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REPRESENTATIONS AND IDENTITIES FOR HOMOGENEOUS TECHNOLOGIES[1]

MIGUEL ESPINOSA
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Abstract

Using up to nine different ways to represent a homogeneous technology, this paper proves explicit one to one identities between most of those different representations of a technology, outlining the homogeneity properties of each representation. These identities, which allow to shift from one representation of a technology to another -and which are summarized in a matrix of identities - can be useful since they provide a tool to obtain explicit functional forms for homogeneous technologies. They can also be useful to simplify computational procedures when different representations of a technology are needed. Finally, the document also refers explicitly to some aspects of producer theory that are often neglected or treated in a marginal way in the literature, such as the inverse supply, the non conditional cost and the inverse input demand functions.

Keywords: Identities, homogeneous production functions and firm theory.

JEL Classification: D20, D21, D24

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REPRESENTACIÓN E IDENTIDADES PARA TECNOLOGÍAS HOMOGÉNEAS[2]

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HERNÁN VALLEJO

Resumen

Usando hasta nueve formas diferentes para representar una tecnología homogénea, este documento prueba identidades uno a uno entre la mayoría de esas diferentes representaciones de una tecnología, resaltando las propiedades de cada representación en términos de su homogeneidad. Estas identidades, que permiten pasar de una representación de la tecnología a otra –y las cuales son resumidas en una matriz de identidades-, proveen una herramienta útil para obtener formas funcionales explícitas de tecnologías homogéneas. También pueden ser útiles para simplificar procedimientos computacionales cuando se requieren diferentes representaciones de una tecnología. Finalmente, el documento también hace referencia explícita a algunos aspectos de la teoría de la firma que son ignorados o tratados de forma marginal en la literatura, tales como la función de oferta inversa, la de costos no condicionada y las de demanda inversa de insumos.

Palabras clave: Identidades, funciones de producción homogéneas y teoría de la firma.

Clasificación JEL: D20, D21, D24

[2] Agradecemos mucho el apoyo de la Universidad de los Andes y su Facultad de Economía. Dirección electrónica de contacto: migu-esp@ uniandes.edu.co, j-bonald@ uniandes.edu.co y hvallejo@ uniandes.edu.co. Los nombres de los autores aparecen en orden aleatorio.

1 Introduction

This paper makes contributions that can be classified in three groups. First, homogeneous technologies are represented in three different ways -the production, cost and conditional input demand functions–, and technologies that are homogeneous of degree less than one are represented in nine different ways -the previous three plus the profit, non conditional input demand, supply, inverse supply, non conditional cost and inverse input demand functions-, supported on duality theory and identities. Homogeneity properties of the different ways in which technologies are represented are also outlined.

Second, one to one identities between the explicit functional forms of most of the representations of the technologies considered in this paper, are proposed and proved, along with two propositions on the homogeneity of cost functions. These identities -which are summarized in a matrix of identities- can be useful in econometric applications, since they provide a tool to obtain explicit functional forms of technologies from observable data on a range of variables. They can also be useful to simplify computational procedures when different representations of a technology are required.

Finally, the document also refers explicitly to some aspects of producer theory that are often neglected or treated in a marginal way in the literature, such as the inverse supply, the non conditional cost, the inverse input demand functions.

The paper is organized as follows: the next section presents a revision of the previous literature on identities within the theory of the firm. This is followed by the theoretical framework including the formal definitions of the nine different ways used to represent a technology, and the presentation of the different identities between representations of a technology and their proposed demonstrations. Then, results are summarized using a matrix of identities. The paper ends with some conclusions.

2 PREVIOUS LITERATURE

Many authors have made important contributions on duality in the theory of the firm. Most of what was done until the mid 1970s has been compiled and explained in detail by Fuss and McFadden [1978]. These authors worked on duality theorems and results linking the production, the profit and the cost functions, which are instruments commonly used in the literature to represent a technology.

Some results on duality between production and costs were obtained by Samuelson [1947], Shephard [1953], Uzawa [1964], Diewert [1973 and 1974] and McFadden [1978a]. Essentially, these authors derived the properties of the cost functions that are obtained by minimization of the total cost given a production set, an input requirement set or a production function, and determined the conditions on the production sets, input requirement sets or production functions, under which they can be uniquely described by the corresponding cost function. Shephard [1953] established well known links between the cost functions and the conditional input demand functions.

Some results between the production set -or the production function- and the profit function have been obtained by Jorgenson and Lau [1974], Lau [1978] and McFadden [1978b]. Hotelling [1932] also found well known results between the profit function and the supply and non-conditional input demand functions. Jorgenson and Lau [1974] studied the case where demands that maximize profits may not be unique, while Lau [1974] and Chambers [1988] presented duality results between cost functions and profit functions.

The standard literature has identified many different ways to represent a technology beyond the production function, the cost function and the profit function. These include the already mentioned production set and the input requirement set, along with representations such as those outlined in McFadden [1978a, p. 24, 37, 77, 92 and 116]: the distance function -as in Shephard [1953, p. 6] and Hanoch [1978 p. 113]-, the factor price requirement set, the Gauge function, the price possibility set, and the indirect production function. Furthermore, McFadden [1978b], Diewert [1973] and Lau [1974] have suggested alternative functional forms for profit functions.

Empirical applications of some duality theorems and results have been made by authors such as Appelbaum and Harris [1977], Woodland [1977], Epstein [1978] and Kohli [1978]. In fact, it is common in the economics theoretical and empirical literature to use homogeneous production functions, for example those included in Fare *et al.* [1989], such as the Transcendental, Translog, Constant Elasticity of Substitution, Cobb Douglas, Leontief and linear production functions.

Identities to shift from some ways to represent a technology to another way to represent such technology have long been proposed in the economic theory literature. Identity maps have been proposed for the theory of the firm and the theory of the household by Madden [1987, p. 347][1]. However, such map does not present all the possible identities between the five representations of technology that are used, and requires in some cases -as is usual in the literature- two representations of a technology to obtain another representation of such technology.

[1]Madden [1987, p. 353] also proposed a similar map for the theory of the household, including the indirect utility function, the expenditure function, the marshallian demands and the hicksian demands. Such map is also reproduced by authors such as Deaton and Muellbauer [1991] and Mas Collel *et al.* [1995, p. 75]

3 THEORETICAL FRAMEWORK

As stated before, this paper represents a homogeneous technology in three different ways: the production, cost, and conditional input demand functions. This paper also represents a technology that is homogeneous of degree less than one in nine different ways: the production, profit, cost, supply, inverse supply, conditional input demand and non-conditional input demand functions.

3.1 General Assumptions

Throughout this paper it is assumed that all good and input markets are perfectly competitive. It is also assumed througout that there are no fixed inputs and that the production process generates only one output.

3.2 Definitions

3.2.1 Production Function

For a firm producing a single output using possibly more than one input, its technology can be described by a production function expressing the maximum level of output that can be achieved by the firm for each vector of inputs.

From now on, it will be assumed that the production function $f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$

satisfies the following conditions:

C.1 $f(\cdot)$ is a continuous function.

C.2 $f(\cdot)$ is homogeneous of degree γ, i.e., for all $t > 0$, $f(t\mathbf{x}) = t^{\gamma} f(\mathbf{x})$, where $\gamma \geq 0$ is the degree of homogeneity.

C.3 $f(\cdot)$ is not decreasing in $\mathbf{x}$, i.e., if $\mathbf{x}_1 \geq \mathbf{x}_2$ then $f(\mathbf{x}_1) \geq f(\mathbf{x}_2)$[2]

C.4 $f(\cdot)$ is a strictly concave function over $\mathbb{R}^n_+$[3].

3.2.2 Cost Function

Given a production function $f(\cdot)$, the cost function expressing the minimum cost at which a firm can achieve a fixed level of production $y \in \mathbb{R}_+$, taking the input prices $\mathbf{w} \in \mathbb{R}^n_{++}$ as given, can be defined as:

[2] For $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, $\mathbf{x}_1 \geq \mathbf{x}_2$ if and only if $\mathbf{x}_{1i} \geq \mathbf{x}_{2i}$ for all $i = 1, ..., n$.

[3] This rules out Leontief and Perfect Substitutes production funcitons, but includes CES production functions, and Leontief and Perfect Substitutes are limits of CES production functions. Although this condition is quite strong, it is used to ensure unicity of the conditional and non conditional input demands.

$$c(y,\mathbf{w}) = \min_{\mathbf{x}\epsilon\mathbb{R}^n_+} \{\mathbf{w}\cdot\mathbf{x} : f(\mathbf{x}) \geq y\} \tag{1}$$

As it will be shown soon, such a minimum always exists, so the cost function is well defined.

3.2.3 Conditional Input Demands

Correspondingly, the input vector that minimizes costs can also be expressed as a function of y and $\mathbf{w}$. Such function is known as the conditional input demands function and can be defined as:

$$\mathbf{x}(y,\mathbf{w}) = \arg\min_{\mathbf{x}\epsilon\mathbb{R}^n_+} \{\mathbf{w}\cdot\mathbf{x} : f(\mathbf{x}) \geq y\} \tag{2}$$

It follows directly from the previous definitions that

$$c(y,\mathbf{w}) = \mathbf{w}\cdot\mathbf{x}(y,\mathbf{w}) \tag{3}$$

It should be noted that this function is well defined, i.e., that a unique minimum exists for all $\mathbf{w} \in\mathbb{R}^n_{++}$ and $y\geq 0$. In fact, since $f(\cdot)$ is continuous, the set $S = \left\{\mathbf{x} \in\mathbb{R}^n_+ : f(\mathbf{x}) \geq y\right\}$ is closed and, since $f(\cdot)$ is not decreasing, it is clear that there exists a $k > 0$ large enough to guarantee that the halfspace described by $\mathbf{w}\cdot\mathbf{x} \leq k$ intersects it. Both S and the halfspace are closed sets and so it is its intersection, which is also bounded, since it is contained in the set $\left\{\mathbf{x} \in\mathbb{R}^n_+ : \mathbf{w}\cdot\mathbf{x} \leq k\right\}$. Then, the set $S \cap \left\{\mathbf{x} \in\mathbb{R}^n_+ : \mathbf{w}\cdot\mathbf{x} \leq k\right\} = \left\{\mathbf{x} \in\mathbb{R}^n_+ : \mathbf{w}\cdot\mathbf{x} \leq k \text{ and } f(\mathbf{x}) \geq y\right\}$ is compact since it is closed and bounded.

The dot product is a continuous function so it attains a minimum $\mathbf{x}^*$ in the compact set

$$\left\{\mathbf{x} \in\mathbb{R}^n_+ : \mathbf{w}\cdot\mathbf{x} \leq k \text{ and } f(\mathbf{x}) \geq y\right\}[4]$$

and, by construction, $\mathbf{w}\cdot\mathbf{x}^* \leq \mathbf{w}\cdot\mathbf{x}$ for $\mathbf{x}$ all such that $f(\mathbf{x}) \geq y$. It follows that $\mathbf{w}\cdot\mathbf{x}$ always attains a minimum at S so the function $c(y,\mathbf{w})$ is well defined. However, the conditional input demands were defined as a function instead of as a correspondence, so it remains to be proved that the solution to the cost minimization problem is unique.

Suppose there are two different vectors $\mathbf{x}$, $\mathbf{x}^* \in \mathbb{R}^n_+$ that minimize costs at given input prices $\mathbf{w}$ and for a fixed level of production $y > 0$, then $f(\mathbf{x}) \geq y$, $f(\mathbf{x}^*) \geq y$. Let $\lambda \in (0,1)$, since $f(\cdot)$ is strictly concave it is also strictly quasiconcave so it follows that $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}^*) > y$, and it is clear that $\mathbf{w}\cdot(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}^*) = \mathbf{w}\cdot\mathbf{x} = \mathbf{w}\cdot\mathbf{x}^*$. It has been assumed that $y > 0$ (the case where $y = 0$ is trivial, in fact, in such case the only input demands that minimize costs are $\mathbf{x} = 0$) so there is some i such that $\lambda x_i + (1-\lambda)x_i^* > 0$. Defining

[4] The extreme value theorem, due to Weierstrass, states that any continuous function from a compact set to the real numbers attains a minimum (and a maximum).

$\mathbf{x}^{\varepsilon} \in \mathbb{R}_{+}^{n}$ as the vector such that $x_j^{\varepsilon} = \lambda x_j + (1-\lambda)\, x_j^*$ for all $i \neq j$ and $x_i^{\varepsilon} = \lambda x_i + (1-\lambda)\, x_i^* - \varepsilon$, then, by the continuity of $f(\cdot)$, there must exist an $\varepsilon > 0$ such that $f(\mathbf{x}^{\varepsilon}) > y$, but clearly $\mathbf{w} \cdot \mathbf{x}^{\varepsilon} < \mathbf{w} \cdot \mathbf{x}$, which contradicts the fact that $\mathbf{x}$ minimizes costs. Thus, the solution to the cost minimization problem is unique.

3.2.4 Profit Function

The profit function expresses the maximum profits that the firm can achieve as a function of the product price and the input prices. If the production function $f(\cdot)$ is known, it can be defined as:

$$\Pi(p, \mathbf{w}) = \max_{\mathbf{x} \epsilon \mathbb{R}_{+}^{n}} pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} \tag{4}$$

In order for this function to be well defined, such a maximum must exist for all $\mathbf{w} \in \mathbb{R}_{++}^{n}$ and $p > 0$, as it will be shown later, but first it is necessary to introduce here a result concerning concave functions taken from the field of convex analisys.

Definition 1 *A direction of recession of a concave function h is a non zero vector $\mathbf{e}$ such that $h(\mathbf{x} + \lambda \mathbf{e}) \geq h(\mathbf{x})$, for all $\mathbf{x}$ in the domain of h and all $\lambda > 0$.*

Rockafeller [1970] states that if $h : \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}$ is a concave function that has no directions of recession then it attains a maximum.

For fixed p and $\mathbf{w} \in \mathbb{R}_{++}^{n}$, define a function $\pi : \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}$ by $\pi(\mathbf{x}) = pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$. Such function is concave, because $f(\cdot)$ and the dot product are concave functions in $\mathbf{x}$. Let $lev_0 = \left\{\mathbf{x} \in \mathbb{R}_{+}^{n} \mid \pi(\mathbf{x}) \geq 0\right\}$ which is closed and convex by the continuity and the concavity of the function π. If $\mathbf{x} \notin lev_0$ then $\pi(\mathbf{0} + \mathbf{x}) = \pi(\mathbf{x}) \leq 0 = \pi(\mathbf{0})$ so it is not a direction of recession of π. If $\mathbf{x} \in lev_0$ and $\mathbf{x} \neq \mathbf{0}$ then $pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} \geq 0$. It is convenient to consider two cases separately, so first assume that $\mathbf{x} \neq \mathbf{0}$ and $pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} = 0$. In such case $p\lambda f(\mathbf{x}) - \lambda \mathbf{w} \cdot \mathbf{x} = 0$ but, if $\lambda > 1$, $f(\lambda \mathbf{x}) = \lambda^{\frac{1}{\gamma}} f(\mathbf{x}) < \lambda f(\mathbf{x})$ and then $pf(\lambda \mathbf{x}) - \mathbf{w} \cdot (\boldsymbol{\lambda} \mathbf{x}) < 0$, so $\mathbf{x}$ is not a direction of recession of π. Finally, suppose that $\mathbf{x} \neq \mathbf{0}$ and $pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} > 0$, then there exists a $k < 1$ such that $kpf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} = 0$ and, for all $\lambda > 0$, $\lambda kpf(\mathbf{x}) - \lambda \mathbf{w} \cdot \mathbf{x} = 0$. Let $\widehat{\lambda} = k^{\frac{\gamma}{1-\gamma}}$, then $\widehat{\lambda} k = \widehat{\lambda}^{\frac{1}{\gamma}}$ and $pf\left(\widehat{\lambda} \mathbf{x}\right) - \mathbf{w} \cdot \left(\widehat{\lambda} \mathbf{x}\right) = \widehat{\lambda} kpf(\mathbf{x}) - \widehat{\lambda} \mathbf{w} \cdot \mathbf{x} = \mathbf{0}$. It follows that $\widehat{\lambda} \mathbf{x}$ is not a direction of recession of π, so, obviously neither is $\mathbf{x}$. Applying the previous theorem, since the concave function $\pi : \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}$ has no directions of recession, it can be concluded that it attains a maximum, and so Π is well defined.

3.2.5 Non Conditional Input Demands

The function that expresses the demands for inputs that maximize profits in terms of the prices of both the product, $p > 0$, and the inputs, $\mathbf{w} \in \mathbb{R}^n_{++}$, called the non conditional input demand functions, can be defined correspondingly as:

$$\mathbf{x}(\mathbf{w}, p) = \underset{\mathbf{x} \epsilon \mathbb{R}^n_+}{\arg\max} pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} \tag{5}$$

It should be shown, given the assumptions on the production function, that the input demands vector that maximizes profits is unique and so the non conditional input demand functions are well defined. In fact, if there where two different vectors $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^n_+$ for which profits attain a maximum then:

$$pf(\mathbf{x}_1) - \mathbf{w} \cdot \mathbf{x_1} = pf(\mathbf{x}_2) - \mathbf{w} \cdot \mathbf{x}_2 \tag{6}$$

And for any $\lambda \in (0,1)$, $\lambda [pf(\mathbf{x}_1) - \mathbf{w} \cdot \mathbf{x_1}] + (1-\lambda)[pf(\mathbf{x}_2) - \mathbf{w} \cdot \mathbf{x}_2] = pf(\mathbf{x}_2) - \mathbf{w} \cdot \mathbf{x}_2$ then $p[\lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)] - \mathbf{w}[\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x}_2] = pf(\mathbf{x}_2) - \mathbf{w} \cdot \mathbf{x}_2$. For a given $\lambda \in (0,1)$, the profits corresponding to the convex combination between $\mathbf{x}_1$ and $\mathbf{x}_2$ are $pf(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) - \mathbf{w}[\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x}_2]$ and since $f(\cdot)$ is strictly concave $pf(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) > p[\lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)]$ and then: $pf(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) - \mathbf{w}[\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x}_2] > pf(\mathbf{x}_2) - \mathbf{w} \cdot \mathbf{x}_2$, which contradicts the fact that $\mathbf{x}_2$ maximizes profits. Thus, the non conditional input demands are well defined.

3.2.6 Supply Function

Given the product and input prices, and the production function, the supply function can be thought as describing the level of product that can be attained if the amount of inputs hired by the firm equals the profit maximizing demands, and it can be defined by:

$$y(p, \mathbf{w}) = f(\mathbf{x}(p, \mathbf{w})) \tag{7}$$

3.2.7 Inverse Supply Function

For a given $\mathbf{w} \in \mathbb{R}^n_{++}$, the supply function can be described as a mapping from D to $\mathbb{R}_{++}$, were $D = \{p \in \mathbb{R}_{++} : \; y(p, \mathbf{w}) > 0\}$ is a restriction of the original domain of the supply function, such that it is injective and so it has an inverse. Then, the inverse supply function can be implicitly defined by:

$$p(y, \mathbf{w}) = p \iff y(p, \mathbf{w}) = y$$

To veriry that this inverse function actually exists, let $y(p_1, \mathbf{w}) = y(p_2, \mathbf{w})$. Since $y(\cdot)$ is homogeneous of degree $\frac{\gamma}{1-\gamma}$ in p, as it will be shown later in corollary 2, then $p_1^{\frac{\gamma}{1-\gamma}}\ y(1, \mathbf{w}) = p_2^{\frac{\gamma}{1-\gamma}} y(1, \mathbf{w})$ and, by assumption, $y(p, \mathbf{w}) > 0$, so $p_1 = p_2$. It follows that, for a given $\mathbf{w} \in \mathbb{R}^n_{++}$, the supply function is injective, so it has an inverse.

3.2.8 Non-Conditional Cost Function

The non-conditional costs $c(p, \mathbf{w})$, can be defined as a function giving the cost corresponding to the input demands for which profits are maximized when the given prices of the output and the inputs are p and $\mathbf{w}$, respectively. Formally,

$$c(p, \mathbf{w}) = \mathbf{w} \cdot \mathbf{x}(p, \mathbf{w}) \tag{8}$$

3.2.9 Inverse Input Demands

The inverse input demand can be defined as a function giving the input price vector for which $\mathbf{x}$ are the input demands that maximize profits, given an output price p. Formally, $\mathbf{w}(p, \mathbf{x}) = \mathbf{w}$ if and only if $\mathbf{x}(p, \mathbf{w}) = \mathbf{x}$.

It must be verified that the function $\mathbf{w}(p, \mathbf{x})$ is well defined. Since it is the inverse of $\mathbf{x}(p, \mathbf{w})$, taking p as a constant, it is enough to show that the non-conditional demands are an injective function of the input price vector. To do so, suppose that $\mathbf{x}(p, \mathbf{w}) = \mathbf{x}(p, \mathbf{v}) = \mathbf{x}$, for $\mathbf{w}, \mathbf{v} \in \mathbb{R}^n_{++}$, then: $pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} \geq pf(\mathbf{z}) - \mathbf{w} \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^n_{+}$, and $pf(\mathbf{x}) - \mathbf{v} \cdot \mathbf{x} \geq pf(\mathbf{z}) - \mathbf{v} \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^n_{+}$. Substracting the last two expressions, it follows that $(\mathbf{v} - \mathbf{w}) \cdot \mathbf{x} \geq (\mathbf{v} - \mathbf{w}) \cdot \mathbf{z}$ and $(\mathbf{w} - \mathbf{v}) \cdot \mathbf{x} \geq (\mathbf{w} - \mathbf{v}) \cdot \mathbf{z}$ so $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{x} - \mathbf{z}) = 0$ for all $\mathbf{z} \in \mathbb{R}^n_{+}$, but this can happens only if $\mathbf{v} = \mathbf{w}$.

Thus, for p constant the function $\mathbf{x}(p, \mathbf{w})$ is injective in $\mathbf{w}$, so it has an inverse function, namely, $\mathbf{w}(p, \mathbf{x})$.

4 IDENTITIES BETWEEN REPRESENTATIONS OF A TECHNOLOGY

Identities are defined in this paper as equations by means of which an explicit functional form of a representation of a technology is expressed as the explicit functional form of other representation(s) of that technology. In order to proof such identities, the following well known propositions are required and proved here for heuristic purposes.

Proposition 1 *If the production function is homogeneous of degree γ (i.e $f(t\mathbf{x}) = t^{\gamma} f(\mathbf{x})$), as has been assumed in this article, the cost function is homogeneous of degree $\frac{1}{\gamma}$ in y (and so it can be written as $c(y, \mathbf{w}) = y^{\frac{1}{\gamma}} c(1, \mathbf{w})$.*

Proof. Taking the price level as fixed, and letting $\mathbf{x}$ be the input vector that minimizes costs for a given level of production y, it must be proved that $t^{\frac{1}{\gamma}}\mathbf{x}$ minimizes the costs for the level of production ty, and so $c(ty, \mathbf{w}) = \mathbf{w}\cdot \left(t^{\frac{1}{\gamma}}\mathbf{x}\right) = t^{\frac{1}{\gamma}}\mathbf{w}\cdot\mathbf{x} = t^{\frac{1}{\gamma}} c(y, \mathbf{w})$. To demonstrate this by contradiction, suppose that $t^{\frac{1}{\gamma}}\mathbf{x}$ does not minimize costs when the level of production is fixed at ty. This is the same as stating that there exists a $\widetilde{\mathbf{x}}$ such that $\mathbf{w}\cdot\widetilde{\mathbf{x}} < \mathbf{w}\cdot \left(t^{\frac{1}{\gamma}}\mathbf{x}\right)$and $f(\widetilde{\mathbf{x}}) \geq ty$. It follows that $\mathbf{w}\cdot\left(\frac{\widetilde{\mathbf{x}}}{t^{\frac{1}{\gamma}}}\right) < \mathbf{w}\cdot\mathbf{x}$, but since the production function is homogeneous of degree γ then $f\left(\frac{\widetilde{\mathbf{x}}}{t^{\frac{1}{\gamma}}}\right) = \frac{1}{\gamma} f(\widetilde{\mathbf{x}}) \geq y$, which contradicts the fact that $\mathbf{x}$ minimizes costs with the level of production y. ■

Defining the average and marginal cost by: $ac(y, \mathbf{w}) = \frac{c(y,\mathbf{w})}{y}$, and $mc(y, \mathbf{w}) = \frac{\partial c(y,\mathbf{w})}{\partial y}$, the next proposition is a straight forward consequence of proposition 1.

Proposition 2 *The cost function is homogeneous of degree $\frac{1}{\gamma}$ in y if and only if the ratio of average to marginal cost equals γ.*

Proof. First, note that if the cost function is homogeneous of degree $\frac{1}{\gamma}$ in y then, $\frac{ac(y,\mathbf{w})}{mc(y,\mathbf{w})} = \frac{y^{\frac{1}{\gamma}-1}c(1,\mathbf{w})}{\frac{1}{\gamma}y^{\frac{1}{\gamma}-1}c(1,\mathbf{w})} = \gamma$. Now, if $\frac{ac(y,\mathbf{w})}{mc(y,\mathbf{w})} = \gamma$ then $\frac{1}{\gamma}c(y, \mathbf{w}) = y\frac{\partial c(y,\mathbf{w})}{\partial y}$ by (the converse of) Euler's theorem[5] the cost function is homogeneous of degree $\frac{1}{\gamma}$.) ■

Theorem 1 *If the production function satisfies conditions C.1 to C.4, then the next identities hold for all $y, p > 0$:*

I.1 $\quad y(p, \mathbf{w}) = \left[\frac{\gamma p}{c(1,\mathbf{w})}\right]^{\frac{\gamma}{1-\gamma}}$

I.1' $\quad c(y, \mathbf{w}) = [y(p, \mathbf{w})]^{\frac{\gamma-1}{\gamma}} y^{\frac{1}{\gamma}} p\gamma$

I.2 $\quad c(y, \mathbf{w}) = \gamma y p(y, \mathbf{w})$

I.3 $\quad \Pi(p, \mathbf{w}) = y(p, \mathbf{w})[1-\gamma] p$

I.4 $\quad p(y, \mathbf{w}) = p\left[\frac{y(p,\mathbf{w})}{y}\right]^{\frac{\gamma-1}{\gamma}}$

[5]Euler's theorem states that a differentiable function is homogeneous of degree γ if and only if $\sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i = \gamma f(\mathbf{x})$, or, in a more concise fashion, $\nabla f(\mathbf{x})\cdot\mathbf{x} = \gamma f(\mathbf{x})$.

I.4’ $p(y,\mathbf{w}) = \left[\frac{y(1,\mathbf{w})}{y}\right]^{\frac{\gamma-1}{\gamma}}$ **or** $y(p,\mathbf{w}) = p\left[\frac{p(1,\mathbf{w})}{p}\right]^{\frac{\gamma}{\gamma-1}}$

I.5 $\Pi(p,\mathbf{w}) = [p(y,\mathbf{w})]^{\frac{\gamma-1}{\gamma}} yp^{\frac{\gamma}{1-\gamma}} [1-\gamma]$

I.6 $c(p,\mathbf{w}) = y(p,\mathbf{w})\, p\gamma$

I.7 $c(p,\mathbf{w}) = [c(y,\mathbf{w})]^{\frac{\gamma}{\gamma-1}} (yp\gamma)^{\frac{1}{1-\gamma}}$

I.8 $c(p,\mathbf{w}) = \frac{\Pi(p,\mathbf{w})}{1-\gamma}\gamma$

I.9 $c(p,\mathbf{w}) = [p(y,\mathbf{w})]^{\frac{\gamma-1}{\gamma}} yp^{\frac{1}{1-\gamma}}\gamma$

I.10 $c(y,\mathbf{w}) = \left[\frac{\Pi(p,\mathbf{w})}{1-\gamma}\right]^{\frac{\gamma-1}{\gamma}} (yp)^{\frac{1}{\gamma}}\gamma$

I.11 $x_i(y,\mathbf{w}) = x_i\left(\left[\frac{\mathbf{w}\cdot\mathbf{x}(p,\mathbf{w})}{yp^{\frac{1}{1-\gamma}}\gamma}\right]^{\frac{\gamma}{\gamma-1}}, \mathbf{w}\right)$

I.12 $x_i(p,\mathbf{w}) = x_i\left(\left[\frac{\mathbf{w}\cdot\mathbf{x}(y,\mathbf{w})}{\left[y^{\frac{1}{\gamma}}p\gamma\right]}\right]^{\frac{\gamma}{\gamma-1}}, \mathbf{w}\right)$

Proof.

Given an output price p, the profit maximizing level of production of a competitive firm is determined by the equation $mc(y,\mathbf{w}) = p$, that can be equivalently expressed as: $\frac{ac(y,\mathbf{w})}{\gamma} = p$ or $\frac{c(y,\mathbf{w})}{y\gamma} = p$, by proposition 2. Evaluating this expression in $y(p,\mathbf{w})$, find:

$$\frac{c(y(p,\mathbf{w}),\mathbf{w})}{y(p,\mathbf{w})\gamma} = p. \tag{9}$$

Assuming that $\gamma < 1$, the production function has decreasing returns to scale and the profit function can be defined as (4). It is easy to see that this equation can be expressed as $\Pi(p,\mathbf{w}) \equiv pf(x(p,\mathbf{w})) - c(y(p,\mathbf{w}),\mathbf{w})$ because $c(p,\mathbf{w}) = c(y(p,\mathbf{w}),\mathbf{w})$. Rearranging terms in equation (9), obtain $\frac{c(y(p,\mathbf{w}),\mathbf{w})}{p\gamma} = y(p,\mathbf{w})$ and then the profit function can be expressed $\Pi(p,\mathbf{w}) = \frac{c(y(p,\mathbf{w}),\mathbf{w})}{\gamma} - c(y(p,\mathbf{w}),\mathbf{w})$ and (10) is easily obtained.

$$\Pi(p,\mathbf{w}) = \frac{1-\gamma}{\gamma}c(y(p,\mathbf{w}),\mathbf{w}) \tag{10}$$

It follows that

$$c(y,\mathbf{w}) = \frac{\gamma\Pi(p(y,\mathbf{w}),\mathbf{w})}{1-\gamma} \tag{11}$$

Evaluating 10 at $p(y,\mathbf{w})$ and solving for the cost function, obtain

$$y(p,\mathbf{w}) = \left[\frac{\gamma p}{c(1,\mathbf{w})}\right]^{\frac{\gamma}{1-\gamma}} \tag{I.1}$$

Given perfectly competitive markets, profits are maximized where the marginal cost equals the price of the product, so the inverse supply function, expressing the profit maximizing price given a production level and an input prices vector, can be identified with the marginal cost function. It follows that $p(y,\mathbf{w}) = \frac{1}{\gamma} y^{\frac{1-\gamma}{\gamma}} c(1,\mathbf{w})$. Evaluating this equation at $y(p,\mathbf{w})$, to obtain $p(y(p,\mathbf{w}),\mathbf{w}) = \frac{1}{\gamma} y(p,\mathbf{w})^{\frac{1-\gamma}{\gamma}} c(1,\mathbf{w})$, and solving for $y(p,\mathbf{w})$, obtain I.1, it follows inmedietely that

$$c(y,\mathbf{w}) = [y(p,\mathbf{w})]^{\frac{\gamma-1}{\gamma}} y^{\frac{1}{\gamma}} p\gamma \tag{I.1'}$$

As pointed out earlier, $mc(y,\mathbf{w}) = p(y,\mathbf{w})$, and $\frac{ac(y,\mathbf{w})}{mc(y,\mathbf{w})} = \gamma$, then $\frac{c(y,\mathbf{w})}{\gamma} = ac(y,\mathbf{w}) = yp(y,\mathbf{w})$ and it follows that

$$c(y,\mathbf{w}) = \gamma yp(y,\mathbf{w}) \tag{I.2}$$

Evaluating I.2 at $y(p,\mathbf{w})$, obtain: $c(y(p,\mathbf{w}),\mathbf{w}) = \gamma y(p,\mathbf{w}) p(y(p,\mathbf{w}),\mathbf{w}) = \gamma y(p,\mathbf{w}) p$. Now substituting this expression into identity 10 and simplifying, find

$$\Pi(p,\mathbf{w}) = y(p,\mathbf{w})[1-\gamma] p \tag{I.3}$$

Note that I.1 can be expressed as $y(p,\mathbf{w}) = \left[\frac{\gamma p}{c(1,\mathbf{w})}\right]^{\frac{\gamma}{1-\gamma}} = \left[\frac{\gamma p}{y^{-\frac{1}{\gamma}} c(y,\mathbf{w})}\right]^{\frac{\gamma}{1-\gamma}}$. Using I.2 and solving for $p(y,\mathbf{w})$ yields

$$p(y,\mathbf{w}) = p\left[\frac{y(p,\mathbf{w})}{y}\right]^{\frac{\gamma-1}{\gamma}} \tag{I.4}$$

It is worth observing that each side of this identity seems to be depending on different variables. However, since the supply and the inverse supply functions are homogeneous of degree $\frac{\gamma}{1-\gamma}$ and $\frac{1-\gamma}{\gamma}$ in p and y, respectively, as will be shown in corollary 2, $y(p,\mathbf{w}) = p^{\frac{\gamma}{1-\gamma}} y(1,\mathbf{w})$ and $p(y,\mathbf{w}) = y^{\frac{1-\gamma}{\gamma}} p(1,\mathbf{w})$, so the identity can be written as:[6]

$$p(y,\mathbf{w}) = \left[\frac{y(1,\mathbf{w})}{y}\right]^{\frac{\gamma-1}{\gamma}} \quad \text{or} \quad y(p,\mathbf{w}) = \left[\frac{p(1,\mathbf{w})}{p}\right]^{\frac{\gamma}{\gamma-1}} \tag{I.4'}$$

Solving for $y(p,\mathbf{w})$ in identity I.4 and substituting in I.3, obtain

$$\Pi(p,\mathbf{w}) = [p(y,\mathbf{w})]^{\frac{\gamma-1}{\gamma}} yp^{\frac{\gamma}{1-\gamma}} [1-\gamma] \tag{I.5}$$

[6]Note that the variables that are not arguments of the corresponding functions are cancelled always due to the degree of homogeneity, as in I.1', I.4, I.5, I.7, I.9, I.10, I.11, I.12, I.14, I.16, I.18, and I.19.

By (4) and I.3,

$$c\left(p,\mathbf{w}\right)=y\left(p,\mathbf{w}\right)p\gamma \tag{I.6}$$

By I.1' and I.6,

$$c\left(p,\mathbf{w}\right)=\left[c\left(y,\mathbf{w}\right)\right]^{\frac{\gamma}{\gamma-1}}\left(yp\gamma\right)^{\frac{1}{1-\gamma}} \tag{I.7}$$

By (11) and I.6, [7]

$$c\left(p,\mathbf{w}\right)=\frac{\Pi\left(p,\mathbf{w}\right)}{1-\gamma}\gamma \tag{I.8}$$

By I.2 and I.8,

$$c\left(p,\mathbf{w}\right)=\left[p\left(y,\mathbf{w}\right)\right]^{\frac{\gamma-1}{\gamma}}yp^{\frac{1}{1-\gamma}}\gamma \tag{I.9}$$

The inverse supply function can be obtained solving from I.2. Solving the supply function from I.3 and replacing these terms in I.4, obtain

$$c\left(y,\mathbf{w}\right)=\left[\frac{\Pi\left(p,\mathbf{w}\right)}{1-\gamma}\right]^{\frac{\gamma-1}{\gamma}}\left(yp\right)^{\frac{1}{\gamma}}\gamma \tag{I.10}$$

By definition, $x_i\left(y\left(p,\mathbf{w}\right),\mathbf{w}\right)=\underset{x\epsilon\mathbb{R}_+^n}{\arg\min}\left\{\mathbf{w}\cdot\mathbf{x}:f\left(\mathbf{x}\right)\geq y\left(p,\mathbf{w}\right)\right\}$ and $\mathbf{x}\left(p,\mathbf{w}\right)=\arg\max\left\{pf\left(\mathbf{x}\right)-\mathbf{w}\cdot\mathbf{x}:\mathbf{x}\in\mathbb{R}_+^n\right\}$, so it must be proved that these two coincide. Demonstrating this by contradiction, suppose that there exists a $\widehat{\mathbf{x}}$ such that $f\left(\widehat{\mathbf{x}}\right)\geq y\left(p,\mathbf{w}\right)$ and $\mathbf{w}\cdot\mathbf{x}<\mathbf{w}\cdot\mathbf{x}\left(p,\mathbf{w}\right)$. Since $f\left(\mathbf{x}\left(p,\mathbf{w}\right)\right)=y\left(p,\mathbf{w}\right)$, then $pf\left(\widehat{\mathbf{x}}\right)-\mathbf{w}\cdot\widehat{\mathbf{x}}\geq pf\left(\mathbf{x}\left(p,\mathbf{w}\right)\right)-\mathbf{w}\cdot\mathbf{x}\left(p,\mathbf{w}\right)$ which contradicts the fact that $\mathbf{x}\left(p,\mathbf{w}\right)=\arg\max\left\{pf\left(\mathbf{x}\right)-\mathbf{w}\cdot\mathbf{x}:\mathbf{x}\in\mathbb{R}_+^n\right\}$.

[7] Equations 10, 11, I.3, and I.8 are similar to those proposed in corollary 1.1 of Lau [1978], except that the profit function in Lau is normalized in the price. Here it is specified that the profit function must be evaluated at the inverse supply function, and that the cost function must be evaluated at the supply function for equations 10 and 11 to hold.

$$x_i(p,\mathbf{w}) = x_i(y(p,\mathbf{w}),\mathbf{w}) \tag{12}$$

Given (12) and identity I.4, it follows directly that $x_i\left(\left[\frac{p(y,\mathbf{w})}{p}\right]^{\frac{\gamma}{\gamma-1}} y, \mathbf{w}\right) = x_i(p,\mathbf{w})$. Since the conditional and the non conditional demands are homogeneous of degree $\frac{1}{\gamma}$ and $\frac{1}{1-\gamma}$ in y and p, respectively, as will be proved later in corollary 2, then $\left[\frac{p(y,\mathbf{w})}{p}\right]^{\frac{1}{\gamma-1}} x_i(y,\mathbf{w}) = x_i(p,\mathbf{w})$ and $x_i(y,\mathbf{w}) = \frac{p^{\frac{1}{1-\gamma}} x_i(1,\mathbf{w})}{\left[\frac{p(y,\mathbf{w})}{p}\right]^{\frac{1}{\gamma-1}}} = x_i(p(y,\mathbf{w}),\mathbf{w})$ .Thus,

$$x_i(y,\mathbf{w}) = x_i(p(y,\mathbf{w}),\mathbf{w}) \tag{13}$$

Substituting I.1 into (13)

$$x_i(p,\mathbf{w}) = x_i\left(\left[\frac{\mathbf{w}\cdot\mathbf{x}(y,\mathbf{w})}{\left[y^{\frac{1}{\gamma}}p\gamma\right]}\right]^{\frac{\gamma}{\gamma-1}}, \mathbf{w}\right) \tag{I.11}$$

Substituting I.9 into (12)

$$x_i(y,\mathbf{w}) = x_i\left(\left[\frac{\mathbf{w}\cdot\mathbf{x}(p,\mathbf{w})}{yp^{\frac{1}{1-\gamma}}\gamma}\right]^{\frac{\gamma}{\gamma-1}}, \mathbf{w}\right) \tag{I.12}$$

Corollary 1 *Given the assumptions of theorem 1, if the representations of a technology are also differentiable in w_i, the following identities hold for $p>0$ y $y>0$:*

I.13 $\quad x_i(p,\mathbf{w}) = -\frac{\partial\Pi(p,\mathbf{w})}{\partial w_i}$

I.14 $\quad x_i(p,\mathbf{w}) = -\frac{\partial[c(y,\mathbf{w})]^{\frac{\gamma}{\gamma-1}}}{\partial w_i}\frac{(1-\gamma)}{\left[(yp)^{\frac{1}{\gamma}}\gamma\right]^{\frac{\gamma}{\gamma-1}}}$

I.15 $\quad x_i(p,\mathbf{w}) = \frac{\partial y(p,\mathbf{w})}{\partial w_i}[1-\gamma]p$

I.16 $\quad x_i(p,\mathbf{w}) = \frac{\partial[p(y,\mathbf{w})]^{\frac{\gamma}{\gamma-1}}}{\partial w_i}yp^{\frac{1}{1-\gamma}}[1-\gamma]$

I.17 $\quad x_i(y,\mathbf{w}) = \frac{\partial c(y,\mathbf{w})}{\partial w_i}$

I.18 $\quad x_i(y,\mathbf{w}) = \frac{\partial[\Pi(p,\mathbf{w})]^{\frac{\gamma-1}{\gamma}}}{\partial w_i}\frac{(yp)^{\frac{1}{\gamma}}\gamma}{(1-\gamma)^{\frac{\gamma-1}{\gamma}}}$

I.19 $\quad x_i(y,\mathbf{w}) = \frac{\partial[y(p,\mathbf{w})]^{\frac{\gamma-1}{\gamma}}}{\partial w_i}\left[y^{\frac{1}{\gamma}}p\gamma\right]$

I.20 $\quad x_i(y,\mathbf{w}) = \gamma y\frac{\partial p(y,\mathbf{w})}{\partial w_i}$

Proof.

According to Hotelling's lemma [1932]

$$x_i\left(p,\mathbf{w}\right)=-\frac{\partial\Pi\left(p,\mathbf{w}\right)}{\partial w_i}\tag{I.13}$$

By Hotelling's Lemma and I.10,

$$x_i\left(p,\mathbf{w}\right)=-\frac{\partial\left[c\left(y,\mathbf{w}\right)\right]^{\frac{\gamma}{\gamma-1}}}{\partial w_i}\frac{\left(1-\gamma\right)}{\left[\left(yp\right)^{\frac{1}{\gamma}}\gamma\right]^{\frac{\gamma}{\gamma-1}}}\tag{I.14}$$

By Hotelling's Lemma and I.3,

$$x_i\left(p,\mathbf{w}\right)=\frac{\partial y\left(p,\mathbf{w}\right)}{\partial w_i}\left[1-\gamma\right]p\tag{I.15}$$

Applying Hotelling's Lemma once more and using I.5

$$x_i\left(p,\mathbf{w}\right)=\frac{\partial\left[p\left(y,\mathbf{w}\right)^{\frac{\gamma}{\gamma-1}}\right]}{\partial w_i}yp^{\frac{1}{1-\gamma}}\left[1-\gamma\right]\tag{I.16}$$

According to Shephard's lemma [1953],

$$x_i\left(y,\mathbf{w}\right)=\frac{\partial c\left(y,\mathbf{w}\right)}{\partial w_i}\tag{I.17}$$

By Shephard's Lemma and using I.10,

$$x_i\left(y,\mathbf{w}\right)=\frac{\partial\left[\Pi\left(p,\mathbf{w}\right)^{\frac{\gamma-1}{\gamma}}\right]}{\partial w_i}\frac{\left(yp\right)^{\frac{1}{\gamma}}\gamma}{\left(1-\gamma\right)^{\frac{\gamma-1}{\gamma}}}\tag{I.18}$$

By Shephard's Lemma and using I.1,

$$x_i\left(y,\mathbf{w}\right)=\frac{\partial\left[y\left(p,\mathbf{w}\right)^{\frac{\gamma-1}{\gamma}}\right]}{\partial w_i}y^{\frac{1}{\gamma}}p\gamma\tag{I.19}$$

Applying Shephard's Lemma and using I.2, obtain

$$x_i\left(y,\mathbf{w}\right)=\gamma y\frac{\partial p\left(y,\mathbf{w}\right)}{\partial w_i}\tag{I.20}$$

5 Properties and identities concerning the inverse input demands

Remark 1 *For a given p, the non-conditional demand function is surjective on $\mathbb{R}^n_+$, i.e., for all $\mathbf{x} \in \mathbb{R}^n_+$, there exists a $\mathbf{w} \in \mathbb{R}^n_{++}$ such that $\mathbf{x}(p, \mathbf{w}) = \mathbf{x}$.*

In fact, let $\mathbf{x} \in \mathbb{R}^n_+$, then, since $f(\cdot)$ is a strictly concave and non decreasing function, $\nabla f(\mathbf{x}) \in \mathbb{R}^n_{++}$, $f(\mathbf{z}) - f(\mathbf{x}) \leq \nabla f(\mathbf{x}) \cdot (\mathbf{z} - \mathbf{x})$ and also $p(f(\mathbf{z}) - f(\mathbf{x})) \leq p\nabla f(\mathbf{x}) \cdot (\mathbf{z} - \mathbf{x})$, for all $\mathbf{z} \in \mathbb{R}^n_+$. It follows that $pf(\mathbf{x}) - p\nabla f(\mathbf{x}) \cdot \mathbf{x} \geq f(\mathbf{z}) - p\nabla f(\mathbf{x}) \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^n_+$, so at input prices $\mathbf{w} = p\nabla f(\mathbf{x}) \in \mathbb{R}^n_{++}$,

the input vector $\mathbf{x}$ maximizes profits, then $\mathbf{x}(p, \mathbf{w}) = \mathbf{x}$.

Remark 2 *By remark 1, $\mathbf{x}(p, p\nabla f(\mathbf{x})) = \mathbf{x}$, so $\mathbf{w}(p, \mathbf{x}) = p\nabla f(\mathbf{x})$, in particular, $\mathbf{w}(1, \mathbf{x}) = \nabla f(\mathbf{x})$. Note further that $\Pi(1, \nabla f(\mathbf{x})) = f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot \mathbf{x}$.*

As stated in the previous remark,

$$\mathbf{w}(p, \mathbf{x}) = p\nabla f(\mathbf{x}) \tag{I.21}$$

Proposition 3 *Let $\mathbf{s} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be defined by $\mathbf{s}(\mathbf{x}) = \arg\min_{\mathbf{s} \in \mathbb{R}^n_+} \{\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s})\}$. Then $\mathbf{w}(p, \mathbf{x}) = p\mathbf{s}(\mathbf{x})$ and, in particular, $\mathbf{w}(1, \mathbf{x}) = \mathbf{s}(\mathbf{x})$.*

This proposition states that the function $\mathbf{s}(\mathbf{x})$ defined following Madden [1987, p 349] is just the inverse input demand function evaluated in $p = 1$.

Proof. Let $\mathbf{s} = \arg\min_{\mathbf{s} \in \mathbb{R}^n_+} \{\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s})\}$, were the dependence on $\mathbf{x}$ has been ignored to simplify notation. It follows directly that $\mathbf{s}(\mathbf{x}) \mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s}) \leq \mathbf{v} \cdot \mathbf{x} + \Pi(1, \mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^n_{++}$. In particular, $\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s}) \leq \nabla f(\mathbf{x}) \cdot \mathbf{x} + \Pi(1, \nabla f(\mathbf{x}))$, but $\Pi(1, \nabla f(\mathbf{x})) = f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot \mathbf{x}$, as it was noted in remark 2, so $\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s}) \leq f(\mathbf{x})$, i.e, $\Pi(1, \mathbf{s}) \leq f(\mathbf{x}) - \mathbf{s} \cdot \mathbf{x}$. However, the definition of the profit function implies that, $\Pi(1, \mathbf{s}) \geq f(\mathbf{x}) - \mathbf{s} \cdot \mathbf{x}$, so $\Pi(1, \mathbf{s}) = f(\mathbf{x}) - \mathbf{s} \cdot \mathbf{x}$. It follows that $\mathbf{x}(1, \mathbf{s}) = \mathbf{x}$ and, as it will be shown later, the non conditional demands function is homogeneous of degree 0 in $(p, \mathbf{w})$, so $\mathbf{x}(p, p\mathbf{s}) = \mathbf{x}$ which implies that $\mathbf{w}(p, \mathbf{x}) = p\mathbf{s}$. ■

Remark 3 $\mathbf{s}(\mathbf{x}) = \nabla f(\mathbf{x})$. *This is an immediate consequence of remark 2 and proposition 3.*

If the production function is homogeneous, it can be easily recovered applying Euler´s theorem to the equation in the previous remark. In fact, $\gamma f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{x} = \mathbf{s}(\mathbf{x}) \cdot \mathbf{x}$, so

$$f(\mathbf{x}) = \frac{\mathbf{s}(\mathbf{x}) \cdot \mathbf{x}}{\gamma} = \frac{\mathbf{w}(p, \mathbf{x}) \cdot \mathbf{x}}{p\gamma} \tag{I.22}$$

If the production function is not homogeneous, it still can be recovered from another representation of the technology applying duality theory. Using such approach, as in Madden [1987, p.349 - p.351], an identity by means of which the production function can be obtained, will be derived next.

The production set $Y = \left\{(y, \mathbf{x}) \in \mathbb{R}_+^{n+1} : y \leq f(\mathbf{x})\right\}$, containing all the vectors of feasible inputs $\mathbf{x}$ and output levels of the single product y, can equivalently be described in terms of the profit function by:

$$Y = \left\{(y, \mathbf{x}) \in \mathbb{R}_+^{n+1} : py - \mathbf{w} \cdot \mathbf{x} \leq \Pi(p, \mathbf{w}), \forall (p, \mathbf{w}) \in \mathbb{R}_{++}^{n+1}\right\}$$

Since the profit function is homogeneous of degree one in p and $\mathbf{w}$, then $\Pi(p, \mathbf{w}) = p\Pi\left(1, \frac{\mathbf{w}}{p}\right)$, for all $p \in \mathbb{R}_{++}^{n+1}$. Let $\mathbf{s} = \frac{\mathbf{w}}{p}$, so $\Pi(p, \mathbf{w}) = p\Pi(1, \mathbf{s})$, and clearly $py - \mathbf{w} \cdot \mathbf{x} \leq \Pi(p, \mathbf{w})$ if and only if $y \leq \mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s})$. The production set can now be expressed as $Y = \left\{(y, \mathbf{x}) \in \mathbb{R}_+^{n+1} : y \leq \mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s}), \forall \mathbf{s} \in \mathbb{R}_{++}^{n+1}\right\}$ and, if $\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s})$ attains a minimum, then

$$Y = \left\{(y, \mathbf{x}) \in \mathbb{R}_+^{n+1} : y \leq \min_{\mathbf{s} \in \mathbb{R}_{++}^{n+1}} \{\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s})\}\right\} \tag{14}$$

Note that $\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s})$ is a convex function in $\mathbf{s}$, so it has a minimum at $\mathbf{s}^*$ if and only if $x_i + \frac{\partial \Pi(1, \mathbf{s}^*)}{\partial s_i} = 0$, for all $i = 1, ..., n$. If the profit function is differentiable in the input prices, Hotelling's Lemma implies that $x_i = x_i(1, \mathbf{s}^*)$, for all $i = 1, ..., n$, where $x_i(1, \mathbf{s}^*)$ is the non conditional input demand function for input i evaluated at $(1, \mathbf{s}^*)$. In other words for a given $\mathbf{x} \in \mathbb{R}_+^n$, the function $\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s})$ attains a minimum, if any, at the vector $\mathbf{s}^*$ that solves this system of n equations. Note further that the previous conclusion is just a restatement of proposition 3.

By the definition of the functions involved, it follows that, $\Pi(1, \mathbf{s}) = y(1, \mathbf{s}) - \mathbf{s} \cdot \mathbf{x}(1, \mathbf{s})$ for all $\mathbf{s} \in \mathbb{R}_{++}^n$, in particular, $y(1, \mathbf{s}^*) = \mathbf{s}^* \cdot \mathbf{x} + \Pi(1, \mathbf{s}^*)$ then, replacing in (14) we get $Y = \left\{(y, \mathbf{x}) \in \mathbb{R}_+^{n+1} : y \leq y(1, \mathbf{s}^*)\right\}$. But we know that $Y = \left\{(y, \mathbf{x}) \in \mathbb{R}_+^{n+1} : y \leq f(\mathbf{x})\right\}$, so $f(\mathbf{x}) = y(1, \mathbf{s}^*)$. Finally, if $\mathbf{s}^*$ is expressed as a function of $\mathbf{x}$, i,e, if we let $\mathbf{s}(\mathbf{x}) = \arg\min_{\mathbf{s} \in \mathbb{R}_+^{\mathbf{n}}} \{\mathbf{s} \cdot \mathbf{x} + \Pi(1, \mathbf{s})\}$ as in proposition 3, then the following identity holds

$$f(\mathbf{x}) = y(1, \mathbf{s}(\mathbf{x})) = y(p, w(p, \mathbf{x})) \tag{I.23}$$

Exploiting the duality between production and costs, the inverse input demands and the cost functions can also be related.

Given a cost function $c(y, \mathbf{w})$, the production set can be expressed as:

$$Y = \left\{ (y, \mathbf{x}) \in \mathbb{R}^{n+1}_{+} : \mathbf{w} \cdot \mathbf{x} \geq c\,(y, \mathbf{w}) \;\; \forall \mathbf{w} \in \mathbb{R}^{n}_{++} \right\}$$

If it is assumed that the production function that is trying to be recovered is homogeneous of degree γ, the corresponding cost function should be homogeneous of degree $\frac{1}{\gamma}$ in y. In such case, $c\,(y, \mathbf{w}) = y^{\frac{1}{\gamma}} c\,(1, \mathbf{w})$ and so,

$$Y = \left\{ (y, \mathbf{x}) \in \mathbb{R}^{n+1}_{+} : \left(\frac{\mathbf{w} \cdot \mathbf{x}}{c\,(1, \mathbf{w})} \right)^{\gamma} \geq y \;\forall \mathbf{w} \in \mathbb{R}^{n}_{++} \right\}$$

Remark 4 *By definition* $Y = \left\{ (y, \mathbf{x}) \in \mathbb{R}^{n+1}_{+} : y \leq f\,(\mathbf{x}) \right\}$*, so using the previous representation of the production set it is easy to prove that* $f\,(\mathbf{x}) \leq \left(\frac{\mathbf{w} \cdot \mathbf{x}}{C(1,\mathbf{w})} \right)^{\gamma}$*, for all* $\mathbf{w} \in \mathbb{R}^{n}_{++}$*.*

Proposition 4 $\left(\frac{\mathbf{s}(\mathbf{x}) \cdot \mathbf{x}}{c(1,\mathbf{s}(\mathbf{x}))} \right)^{\gamma} \leq \left(\frac{\mathbf{w} \cdot \mathbf{x}}{c(1,\mathbf{w})} \right)^{\gamma}$ *, for all* $\mathbf{w} \in \mathbb{R}^{n}_{++}$

Proof. Equation I.23 states that $f\,(\mathbf{x}) = y\,(1, \mathbf{s}\,(\mathbf{x}))$ so, if $p = 1$, $\mathbf{w} = \mathbf{s}\,(\mathbf{x})$ and $y = f\,(\mathbf{x})$, I.1' implies that $c\,(f\,(\mathbf{x}), \mathbf{s}\,(\mathbf{x})) = [y\,(1, \mathbf{s}\,(\mathbf{x}))]^{\frac{\gamma-1}{\gamma}} \left[f\,(\mathbf{x})^{\frac{1}{\gamma}} \gamma \right] = \gamma f\,(\mathbf{x})$. Applying remark 3 and Euler's theorem, it follows that $c\,(f\,(\mathbf{x}), \mathbf{s}\,(\mathbf{x})) = \gamma f\,(\mathbf{x}) = \nabla f\,(\mathbf{x}) \cdot \mathbf{x} = \mathbf{s}\,(\mathbf{x}) \cdot \mathbf{x}$, but, since the cost function is homogeneous of degree $\frac{1}{\gamma}$ in y, $f\,(\mathbf{x})^{\frac{1}{\gamma}}\, c\,(1, \mathbf{s}\,(\mathbf{x})) = c\,(f\,(\mathbf{x}), \mathbf{s}\,(\mathbf{x})) = \mathbf{s}\,(\mathbf{x}) \cdot \mathbf{x}$, and it follows immediately that:

$$\left(\frac{\mathbf{s}\,(\mathbf{x}) \cdot \mathbf{x}}{c\,(1, \mathbf{s}\,(\mathbf{x}))} \right)^{\gamma} = f\,(\mathbf{x})$$

then, by remark 4, $\left(\frac{\mathbf{s}(\mathbf{x}) \cdot \mathbf{x}}{c(1,\mathbf{s}(\mathbf{x}))} \right)^{\gamma} \leq \left(\frac{\mathbf{w} \cdot \mathbf{x}}{c(1,\mathbf{w})} \right)^{\gamma}$, for all $\mathbf{w} \in \mathbb{R}^{n}_{++}$.

Note that proposition 4 states that, for a given $\mathbf{x} \in \mathbb{R}^{n}_{+}$, $\left(\frac{\mathbf{w} \cdot \mathbf{x}}{c(1,\mathbf{w})} \right)^{\gamma}$, thought as a function of $\mathbf{w}$, attains a minimum at $\mathbf{s}\,(\mathbf{x}) \in \mathbb{R}^{n}_{++}$. However, this minimum is not unique, in fact $\left(\frac{(t\mathbf{w}) \cdot \mathbf{x}}{c(1,(t\mathbf{w}))} \right)^{\gamma} = \left(\frac{t\mathbf{w} \cdot \mathbf{x}}{tc(1,(\mathbf{w}))} \right)^{\gamma} = \left(\frac{\mathbf{w} \cdot \mathbf{x}}{c(1,(\mathbf{w}))} \right)^{\gamma}$, for all $t > 0$ and $\mathbf{w} \in \mathbb{R}^{n}_{++}$, because the cost function is homogeneous of degree 1 in $\mathbf{w}$, as it will be shown in the next section, so $\left(\frac{\mathbf{s}(\mathbf{x}) \cdot \mathbf{x}}{c(1,\mathbf{s}(\mathbf{x}))} \right)^{\gamma} = \left(\frac{(t\mathbf{s}(\mathbf{x})) \cdot \mathbf{x}}{c(1,t\mathbf{s}(\mathbf{x}))} \right)^{\gamma}$ and then, $\left(\frac{\mathbf{w} \cdot \mathbf{x}}{c(1,\mathbf{w})} \right)^{\gamma}$ also attains a minimum at $t\mathbf{s}\,(\mathbf{x})$ for all $t > 0$. ∎

6 Homogeneity Properties of Representations of a Technology

Corollary 2 *If the production function* $f : \mathbb{R}^{n}_{+} \rightarrow \mathbb{R}_{+}$ *is a homogenous function of degree* $0 < \gamma < 1$*, and the cost and profit functions are differentiable in their parameters, the properties included in table 1 follow from theorem 1:*

Table 1

Homogeneity Properties of the Representations of a Technology Considered in this Paper

R.O.T		$\mathbf{w}$ or $\mathbf{x}$	y	p	y and $\mathbf{w}$ p and $\mathbf{w}$ p and $\mathbf{x}$
Cost Function	$c(y,\mathbf{w})$	1	$\frac{1}{\gamma}$		$\frac{1+\gamma}{\gamma}$
Conditional Input Demands	$\mathbf{x}(y,\mathbf{w})$	0	$\frac{1}{\gamma}$		$\frac{1}{\gamma}$
Profit Function	$\Pi(p,\mathbf{w})$	$\frac{\gamma}{\gamma-1}$		$\frac{1}{1-\gamma}$	1
Non Conditional Input Demands	$\mathbf{x}(p,\mathbf{w})$	$\frac{1}{\gamma-1}$		$\frac{1}{1-\gamma}$	0
Supply Function	$y(p,\mathbf{w})$	$\frac{\gamma}{\gamma-1}$		$\frac{\gamma}{1-\gamma}$	0
Inverse Supply Function	$p(y,\mathbf{w})$	1	$\frac{1-\gamma}{\gamma}$		$\frac{1}{\gamma}$
Non Conditional Cost Function	$c(p,\mathbf{w})$	$\frac{\gamma}{\gamma-1}$		$\frac{1}{1-\gamma}$	1
Inverse Input Demands	$\mathbf{w}(p,\mathbf{x})$	$\gamma-1$		1	γ

Proof. It is a well known fact that the cost function and the conditional demands are homogeneous of degree 1 and 0 in $\mathbf{w}$, respectively (by Shephard´s Lemma). On the other hand, the degree of homogeneity of the cost funtion in y was previosly stated in proposition 1 and, again by Shephard´s Lemma, it follows that $x_i(ty,\mathbf{w}) = \frac{\partial c(ty,\mathbf{w})}{\partial w_i} = t^{\frac{1}{\gamma}} \frac{\partial c(y,\mathbf{w})}{\partial w_i} = t^{\frac{1}{\gamma}} x_i(y,\mathbf{w})$.

The degree of homogeneity of the inverse supply function in both its arguments follows directly from the corresponding properties of the cost function and I.2. Similarly, the degree of homogeneity in $\mathbf{w}$ of the profit function, the supply function and the non conditional cost function can be obtained applying identities I.10, I.1 and I.7, respectively. Then, equation (13) can be applied to derive such property for the non conditional input demands, since homogeneity is already known for the supply function and the conditional demands.

If the profit function is differentiable in p, Hotelling‘s lemma states that $\frac{\partial \Pi(p,\mathbf{w})}{\partial p} = y(p,\mathbf{w})$, and then, by I.3, $\frac{\partial \Pi(p,\mathbf{w})}{\partial p} = \frac{\Pi(p,\mathbf{w})}{[1-\gamma]p}$. Applying Euler‘s theorem, It follows that the profit function is homogeneous of degree $\frac{1}{1-\gamma}$ in p, and then, the corresponding properties of the non conditional input demands, the supply function and the non conditional cost function are a direct consequence of identities I.13, I.3 and I.8, respectively [8]

Finally, the degrees of homogeneity of the inverse supply function follow directly from I.21: $\mathbf{w}(p,\mathbf{x}) = p\nabla f(\mathbf{x})$. ■

[8] Note that if $f(\mathbf{x},\mathbf{y}) : \mathbb{R}_+^{n+m} \to \mathbb{R}_+$, $f(t\mathbf{x},\mathbf{y}) = t^{\alpha}\ f(\mathbf{x},\mathbf{y})$ and $f(\mathbf{x},t\mathbf{y}) = t^{\beta} f(\mathbf{x},\mathbf{y})$, then $f(t\mathbf{x},t\mathbf{y}) = t^{\alpha+\beta}\ f(\mathbf{x},\mathbf{y})$.
For that reason, the last column of table 1 is the sum of the three previous columns., and the conditional input demands and inverse supply functions are homogeneous in y of a degree equal to the degree of the cost function in y minus one, while the non conditional input demand and supply functions are homogeneous in w and p respectively of a degree equal to the degree of the profit function in w and p respectively, minus one.

7 MATRIX OF IDENTITIES

The identities presented in theorem 1, corollary 1 and the remarks regarding the inverse input demands, can be summarized using the matrix of identities shown in table 2, while a matrix that summarizes these results in terms of the explicit functional forms is included in table A.1 of Appendix 1.

Table 2

Summarized Matrix of Identities for Theorem 1

R.O.T.	$f(\mathbf{x})$	$c(y,\mathbf{w})$	$\mathbf{x}(y,\mathbf{w})$	$\Pi(p,\mathbf{w})$	$\mathbf{x}(p,\mathbf{w})$	$y(p,\mathbf{w})$	$p(y,\mathbf{w})$	$c(p,\mathbf{w})$	$\mathbf{w}(p,\mathbf{x})$
$f(\mathbf{x})$	/					**I.23**			**I.22**
$c(y,\mathbf{w})$		/	**(3)**	**I.10**	**I.7**	**I.1'**	**I.2**	**I.7**	
$\mathbf{x}(y,\mathbf{w})$		**I.17**	/	**I.18**	**I.12**	**I.19**	**I.20**	**I.12**	
$\Pi(p,\mathbf{w})$		**I.10**	**I.10**	/	**I.8**	**I.3**	**I.5**	**I.8**	
$\mathbf{x}(p,\mathbf{w})$		**I.14**	**I.12, I.14**	**I.13**	/	**I.15**	**I.16**		
$y(p,\mathbf{w})$	**(7)**	**I.1'**	**I.1'**	**I.3**	**(7)**	/	**I.4**	**I.6**	
$p(y,\mathbf{w})$		**I.2**	**I.2**	**I.5**	**I.9**	**I.4**	/	**I.9**	
$c(p,\mathbf{w})$		**I.7**	**I.7**	**I.8**	**(8)**	**I.6**	**I.9**	/	
$\mathbf{w}(p,\mathbf{x})$	**I.21**								/

Note that these matrices (Table 2 and Table A.1) show how to obtain the functions that are on the vertical axis at the left, using the functions that appear on the horizontal axis at the top. Note also that these matrices are not symmetrical, since identities are sometimes bidirectional -such as identity I.10 and sometimes are unidirectional -such as identity I.13. These representations and identities can be useful in econometric applications, since they provide a tool to obtain explicit functional forms of technologies from observable data on a range of variables. They can also be useful to simplify computational procedures when different representations of a technology are required.

8 CONCLUSIONS

In this paper, one to one identities that allow to shift between most of up to nine different ways of representing a homogeneous technology, were derived. The homogeneity properties of those representations of a technology have also been outlined. These results, which have been summarized using matrices and tables, can be useful in econometric estimations, and to simplify computational procedures when different representations of a technology are required.

Further work on this topic could focus on generalizing the results presented here, for example, to multi-output and non-homogeneous technologies.

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Appendix

MATRIX OF IDENTITIES

This appendix presents the matrix of the identities proposed and proved in this paper in terms of explicit functional forms of the representations of the technology.

Table A.1.

Summarized Matrix of Identities

R.O.T.	$f(\mathbf{x})$	$c(y,\mathbf{w})$	$\mathbf{x}(y,\mathbf{w})$
$f(\mathbf{x})$	/		
$c(y,\mathbf{w})$		/	$c(y,\mathbf{w}) = \mathbf{w} \cdot \mathbf{x}(y,\mathbf{w})$
$\mathbf{x}(y,\mathbf{w})$		$\frac{\partial c(y,\mathbf{w})}{\partial w_i}$	/
$\Pi(p,\mathbf{w})$		$\left[\frac{c(y,\mathbf{w})}{(yp)^{\frac{1}{\gamma}}\gamma}\right]^{\frac{\gamma}{\gamma-1}}(1-\gamma)$	$\left[\frac{\mathbf{w}\cdot\mathbf{x}(y,\mathbf{w})}{(yp)^{\frac{1}{\gamma}}\gamma}\right]^{\frac{\gamma}{\gamma-1}}(1-\gamma)$
$\mathbf{x}(p,\mathbf{w})$		$-\frac{\partial[c(y,\mathbf{w})]^{\frac{\gamma}{\gamma-1}}}{\partial w_i}\frac{(1-\gamma)}{\left[(yp)^{\frac{1}{\gamma}}\gamma\right]^{\frac{\gamma}{\gamma-1}}}$	$x_i\left(\left[\frac{\mathbf{w}\cdot\mathbf{x}(y,\mathbf{w})}{\left[y^{\frac{1}{\gamma}}p\gamma\right]}\right]^{\frac{\gamma}{\gamma-1}},\mathbf{w}\right); -\frac{\partial[\mathbf{w}\cdot\mathbf{x}(y,\mathbf{w})]^{\frac{\gamma}{\gamma-1}}}{\partial w_i}\frac{(1-\gamma)}{\left[(yp)^{\frac{1}{\gamma}}\gamma\right]^{\frac{\gamma}{\gamma-1}}}$
$y(p,\mathbf{w})$	$f(\mathbf{x}(p,\mathbf{w}))$	$\left[\frac{c(y,\mathbf{w})}{y^{\frac{1}{\gamma}}p\gamma}\right]^{\frac{\gamma}{\gamma-1}}$	$\left[\frac{\mathbf{w}\cdot\mathbf{x}(y,\mathbf{w})}{y^{\frac{1}{\gamma}}p\gamma}\right]^{\frac{\gamma}{\gamma-1}}$
$p(y,\mathbf{w})$		$\frac{c(y,\mathbf{w})}{\gamma y}$	$\frac{\mathbf{w}\cdot\mathbf{x}(y,\mathbf{w})}{\gamma y}$
$c(p,\mathbf{w})$		$[c(y,\mathbf{w})]^{\frac{\gamma}{\gamma-1}}(yp\gamma)^{\frac{1}{1-\gamma}}$	$[\mathbf{w}\cdot\mathbf{x}(y,\mathbf{w})]^{\frac{\gamma}{\gamma-1}}(yp\gamma)^{\frac{1}{1-\gamma}}$
$\mathbf{w}(p,\mathbf{x})$	$p\nabla f(\mathbf{x})$		

R.O.T.	$\Pi(p,\mathbf{w})$	$\mathbf{x}(p,\mathbf{w})$	$y(p,\mathbf{w})$
$f(\mathbf{x})$			$y(p,w(p,\mathbf{x}))$
$c(y,\mathbf{w})$	$\left[\frac{\Pi(p,\mathbf{w})}{1-\gamma}\right]^{\frac{\gamma-1}{\gamma}}(yp)^{\frac{1}{\gamma}}\gamma$	$\mathbf{w}\cdot\mathbf{x}(p,\mathbf{w})\left[(p\gamma y)^{\frac{1}{\gamma}}\right]$	$[y(p,\mathbf{w})]^{\frac{\gamma-1}{\gamma}}y^{\frac{1}{\gamma}}p\gamma$
$\mathbf{x}(y,\mathbf{w})$	$\frac{\partial\left[\Pi(p,\mathbf{w})^{\frac{\gamma-1}{\gamma}}\right]}{\partial w_i}\frac{(yp)^{\frac{1}{\gamma}}\gamma}{(1-\gamma)^{\frac{\gamma-1}{\gamma}}}$	$x_i\left(\left[\frac{\mathbf{w}\cdot\mathbf{x}(p,\mathbf{w})}{yp^{\frac{1}{1-\gamma}}\gamma}\right]^{\frac{\gamma}{\gamma-1}},\mathbf{w}\right)$	$\frac{\partial\left[y(p,\mathbf{w})^{\frac{\gamma-1}{\gamma}}\right]}{\partial w_i}y^{\frac{1}{\gamma}}p\gamma;$
$\Pi(p,\mathbf{w})$	/	$\frac{\mathbf{w}\cdot\mathbf{x}(p,\mathbf{w})[1-\gamma]}{\gamma}$	$y(p,\mathbf{w})[1-\gamma]p$
$\mathbf{x}(p,\mathbf{w})$	$-\frac{\partial\Pi(p,\mathbf{w})}{\partial w_i}$	/	$\frac{\partial y(p,\mathbf{w})}{\partial w_i}[1-\gamma]p$
$y(p,\mathbf{w})$	$\frac{\Pi(p,\mathbf{w})}{[1-\gamma]p}$	$\frac{\mathbf{w}\cdot\mathbf{x}(p,\mathbf{w})}{p\gamma}$	/
$p(y,\mathbf{w})$	$\left[\frac{\Pi(p,\mathbf{w})}{yp^{\frac{\gamma}{1-\gamma}}[1-\gamma]}\right]^{\frac{\gamma}{\gamma-1}}$	$\left[\frac{\mathbf{w}\cdot\mathbf{x}(p,\mathbf{w})}{yp^{\frac{1}{1-\gamma}}\gamma}\right]^{\frac{\gamma}{\gamma-1}}$	$\left[\frac{y(p,\mathbf{w})}{y}\right]^{\frac{\gamma-1}{\gamma}}p$
$c(p,\mathbf{w})$	$\frac{\Pi(p,\mathbf{w})\gamma}{1-\gamma}$	$\mathbf{w}\cdot\mathbf{x}(p,\mathbf{w})$	$y(p,\mathbf{w})p\gamma$
$\mathbf{w}(p,\mathbf{x})$			

R.O.T.	$p(y,\mathbf{w})$	$c(p,\mathbf{w})$	$\mathbf{w}(p,\mathbf{x})$
$f(\mathbf{x})$			$\frac{\mathbf{w}(p,\mathbf{x})\cdot\mathbf{x}}{p\gamma}$
$c(y,\mathbf{w})$	$\gamma yp(y,\mathbf{w})$	$c(p,\mathbf{w})\left[(p\gamma y)^{\frac{1}{\gamma}}\right]$	
$\mathbf{x}(y,\mathbf{w})$	$\gamma y\frac{\partial p(y,\mathbf{w})}{\partial w_i}$	$x_i\left(\left[\frac{c(p,\mathbf{w})}{yp^{\frac{1}{1-\gamma}}\gamma}\right]^{\frac{\gamma}{\gamma-1}},\mathbf{w}\right)$	
$\Pi(p,\mathbf{w})$	$[p(y,\mathbf{w})]^{\frac{\gamma-1}{\gamma}}yp^{\frac{\gamma}{1-\gamma}}[1-\gamma]$	$\frac{c(p,\mathbf{w})[1-\gamma]}{\gamma}$	
$\mathbf{x}(p,\mathbf{w})$	$\frac{\partial\left[p(y,\mathbf{w})^{\frac{\gamma}{\gamma-1}}\right]}{\partial w_i}yp^{\frac{1}{1-\gamma}}[1-\gamma]$		
$y(p,\mathbf{w})$	$y\left[\frac{p(y,\mathbf{w})}{p}\right]^{\frac{\gamma}{\gamma-1}}$	$\frac{c(p,\mathbf{w})}{p\gamma}$	
$p(y,\mathbf{w})$	/	$\left[\frac{c(p,\mathbf{w})}{yp^{\frac{1}{1-\gamma}}\gamma}\right]^{\frac{\gamma}{\gamma-1}}$	
$c(p,\mathbf{w})$	$[p(y,\mathbf{w})]^{\frac{\gamma-1}{\gamma}}yp^{\frac{1}{1-\gamma}}\gamma$	/	
$\mathbf{w}(p,\mathbf{x})$			/