an easy computation shows that the associated Slutsky matrix is not symmetric. (Incidentally, demand functions with non symmetric Slutsky matrices do not satisfy the strong axiom of revealed preferences [21].)

5. Complete non-transitive preferences

5.1. From non-complete to complete preferences

In this section, we relate the assumptions of the previous section on non-transitive and not necessarily complete preferences to Shafer's characterization of non-transitive complete preference [30].

Non-complete preferences can always be extended into complete preferences without modifying the consumer's demand function. It suffices to make equivalent non comparable consumption bundles as follows. Define the relation $x_i \tilde{\succeq}_i y_i$ if either $x_i \succeq_i y_i$ or, $x_i \not\succeq_i y_i$ and $y_i \not\succeq_i x_i$. The relation $\tilde{\succeq}_i$ is obviously complete. In addition, x_i maximizes $\tilde{\succeq}_i$ for the price-income vector $(p, w_i) \in S \times \mathbb{R}_{++}$ if x_i maximizes the preference relation $\succeq_i$ for the same $(p, w_i) \in S \times \mathbb{R}_{++}$ and conversely.

5.2. Nontransitive complete preferences

Let now $\succeq_i$ be a complete non-transitive relation. This relation is said to be strictly convex if the set $\succeq_i (x_i) = \{y_i \in X \mid y_i \succeq_i x_i\}$ of commodity bundles preferred to x_i is strictly convex for all $x_i \in X$. The relation $\succeq_i$ is said to be continuous if the two sets $\{y_i \in X \mid y_i \succeq_i x_i\}$ and $\{y_i \in X \mid x_i \succeq_i y_i\}$ are closed for every $x_i \in X$. We also denote by $\succ_i$ the strict preference relation associated with $\succeq_i$: $x_i \succ_i y_i$ is equivalent to the combination of $x_i \succeq_i y_i$ and $y_i \not\succeq_i x_i$.

Shafer's antisymmetric representation

By Theorem 1 in [30], there exists a function $k_i : X \times X \to \mathbb{R}$ such that $k_i(x_i, y_i) > 0$ is equivalent to $x_i \succ_i y_i$ and $k_i(x_i, y_i) < 0$ to $y_i \succ_i x_i$. In addition, the function is antisymmetric in x_i and y_i, i.e., $k_i(y_i, x_i) = -k_i(x_i, y_i)$.

We define the following properties for the function $k_i(x_i, y_i)$.

(B.1) $k_i : X \times X \to \mathbb{R}$ is smooth;

(B.2) $D_1 k_i(x_i, x_i) \in X$ for all $x_i \in X$;

(B.3) For all $(x_i, y_i) \in X \times X$, the inequality

$$z^T D_{11}^2 k_i(x_i, y_i) z < 0$$

is satisfied for $z \in \mathbb{R}^\ell \setminus \{0\}$ such that $z^T D_1 k_i(x_i, y_i) = 0$.

(B.4) The set $\{y_i \in X \mid k_i(y_i, x_i) \geq 0\}$ is closed in $\mathbb{R}^\ell$ for all $x_i \in X$.

These properties are obvious generalizations of similar properties of smooth utility functions. See for example [17].

The associated vector field

Let $Q_i : X \times X \to \mathbb{R}^\ell$ be the map $Q_i(x_i, y_i) = D_1 k(x_i, y_i)$. Define the vector field $q_i : X \to \mathbb{R}^\ell$ by

$$q_i(x_i) = Q_i(x_i, x_i).$$

Proposition 6. *The preference relation $\succeq_i$ defined by the function $k_i(x_i, y_i)$ satisfying Properties (B.1) to (B.4) is represented by the vector field $q_i : X \to \mathbb{R}^\ell$.*

Proof. Follows from the observation that $x_i \in X$ maximizes $\succeq_i$ on the budget set $p \cdot x_i \leq w_i$ if and only if $D_1 k_i(x_i, x_i)$ is collinear with $p \in S$. $\square$

We now relate the properties of the vector field $q_i(x_i)$ to those of the antisymmetric function à la Shafer $k_i(x_i, y_i)$.

Theorem 16. *The vector field $q_i(x_i)$ on X satisfies Properties (A.1) to (A.4) of section 4.1 if the antisymmetric function $k_i(x_i, y_i)$ satisfies Properties (B.1) to (B.4).*

Proof. It is obvious that (B.1) and (B.2) imply (A.1) and (A.2) respectively.

By the chain rule, we have

$$D_1 q_i(x_i) = D_{11}^2 k_i(x_i, x_i) + D_{12}^2 k_i(x_i, x_i).$$

Let us show that we have

$$z^T D_{12}^2 k_i(x_i, x_i) z = 0$$

for any $z \in \mathbb{R}^\ell$.

By the antisymmetry $k_i(x_i, y_i)+k_i(y_i, x_i) = 0$, we have $D^2_{21}k_i(x_i, y_i)+D^2_{12}k_i(y_i, x_i) = 0$, hence

$$z^T D^2_{21}k_i(x_i, y_i)z = -z^T D^2_{12}k_i(y_i, x_i)z.$$

The symmetry of the Hessian matrix of second order derivatives implies the equality

$$D^2_{12}k_i(x_i, y_i) = D^2_{21}k_i(x_i, y_i)^T,$$

which implies

$$z^T D^2_{12}k_i(x_i, y_i)z = z^T \left(D^2_{21}k_i(x_i, y_i)\right)^T z,$$

hence

$$z^T D^2_{12}k_i(x_i, y_i)z = -z^T D^2_{12}k_i(y_i, x_i)z,$$

and

$$z^T D^2_{12}k_i(x_i, x_i)z = 0.$$

This proves that (B.3) implies (A.3).

Let now $x_i^n \in X$ be a sequence that converges to $x_i \in \partial X$. Let $q^n = q_i(x_i^n)$ and let q be a limit point of that sequence. Let us show that we have $q \cdot x_i = 0$. Pick $\omega_i \in X$ arbitrarily. By (B.4), the set $\{y_i \in X \mid k_i(y_i, \omega_i) \geq 0\}$ is closed in $\mathbb{R}^\ell$. For n large enough, x_i^n does not belong to that set. Otherwise, the limit $x_i \in \partial X$ of the sequence x_i^n would belong to that set and therefore to X, a contradiction. Therefore, there exists an integer N such that, for $n \geq N$, $k_i(x_i^n, \omega_i)$ is < 0. The budget constraint associated with q^n and x_i^n cannot be satisfied by $\omega_i \in X$, i.e., $q^n \cdot x_i^n < q^n \cdot \omega_i$. Going to the limit yields the inequality $q \cdot x_i \leq q \cdot \omega_i$. Since ω_i is arbitrary in X, this implies the equality $q \cdot x_i = 0$, which is (A.4).

□

Remark 6. Following Shafer, the complete non-transitive preference relation $\succeq_i$ can be represented by the price-income dependent "utility function" $u_i(x_i, p) = k_i(x_i, f_i(p.p \cdot x_i))$, a terminology justified by the property that the demand $f_i(p, w_i)$ of consumer i given $(p, w_i) \in S \times \mathbb{R}_{++}$ maximizes $u_i(x_i, p)$ subject to the budget constraint $p \cdot x_i = w_i$ [30].

Note that the demand functions $f_i : S \times \mathbb{R}_{++} \to X$ associated with general price-dependent utility functions $u_i(x_i, p)$ (or preferences) do not satisfy (NQD) nor even (WRP). In that more general setup, the general equilibrium model still retains the smooth manifold structure of the equilibrium manifold, the diffeomorphism of the set of no-trade equilibria with a Euclidean space and its corollary, the diffeomorphism of the equilibrium manifold with a Euclidean space [11]. The other properties, namely, the openness and genericity of the set of regular equilibria as a subset of the equilibrium manifold, the uniqueness, regularity and stability of equilibrium (for the two dynamics considered in this paper) for small trade vectors, and the pathconnectedness of the set of stable equilibria (again for the two price adjustment dynamics) are then lost.

6. Conclusion

Ordered preferences are not necessary for the standard properties of the general equilibrium model to be satisfied. Preferences can be weakened to account for incompleteness and nontransitivity. But we have seen that the properties of the demand functions derived from the maximization of such non-ordered preferences are still unnecessarily strong from the perspective of the general equilibrium model. If preference maximization can be thought of as a definition of consumer's rationality, then the properties of the general equilibrium are robust to a significant dose of irrationality. It would certainly be interesting to get a better understanding of this gray area.

Preliminary research suggests that the results on the equilibrium manifold extend to the case of fixed total endowments. However, the crucial Theorem 4 on the global structure of the set of no-trade equilibria requires a completely new proof, the currently known proofs working only for demand functions derived from utility maximization.

Last, the list of properties of the general equilibrium model that we have selected reflects our views of their importance. This list, however, is not exhaustive. For example, we haven't included the asymptotic behavior of the size of economies with more than n equilibria when n tends to infinity, a property that is nevertheless a consequence of the ramified covering property of the natural projection and is therefore satisfied under the assumptions of Section 3.2. More generally, quite a few properties are known for more specialized versions of the general equilibrium model like the growth and the sunspot models or the models of international trade. It would certainly be interesting to know how the properties of these models fare under the more general setting of non-ordered preferences considered in this paper.

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