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Luciano I. de Castro
Álvaro Riascos

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Carrera 1 No. 18 A – 10, Bloque C.
Bogotá, D. C., Colombia
Teléfonos: 3394949- 3394999, extensiones 2400, 2049, 2474
infcede@uniandes.edu.co
http://economia.uniandes.edu.co

Ediciones Uniandes
Carrera 1 No. 19 – 27, edificio Aulas 6, A. A. 4976
Bogotá, D. C., Colombia
Teléfonos: 3394949- 3394999, extensión 2133, Fax: extensión 2158
infeduni@uniandes.edu.co
http://ediciones.uniandes.edu.co/

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Caracterización de las Estrategias Óptimas en Subastas Multiunidades y Aplicaciones

Luciano I. de Castro
University of Illinois at Urbana-Champaign

Alvaro Riascos
Universidad de los Andes

Resumen

Este artículo reporta condiciones de primer orden para una clase muy general de subastas unitarias o multi-unidades. Bajo condiciones débiles caracterizamos el comportamiento óptimo de los agentes y generalizamos resultados estándar en la literatura. Como una aplicación obtenemos condiciones suficientes para revelación de la verdad, estrategias óptimas monótonas y algunos resultados de identificación para subastas multi-unidades.

Clasificación **JEL**: C62, C72, D44, D82.

Palabras Clave: Subatas multiunidades, condiciones de primer orden, revelación de la verdad, identificación, respuesta óptima monótona.

Characterization of Bidding Behavior in Multi-Unit Auctions and Applications[‡]

Luciano I. de Castro[‡] Alvaro Riascos[§]

November 2, 2007

Abstract

This paper provides first order conditions for a very general class of single or multi-unit auctions. Under mild conditions we characterize bidding-behavior and generalize previous standard results in the literature. As an application we obtain sufficient conditions for truth-telling, monotonic best reply strategies and identification results for multi-unit auctions.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: Multi-unit auctions, first order conditions, truth-telling, identification, monotonic best-reply.

1 Introduction

Many experimental and empirical work suggest that the participants of auctions do (or at least may) not follow their equilibrium strategies.¹ Although there is a considerable debate about this point, it highlights the assumption that equilibrium behavior might be too strong. An alternative approach is to assume only that

*This is an extended and improved version of the second chapter of de Castro (2004).

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‡University of Illinois at Urbana-Champaign 185 Wohlers Hall, 1206 S 6th. St. Champaign, IL

§Corresponding Author: Universidad de los Andes, Facultad de Economía, Universidad de los Andes; Carrera 1 No 18A-70, Bloque C, Bogota D.C, Colombia. Telephone: 57+2+3394949, Fax: 57+2+3324492. E-mail address: ariascos@uniandes.edu.co

¹For a survey of experimental works, see Kagel (1995) and for the empirical literature on auction data, see Laffont (1997).

the players follow rationalizable strategies, instead of equilibrium strategies. Pursuing this idea, Battigalli and Siniscalchi (2003) show that some empirical and experimental findings can be explained. Nevertheless, they still assume what Harsanyi (1967-8) calls consistency of beliefs, that is, the subjective probability that players attribute to the distribution of signals of the opponents is just a conditional distribution and the conditional distribution of all players comes from the same prior distribution.² This is almost always assumed in game theory and one may think that nothing can be said without this basic assumption. To the contrary, we show that if we adhere to the even weaker assumption that bidders are rational, we can still characterize their strategic behavior.

Our model encompasses a very general class of seal-bid auction models. We allow for interdependent values, asymmetric valuations, any attitude towards risk, non-monotonic valuations, non-separable transfers, dependent signals of any dimension (universal type spaces), unitary or multiple unit demands auctions with just sellers or buyers or both. Under general conditions we prove what we call the basic principle of bidding. This formalizes in a general setup an intuition that auction theorists already have. That is “a rational bidder bids in order to equalize the marginal benefit of bidding (the utility that she obtains in case of winning) to the marginal cost of bidding”.

In one sense, it is obvious that in smooth optimization problems, at the optimum the marginal benefit (derivative of the objective function) equals to the marginal cost (shadow price) of the constraints. Nevertheless, this is not exactly the case for auctions, where the marginal cost does not come from a constraint. This is also different from the classic firm’s problem: marginal revenue equals to marginal cost, because we do not need to assume separability of revenues and costs. In auctions, the marginal costs and benefits come from another source. The basic trade-off that a bidder faces is that a higher bid, although it increases the probability of winning, it may also decrease the payoff in case of winning. Using the Leibniz rule, to differentiate an integral that depends on the variable both in the region of integration and in the integrand, we obtain two terms. These two terms can be interpreted as marginal benefit and marginal cost.

Although the prove is reminiscent of Leibniz rule in differential calculus, we rely on the differential theory of measures. When we introduce additional assumptions, i.e., continuously differentiability of payoffs with respect to bids, we provide first order conditions that generalize those obtained by Milgrom and Weber (1982) for first- and second-price auctions, Krishna and Morgan (1997) for the all-pay auction and war of attrition, and Williams (1991) for buyers’-bids double auctions. When one introduces the additional hypotheses of risk neutrality, symmetry and monotonicity of the utility function, the characterization provided reduces to

²This is also called common prior assumption.

the ones on those papers. In addition, we provide first order conditions for the multi-unit discriminatory, uniform and Vickrey auction.

The payoff characterization lemma, which is the main result of this paper, and which is valid in the most general setting, interdependent values, asymmetric valuations, any attitude towards risk, non-monotonic valuations or separability of transfers, dependent signals of any dimension and unitary or multiple unit demands, opens the way to a general approach to equilibrium existence for general auction models like in Araujo and De Castro (2005). It can also provide insights for empirical and experimental studies, since every bid (even the initial or the apparently inconsistent ones in a repeated game) bears valuable information about the players' beliefs and the first order conditions is a first step towards characterizing rational behavior in general auctions. Also, as the recent literature on econometric identification of auction models has pointed out, characterizing best reply bidding strategies allows for identification in many standard auction formats.³ Along this approach, our result provides ground for general econometric identification of multidimensional auction models. We present some of these results in subsection 4.3. We also use our results to give a simple characterization of truth-telling in multi-unit auctions. After our results, the proof that Vickrey auction is truth-telling is immediate.

The paper is organized as follows. Section 2 presents the model and notation. Section 3 contains the main results and some examples of direct applications. Section 4 uses the results to obtain a proof of the truth-telling property of Vickrey auctions. We also prove a monotonic-best reply result that generalizes for multiunits auctions a result of Araujo and de Castro (2007), which is the key result for their proof of equilibrium existence in single-object auctions. In this section we also report results on identification in multiunit auctions, some of which are new, to the best of our knowledge. Section 5 is the appendix, contains some proofs.

2 The Model

2.1 Players and Information

There are N strategic players.⁴ we denote by $\mathcal{N} = \{1, \dots, N\}$ the set of strategic players. Player $i \in \mathcal{N}$ receives a signal (i.e., private information), $t_i \in T_i$ where T_i is the information set of player i . We denote by $t = (t_1, t_2, \dots, t_N) = (t_i, t_{-i})$ the vector of all players' information, where $t_{-i} = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_N)$, as

³See Athey and Haile (2005) for a survey on the main issues regarding the econometric identification of auction models.

⁴Our model is inspired in auction games, although it can encompass a general class of discontinuous games. For convenience and easy understanding, we will use the terminology of auction theory, such as "bidding functions" and "bids" for strategies and actions, respectively.

usual. Let \mathfrak{S}_i be a σ -field of subsets of T_i and define $T \equiv \prod_{i=1}^N T_i$ and the product σ -field over T , $\mathfrak{S} = \prod_{i=1}^N \mathfrak{S}_i$. We assume that there is a probability space (T, \mathfrak{S}, τ) . Define the probability spaces, $(T_i, \mathfrak{S}_i, \tau_i)$, where each τ_i is the marginal probability of τ (i.e., $\tau_i(A) = \tau\{t \in T : t_i \in A\}$). For notational simplicity we assume that τ is the same for every agent but, nothing that follows depends on that. That is, our results do not require the common prior assumption. If g is a function of t_{-i} , we denote the expectation of g with respect to τ_{-i} , given t_i , by $E[g|t_i]$.

Notice that individual signals may be dependent and of arbitrary dimension. We allow for the existence of an uninformed and non-strategic player, named 0. This is the seller in traditional auctions. For double auctions, there is no such player. We denote by $\mathcal{N}_0 = \{0, 1, \dots, N\}$ the set of all players (strategic and non-strategic). Also, \mathcal{N}_{0-i} denotes the set of strategic and non-strategic players except for agent i , and similarly for \mathcal{N}_{-i} .

2.2 Objects and Bidding

There are K identical indivisible objects. Each player $i \in \mathcal{N}_0$ comes to the auction with $e_i \in \{0, 1, 2, \dots\}$ units of the same object, and $\sum_{i=0}^N e_i = K$. After receiving its signal, a strategic player submits a sealed proposal, that is, a bid (or offer) that is a vector of real numbers, $b_i \in B \subseteq \mathbb{R}^K$ where B denotes the set of valid bids, that is, $B = \{b \in \mathbb{R}^K : b_k \geq b_{k+1} \text{ for } k = 1, \dots, K\} \cap [\underline{b}, \bar{b}]$, $b_{i,k}$ is the maximum value that bidder i is willing to pay for the k 'th unit, given that he is receiving $k - 1$ units; and $[\underline{b}, \bar{b}]$ denotes a K dimensional rectangle that bounds the set of all bids. Since bids are non-increasing we are implicitly assuming that there are no complementarity among objects. Bids are in units of account (i.e., dollars). The non-strategic player 0 also places a bid $b_0 \in B$, meaning that there is a reserve price for each unit.⁵ For instance, in a one-object auction ($K = 1$) where all players are buyers, if $\max_{j=1, \dots, N} b_{j,1} < b_{0,1}$, this means that none of the bidders are willing to pay the reserve price. The difference is that b_0 is known for everyone at the time the auction takes place, while b_j , $j \neq 0$, is not known for bidder $i \neq j$, $i \in \mathcal{N}_0$.⁶ We denote by b the vector of all players' bids, $b \in \mathbb{R}^{(N+1)K}$.

⁵If the model does not specify a reserve price it is usual to assume $b^{\min} = 0$.

⁶Unknown reserve prices can be modeled as the bid of a strategic bidder.

2.3 Allocation and Payoffs

The ‘‘auction house’’ computes the bids and determines how many units each player receives. If player i wins a k ’th unit, his payoff is increased by $u_{i,k}(t, b)$, where $u_{i,k} : T \times \mathbb{R}^{(N+1)K} \rightarrow \mathbb{R}$.⁷ Thus, if player $i \in \mathcal{N}$ ends the auction with exactly $m_i \in \{0, 1, \dots, K\}$ units, his payoff is $\sum_{k=0}^{m_i} u_{i,k}(t, b)$.⁸ In the examples we shall restrict to separable transfers so, for later reference, for each player i and unit k , let $v_{i,k} : T \rightarrow \mathbb{R}$ be a function such that $v_{i,k}(t)$ represents the (marginal) value, in units of account, of the k ’th unit for player i when the vector of signals is $t \in T$.

If $m_i < e_i$, the player has sold $e_i - m_i$ units in the auction and if $m_i > e_i$, the player has bought $m_i - e_i$ units in the auction. No negotiation was made if $m_i = e_i$.

Given b_{-i} , let $s_i = (s_{i,1}, s_{i,2}, \dots, s_{i,K})$, with $s_{i,1} \leq s_{i,2} \leq \dots \leq s_{i,K}$, denote the (inverse) residual supply curve facing bidder i . That is, $s_{i,K}$ is the highest of the bids by players $j \neq i$, $s_{i,K-1}$ is the second highest and so on. Thus, for getting (for sure) at least one unit, bidder i ’s highest bid must be above $s_{i,1}$, that is, $b_{i,1} > s_{i,1}$. For bidder i earning at least two units, it is necessary $b_{i,2} > s_{i,2}$ and so on. Figure 1 illustrates this.

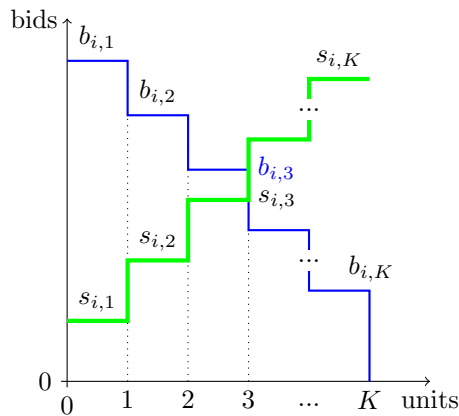


Figure 1: Bid (b_i) and supply (s_i) curves for bidder i . In the situation displayed, bidder i receives three units, because $b_{i,3} > s_{i,3}$ but $b_{i,4} < s_{i,4}$.

In order to decide who wins an object, we will assume that the auction house uses an allocation (or tie-breaking) rule.

Definition 1 *An allocation rule is any function $a : \mathbb{R}^{(N+1)K} \rightarrow [0, 1]^{(N+1)K}$ such that:*

⁷We consider the dependence on b instead of b_i because we want to include in our results auctions where the payoff depends on bids of the opponents, such as the second-price auction, for instance. Also, this allows the study of ‘‘exotic’’ auctions, i.e., auctions where the payment is an arbitrary function of all bids.

⁸More precisely, his payoff would be $u_{i,0}(t, b) + \sum_{k=1}^{m_i} u_{i,k}(t, b) \delta_{i,k}$ where $\delta_{i,k} = 0$ if $k \leq e_i$ and $\delta_{i,k} = 1$ if $k > e_i$. To simplify notation we just write, without loss of generality: $\sum_{k=0}^{m_i} u_{i,k}(t, b)$.

1. If $b_{i,k} < s_{i,k}$ then $a_{i,k}(b) = 0$.
2. If $b_{i,k} > s_{i,k}$ then $a_{i,k}(b) = 1$.
3. If for some k , $a_{i,k}(b) = 1$ then for all $k' \leq k$, $a_{i,k'}(b) = 1$.
4. $\sum_{i=0}^N \sum_{k=1}^K a_{i,k}(b) = K$.

The interpretation is the following. If $a_{i,k}(b) = 1$ then player i wins at least k objects. If $a_{i,k}(b) = 0$ then player i wins at most $k - 1$ objects. Formally, the first condition says that if player i 's k -th bid is lower than the $K - k + 1$ highest competing bid he will not be awarded the k -th object. The second condition says that if player i bids higher for unit k than the $K - k + 1$ highest competing bids then he will win at least k objects. The third says that if he wins at least k objects then he must also win at least $1, \dots, k - 1$ objects. The fourth says that at most K units are allocated among the N agents.

Observe that in the definition of allocation rules, there is freedom to define the rule only when $b_{i,k} = s_{i,k}$, provided the other conditions are satisfied. Thus, it is sufficient to define the rule for ties.

This setting is very general and applies to a broad class of discontinuous games, as we exemplify below.⁹

Allocation Rules

Example 1 (*Nominal Allocation Rule*) Let us suppose that the bidders are numbered following a given order (say, the lexicographic order for their names). We can define that, in the case of a tie, the bidder with the least number, among those that are tying, gets the object. It is easy to see that this rule satisfies all conditions in definition 1.

Another example of allocation rule is the standard one, that splits randomly the objects.

Example 2 (*Standard Allocation Rule*) In the case of a tie, the objects involved in the tie are randomly divided among the tying bidders. Formally: if $b_{i,k} = s_{i,k}$ then $a_{i,k}(b) = p/q$ where p is the number of the objects to be allocated in the tie, that is, $p = K - \#\{(j, \tilde{k}) : \text{such that } b_{j,\tilde{k}} > b_{i,k}\}$ and q is the number of tying bids, that is, $q = \#\{(j, \tilde{k}) : \text{such that } b_{j,\tilde{k}} = b_{i,k}\}$.

Auctions

⁹Obviously, the utility function is specified only for bidders, that is, for $i \neq 0$.

Example 3 (Single unit auctions). $u_{i,1}(t, b) = U_i(v_{i,1}(t) - b_{i,1})$ and $u_{i,0}(t, b) = 0$ corresponds to a first-price auction with risk aversion or risk loving.¹⁰ If $U_i(x) = x$, we have risk neutrality. If $u_{i,1}(t, b) = v_{i,1}(t)$ and $u_{i,0}(t, b) = -b_{i,1}$ we have the all-pay auction. If $u_{i,1}(t, b) = v_{i,1}(t) - \mathbf{s}_{i,1}$ and $u_{i,0}(t, b) = 0$ we have the second-price auction. If $u_{i,1}(t, b) = v_i(t) - \mathbf{s}_{i,2}$ and $u_{i,0}(t, b) = 0$ we have the third-price auction. If $u_{i,1}(t, b) = v_{i,1}(t) + b_i - \mathbf{s}_{i,1}$ and $u_{i,0}(t, b) = -b_i$ we have the war of attrition. We can have also combinations of these games. For example, $u_{i,1}(t, b) = v_{i,1}(t) - \alpha b_i - (1 - \alpha) \mathbf{s}_{i,1}$ and $u_{i,0}(t, b) = 0$, with $\alpha \in (0, 1)$, gives a combination of the first- and second-price auctions.

Example 4 (Multi-unit auction with unitary demand). It is also useful to consider K -unit auctions with unitary demand, among N buyers, $1 < K < N$. In this case, $b_{j,k} < b_{0,1}$, for all $j = 1, \dots, N$ and $k = 2, \dots, K$. Then, a pay-your-bid auction is given by $u_{i,1}(t, b) = v_{i,1}(t) - b_{i,1}$ and $u_{i,0}(t, b) = 0$. If it is a uniform price with the price determined by the highest loser's bid, $u_{i,1}(t, b) = v_{i,1}(t) - \mathbf{s}_{i,K}$ and $u_{i,0}(t, b) = 0$. If it is a uniform price with the price determined by the lowest winner's bid, $u_{i,0}(t, b) = 0$, $u_i(t, b) = v_{i,1}(t) - \max\{b_{i,1}, \mathbf{s}_{i,K}\}$.

Example 5 (Multi-unit auctions with multi-unit demand). $u_{i,1}(t, b) = v_{i,1}(t) - b_{i,1}, \dots, u_{i,K}(t, b) = v_{i,K}(t) - b_{i,K}$ and $u_{i,0}(t, b) = 0$ corresponds to a multiple unit auction with discriminatory price. If $u_{i,1}(t, b) = v_{i,1}(t) - p(b), \dots, u_{i,K}(t, b) = v_{i,K}(t) - p(b)$ and $u_{i,0}(t, b) = 0$ it correspond to a uniform multiple unit auction. There are two different uniform price auctions: $p(b)$ can be the lowest winner's bid (as in some actual treasury bills auctions) or $p(b)$ can be the highest loser's bid (as described by Krishna 2002). If $u_{i,1}(t, b) = v_{i,1}(t) - \mathbf{s}_{i,1}, \dots, u_{i,k}(t, b) = v_{i,k}(t) - \mathbf{s}_{i,k}, u_{i,K}(t, b) = v_{i,K}(t) - \mathbf{s}_{i,K}$ and $u_{i,0}(t, b) = 0$ we have Vickrey auction.

Strategies and Order Statistics

The strategy of a bidder $i \in \mathcal{N}$ is a bidding function $\mathbf{b}_i : T_i \rightarrow B$. We will use bold type for bidding functions. Notice that we do not specify a strategy for the non-strategic player, $i = 0$. We will restrict to integrable strategies, that is, we assume that the vector of strategies is $\mathbf{b} = (\mathbf{b}_i)_{i \in \mathcal{N}} \in \prod_{i \in \mathcal{N}} \mathbb{L}^1(T_i, B)$. For a vector of strategies $\mathbf{b} = (\mathbf{b}_i)_{i \in \mathcal{N}}$, let \mathbf{b}_{-i} be the vector of strategies of all strategic players except player i , we denote by \mathbf{s}_i , for a fixed \mathbf{b}_{-i} , the function $\mathbf{s}_i : T_{-i} \rightarrow \mathbb{R}^{KN}$ that orders the KN vector $(b_0, \mathbf{b}_{-i}(t_{-i}))$ from the highest to the lowest bid. Notice that we include the non-strategic bid b_0 . Given \mathbf{b}_{-i} and j , $1 \leq j \leq KN$, define the distribution function, $F_{\mathbf{s}_{i,j}}(\cdot | t_i)$ on \mathbb{R} , by $F_{\mathbf{s}_{i,j}}(\beta | t_i) \equiv \tau_{-i}(\{t_{-i} \in T_{-i} : \mathbf{s}_{i,j}(t_{-i}) < \beta\} | t_i)$ and let

¹⁰If we put $u_i(t, b) = U_i(v_i(t) - b_{i,1})$ we can have any attitude towards risk.

$f_{\mathbf{s}_{i,j}}(b|t_i)$ be its Radon-Nykodim derivative with respect to the Lebesgue measure. We denote by $F_{\mathbf{s}_{i,j}}^\perp(b|t_i)$ the singular part of $F_{\mathbf{s}_{i,j}}(b|t_i)$.

In the examples, sometimes we will restrict to monotone strategies. In such cases we will implicitly assume that $T_i = \mathbb{R}$ and use the following notation. Given $t \in T$, we define $t_{(-i)}$ as $t_{(-i)} \equiv \max_{j \neq i} t_j$.

Expected Payoff

In order to simplify notation below, we will write (\cdot) in the place of $(t_i, t_{-i}, b_0, b_i, \mathbf{b}_{-i}(t_{-i}))$, (β, \cdot) in the place of $(t_i, t_{-i}, b_0, (\beta, b_{i,-j}), \mathbf{b}_{-i}(t_{-i}))$ and (\circ) in the place of $(b_0, b_i, \mathbf{b}_{-i}(t_{-i}))$. Thus, if the bid b_0 and the profile of bidding functions \mathbf{b}_{-i} are fixed, the expected payoff of bidder i of type t_i , when bidding b_i , is:

$$\begin{aligned} \Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i}) &\equiv \int_{T_{-i}} u_{i,0}(\cdot) \tau_{-i}(dt_{-i}|t_i) + \\ &\sum_{k=1}^K \int_{T_{-i}} a_{i,k}(\circ) u_{i,k}(\cdot) \tau_{-i}(dt_{-i}|t_i), \end{aligned}$$

which is equivalent to:

$$\begin{aligned} \Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i}) &= \int_{T_{-i}} u_{i,0}(\cdot) \tau_{-i}(dt_{-i}|t_i) + \\ &\sum_{k=1}^K \int_{T_{-i}} u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) + \\ &\sum_{k=1}^K \int_{T_{-i}} a_{i,k}(\circ) u_{i,k}(\cdot) 1_{[b_{i,k} = \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i). \end{aligned} \tag{1}$$

Remark 1 *There are two important ways in which the third term in the above expression may be omitted. If for all $k = 1, \dots, K$, the distribution $F_{\mathbf{s}_{i,k}}(-|t_i)$ has no atoms and therefore, the tie-breaking rule (i.e., allocation rule a) is not important and, if the auctioneer keeps the objects in case of ties. That is, when $a_{i,k}(\circ) = 0$ whenever $b_{i,k} = \mathbf{s}_{i,k}$.*

3 Bidding Behavior

Our first result is a characterization of the payoff through its derivative with respect to the bid given by an integral expression (i.e., a kind of first fundamental theorem of calculus). For this, we will need the following

assumption:

Condition 1 $u_{i,k} : T \times \mathbb{R}^{K(N+1)} \rightarrow \mathbb{R}$, $k = 0, 1, \dots, K$ are absolutely continuous on $b_{i,k}$ and $\partial_{b_{i,k}} u_{i,k}$ is essentially bounded.¹¹

Our main result is the following:

Lemma 1 (Payoff Characterization) Assume condition 1. Fix b_0 and a profile of bidding functions \mathbf{b}_{-i} .

Then, for all $j = 1, \dots, K$ the payoff of bidder i when bidding b_i can be expressed as:

$$\begin{aligned} \Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i}) &= \Pi_i(t_i, b_0, b_0, \mathbf{b}_{-i}) \\ &+ \int_{[b_0, j, b_{i, j}]} \partial_{b_{i, j}} \Pi_i(t_i, b_0, (\beta, b_{i, -j}), \mathbf{b}_{-i}) d\beta \\ &+ \int_{[b_0, j, b_{i, j}]} E[u_{i, j}(\cdot) | t_i, \mathbf{s}_{i, k} = \beta] F_{\mathbf{s}_{i, j}}^\perp(d\beta | t_i) \\ &+ \sum_{k=1}^K E[a_{i, k}(\circ) u_{i, k}(\cdot) \mathbf{1}_{[b_{i, k} = \mathbf{s}_{i, k}]} | t_i] \end{aligned}$$

where, $E[\cdot | t_i]$ is the expectation with respect to the measure $\tau_{-i}(-|t_i)$, and for almost all $b_{i, j}$:

$$\begin{aligned} \partial_{b_{i, j}} \Pi_i(t_i, b_0, (\beta, b_{i, -j}), \mathbf{b}_{-i}) &= E[\partial_{b_{i, j}} u_{i, 0}(\beta, \cdot) | t_i] \\ &+ \sum_{k \neq j} E[\partial_{b_{i, j}} u_{i, k}(\beta, \cdot) \mathbf{1}_{[b_{i, k} > \mathbf{s}_{i, k}]} | t_i] \\ &+ E[\partial_{b_{i, j}} u_{i, j}(\beta, \cdot) \mathbf{1}_{[\beta > \mathbf{s}_{i, j}]} | t_i] \\ &+ E[u_{i, j}(\cdot) | t_i, \mathbf{s}_{i, j} = \beta] f_{\mathbf{s}_{i, j}}(\beta | t_i) \end{aligned}$$

Proof. See appendix. ■

The most important part of Lemma 1 is the expression of $\partial_{b_{i, j}} \Pi_i(t_i, b_0, (\beta, b_{i, -j}), \mathbf{b}_{-i})$. When there is no tie with positive probability at b_i (i.e., $F_{\mathbf{s}_{i, k}}(-|t_i)$ has no atoms) $\partial_{b_{i, j}} \Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i})$ is, for almost all $b_{i, j}$, the partial derivative of $\Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i})$ (see section 5). It is useful to observe that in the expression of $\partial_{b_{i, j}} \Pi_i(t_i, b_0, (\beta, b_{i, -j}), \mathbf{b}_{-i})$ above, the first three lines capture only the impact of the changing of $b_{i, j} = \beta$ in the payoff (payment) of each unit, while the last line captures the impact of such a change in the probability

¹¹Absolute continuity with respect to $b_{i, k}$ implies that $\partial_{b_{i, k}} u_{i, k}$ exists almost everywhere (with respect to Lebesgue measure) and

$$u_{i, 0}(b_{i, j}, \cdot) - u_{i, 0}(b_0, j, \cdot) = \int_{[b_0, j, b_{i, j}]} \partial_{b_{i, j}} u_{i, 0}(\beta, \cdot) d\beta.$$

Essentially bounded is used to invoke Lebesgue dominated convergence theorem.

of winning the unit j . Note also the difference in the events in the second and the third line: $[b_{i,k} > \mathbf{s}_{i,k}]$ and $[\beta > \mathbf{s}_{i,j}]$.

The following corollary characterizes best response bids in an intuitive way.

Corollary 1 (Basic Principle of Bidding) . *Under condition 1, the optimum bid is such that the marginal cost of bidding is equal to the marginal utility from bidding. More formally, if $\Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i})$ is differentiable at $b_i \in \arg \max_{b \in B} \Pi_i(t_i, b_0, b, \mathbf{b}_{-i})$, b_i in the interior of B and there is no tie with positive probability at b_i i.e., $(F_{\mathbf{s}_{i,k}}(-|t_i)$ has no atoms), then for all j ,*

$$\begin{aligned} & E[u_{i,j}(\cdot) | t_i, \mathbf{s}_{i,j} = b_{i,j}] f_{\mathbf{s}_{i,k}}(\beta | t_i) + \sum_{k \neq j}^K E[\partial_{b_{i,j}} u_{i,k}(b_{i,j}, \cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}] | t_i}] \\ &= E[-\partial_{b_{i,j}} u_{i,0}(b_{i,j}, \cdot) | t_i] + E[-\partial_{b_{i,j}} u_{i,j}(b_{i,j}, \cdot) 1_{[b_{i,j} > \mathbf{s}_{i,j}] | t_i}] \end{aligned} \quad (2)$$

Proof. If $F_{\mathbf{s}_{i,k}}(-|t_i)$ has no atoms then $F_{\mathbf{s}_{i,k}}^\perp(-|t_i) = 0$ almost everywhere. Therefore, by the payoff characterization lemma:

$$\Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i}) = \Pi_i(t_i, b_0, b_0, \mathbf{b}_{-i}) + \int_{[b_0, b_i]} \partial_{b_{i,j}} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), \mathbf{b}_{-i}) d\beta.$$

If $\Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i})$ is differentiable at $b_i \in \arg \max_{b \in B} \Pi_i(t_i, b_0, b, \mathbf{b}_{-i})$ then

$$\partial_{b_{i,j}} \Pi_i(t_i, b_0, (b_{i,j}, b_{i,-j}), \mathbf{b}_{-i}) = 0$$

■

Observe that $E[u_{i,j}(\cdot) | t_i, \mathbf{s}_{i,k} = b_{i,j}] f_{\mathbf{s}_{i,k}}(b_{i,j} | t_i)$ represents the marginal benefit of winning unit j , that is, bidder i 's marginal gain from losing to winning unit j . $E_{\tau_{-i}}[\partial_{b_{i,j}} u_{i,j}(b_{i,j}, \cdot) 1_{[b_{i,j} > \mathbf{s}_{i,j}] | t_i}]$ represents the marginal benefit of changing the bid if the j unit is an inframarginal winning bid. On the other hand, $E[-\partial_{b_{i,j}} u_{i,0}(b_{i,j}, \cdot) | t_i]$ represents the marginal cost of participation and the last term represents the marginal cost of changing j bid for the other inframarginal winning bids. Note that we do not require separability in the monetary transfer (risk neutrality) to reach such an interpretation.

This interpretation is useful for understanding bidding behavior. In first-price auctions, the marginal cost of bidding is what implies a decreasing bid in the way bidders bid. In second-price auctions, the marginal cost of bidding is zero, so that each bidder bids until its marginal utility of bidding becomes zero.

The following corollary will be used later to prove a monotone best-reply result.

Corollary 2 (Payoff Characterization as a Line Integral) *Assume condition 1 and suppose for all i and k , $\partial_{b_{i,k}} \Pi_i$ exists and is continuous in b_i . Fix b_0 , a profile of bidding functions \mathbf{b}_{-i} and a smooth curve $\alpha : [0, 1] \rightarrow B$ such that $\alpha(0) = b_0$ and $\alpha(1) = b_i$ then, for all $j = 1, \dots, K$ the payoff of bidder i when bidding b_i can be expressed as:*

$$\begin{aligned} \Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i}) &= \Pi_i(t_i, b_0, b_0, \mathbf{b}_{-i}) \\ &+ \int_{[0,1]} \nabla_{b_i} \Pi_i(t_i, b_0, \alpha(s), \mathbf{b}_{-i}) \cdot \alpha'(s) ds \end{aligned}$$

where $\nabla_{b_i} \Pi_i(t_i, b_0, \alpha(s), \mathbf{b}_{-i}) = (\partial_{b_{i,j}} \Pi_i(t_i, b_0, \alpha(s), \mathbf{b}_{-i}))_{j=1, \dots, K}$

3.1 Examples

The example below show that corollary 1 is a generalization of the necessary first-order conditions for the first and second-price auctions presented in Milgrom and Weber (1982), for the war of attrition and all-pay auctions presented in Krishna and Morgan (1997). The example on double auctions shows that the Basic Principle of Bidding is concise. Such an example is the application of Corollary 1 for double auctions and it presents a comparison with the equivalent expression obtained by Williams (1991).

Example 6 (First Price - Single Object Auction). *When we restrict ourselves to the case of the first-price single object auction with risk neutrality: $K = 1$, $u_{i,0} = 0$ and $u_{i,1}(t, b_0, b) = v_{i,1}(t) - b_i$, then $\partial_{b_i} u_{i,1}(t, b_0, b) = -1$. The condition of corollary 1 becomes:*

$$b_i = E[v_{i,1}|t_i, \mathbf{s}_{i,1} = b_i] - \frac{F_{\mathbf{s}_{i,1}}(b_i|t_i)}{f_{\mathbf{s}_{i,1}}(b_i|t_i)}. \quad (3)$$

This (necessary) first-order condition provides a useful way to determine best-reply bids. Note that this expression admits non-monotonic bidding functions, contrary to Milgrom and Weber's model. It also encompasses asymmetries in valuations and distribution of types. Assuming affiliation and monotonic utilities, Milgrom and Weber (1982) can restrict themselves to the space of monotone symmetric bidding functions (i.e., $\mathbf{b}_i = \mathbf{b}$, for all $i \in N$). Thus,

$$\mathbf{s}_{i,1}(\mathbf{b}_{-i}(t_{-i})) = x \iff \max_{j \neq i} \mathbf{b}(t^j) = x \iff \max_{j \neq i} t^j = (\mathbf{b})^{-1}(x),$$

where, in the last equation $(\mathbf{b})^{-1}$ stands for the inverse (generalized) of \mathbf{b} . This equation says that con-

conditioning on $\mathbf{s}_{i,1} = b_i$ is the same to conditioning on $\max_{j \neq i} t^j = t_i$. Recall that $t^{(-i)} \equiv \max_{j \neq i} t^j$. Then $f_{\mathbf{s}_{i,1}}(s|t_i) = \frac{f_{t^{(-i)}}(s|t_i)}{\mathbf{b}'(s)}$ and $F_{\mathbf{s}_{i,1}}(s|t_i) = F_{t^{(-i)}}(s|t_i)$. With this, (3) becomes

$$\frac{d\mathbf{b}}{dt}(t_i) = \left(E \left[v_{i,1} | t_i, t^{(-i)} = t_i \right] - \mathbf{b}(t_i) \right) \frac{f_{t^{(-i)}}(t_i|t_i)}{F_{t^{(-i)}}(t_i|t_i)} \quad (4)$$

whose solution is shown to be an equilibrium under affiliation.

Example 7 (Second Price - Single Object Auction). In the second price single object auction, Milgrom and Weber's model is equivalent to $K = 1$, $u_{i,1}(t, b) = v_i(t) - \mathbf{s}_{i,1}$ and $u_{i,0} = 0$. Then, $\partial_{b_i} u_{i,1}(t, b) = 0$ and the condition in corollary 1 reduces to $E_{\tau_{-i}}[v_i - b_i | t_i, \mathbf{s}_{i,1} = b_i] f_{\mathbf{s}_{i,1}}(b_i | t_i) = 0$ which can be simplified to

$$b_i = E[v_{i,1} | t_i, \mathbf{s}_{i,1} = b_i].$$

Again, with monotonicity and symmetry assumptions, Milgrom and Weber's expression for the equilibrium bid function can be obtained:

$$\mathbf{b}(t_i) = E \left[v_{i,1} | t_i, t^{(-i)} = t_i \right] \equiv \bar{v}(t_i, t_i).$$

Example 8 (All Pay - Single Object Auction). Krishna and Morgan (1997) extend the method of Milgrom and Weber (1982) to the cases of war of attrition and all-pay auctions. In the all-pay auction, their model is equivalent to $u_{i,1}(t, b) = v_i(t) - \mathbf{s}_{i,1}$ and $u_{i,0}(t, b) = -b_i$. Then, $\partial_{b_i} u_{i,1}(t, b) = 0$ and $\partial_{b_i} u_{i,0}(t, b) = -1$. So, the condition in corollary 1 reduces to

$$E[v_{i,1} | t_i, \mathbf{s}_{i,1} = b_i] f_{\mathbf{s}_{i,1}}(b_i | t_i) = 1.$$

Under the same hypothesis of monotonicity and symmetry, they find the following differential equation:

$$\frac{d\mathbf{b}}{dt}(t_i) = E \left[v_{i,1} | t_i, t^{(-i)} = t_i \right] f_{t^{(-i)}}(t_i | t_i),$$

whose solution they show to be an equilibrium under affiliation.

Example 9 (War of Attrition - Single Object Auction). In the war of attrition, Krishna and Morgan (1997) model is equivalent to $u_{i,1}(t, b) = v_i(t) + b_i - \mathbf{s}_{i,1}$ and $u_{i,0}(t, b) = -b_i$. Then, $\partial_{b_i} u_{i,1}(t, b) = 1$ and $\partial_{b_i} u_{i,0}(t, b) = -1$. So, the condition in corollary 1 reduces to

$$E[v_{i,1} | t_i, \mathbf{s}_{i,1} = b_i] f_{\mathbf{s}_{i,1}}(b_i | t_i) = 1 - F_{\mathbf{s}_{i,1}}(b_i | t_i).$$

Again, with monotonicity and symmetry, they derive the equation

$$\frac{d\mathbf{b}}{dt}(t_i) = E \left[v_{i,1} | t_i, t^{(-i)} = t_i \right] \frac{1 - F_{t^{(-i)}}(t_i | t_i)}{f_{t^{(-i)}}(t_i | t_i)},$$

and the equilibrium is shown to exist under affiliation.

Example 10 (Double Auction). In the analysis of a double auction with private values, risk neutrality, independent types and symmetry among buyers and sellers, Williams (1991) assumes that the payment is determined by the buyer's bid. So, it is optimum for the seller to bid her value. To analyze the behavior of the buyer i , Williams (1991) reaches the following expression:

$$\begin{aligned} \partial_{b_i} \Pi_i(v, \beta) &= \left[n f_1(\beta) K_{n,m}(\mathbf{b}^{-1}(\beta), \beta) \right. \\ &\quad \left. + (m-1) \frac{f_2(\mathbf{b}^{-1}(\beta))}{b'(\beta)} L_{n,m}(\mathbf{b}^{-1}(\beta), \beta) \right] (v - \beta) \\ &\quad - M_{n,m}(\mathbf{b}^{-1}(\beta), \beta) \end{aligned} \quad (5)$$

where \mathbf{b} denotes here the symmetric bidding function followed by all buyers, f_1 is the common density function of sellers, f_2 is the common density function of buyers, n is the number of sellers, m is the number of buyers and $M_{n,m}(\cdot, \cdot)$ is given by:¹²

$$\begin{aligned} M_{n,m}(v, \beta) &\equiv \sum_{\substack{i+j=m, \\ 0 \leq i \leq m-1}} \binom{n}{j} \binom{m-1}{i} F_1(\beta)^j F_2(v)_i (1 - F_1(\beta))^{n-j} \\ &\quad \cdot (1 - F_2(v))^{m-1-i}; \\ K_{n,m}(v, \beta) &\equiv \sum_{\substack{i+j=m-1, \\ 0 \leq i \leq m-1}} \binom{n-1}{j} \binom{m-1}{i} F_1(\beta)^j F_2(v)_i \\ &\quad \cdot (1 - F_1(\beta))^{n-1-j} (1 - F_2(v))^{m-1-i}; \\ L_{n,m}(v, \beta) &\equiv \sum_{\substack{i+j=m-1, \\ 0 \leq i \leq m-1}} \binom{n}{j} \binom{m-2}{i} F_1(\beta)^j F_2(v)_i (1 - F_1(\beta))^{n-j} \\ &\quad \cdot (1 - F_2(v))^{m-2-i}. \end{aligned}$$

The expression (5) is just a special case of (2). To see this, observe that $F_{b_{(-i)}}(\beta)$, the probability that the

¹²To obtain $K_{n,m}(\cdot, \cdot)$ just substitute $n-1$ for n where it occurs in $M_{n,m}(\cdot, \cdot)$. To obtain $L_{n,m}(\cdot, \cdot)$, substitute $m-2$ for $m-1$ where it occurs in $M_{n,m}(\cdot, \cdot)$.

threshold bid is less or equal to β , is given by the probability of the union of following disjoint events: there are i bids of buyers and j bids of sellers below or equal to β and $i + j = m$ (because the m -th bid determines the threshold between winning and losing). Thus,

$$F_{\mathbf{s}_{i,1}}(\beta) = \sum_{\substack{i+j=m, \\ 0 \leq i \leq m-1, 0 \leq j \leq n}} \binom{n}{j} \binom{m-1}{i} F_1(\beta)^j \cdot F_2(\mathbf{b}^{-1}(\beta))_i (1 - F_1(\beta))^{n-j} (1 - F_2(\mathbf{b}^{-1}(\beta)))^{m-1-i},$$

which is equal to $M_{n,m}(\mathbf{b}^{-1}(\beta), \beta)$ above. Now, it is a matter of length but elementary derivation to confirm that

$$f_{\mathbf{s}_{i,1}}(\beta) = n f_1(\beta) K_{n,m}(\mathbf{b}^{-1}(\beta), \beta) + (m-1) \frac{f_2(\mathbf{b}^{-1}(\beta))}{b'(\beta)} L_{n,m}(\mathbf{b}^{-1}(\beta), \beta),$$

which concludes the proof of the claim.

Example 11 (Multiple Object Discriminatory Auction). Let $u_{i,0} = 0$, $u_{i,k}(t, b) = v_{i,k}(t) - b_{i,k}$. Then $\partial_{b_{i,j}} u_{i,k}(t, b) = 0$ if $j \neq k$ and -1 if $j = k$. It is easy to show that the condition in corollary 1 reduces to:

$$b_{i,k} = E[v_{i,k}(t) | t_i, \mathbf{s}_{i,k} = b_{i,k}] - \frac{F_{\mathbf{s}_{i,k}}(b_{i,k} | t_i)}{f_{\mathbf{s}_{i,k}}(b_{i,k} | t_i)}.$$

Assuming monotonic and symmetric bidding functions the first order condition reduces to:

$$\frac{d\mathbf{b}_k}{dt}(t_i) = (E_{\tau_{-i}}[v_{i,k}(t) | t_i, \mathbf{s}_{i,k} = b_{i,k}] - \mathbf{b}_k(t_i)) \frac{f_{t^{(-i)}}(b_{i,k} | t_i)}{F_{t^{(-i)}}(b_{i,k} | t_i)}$$

Example 12 (Multiple Object Vickrey Auction). Let $u_{i,0} = 0$, $u_{i,k}(t, b) = v_{i,k}(t) - \mathbf{s}_{i,k}$. Then $\partial_{b_{i,j}} u_{i,k}(t, b) = 0$. Therefore the condition in corollary 1 reduces to:

$$b_{i,k} = E[v_{i,k}(t) | t_i, \mathbf{s}_{i,k} = b_{i,k}].$$

This makes clear that the Vickrey Auction implies revelation of the truth in a general setting.

Example 13 (Uniform price auction) Let $u_{i,0} = 0$, $u_{i,k}(t, b) = v_{i,k}(t) - p$, where p is the payment, which is equal for all units and bidders. There are two common rules for the uniform price auction. One is the highest looser bid, which is the uniform price auction described by Krishna (2002). In this case, the payment is equal to the highest bid among those bids that do not receive the object. A variant is to put the payment

equal to the lowest winning bid. We treat both below. Note that for any k ,

$$\partial_{b_{i,j}} u_{i,j}(b_{i,j}, \cdot) = -\partial_{b_{i,j}} p(b) = \begin{cases} -1, & \text{if } b_{i,j} \text{ determines the payment} \\ 0, & \text{otherwise} \end{cases}$$

In the case of the lowest winning bid, $b_{i,j}$ determines the payment if $b_{i,j} > s_{i,j}$, $b_{i,j+1} < s_{i,j+1}$ and $b_{i,j} < s_{i,j+1}$. This event contains the event $[b_{i,k} > s_{i,k}]$ if and only if $k \leq j$. Thus, the first order condition becomes:

$$b_{i,j} = E[v_{i,j}(t) | t_i, \mathbf{s}_{i,j} = b_{i,j}] - j \frac{\Pr[s_{i,j+1} > b_{i,j} > s_{i,j}, b_{i,j+1} < s_{i,j+1}]}{f_{\mathbf{s}_{i,j}}(b_{i,j} | t_i)}.$$

In the case of the highest losing bid, $b_{i,j}$ determines the payment if $b_{i,j} < s_{i,j}$, $b_{i,j-1} > s_{i,j-1}$ and $b_{i,j} > s_{i,j-1}$. Similarly,

$$b_{i,j} = E[v_{i,j}(t) | t_i, \mathbf{s}_{i,j} = b_{i,j}] - (j-1) \frac{\Pr[s_{i,j-1} < b_{i,j} < s_{i,j}, b_{i,j-1} > s_{i,j-1}]}{f_{\mathbf{s}_{i,j}}(b_{i,j} | t_i)}.$$

4 Applications

Here we point out some potential applications and how our main result can be used to give a simple prove of some useful facts about auctions. Some of these results are new.

For all results below, we assume that the strategies \mathbf{b}_{-i} of bidder i 's opponents are such that the distribution of \mathbf{s}_i is absolutely continuous with respect to the Lebesgue measure. Thus, the payoff is given only by the integral of its derivative.

4.1 Sufficient Conditions for Truth-telling

It is widely know that second price auctions lead to bidding equal to the truthful expected value by the bidder. This can be easily seen from the first order condition for this auction:

$$b_i = E[v_{i,1} | t_i, \mathbf{s}_{i,1} = b_i].$$

The other terms in the first order condition disappear because they are identically zero: $\partial_{b_{i,1}} u_{i,1}(t, b) = 0$. The reason for that is that the payment does not depend on the own bidder's bid. It turns out that this last property is exactly what is need to obtain truthful bidding, as we illustrate below.

Proposition 1 *Consider an auction where the bid $b_{i,j}$ never modifies the payment of any unit, more pre-*

cisely, $\partial_{b_{i,j}} u_{i,k}(t, b) = 0$ for all k and (t, b) . Then, it is optimal for bidder i to bid $b_{i,j}$ such that:

$$E[u_{i,j}(\cdot) | t_i, \mathbf{s}_{i,j} = b_{i,j}] = 0.$$

If the bidder is risk neutral in the unit j , that is, $u_{i,j}(t, b) = v_{i,j}(t) - p_{i,j}(b)$, and the payment $p(b)$ is $b_{i,j}$ in case of a relevant tie at $b_{i,j} = s_{i,k}$, then the optimal bid is to bid the expected value of the unit:

$$b_{i,j} = E[v_{i,j}(t) | t_i, \mathbf{s}_{i,j} = b_{i,j}].$$

Proof. It is sufficient to examine the expression of $\partial_{b_{i,j}} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), \mathbf{b}_{-i})$ and to observe that in the event $\mathbf{s}_{i,j} = b_{i,j}$, the payment $p(b) = b_{i,j}$. ■

Some known results are immediate corollaries:

Corollary 3 *The first (highest) bid in the uniform price auction (with payment equal to the highest looser bid) is truthful.*

Proof. *The first bid cannot affect the payment of a winning bidder.* ■

Corollary 4 *The bids in the Vickrey auction are truthful.*

Proof. *In the Vickrey auction, the payments depend only on the others' bids.* ■

4.2 Sufficient Conditions for Increasing Best Reply

Let

$$V_i(\mathbf{b}_i, \mathbf{b}_{-i}) = \int \Pi_i(t_i, \mathbf{b}_i(t_i), \mathbf{b}_{-i}) dt_i$$

be the ex-ante payoff. We define the interim and the ex-ante best-reply correspondence, respectively, by

$$\Theta_i(t_i, \mathbf{b}_{-i}) \equiv \arg \max_{\beta \in \mathcal{B}} \Pi_i(t_i, \beta, \mathbf{b}_{-i}),$$

and

$$\Gamma_i(\mathbf{b}_{-i}) \equiv \arg \max_{\mathbf{b}_i \in \mathcal{L}^1([0,1], \mathcal{B})} V_i(\mathbf{b}_i, \mathbf{b}_{-i}).$$

We need the following:

Definition 2 Given a partial order \succsim on T_i , we say that a function $g(t, b)$ is strictly increasing (non-decreasing) in t_i if $t_i^2 \succ t_i^1$ ($t_i^2 \succsim t_i^1$) implies $g(t_i^2, t_{-i}, b) > (\geq) g(t_i^1, t_{-i}, b)$ for all t_{-i}, b .¹³

Let \geq denote the coordinate-wise partial order in B , that is: $b_i^1 \geq b_i^0$ if $b_{i,j}^1 \geq b_{i,j}^0$ for all $j = 1, \dots, K$. We write $b_i^1 > b_i^0$ if $b_i^1 \geq b_i^0$ and $b_i^1 \neq b_i^0$.

Proposition 2 Assume we are under the conditions of Corollary 2. Let \succsim be a partial order on T_i . For all $k, j = 1, \dots, K$, assume that $u_{i,k}(t, b)$ is absolutely continuous in t and b , and strictly increasing in t_i ; and $\partial_{b_{i,j}} u_{i,k}(t, b)$ is non-decreasing in t_i (except, possibly in a set of null measure). Then the following holds:

1. For each t_i , $\Theta_i(t_i, \mathbf{b}_{-i})$ is non-empty.
2. Consider two types t_i^1, t_i^2 , $t_i^2 \succ t_i^1$, and best reply bids for them, that is, $b_i^1 \in \Theta_i(t_i^1, \mathbf{b}_{-i})$, $b_i^2 \in \Theta_i(t_i^2, \mathbf{b}_{-i})$ and assume that these bids imply different probability of winning, i.e.,

$$\Pr(\{t_{-i} : \exists k \text{ such that } a_{i,k}(b_i^1, \mathbf{b}_{-i}(t_{-i})) \neq a_{i,k}(b_i^2, \mathbf{b}_{-i}(t_{-i}))\}) > 0.$$

Then $\sim (b_i^1 > b_i^2)$.

Proof. See Appendix. ■

The unidimensional version of the above theorem was used by Araujo and de Castro (2007) to prove equilibrium existence in single unit auctions. The main role of this result in their equilibrium proof is to restrict the set of strategies to a compact set (the set of non-decreasing functions). Restricted to this strategy set, they obtained approximated equilibria of perturbed games, used compactness to obtain a converging subsequence and proved that the limit is equilibrium of the original auction. Maybe the above theorem could be equally useful in obtaining new equilibrium existence results for multiunit auctions, but such results are out of the scope of this paper.

4.3 Identification of Multi-Unit Auctions

There is large literature on structural identification of unitary auctions (see Athey and Haile (2002), (2005)). The case of multiple unit demand auctions has been recently the focus of attention. The problem is of

¹³Given the partial order \succsim , we write $x \succ y$ if $x \succsim y$ but $\sim (y \succ x)$.

interest in the applied literature because many important markets rely on auction mechanisms to allocate goods or services and these are naturally modeled as markets for the allocation of multiple units. Prominent examples are the markets for treasury auctions and for the demand and supply of electricity (see Hortacsu (2002) for treasury auctions, Hortacsu and Puller (2007) and Wolak (2006) for electricity markets). With very few exceptions (for example McAdams (2007) or Wolak (2006)), most applied work rely on Wilson (1979) share model. One of salient features of this approach is the use of continuous bid functions that are hardly found in real markets. Therefore, it is of interest to study identification when bids are discrete. The particular institutional settings in which some of these auctions are carried in real markets make the continuous bid assumption unattainable. For example, in England spot electricity market generators make three bids out of their supply function. In the Colombian spot markets, generators make price offers for generating a fixed (although different among plants) amount of energy per generating unit (see de Castro, Espinosa and Riascos (2007) for an application of our identification results to the Colombian electricity spot market). Below we provide identification results for the two most important multi-unit auctions, the discriminatory auction and the uniform auction. To the extend of our knowledge, the discrete case result for the uniform auction is new (Hortacsu and Puller (2007) report a similar result for continuous bid functions). In all cases, estimation methods assume that observables are generated by a Bayesian-Nash equilibrium.

Recall example 13 where we derived the first order conditions for the multi-unit uniform auction. Consider the case in which agents pay the lowest winning bid.

Proposition 3 *Consider the uniform price auction (see example 13). Then if values are private, the marginal utility of an additional unit is nonparametrically identified from agents bid. Formally,*

$$v_{i,j}(t_i) = b_{i,j} + j \frac{\Pr[s_{i,j+1} > b_{i,j} > s_{i,j}, b_{i,j+1} < s_{i,j+1}]}{f_{s,j}(b_{i,j}|t_i)}. \quad (6)$$

Proof. To the extend that all agents' bids are observable and one is able to estimate the second term from the right hand side, identification follows. ■

Remark 2 *Equation 6 is analogous to equation (2) in Hortacsu and Puller (2007).*

Remark 3 *To estimate the right hand side of equation 6 one can follow the same estimation methods of Hortacsu (2002).*

Proposition 4 *Consider the discriminatory multi-unit auction (see example 11). Then if values are private,*

the marginal utility of additional unit is nonparametrically identified from agents bid. Formally,

$$v_{i,j}(t_i) = b_{i,j} + \frac{F_{\mathbf{s}_{i,j}}(b_{i,j}|t_i)}{f_{\mathbf{s}_{i,j}}(b_{i,j}|t_i)}. \quad (7)$$

Proof. The same argument as before. ■

Remark 4 Equation 7 is analogous to equation (2) in Hortacsu (2002). Notice that Hortacsu (2002) also studies a discrete version in which prices are restricted to lie in on a discrete grid but the divisibility assumption of goods is still assumed.

Remark 5 To estimate the right hand side of equation 7 one can follow the same estimation methods of Hortacsu (2002).

5 Appendix: Proofs

5.1 Payoff Characterization Lemma

The proof follows the demonstration of the Leibiniz rule. The main point is the use of a well known theorem on the derivatives of measures and its integral expression. The theorem we use is in Rudin (1966).

Recall the expression for agents expected payoff:

$$\Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i}) = \int_{T_{-i}} u_{i,0}(\cdot) \tau_{-i}(dt_{-i}|t_i) + \quad (8)$$

$$\sum_{k=1}^K \int_{T_{-i}} u_{i,k}(\cdot) \mathbb{1}_{[b_{i,k} > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) + \quad (9)$$

$$\sum_{k=1}^K \int_{T_{-i}} a_{i,k}(\circ) u_{i,k}(\cdot) \mathbb{1}_{[b_{i,k} = \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i).$$

Fix j , and consider each term above separately.

1. The first one has a derivative with respect to $b_{i,j}$ almost everywhere and is equal to $E[\partial_{b_{i,j}} u_{i,0}(\cdot)|t_i]$.

Also,

$$E[u_{i,0}(\cdot)|t_i] = \int_{[b_{0,j}, b_{i,j})} E[\partial_{b_{i,j}} u_{i,0}(\beta, \cdot)|t_i] d\beta$$

2. If the distribution $F_{\mathbf{s}_{i,k}}(-|t_i)$ has no atoms, the third term is equal to zero and its derivative exists and it's zero.

3. Now consider the second term, $\int_{T_{-i}} u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i)$. There are two cases, $j \neq k$ and $j = k$.

In the first case ($j \neq k$), let $a^n \rightarrow (b_{i,j})^+$ (i.e., $a^n > b_{i,j}$; the other case is analogous). We have

$$\begin{aligned} & \int_{T_{-i}} u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-j}), \mathbf{b}_{-i}(t_{-i})) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) \\ & - \int_{T_{-i}} u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) \\ & = \int_{T_{-i}} (u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-j}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) \end{aligned}$$

Since u_i has bounded derivative with respect to almost all $b_{i,j}$,

$$\lim_{a^n \rightarrow (b_{i,j})^+} \frac{u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-j}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,j}} = \partial_{b_{i,j}} u_{i,k}(\cdot),$$

for almost all b_i . By Lebesgue Theorem, the integral converges, that is,

$$\lim_{a^n \rightarrow (b_i)^+} \int_{T_{-i}} \frac{u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-j}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,j}} 1_{[b_{i,k} > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i)$$

exists and it is equal to $E[\partial_{b_{i,j}} u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} | t_i]$. Also,

$$\begin{aligned} & E[u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{c}_{i,k}]} | t_i] = \\ & \int_{[b_0, j, b_{i,j}]} E[\partial_{b_{i,j}} u_{i,k}(\beta, \cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} | t_i] \end{aligned}$$

In the second case ($j = k$), let $a^n \rightarrow (b_{i,k})^+$ (i.e., $a^n > b_{i,k}$; the other case is analogous). Then,

$$\begin{aligned}
& \int_{T_{-i}} u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) 1_{[a^n > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) - \\
& \int_{T_{-i}} u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) \\
&= \int_{T_{-i}} (u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)) 1_{[a^n > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) + \\
& \int_{T_{-i}} u_{i,k}(\cdot) (1_{[a^n > \mathbf{s}_{i,k}]} - 1_{[b_{i,k} > \mathbf{s}_{i,k}]}) \tau_{-i}(dt_{-i}|t_i) \\
&= \int_{T_{-i}} (u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)) 1_{[a^n > \mathbf{s}_{i,k}]} \tau_{-i}(dt_{-i}|t_i) + \\
& \int_{T_{-i}} u_{i,k}(\cdot) 1_{[a^n > \mathbf{s}_{i,k} \geq b_{i,k}]} \tau_{-i}(dt_{-i}|t_i)
\end{aligned}$$

Since u_i has bounded derivative with respect to almost all $b_{i,k}$,

$$\lim_{a^n \rightarrow (b_{i,k})^+} \frac{u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,k}} = \partial_{b_{i,k}} u_{i,k}(\cdot),$$

for almost all $b_{i,k}$. Also, $1_{[a^n > \mathbf{s}_{i,k}]} \rightarrow 1_{[b_{i,k} > \mathbf{s}_{i,k}]}$. These imply that:

$$\begin{aligned}
& \lim_{a^n \rightarrow (b_{i,k})^+} \frac{u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,k}} 1_{[a^n > \mathbf{s}_{i,k}]} \\
&= \partial_{b_{i,k}} u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{c}_{i,k}]}
\end{aligned}$$

for almost all $b_{i,k}$ and these functions are (almost everywhere) bounded. By Lebesgue Theorem, the integral converges, that is,

$$\lim_{a^n \rightarrow (b_{i,k})^+} \int_{T_{-i}} \frac{(u_{i,k}(t_i, t_{-i}, b_0, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot))}{a^n - b_{i,k}} 1_{[a^n > \mathbf{c}_{i,k}]} \tau_{-i}(dt_{-i}|t_i)$$

exists and it is equal to $E[\partial_{b_{i,k}} u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} | t_i]$. Also,

$$\begin{aligned}
& E[u_{i,k}(\cdot) 1_{[b_{i,k} > \mathbf{c}_{i,k}]} | t_i] \\
&= \int_{[b_0, b_{i,k}]} E[\partial_{b_{i,k}} u_{i,k}(t_i, t_{-i}, b_0, (\beta, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) 1_{[\beta > \mathbf{s}_{i,k}]} | t_i] d\beta
\end{aligned}$$

Now, we want to determine:

$$\lim_{a^n \rightarrow (b_{i,k})^+} \frac{1}{a^n - b_{i,k}} \int_{T_{-i}} u_{i,k}(\cdot) 1_{[a^n > \mathbf{s}_{i,k} \geq b_{i,k}]} \tau_{-i}(dt_{-i}|t_i)$$

For each each $t_i \in T_i$ and b_i fixed, define the signed measure ρ over \mathbb{R} by¹⁴

$$\rho(V; t_i, b_0, b_i, \mathbf{b}_{-i}) \equiv \int_{T_{-i}} u_{i,k}(\cdot) 1_{[\mathbf{s}_{i,k} \in V]} \tau_{-i}(dt_{-i}|t_i).$$

Then,

$$\begin{aligned} & \lim_{a^n \rightarrow (b_{i,k})^+} \frac{1}{a^n - b_{i,k}} \int_{T_{-i}} u_{i,k}(\cdot) 1_{[a^n > \mathbf{s}_{i,k} \geq b_{i,k}]} \tau_{-i}(dt_{-i}|t_i) \\ &= \lim_{a^n \rightarrow (b_{i,k})^+} \frac{1}{a^n - b_{i,k}} \rho([b_{i,k}, a^n]; t_i, b_0, b_i, \mathbf{b}_{-i}) \\ &= \lim_{a^n \rightarrow (b_{i,k})^+} \frac{\rho([b_{i,k}, a^n]; t_i, b_0, b_i, \mathbf{b}_{-i})}{m([b_{i,k}, a^n])}, \end{aligned}$$

where m is Lebesgue measure over \mathbb{R} . By Theorem 8.6 of Rudin [1966] this limit exists m -almost everywhere in $b_{i,k}$ and we call it $D\rho(t_i, b_0, b_i, \mathbf{b}_{-i})$. Also, $D\rho$ coincides almost everywhere with the Radon-Nikodym derivative $\frac{d\rho}{dm}(t_i, b_0, b_i, \mathbf{b}_{-i})$. Therefore,

$$\rho(V; t_i, b_0, b_i, \mathbf{b}_{-i}) = \int_V \frac{d\rho}{dm}(t_i, b_0, b_i, \mathbf{b}_{-i}) dm + \rho^\perp(V; t_i, b_0, b_i, \mathbf{b}_{-i}).$$

where ρ^\perp denotes the singular part of ρ , and it has the property

$$\lim_{a^n \rightarrow (b_{i,k})^+} \frac{\rho^\perp([b_{i,k}, a^n]; t_i, b_0, b_i, \mathbf{b}_{-i})}{m([b_{i,k}, a^n])} = 0,$$

by the same theorem.

It is easy to see that ρ is absolutely continuous with respect to the distribution $F_{\mathbf{s}_{i,k}}(\cdot|t_i)$. The Radon-Nikodym Theorem guarantees the existence of the Radon-Nikodym derivative of ρ with respect to the distribution of $F_{\mathbf{s}_{i,k}}(\cdot|t_i)$, which we denote by g . Therefore, g is such that

$$\rho(V; t_i, b_0, b_i, \mathbf{b}_{-i}) = \int_V g(\beta) F_{\mathbf{s}_{i,k}}(d\beta|t_i) = \int_V g(\beta) f_{\mathbf{s}_{i,k}}(\beta|t_i) d\beta$$

¹⁴On a σ -field this is synonymous with a countably additive set function.

and by definition:

$$\begin{aligned}
\rho(V; t_i, b_0, b_i, \mathbf{b}_{-i}) &\equiv \int_{T_{-i}} u_{i,k}(\cdot) 1_{[\mathbf{s}_{i,k} \in V]} \tau_{-i}(dt_{-i}|t_i) \\
&= E[u_{i,k}(\cdot) 1_{[\mathbf{s}_{i,k} \in V]} | t_i] \\
&= E_{\tau_{-i}, t_i} [u_{i,k}(\cdot) 1_{[\mathbf{c}_{i,k} \in V]}] \\
&= \int_{[b_0, K-k+1, \infty)} E[u_{i,k}(\cdot) 1_{[\mathbf{s}_{i,k} \in V]} | t_i, \mathbf{c}_{i,k} = \beta] F_{\mathbf{s}_{i,k}}(d\beta | t_i) \\
&= \int_{[b_0, K-k+1, \infty)} E_{\tau_{-i}, t_i} [u_{i,k}(\cdot) 1_{[\mathbf{s}_{i,k} \in V]} | \mathbf{s}_{i,k} = \beta] F_{\mathbf{s}_{i,k}}(d\beta | t_i) \\
&= \int_V E[u_{i,k}(\cdot) | t_i, \mathbf{s}_{i,k} = \beta] F_{\mathbf{s}_{i,k}}(d\beta | t_i) \\
&= \int_V E_{\tau_{-i}, t_i} [u_{i,k}(\cdot) | \mathbf{s}_{i,k} = \beta] F_{\mathbf{s}_{i,k}}(d\beta | t_i) \\
&= \int_V E[u_{i,k}(\cdot) | t_i, \mathbf{s}_{i,k} = \beta] f_{\mathbf{s}_{i,k}}(\beta | t_i) d\beta \\
&= \int_V E_{\tau_{-i}, t_i} [u_{i,k}(\cdot) | \mathbf{s}_{i,k} = \beta] f_{\mathbf{s}_{i,k}}(\beta | t_i) d\beta
\end{aligned}$$

therefore, by the unicity of the Radom Nikodyn derivative of ρ with respect to Lebesgue measure m , we have that:

$$g(\beta) = E_{\tau_{-i}, t_i} [u_{i,k}(\cdot) | \mathbf{s}_{i,k} = \beta] f_{\mathbf{s}_{i,k}}(\beta | t_i),$$

m -almost everywhere in β .

Thus,

$$\begin{aligned}
\partial_{b_{i,j}} \Pi_i(t_i, b_0, (\beta, b_{i,-j}), \mathbf{b}_{-i}) &= E[\partial_{b_{i,j}} u_{i,0}(\beta, \cdot) | t_i] \\
&\quad + \sum_{\substack{k \\ k \neq j}}^K E[\partial_{b_{i,j}} u_{i,k}(\beta, \cdot) 1_{[b_{i,k} > \mathbf{s}_{i,k}]} | t_i] + \\
&\quad E[\partial_{b_{i,j}} u_{i,j}(\beta, \cdot) 1_{[\beta > \mathbf{s}_{i,k}]} | t_i] \\
&\quad E[u_{i,j}(\cdot) | t_i, \mathbf{s}_{i,j} = \beta] f_{\mathbf{s}_{i,k}}(\beta | t_i)
\end{aligned}$$

Finally, by the Lebesgue Theorem,

$$\begin{aligned} \Pi_i(t_i, b_0, b_i, \mathbf{b}_{-i}) &= \int_{[b_0, j, b_{i, j})} \partial_{b_{i, j}} \Pi_i(t_i, b_0, (\beta, b_{i, -j}), \mathbf{b}_{-i}) d\beta + \\ &\int_{[b_0, j, b_{i, j})} E[u_{i, j}(\cdot) | t_i, \mathbf{s}_{i, k} = \beta] F_{\mathbf{s}_{i, j}}^\perp(\beta | t_i) \\ &\sum_{k=1}^K \int_{T_{-i}} a_{i, k}(\circ) u_{i, k}(\cdot) 1_{[b_{i, k} = \mathbf{s}_{i, k}]} \tau_{-i}(dt_{-i} | t_i) \end{aligned}$$

This concludes the proof.

5.2 Proof of Proposition 2

Proof. (i) Since B is compact and $\Pi_i(t_i, \cdot, \mathbf{b}_{-i})$ is continuous if $F_{\mathbf{b}_{-i}}(\cdot)$ is absolutely continuous, the conclusion is immediate.

(ii) We will make use of the expression:

$$\begin{aligned} \partial_{b_{i, j}} \Pi_i(t_i, b_0, (\beta, b_{i, -j}), \mathbf{b}_{-i}) &= E[\partial_{b_{i, j}} u_{i, 0}(\beta, \cdot) | t_i] \\ &+ \sum_{k \neq j} E[\partial_{b_{i, j}} u_{i, k}(\beta, \cdot) 1_{[b_{i, k} > \mathbf{s}_{i, k}]} | t_i] \\ &+ E[\partial_{b_{i, j}} u_{i, j}(\beta, \cdot) 1_{[\beta > \mathbf{s}_{i, j}]} | t_i] \\ &+ E[u_{i, j}(\cdot) | t_i, \mathbf{s}_{i, j} = \beta] f_{\mathbf{s}_{i, j}}(\beta | t_i). \end{aligned}$$

Since $b_i^1 > b_i^2$, we can choose a curve $\alpha : [0, 1] \rightarrow B$, such that $\alpha(0) = b_i^2$, $\alpha(1) = b_i^1$, such that

$$\alpha'_j(s) \geq 0, \forall j, s \in [0, 1] \text{ and } \exists j \text{ such that } \alpha'_j(s) > 0, \forall s \in [0, 1]. \quad (10)$$

By Assumption, $\partial_{b_{i, j}} u_{i, k}(t_i^0, \cdot) \leq \partial_{b_{i, j}} u_{i, k}(t_i^1, \cdot)$ for all k . Thus,

$$E[\partial_{b_{i, j}} u_{i, k}(t_i^0, \cdot) 1_{[\alpha_k(s) > \mathbf{s}_{i, k}]}] \leq E[\partial_{b_{i, j}} u_{i, k}(t_i^1, \cdot) 1_{[\alpha_k(s) > \mathbf{s}_{i, k}]}]. \quad (11)$$

Since $[0, 1]^{n-1}$ and \mathcal{B}^n are compact and u_i is (absolutely) continuous, there exists $\delta > 0$ such that $u_{i, k}(t_i^1, t_{-i}, b) +$

$2\delta < u_{i,k}(t_i^2, t_{-i}, b)$ for all $t_{-i} \in [0, 1]^{n-1}$, all $b \in B^n$ and all k . For fixed bid $\beta \in B$ and j , define the functions

$$\begin{aligned} g^1(t_{-i}) &= u_{i,j}(t_i^1, t_{-i}, \beta, \mathbf{b}_{-i}(t_{-i})), \text{ and} \\ g^2(t_{-i}) &= u_{i,j}(t_i^2, t_{-i}, \beta, \mathbf{b}_{-i}(t_{-i})). \end{aligned}$$

Then, $g^1(t_{-i}) + 2\delta < g^2(t_{-i})$. By the positivity of conditional expectations,¹⁵

$$E[g^2 - g^1 - 2\delta | \mathbf{s}_{i,j} = \beta] \geq 0.$$

Thus, from the independence of types, we conclude that

$$E[u_{i,j}(t_i^1, \cdot) | t_i, \mathbf{s}_{i,j} = \beta] + \delta < E[u_{i,j}(t_i^2, \cdot) | t_i, \mathbf{s}_{i,j} = \beta]. \quad (12)$$

Then, (11), (12), and the expression of $\partial_{b_i} \Pi_i(t_i, \beta, \mathbf{b}_{-i})$ given by the characterization Lemma imply that for almost all β ,

$$\nabla_{b_i} \Pi_i(t_i^2, \alpha(s), \mathbf{b}_{-i}) > \nabla_{b_i} \Pi_i(t_i^1, \alpha(s), \mathbf{b}_{-i}) + \delta f_{\mathbf{s}}(\alpha(s)), \quad (13)$$

where $f_{\mathbf{s}}(\alpha(s))$ denotes the vector $(f_{\mathbf{s}_{i,1}}(\alpha_1(s)), \dots, f_{\mathbf{s}_{i,K}}(\alpha_K(s)))$. The assumption on the distribution implied by \mathbf{b}_{-i} allow to write the difference $\Pi_i(t_i^2, b_i^1, \mathbf{b}_{-i}) - \Pi_i(t_i^2, b_i^2, \mathbf{b}_{-i})$ as an integral:

$$\begin{aligned} & \Pi_i(t_i^2, b_i^1, \mathbf{b}_{-i}) - \Pi_i(t_i^2, b_i^2, \mathbf{b}_{-i}) \\ &= \int_{[0,1]} \nabla_{b_i} \Pi_i(t_i^2, b_0, \alpha(s), \mathbf{b}_{-i}) \cdot \alpha'(s) ds \\ &> \int_{[0,1]} \nabla_{b_i} \Pi_i(t_i^1, b_0, \alpha(s), \mathbf{b}_{-i}) \cdot \alpha'(s) ds + \delta \sum_j \int_{[0,1]} f_{\mathbf{s}_{i,j}}(\alpha_j(s)) \alpha_j'(s) ds \\ &\geq \delta \sum_j \int_{[0,1]} f_{\mathbf{s}_{i,j}}(\alpha_j(s)) \alpha_j'(s) ds \\ &\geq 0, \end{aligned}$$

where the first inequality comes from (10) and (13); the second comes from the fact that $b_i^1 \in \Theta_i(t_i^1, \mathbf{b}_{-i})$, that is,

$$\Pi_i(t_i^1, b_i^1, \mathbf{b}_{-i}) - \Pi_i(t_i^1, b_i^2, \mathbf{b}_{-i}) = \int_{[0,1]} \nabla_{b_i} \Pi_i(t_i^1, b_0, \alpha(s), \mathbf{b}_{-i}) \cdot \alpha'(s) ds \geq 0;$$

¹⁵See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.

and the third comes from $\alpha'_j(s) \geq 0$ for all j . Now, this implies that $\Pi_i(t_i^2, b_i^1, \mathbf{b}_{-i}) > \Pi_i(t_i^2, b_i^2, \mathbf{b}_{-i})$, which contradicts the fact that $b_i^2 \in \Theta_i(t_i^2, \mathbf{b}_{-i})$. ■

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